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Stability and Perturbations of Generalized Heteroclinic Loops in Piecewise Smooth Systems

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Abstract We investigate a class of planar piecewise smooth systems with a generalized heteroclinic loop (a closed curve composed of hyperbolic saddle points, generalized singular points and regular orbits). We give conditions for the stability of the generalized heteroclinic loop and provide some sufficient conditions for the maximum number of limit cycles that bifurcate from the heteroclinic connection. The discussions rely on the approximation of the Poincaré map, which is constructed near the generalized heteroclinic loop. To obtain it, we introduce the Dulac map and use Melnikov method. By analyzing the fixed point of the Poincaré map, we get the number of limit cycles, which can be produced from the generalized heteroclinic loop. As applications to our theories, we give an example to show that two limit cycles can appear.

Keywords Bifurcation · Piecewise smooth system · Limit cycle · Generalized heteroclinic loop

Mathematics Subject Classification 34A36 · 34C05 · 34C37

1 Introduction

Piecewise smooth (abbreviated as PWS) systems are frequently encountered in applied science and engineering, such as control theory, mechanical engineering, power electronic circuits and so on (see for instance [4,8,14,19,30] and the references given

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there). For the details about the fundamental theory of PWS systems, we refer to the monographs [9,13,22,27].

As known in [2,6], studying bifurcations of limit cycles is one of main problems in smooth systems. This problem in PWS systems has attracted considerable attentions in the past tens of years. Hopf bifurcation and periodic bifurcation for Filippov systems have been studied in [1,7,11,12,18,23,24]. Homoclinic bifurcation for PWS systems was investigated in [5,23,24]. Attentions also have been paid to limit cycles bifurcate from heteroclinic loops. In this respect, bifurcations of limit cycles in planar PWS systems with two zones, which are separated by a straight line and contain two real saddles in each zones, were studied, for example, in [3,20,26] for PWS linear systems, in [15] for PWS Hamiltonian systems. Recently, heteroclinic bifurcation in PWS systems with multiple zones was considered in [25,29]. Liang, Han and Zhang [24] in 2013 once studied bifurcation of limit cycles from generalized homoclinic loops, which have generalized singular points on the switching manifold. We remark that generalized singular points are also referred to as sliding points (see for instance [9,13]). In this work, we are concerned with a planar PWS system defined in two domains which are separated by a switching manifold and assume that the system has a generalized heteroclinic loop, which has a real saddle in one subsystem and a generalized singular point on the switching manifold.

More precisely, we assume that the plane is divided into

$$\Omega_+ = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0\},$$
$$\Omega_- = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 < 0\}$$

with the switching maniflold

$$\Omega_0 = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 = 0\}.$$

Consider a planar PWS system in the form

$$\dot{x} = f_+(x) + \varepsilon g_+(x) := F_+(x, \varepsilon), \quad x \in \Omega_+, \tag{1a}$$
$$\dot{x} = f_-(x) + \varepsilon g_-(x) := F_-(x, \varepsilon), \quad x \in \Omega_-, \tag{1b}$$

where

$$f_\pm(x) = \begin{pmatrix} p^\pm(x) \\ q^\pm(x) \end{pmatrix}, \quad g_\pm(x) = \begin{pmatrix} j^\pm(x) \\ k^\pm(x) \end{pmatrix}$$

with $p^\pm, q^\pm, j^\pm, k^\pm \in C^n(\Omega_\pm \cup \Omega_0, \mathbb{R}^2)$ $(n \geq 3)$ and $|\varepsilon| < \varepsilon_0 \ll 1$ for some $\varepsilon_0 > 0$. When $\varepsilon = 0$, the system is reduced to

$$\dot{x} = f_+(x), \quad x \in \Omega_+, \tag{2a}$$
$$\dot{x} = f_-(x), \quad x \in \Omega_-. \tag{2b}$$

As indicated in [18], the generalized singular point of system (2) is a point $P \in \Omega_0$ satisfying $q^+(P)q^-(P) \leq 0$. We define a closed curve Γ as a generalized heteroclinic

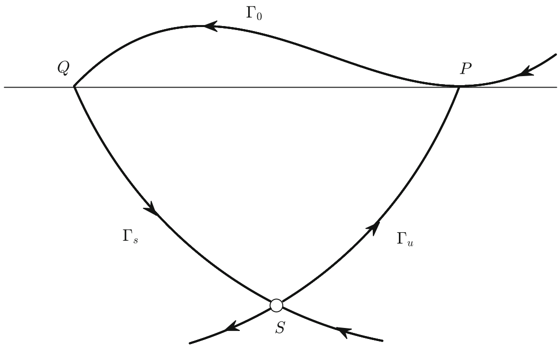

Fig. 1 The generalized heteroclinic loop Γ of the unperturbed system (2)

loop if it consists of at least one generalized singular point on Ω_0 and one singular point in $\Omega_\pm$. We assume system (2) satisfies the following hypotheses:

(H) The system (2) has a counterclockwise generalized heteroclinic loop Γ, which has a hyperbolic saddle point $S \in \Omega_-$ and a generalized singular point $P \in \Omega_0$ satisfying

$$p^+(P) < 0, \quad q^+(P) = 0, \quad q^+_{x_1}(P) < 0, \quad q^-(P) > 0. \tag{3}$$

Besides the point P, the generalized heteroclinic loop Γ intersects Ω_0 at the other point Q, which is a crossing point satisfying $q^+(Q) < 0$ and $q^-(Q) < 0$. See Fig. 1.

Without loss of generality, in what follows we always assume that the generalized singular point P is at the origin point.

The aim of this paper is to study the stability and perturbations of the generalized heteroclinic loop Γ. Among various approaches for determining the number of limit cycles bifurcate from periodic orbits, homoclinic loops or heteroclinic loops in smooth systems, the Melnikov method is treated as one of the efficient techniques (see [16,28]). In the recent decades, the efforts in extending this method to PWS system have been made, see for instance in [5,12,15,22,24,25,29]. We will use the Melnikov method and introduce the Dulac map in a small neighborhood of S to estimate the Poincaré map, from which we get stability of Γ. By analyzing zeros of the successor function of the perturbed system in the neighborhood of Γ, which will be defined in Sect. 3, we can obtain the number of limit cycles bifurcate from the generalized heteroclinic loop Γ.

The present paper is built up as follows. Some necessary preparations are presented in Sect. 2. We construct the Poincaré map near the generalized heteroclinic loop and give conditions for stability of it in Sect. 3. Section 4 is devoted to perturbations of the generalized heteroclinic loop. As applications of main results, an example is given in the final section.

2 Preliminaries

We firstly introduce some notations used repeatedly below. Let $\langle a, b\rangle = a^T b$, $\|a\| = \sqrt{\langle a, a\rangle}$, $a \wedge b = a_1 b_2 - a_2 b_1$ and $a^\perp = (-a_2, a_1)^T$ for $a = (a_1, a_2)^T$, $b = (b_1, b_2)^T$ in $\mathbb{R}^2$. The vector n is given by $n = (0, 1)^T$. div X and DX respectively denote the divergence and the Jacobian matrix of a smooth vector field $X(x) = (X_1(x), X_2(x))^T$.

Let real constants λ^- and λ^+ with $\lambda^- < 0 < \lambda^+$ be the eigenvalues of the matrix $Df_-(S)$ and $\lambda_0 = -\lambda^-/\lambda^+ > 0$. The stable and unstable manifolds of the hyperbolic saddle point S are respectively denoted by Γ_s and Γ_u, the branch of Γ in Ω_+ is defined by Γ_0. See Fig. 1. As the solution defined in [13,18], we define $\Gamma_s := \{\gamma_s(t) : t \in [t_Q, +\infty)\}$, $\Gamma_u := \{\gamma_u(t) : t \in (-\infty, t_P)\}$ and $\Gamma_0 := \{\gamma_0(t) : t \in (t_P, t_Q]\}$, furthermore, $\gamma_s(t_Q) = \gamma_0(t_Q) = Q$, $\lim_{t\to t_P^-} \gamma_u(t) = P$ and $\lim_{t\to t_P^+} \gamma_0(t) = P$. Note that $S \in \Omega_-$ is a hyperbolic saddle point, then subsystem (1b) has a hyperbolic saddle point S_ε near the point S for sufficiently small $|\varepsilon|$. Let $\lambda^-(\varepsilon)$ and $\lambda^+(\varepsilon)$ with $\lambda^-(\varepsilon) < 0 < \lambda^+(\varepsilon)$ be the eigenvalues of the matrix $DF_-(S_\varepsilon, \varepsilon)$. Under the perturbation, Γ_s (resp., Γ_u) becomes Γ_s^ε (resp., Γ_u^ε), the stable (resp., unstable) manifold of S_ε.

In order to approximate the Poincaré map, which will be constructed below, we introduce the Dulac map in the following. As known in [21], there exists a local C^{n-1} diffeomorphism T_ε, which transforms subsystem (1b) into the following normal form:

$$\dot{u} = \lambda^+(\varepsilon)u(1 + h_1(u, v, \varepsilon)), \quad \dot{v} = \lambda^-(\varepsilon)v(1 + h_2(u, v, \varepsilon)), \tag{4}$$

where $h_i(u, v, \varepsilon) = uvh_{i0}(u, v, \varepsilon)$ with $h_{i0} \in C^{n-2}$ for $i = 1, 2$. For sufficiently small ρ, we take two sections of form

$$l_1' = \{(u, v)^T \mid v = \rho,\ 0 \le u \le \rho\}, \quad l_2' = \{(u, v)^T \mid u = \rho,\ 0 \le v \le \rho\},$$

then the flow of system (4) induces the Dulac map $\mathcal{D}_0 := \mathcal{D}_0(\cdot\,, \varepsilon)$ from l_1' to l_2': $\mathcal{D}_0(\cdot\,, \varepsilon) : [0, \rho] \to [0, \rho]$. See Fig. 2.

Let the sections l_1 and l_2 cross Γ_s and Γ_u at the points $A_s := T_0^{-1}((0, \rho)^T)$ and $B_u := T_0^{-1}((\rho, 0)^T)$ along the vectors

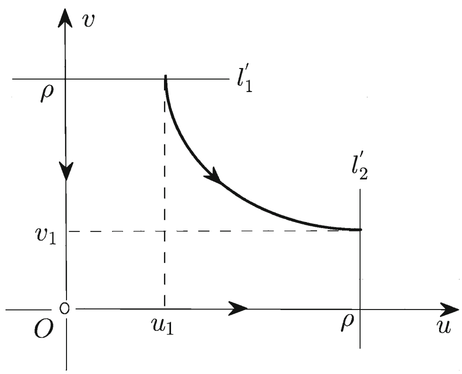

Fig. 2 The Dulac map $\mathcal{D}_0$ of system (4) near the hyperbolic sadddle

$$n_{A_s} := \frac{f_-^{\perp}(A_s)}{||f_-(A_s)||}, \quad n_{B_u} := \frac{f_-^{\perp}(B_u)}{||f_-(B_u)||},$$

respectively. Then for sufficiently small $|\varepsilon|$, the section l_1 (resp., l_2) can intersect Γ_s^ε (resp., Γ_u^ε) transversally at $A_s^\varepsilon := A_s + a^s(\varepsilon)n_{A_s}$ (resp., $B_u^\varepsilon := B_u + b^u(\varepsilon)n_{B_u}$). Given that $A := A_s + an_{A_s}$ for some small $a > a^s(\varepsilon)$, then the flow of subsystem (1b) from A crosses l_2 at $B := B_u + bn_{B_u}$, which induces the Dulac map $\mathcal{D} := \mathcal{D}(\cdot, \varepsilon)$, that is, $b = \mathcal{D}(a, \varepsilon)$. The expression of the Dulac map $\mathcal{D}$ can be obtained by [17, Lemma 3.5, p.302] and [17, Lemma 3.8, p.308]. The compendium of them is shown in the following lemma.

Lemma 1 ([17]) *Let the notations be given above. Then for sufficiently small $|\varepsilon|$, the following assertions hold:*

(i) Let $\lambda(\varepsilon) = -\lambda^-(\varepsilon)/\lambda^+(\varepsilon) > 0$. Then for any $k \in (0, \lambda_0/(1 + \lambda_0))$, we have

$$\mathcal{D}_0(u, \varepsilon) = \rho^{1-\lambda(\varepsilon)} u^{\lambda(\varepsilon)}(1 + \varphi_0(u, \varepsilon)),$$
$$\frac{\partial \mathcal{D}_0}{\partial u}(u, \varepsilon) = \lambda(\varepsilon)\rho^{1-\lambda(\varepsilon)} u^{\lambda(\varepsilon)-1}(1 + \varphi_1(u, \varepsilon)),$$

where $\varphi_0(u, \varepsilon) = o(u^k)$, $\varphi_1(u, \varepsilon) = o(u^k)$ and $u\frac{\partial \varphi_0}{\partial u} = o(u^k)$.

(ii) Let $\beta_1 = \|T_0^{-1}((1, 0)^T) - P_0\|$, $\beta_2 = \|T_0^{-1}((0, 1)^T) - P_0\|$. Then there exist C^{n-1} functions

$$W_1(u, \rho, \varepsilon) = a^s(\varepsilon) + M_1(\rho, \varepsilon)u + O(u^2),$$
$$W_2(u, \rho, \varepsilon) = b^u(\varepsilon) + M_2(\rho, \varepsilon)u + O(u^2),$$

such that if $a(\varepsilon) = W_1(u, \rho, \varepsilon)$, then $\mathcal{D}(a(\varepsilon), \varepsilon) = W_2(\mathcal{D}_0(u, \varepsilon), \rho, \varepsilon)$, furthermore, $M_i(\rho, 0) \to \beta_i \sin\theta$ as $\rho \to 0$, $i = 1, 2$, where θ is the angle between eigenvectors of λ^+ and λ^-.

3 Stability of Generalized Heteroclinic Loops

To get the stability of the generalized heterocinic loop Γ with the assumption **(H)**, we will construct the Poincaré map near Γ and analyze the properties of the successive function. Precisely, we take l_1 as the Poincaré section, where l_1 is the same as defined in Sect. 2. Let $A = A_s + \delta n_{A_s} \in l_1$ for sufficiently small $\delta > 0$. Then the flow of subsystem (1b) starting from A intersects l_2 at $B := B_u + \mathcal{D}(\delta, 0)n_{B_u}$. As stated in Lemma 1, we can have the fact that if δ is small, so is $\mathcal{D}(\delta, 0)$. Then by the continuous dependency on initial values, for sufficiently small δ the flow starting from B intersects the switching manifold Ω_0 at C and D successively. The flow returns to the Poincaré section l_1 for the first time at $E := A_s + h(\delta)n_{A_s}$. See Fig. 3. Let t_X be the time, when the flow reaches X, $X = A, B, C, D, E$. Then we can define the map $\mathcal{P}$ from l_1 to itself by $\mathcal{P}(A) = E$, which is called the Poincaré map.

To approximate the Poincaré map, we need to make some preparations. Due to the assumption **(H)**, the orbit Γ_0 of subsystem (2a) intersects Ω_0 at exactly two points P

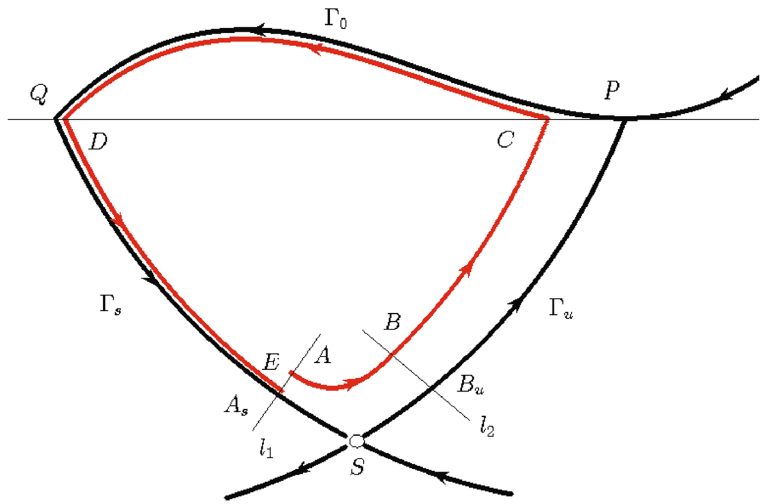

Fig. 3 The Poincaré map near the generalized heteroclinic loop Γ

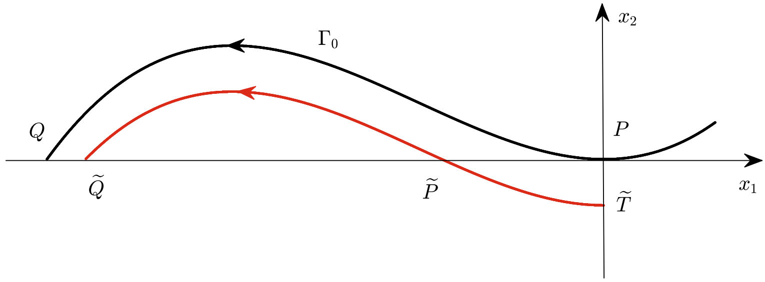

Fig. 4 The flow of system (2a) with the initial value near the generalized singular point P

and Q, furthermore, $q^+(Q) < 0$ and (3) holds. Suppose that the orbit $\widetilde{\Gamma}_0$ of subsystem (2a) crosses the point $\widetilde{P} = (-d, 0)^T$ for sufficiently small $d > 0$, then by the continuous dependency on initial value, $\widetilde{\Gamma}_0$ intersects Ω_0 at $\widetilde{Q} = Q + (\widetilde{d}, 0)^T$ and x_2-axis at $\widetilde{T} = (0, -d_0)^T$ if the domain of subsystem (2a) is extended to $\mathbb{R}^2$. See Fig. 4. Then the following result gives the relationship between d and $\widetilde{d}$.

Lemma 2 *Let notations be given above. Then for sufficiently small d, we have*

$$\widetilde{d} = \frac{q^+_{x_1}(P)}{2q^+(Q)} \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(s))ds\right) d^2 + O(d^3). \tag{5}$$

Proof Since subsystem (2a) satisfies (3), then near the point $P = (0, 0)^T$,

$$\frac{dx_2}{dx_1} = \frac{q^+_{x_1}(P)}{p^+(P)} x_1 + \frac{q^+_{x_2}(P)}{p^+(P)} x_2 + \varphi(x_1, x_2), \tag{6}$$

where $\varphi(x_1, x_2)$ is the high order term of x_1 and x_2. Consider Eq. (6) with the initial value $x_2(0) = -d_0$, then

$$d_0 = \frac{q^+_{x_1}(P)}{2p^+(P)}d^2 + \frac{q^+_{x_2}(P)}{p^+(P)}\int_0^{-d} x_2 dx_1 + \int_0^{-d} \varphi(x_1, x_2)dx_1, \tag{7}$$

note that $x_2(-d) = 0$ and through partial integration, we can obtain

$$\int_0^{-d} x_2 dx_1 = -\int_0^{-d} x_1 \left(\frac{q^+_{x_1}(P)}{p^+(P)}x_1 + \frac{q^+_{x_2}(P)}{p^+(P)}x_2 + \varphi(x_1, x_2)\right) dx_1 = O(d^3), \tag{8}$$

and it is clear that for sufficiently small d,

$$\int_0^{-d} \varphi(x_1, x_2)dx_1 = O(d^3). \tag{9}$$

From (7) to (9) it follows that

$$d_0 = \frac{q^+_{x_1}(P)}{2p^+(P)}d^2 + O(d^3). \tag{10}$$

Therefore, in order to obtain the relationship between d and $\widetilde{d}$, it is only necessary to get the function of $\widetilde{d}$ in d_0.

Let $x_0^+(t; t_0, X)$ be the solution of (2a) with $x_0^+(t_0) = X$ for any $X \in \Omega_+ \bigcup \Omega_0$. By the C^n dependency on initial values, for sufficiently small d_0, we can expand $x_0^+(t; t_P, \widetilde{T})$ as

$$x_0^+(t; t_P, \widetilde{T}) = \gamma_0(t) + \Psi_1(t; P)(\widetilde{T} - P) + O(\|\widetilde{T} - P\|^2), \tag{11}$$

where $\Psi_1(t; P)$ satisfies the variational equation

$$\dot{\Psi}_1(t; P) = Df_+(\gamma_0(t))\Psi_1(t; P), \quad \Psi_1(t_P; P) = I.$$

Assume that $x_0^+(t_{\widetilde{Q}}; t_P, \widetilde{T}) = \widetilde{Q} \in \Omega_0$, then from the C^n dependency on initial values it follows that

$$t_{\widetilde{Q}} = t_Q + \tau_1 + O(\|\widetilde{T} - P\|^2), \tag{12}$$

where $\tau_1 = O(\|\widetilde{T} - P\|)$. Note that $n^T Q = n^T \widetilde{Q} = 0$, substituting (12) into (11) yields

$$\tau_1 = -\frac{n^T \Psi_1(t_Q; P)(\widetilde{T} - P)}{n^T f_+(Q)}. \tag{13}$$

If we plug (12) and (13) back into (11), we get

$$\widetilde{Q} = Q + \frac{f_+(Q) \wedge [\Psi_1(t;P)(\widetilde{T} - P)]}{q^+(Q)} n^\perp + O(\|\widetilde{T} - P\|^2). \tag{14}$$

Take $\omega(t; P) := f_+(\gamma_0(t)) \wedge [\Psi_1(t; P)(\widetilde{T} - P)]$, we can check that $\omega(t; P)$ satisfies

$$\frac{d}{dt}\omega(t;P) = \operatorname{div} f_+(\gamma_0(t))\omega(t;P), \quad \omega(t_P;P) = -d_0 p^+(P),$$

which yields

$$\omega(t_Q; P) = -d_0 p^+(P) \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(s))ds\right). \tag{15}$$

Consequently, from (14) and (15) it follows that

$$\widetilde{d} = \frac{p^+(P)}{q^+(Q)} \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(s))ds\right) d_0 + O(d_0^2). \tag{16}$$

Then, substituting (10) into (16) yields (5). Thus the proof is complete. □

Lemma 3 *Suppose that δ and ρ are sufficiently small, then we have*

$$h(\delta) = K_1(\rho)\mathcal{D}^2(\delta, 0) + O(\mathcal{D}^3(\delta, 0)), \tag{17}$$

where

$$K_1(\rho) = -\frac{q^+_{x_1}(P)q^-(Q)}{2q^+(Q)(q^-(P))^2}\left(-\frac{\lambda_+^2\beta_1^2}{\lambda_-\beta_2}\rho + O(\rho^2)\right)\exp(H(\rho)),$$

$$H(\rho) = 2\int_{t_{B_u}}^{t_P} \operatorname{div} f_-(\gamma_u(\tau))d\tau + \int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(\tau))d\tau + \int_{t_Q}^{t_{A_s}} \operatorname{div} f_-(\gamma_s(\tau))d\tau.$$

Proof By the similar method used to obtain the formula (14), we can obtain

$$C = P + \frac{f_-(B_u) \wedge (B - B_u)}{f_-(P) \wedge n^\perp} \exp\left(\int_{t_{B_u}}^{t_P} \operatorname{div} f_-(\gamma_u(\tau))d\tau\right) n^\perp + O(\|B - B_u\|^2), \tag{18}$$

$$E = A_s + \frac{f_-(Q) \wedge (D - Q)}{f_-(A_s) \wedge n_{A_s}} \exp\left(\int_{t_Q}^{t_{A_s}} \operatorname{div} f_-(\gamma_s(\tau))d\tau\right) n_{A_s} + O(\|D - Q\|^2), \tag{19}$$

and it is clear to check that

$$f_-(B_u) \wedge (B - B_u) = \|f_-(B_u)\|\|B - B_u\|, \quad f_-(A_s) \wedge n_{A_s} = \|f_-(A_s)\|.$$

From (5), (18) and (19), we can obtain

$$h(\delta) = -\frac{q^{+}_{x_1}(P)q^{-}(Q)}{2q^{+}(Q)(q^{-}(P))^2} \frac{\|f_{-}(B_u)\|^2}{\|f_{-}(A_s)\|} \mathcal{D}^2(\delta, 0)\exp(H(\rho)) + O(\mathcal{D}^3(\delta, 0)). \quad (20)$$

By [5, Lemma 3], we have the fact that

$$\|f_{-}(A_s)\| = -\rho\lambda_{-}\beta_2 + O(\rho^2), \quad (21)$$

$$\|f_{-}(B_u)\| = \rho\lambda_{+}\beta_1 + O(\rho^2). \quad (22)$$

Then, substituting (21) and (22) into (20) yields (17). Therefore, the proof is now complete. □

Theorem 1 *Suppose that system* (2) *has a generalized heteroclinic loop Γ with the assumption* **(H)**. *Given that $\lambda_0 \neq 1/2$, then Γ is asymptotically stable if $\lambda_0 > 1/2$, unstable if $\lambda_0 < 1/2$.*

Proof For sufficiently small δ, using result **(ii)** in Lemma 1, we can obtain

$$u = \frac{\delta}{M_1(\rho, 0)}(1 + O(\delta)), \quad (23)$$

$$\mathcal{D}(\delta, 0) = M_2(\rho, 0)\mathcal{D}_0(u, 0) + O\left(\mathcal{D}_0^2(u, 0)\right), \quad (24)$$

thus from (23), (24) and result **(i)** in Lemma 1, it follows that

$$\begin{aligned} \mathcal{D}(\delta, 0) &= \rho^{1-\lambda_0} \frac{M_2(\rho, 0)}{M_1^{\lambda_0}(\rho, 0)} \left(1 + o\left(\delta^{k_0}\right)\right) \delta^{\lambda_0} + O\left(\delta^{2\lambda_0}\right) \\ &= K_2(\rho)\left(1 + o\left(\delta^{k_0}\right)\right) \delta^{\lambda_0} + O\left(\delta^{2\lambda_0}\right), \end{aligned} \quad (25)$$

where the constant $k_0 \in (0, \lambda_0/(1 + \lambda_0))$,

$$K_2(\rho) = (\rho \sin\theta)^{1-\lambda_0} \frac{\beta_2}{\beta_1^{\lambda_0}}(1 + \varphi(\rho)),$$

and the function $\varphi(\rho)$ with $\varphi(\rho) \to 0$ as $\rho \to 0$. Take sufficiently small ρ fixed, substituting (25) into (17) yields

$$h(\delta) = K_1(\rho)K_2^2(\rho)(1 + o(\delta^{k_0}))\delta^{2\lambda_0} + O\left(\delta^{3\lambda_0}\right).$$

Consequently, for sufficiently small $\delta > 0$, we have

$$\frac{h(\delta)}{\delta} = K_1(\rho)K_2^2(\rho)(1 + o(\delta^{k_0}))\delta^{2\lambda_0 - 1} + O\left(\delta^{3\lambda_0 - 1}\right). \quad (26)$$

If $\lambda_0 > 1/2$, from (26) it follows that $h(\delta)/\delta \to 0$ as $\delta \to 0$. Thus the generalized heteroclinic loop Γ is asymptomatically stable.

If $\lambda_0 < 1/2$, from (26) we have

$$\frac{h(\delta)}{\delta} = \left(K_1(\rho)K_2^2(\rho)(1 + o(\delta^{k_0})) + O\left(\delta^{\lambda_0}\right)\right)\delta^{2\lambda_0 - 1},$$

where $K_1(\rho)K_2^2(\rho) > 0$, $k_0 > 0$ and $\lambda_0 > 0$. Then we have that $h(\delta)/\delta \to +\infty$ as $\delta \to 0$. Thus the generalized heteroclinic loop Γ is unstable. Therefore, the proof is now complete. □

4 Perturbations of Generalized Heteroclinic Loops

We assume that Γ_s^ε (resp., Γ_u^ε), the stable (resp., unstable) manifold of S_ε, intersects Ω_0 at Q_ε^- (resp., P_ε^-). Under the condition (3) in **(H)**, we will prove that there exists a generalized singular point $P_\varepsilon^+ \in \Omega_0$ of system (1) near P. The flow of subsystem (1a) starting from P_ε^+ crosses Ω_0 transversally at point Q_ε^+. The next lemma will give the locations of $P_\varepsilon^\pm$ and $Q_\varepsilon^\pm$.

Lemma 4 *Suppose that system* (2) *has a generalized heteroclinic loop Γ with the assumption* **(H)**, *then for sufficiently small $|\varepsilon|$, system* (1) *has a generalized singular point $P_\varepsilon^+ \in \Omega_0$, furthermore,*

$$\begin{aligned}
P_\varepsilon^+ &:= P + \delta_1^+(\varepsilon)n^\perp = P + \frac{k^+(P)}{q_{x_1}^+(P)}n^\perp\varepsilon + O(\varepsilon^2),\\
P_\varepsilon^- &:= P + \delta_1^-(\varepsilon)n^\perp = P + \frac{M_u}{q^-(P)}n^\perp\varepsilon + O(\varepsilon^2),\\
Q_\varepsilon^- &:= Q + \delta_2^-(\varepsilon)n^\perp = Q - \frac{M_s}{q^-(Q)}n^\perp\varepsilon + O(\varepsilon^2),\\
Q_\varepsilon^+ &:= Q + \delta_2^+(\varepsilon)n^\perp = Q + \left\{\frac{k^+(P)q^+(P)}{q_{x_1}^+(P)q^+(Q)} + \frac{M_0}{q^+(Q)}\right\}\\
&\qquad\qquad \times \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(\tau))d\tau\right)n^\perp\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where

$$\begin{aligned}
M_u &:= \int_{-\infty}^{t_P} f_-(\gamma_u(\tau)) \wedge g_-(\gamma_u(\tau)) \exp\left(-\int_{t_P}^{\tau} \operatorname{div} f_-(\gamma_u(s))ds\right)d\tau,\\
M_s &:= \int_{t_Q}^{+\infty} f_-(\gamma_s(\tau)) \wedge g_-(\gamma_s(\tau)) \exp\left(-\int_{t_Q}^{\tau} \operatorname{div} f_-(\gamma_s(s))ds\right)d\tau,\\
M_0 &:= \int_{t_P}^{t_Q} f_+(\gamma_0(\tau)) \wedge g_+(\gamma_0(\tau)) \exp\left(-\int_{t_P}^{\tau} \operatorname{div} f_+(\gamma_0(s))ds\right)d\tau.
\end{aligned}$$

Proof We define a function $\psi(\delta, \varepsilon) := q^+(-\delta, 0) + \varepsilon k^+(-\delta, 0)$. Clearly, the function ψ is continuously differentiable in δ and ε. Note that $\psi(0, 0) = q^+(P) = 0$ and $\psi_\delta(0, 0) = -q^+_{x_1}(P) > 0$, using the implicit function theorem yields that there exists a unique C^n function

$$\delta_1^+(\varepsilon) = \frac{k^+(P)}{q^+_{x_1}(P)}\varepsilon + O(\varepsilon^2) \tag{27}$$

satisfying $\psi(\delta_1^+(\varepsilon), \varepsilon) = 0$ for small $|\varepsilon|$. We can take $P_\varepsilon^+ = (-\delta_1^+(\varepsilon), 0)^T \in \Omega_0$, which satisfies $q^+(P_\varepsilon^+) + \varepsilon k^+(P_\varepsilon^+) = 0$. Thus, the point P_ε^+ is a generalized singular point of system (1). Note that $n^\perp = (-1, 0)^T$, then we prove the existence and location of P_ε^+.

From [5, Lemma 5], we can obtain the expressions of P_ε^- and Q_ε^-. As follows, we only carry out the proof for Q_ε^+.

We define $x_\varepsilon^+(t; t_0, X)$ to be the solution of (1a) with $x_\varepsilon^+(t_0) = X$ for any $X \in \Omega_+ \bigcup \Omega_0$. Using (27) and the C^n dependency on initial values and parameters yields that for sufficiently small $|\varepsilon|$, we can write $x_\varepsilon^+(t; t_P, P_\varepsilon^+)$ as

$$x_\varepsilon^+(t; t_P, P_\varepsilon^+) = \gamma_0(t) + \alpha(t)\varepsilon + O(\varepsilon^2), \tag{28}$$

where $\alpha = \frac{\partial x_\varepsilon^+}{\partial \varepsilon}|_{\varepsilon=0}$. Note that x_ε^+ is C^n with respect to (t, ε) for $t \in \mathbb{R}$ and sufficiently small $|\varepsilon|$, then we have

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial x_\varepsilon^+}{\partial \varepsilon}|_{\varepsilon=0}\right) &= \frac{\partial^2 x_\varepsilon^+}{\partial t \partial \varepsilon}|_{\varepsilon=0} = \frac{\partial^2 x_\varepsilon^+}{\partial \varepsilon \partial t}|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left(f_+(x_\varepsilon^+) + \varepsilon g_+(x_\varepsilon^+)\right)|_{\varepsilon=0} \\ &= \left(Df_+(x_\varepsilon^+)\frac{\partial x_\varepsilon^+}{\partial \varepsilon} + g_+(x_\varepsilon^+) + \varepsilon Dg_+(x_\varepsilon^+)\frac{\partial x_\varepsilon^+}{\partial \varepsilon}\right)|_{\varepsilon=0} \\ &= Df_+(\gamma_0(t))\alpha(t) + g_+(\gamma_0(t)), \end{aligned} \tag{29}$$

where the first equality follows from the result in [10, Exercise 3211.1] and the others can be easily checked. Letting $t = t_P$ in (28) and using (29) yield that α satisfies

$$\dot{\alpha}(t) = Df_+(\gamma_0(t))\alpha(t) + g_+(\gamma_0(t)), \quad \alpha(t_P) = \frac{k^+(P)}{q^+_{x_1}(P)}n^\perp.$$

Assume that $x_\varepsilon^+(t_{Q_\varepsilon^+}; t_P, P_\varepsilon^+) = Q_\varepsilon^+ \in \Omega_0$, from the C^n dependency on initial values and parameters it follows that $t_{Q_\varepsilon^+}$ can be expanded as

$$t_{Q_\varepsilon^+} = t_Q + T_1\varepsilon + O(\varepsilon^2). \tag{30}$$

Note that $n^T Q_\varepsilon^+ = n^T Q = 0$, by substituting (30) into (28), we can obtain

$$T_1 = -\frac{n^T \alpha(t_Q)}{n^T f_+(Q)}. \tag{31}$$

Thus, substituting (30) and (31) into (28) yields

$$Q_\varepsilon^+ = Q + \frac{f_+(Q) \wedge \alpha(t_Q)}{n^T f_+(Q)} n^\perp \varepsilon + O(\varepsilon^2). \tag{32}$$

To get the expansion of Q_ε^+, it is only necessary to obtain $f_+(Q) \wedge \alpha(t_Q)$. We define $\zeta(t) := f_+(x_0^+(t; t_P, P)) \wedge \alpha(t)$, and we can check that $\zeta(t)$ satisfies

$$\dot{\zeta}(t) = \operatorname{div} f_+(\gamma_0(t))\zeta(t) + f_+(\gamma_0(t)) \wedge g_+(\gamma_0(t)), \quad \zeta(t_P) = \frac{k^+(P)n^T f_+(P)}{q_{x_1}^+(P)},$$

which implies

$$\zeta(t_Q) = \left\{\frac{k^+(P)n^T f_+(P)}{q_{x_1}^+(P)} + M_0\right\} \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(\tau))d\tau\right). \tag{33}$$

Then, from (32) and (33) we can obtain the expression of Q_ε^+. Therefore, the proof is now complete. □

Consider the following sets:

$$V_1(\varepsilon) := \{\varepsilon \in \mathbb{R} : |\varepsilon| \le \varepsilon_0\,, \delta_2^-(\varepsilon) \ge \delta_2^+(\varepsilon)\},$$
$$V_2(\varepsilon) := \{\varepsilon \in \mathbb{R} : |\varepsilon| \le \varepsilon_0\,, \delta_2^-(\varepsilon) < \delta_2^+(\varepsilon)\},$$

where $\delta_2^\pm(\varepsilon)$ are defined in Lemma 4. Clearly, if $\varepsilon \in V_1(\varepsilon)$, then Q_ε^+ is at the right side of Q_ε^-, otherwise, Q_ε^+ is at the left side of Q_ε^-. See Fig. 5.

Take l_i, $i = 1, 2$, to be the sections as those defined in Sect. 2. We assume that the flow of subsystem (1b) starting from Q_ε^+ intersects l_1 for the first time at $\overline{A}_\varepsilon := A_s + \overline{a}(\varepsilon)n_{A_s}$. Let $A_\varepsilon := A_s + \delta n_{A_s}$ with $\delta > \max\{a^s(\varepsilon), \overline{a}(\varepsilon)\}$. The forward flow of system (1) from A_ε intersects l_2 and Ω_0 at B_ε^1 and C_ε^1 respectively, the backward flow crosses Ω_0 at B_ε^2 and C_ε^2 in order. See Fig. 5. Then we can define the Poincaré map $\mathcal{P}(\cdot\,, \varepsilon)$ in the form $\mathcal{P}(C_\varepsilon^2, \varepsilon) = C_\varepsilon^1$. Let

$$\|A_\varepsilon - A_s^\varepsilon\| = d_{1\varepsilon},\ \|B_\varepsilon^1 - B_u^\varepsilon\| = d_{2\varepsilon},\ \|C_\varepsilon^1 - P_\varepsilon^-\| = d_{3\varepsilon},$$
$$\|A_\varepsilon - \overline{A}_\varepsilon\| = \delta_{1\varepsilon},\ \|B_\varepsilon^2 - Q_\varepsilon^+\| = \delta_{2\varepsilon},\ \|C_\varepsilon^2 - P_\varepsilon^+\| = \delta_{3\varepsilon}.$$

Clearly, $d_{1\varepsilon} \ge \delta_{1\varepsilon}$ if $\varepsilon \in V_1(\varepsilon)$, $d_{1\varepsilon} < \delta_{1\varepsilon}$ if $\varepsilon \in V_2(\varepsilon)$.

Theorem 2 *Suppose that system (2) has a generalized heteroclinic loop Γ with the assumption* **(H)**. *If $\lambda_0 > 1$, then there exists a neighborhood U of Γ such that for sufficiently small $|\varepsilon|$, system* (1) *has at most one limit cycle in U. If $0 < \lambda_0 \le 1$ and $\lambda_0 \ne 1/2$, then there exists a neighborhood U of Γ such that for sufficiently small $|\varepsilon|$, system* (1) *has at most two limit cycles in U.*

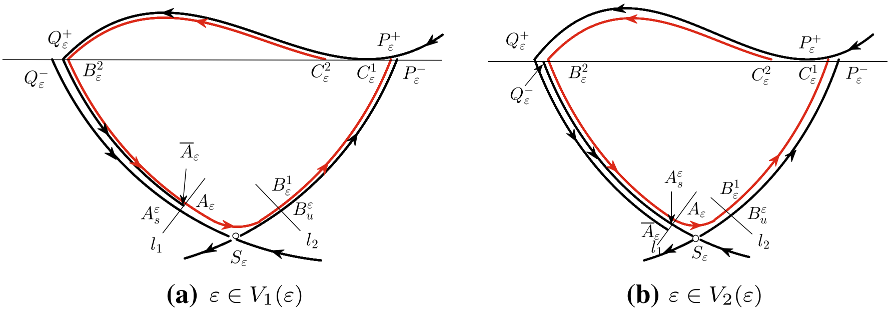

Fig. 5 The generalized heteroclinic loop under perturbations

Proof As stated above, we have

$$d_{1\varepsilon} = \delta - a^s(\varepsilon), \quad d_{2\varepsilon} = \mathcal{D}(\delta, \varepsilon) - b^u(\varepsilon), \tag{34}$$

where $a^s(\varepsilon)$ and $b^u(\varepsilon)$ satisfy $A_s^\varepsilon = A_s + a^s(\varepsilon)n_{A_s}$ and $B_u^\varepsilon = B_u + b^u(\varepsilon)n_{B_u}$, respectively, and the Dulac map $\mathcal{D}$ is defined in Sect. 2. Then from (34) and Lemma 1, we have

$$\begin{aligned} u &= \frac{d_{1\varepsilon}}{M_1(\rho, \varepsilon)}(1 + O(d_{1\varepsilon})), \\ \mathcal{D}_0(u, \varepsilon) &= \frac{\rho^{1-\lambda(\varepsilon)}}{M_1^{\lambda(\varepsilon)}(\rho, \varepsilon)}(1 + o(d_{1\varepsilon}^{k_0}))d_{1\varepsilon}^{\lambda(\varepsilon)} + O(d_{1\varepsilon}^{2\lambda(\varepsilon)}), \end{aligned} \tag{35}$$

where $k_0 \in (0, \lambda_0/(1 + \lambda_0))$ is fixed. Then from (34), (35) and result **(ii)** in Lemma 1 it follows

$$d_{2\varepsilon} = \frac{M_2(\rho, \varepsilon)\rho^{1-\lambda(\varepsilon)}}{M_1^{\lambda(\varepsilon)}(\rho, \varepsilon)}(1 + o(d_{1\varepsilon}^{k_0}))d_{1\varepsilon}^{\lambda(\varepsilon)} + O(d_{1\varepsilon}^{2\lambda(\varepsilon)}). \tag{36}$$

By the same argument used in the proof of (14), we have

$$d_{3\varepsilon} = K_{1\varepsilon}d_{2\varepsilon} + O(d_{2\varepsilon}^2), \tag{37}$$

$$\delta_{2\varepsilon} = K_{2\varepsilon}\delta_{1\varepsilon} + O(\delta_{1\varepsilon}^2), \tag{38}$$

where

$$K_{1\varepsilon} = \frac{F_-(B_u^\varepsilon, \varepsilon) \wedge n_{B_u}}{F_-(P_\varepsilon^-, \varepsilon) \wedge n^\perp} \exp\left(\int_{t_{B_u^\varepsilon}}^{t_{P_\varepsilon^-}} \operatorname{div} F_-(x_\varepsilon^-(s; t_{B_u^\varepsilon}, B_u^\varepsilon), \varepsilon)ds\right),$$

$$K_{2\varepsilon} = -\frac{F_-(\overline{A}_\varepsilon, \varepsilon) \wedge n_{A_s}}{F_-(Q_\varepsilon^+, \varepsilon) \wedge n^\perp} \exp\left(\int_{t_{\overline{A}_\varepsilon}}^{t_{Q_\varepsilon^+}} \operatorname{div} F_-(x_\varepsilon^-(s; t_{\overline{A}_\varepsilon}, \overline{A}_\varepsilon), \varepsilon)ds\right).$$

As stated in Lemma 2, we have

$$\delta_{2\varepsilon} = K_{3\varepsilon}\delta_{3\varepsilon}^2 + O(\delta_{3\varepsilon}^3), \tag{39}$$

where

$$K_{3\varepsilon} = \frac{q_{x_1}^+(P_\varepsilon^+) + \varepsilon k_{x_1}^+(P_\varepsilon^+)}{2q^+(Q_\varepsilon^+) + 2\varepsilon k^+(Q_\varepsilon^+)} \exp\left(\int_{t_{P_\varepsilon^+}}^{t_{Q_\varepsilon^+}} \operatorname{div} F_+(x_\varepsilon^+(s; t_{P_\varepsilon^+}, P_\varepsilon^+), \varepsilon)ds\right).$$

Furthermore, we can check the fact that as $\varepsilon \to 0$,

$$\begin{aligned}
K_{1\varepsilon} &\to \frac{\|f_-(B_u)\|}{q^-(P)} \exp\left(\int_{t_{B_u}}^{t_P} \operatorname{div} f_-(\gamma_u(\tau))d\tau\right),\\
K_{2\varepsilon} &\to -\frac{\|f_-(A_s)\|}{q^-(Q)} \exp\left(\int_{t_{A_s}}^{t_Q} \operatorname{div} f_-(\gamma_s(\tau))d\tau\right),\\
K_{3\varepsilon} &\to \frac{q_{x_1}^+(P)}{2q^+(Q)} \exp\left(\int_{t_P}^{t_Q} \operatorname{div} f_+(\gamma_0(\tau))d\tau\right).
\end{aligned}$$

Therefore, from (36) to (39) it follows that

$$d_{3\varepsilon} = N_1(\varepsilon)(1 + o(d_{1\varepsilon}^{k_0}))d_{1\varepsilon}^{\lambda(\varepsilon)} + O(d_{1\varepsilon}^{2\lambda(\varepsilon)}), \tag{40}$$

$$\delta_{3\varepsilon} = N_2(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right)\delta_{1\varepsilon}^{\frac{1}{2}}, \tag{41}$$

where

$$N_1(\varepsilon) = \frac{K_{1\varepsilon}M_2(\rho,\varepsilon)\rho^{1-\lambda(\varepsilon)}}{M_1^{\lambda(\varepsilon)}(\rho,\varepsilon)} \text{ and } N_2(\varepsilon) = K_{2\varepsilon}^{\frac{1}{2}}K_{3\varepsilon}^{\frac{1}{2}}.$$

In order to get the number of limit cycles bifurcate from Γ, we take sufficiently small $\rho > 0$ fixed and consider the function $h(\delta, \varepsilon)$, which satisfies

$$\begin{aligned}
h(\delta,\varepsilon)n^\perp &= \mathcal{P}(C_\varepsilon^2, \varepsilon) - C_\varepsilon^2 = C_\varepsilon^1 - C_\varepsilon^2\\
&= (C_\varepsilon^1 - P_\varepsilon^-) + (P_\varepsilon^- - P_\varepsilon^+) + (P_\varepsilon^+ - C_\varepsilon^2)\\
&= (d_{3\varepsilon} + \varphi(\varepsilon) - \delta_{3\varepsilon})n^\perp,
\end{aligned} \tag{42}$$

where the function $|\varphi(\varepsilon)| = \|P_\varepsilon^- - P_\varepsilon^+\|$ only depends on the parameter ε. From (40) to (42) it follows that

$$\begin{aligned}
\frac{\partial}{\partial\delta}h(\delta,\varepsilon) = \lambda(\varepsilon)N_1(\varepsilon)d_{1\varepsilon}^{\lambda(\varepsilon)-1}&\left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right)\\
&-\frac{N_2(\varepsilon)}{2}\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right)\delta_{1\varepsilon}^{-\frac{1}{2}}.
\end{aligned}$$

Suppose that $\lambda_0 > 1$, then for sufficiently small $|\delta| + |\varepsilon|$, we have $\lambda(\varepsilon) - 1 > 0$ and $\frac{\partial h}{\partial \delta}(\delta, \varepsilon) < 0$. Therefore, if $\lambda_0 > 1$, then there exists a neighborhood U of Γ such that for sufficiently small $|\varepsilon|$, system (1) has at most one limit cycle in U.

The proof of the case $0 < \lambda_0 \leq 1$ and $\lambda_0 \neq 1/2$ will be divided into two different cases, which rely on the relative location between Q_ε^+ and Q_ε^-.

Case (i) Suppose that $\varepsilon \in V_1(\varepsilon)$, that is, Q_ε^- is at the left side of Q_ε^+ (see Fig. 5a). Note that $\frac{\partial h}{\partial \delta}(\delta, \varepsilon) = 0$ is equivalent to

$$\lambda(\varepsilon) N_1(\varepsilon) \delta_{1\varepsilon}^{\frac{1}{2}} d_{1\varepsilon}^{\lambda(\varepsilon)-1} \left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) - \frac{N_2(\varepsilon)}{2} \left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right) = 0. \tag{43}$$

Suppose that $\lambda_0 > 1/2$, we can rewrite (43) as

$$\lambda(\varepsilon) N_1(\varepsilon) d_{1\varepsilon}^{\lambda(\varepsilon)-\frac{1}{2}} \left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) (\delta_{1\varepsilon}^{\frac{1}{2}} d_{1\varepsilon}^{-\frac{1}{2}}) = \frac{N_2(\varepsilon)}{2} \left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right). \tag{44}$$

Since $\varepsilon \in V_1(\varepsilon)$, we have $d_{1\varepsilon} \geq \delta_{1\varepsilon}$. Thus, for $\varepsilon \in V_1(\varepsilon)$,

$$\delta_{1\varepsilon}^{\frac{1}{2}} d_{1\varepsilon}^{-\frac{1}{2}} \leq 1, \quad d_{1\varepsilon}^{\lambda(\varepsilon)-\frac{1}{2}} \to 0, \quad \text{as } |\delta| + |\varepsilon| \to 0. \tag{45}$$

Therefore, from (44) and (45) we can get the result that for sufficiently small $|\delta| + |\varepsilon|$ and $\varepsilon \in V_1(\varepsilon)$, $h(\delta, \varepsilon)$ has at most one zero.

Suppose that $0 < \lambda_0 < 1/2$, note that $\frac{\partial h}{\partial \delta}(\delta, \varepsilon) = 0$ is equivalent to

$$(\lambda(\varepsilon) N_1(\varepsilon))^{-2} d_{1\varepsilon}^{2-2\lambda(\varepsilon)} \left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) = 4 N_2^{-2}(\varepsilon) \left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right) \delta_{1\varepsilon}.$$

Let

$$h_1(\delta, \varepsilon) = d_{1\varepsilon}^{2-2\lambda(\varepsilon)} \left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) - N(\varepsilon) \left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right) \delta_{1\varepsilon},$$

where $N(\varepsilon) = 4\lambda^2(\varepsilon) N_1^2(\varepsilon) N_2^{-2}(\varepsilon)$. Since

$$\frac{\partial}{\partial \delta} h_1(\delta, \varepsilon) = d_{1\varepsilon}^{1-2\lambda(\varepsilon)} \left(2 - 2\lambda(\varepsilon) + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) \\ - N(\varepsilon) \left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right),$$

and $2 - 2\lambda(\varepsilon) > 0$, $1 - 2\lambda(\varepsilon) < 0$ for sufficiently small $|\varepsilon|$, then for sufficiently small $|\delta| + |\varepsilon|$, $h_1(\delta, \varepsilon)$ has at most one zero. Therefore, by Rolle's Theorem, $h(\delta, \varepsilon)$ has at most two zeros for sufficiently small $|\delta| + |\varepsilon|$.

Case (ii) Suppose that $\varepsilon \in V_2(\varepsilon)$, that is, Q_ε^+ is at the left side of Q_ε^- (see Fig. 5b).

Suppose that $0 < \lambda_0 < 1/2$, the condition $\varepsilon \in V_2(\varepsilon)$ implies $d_{1\varepsilon} < \delta_{1\varepsilon}$. Thus for $\varepsilon \in V_2(\varepsilon)$,

$$\delta_{1\varepsilon}^{\frac{1}{2}} d_{1\varepsilon}^{-\frac{1}{2}} > 1, \quad d_{1\varepsilon}^{\lambda(\varepsilon)-\frac{1}{2}} \to +\infty, \quad \text{as } |\delta| + |\varepsilon| \to 0. \tag{46}$$

Therefore, from (44) and (46) it follows that $h(\delta, \varepsilon)$ has at most one zero for sufficiently small $|\delta| + |\varepsilon|$ and $\varepsilon \in V_1(\varepsilon)$.

Suppose that $1/2 < \lambda_0 \leq 1$, set $\alpha(\varepsilon) := 2 - 2\lambda(\varepsilon)$, $\omega(\delta, \varepsilon) := (d_{1\varepsilon}^{\alpha(\varepsilon)} - 1)/\alpha(\varepsilon)$ for $\alpha(\varepsilon) \neq 0$ and $\omega(\delta, \varepsilon) := \ln d_{1\varepsilon}$ for $\alpha(\varepsilon) = 0$, then we can rewrite $h_1(\delta, \varepsilon)$ as

$$h_1(\delta, \varepsilon) = (\alpha(\varepsilon)\omega(\delta, \varepsilon) + 1)\left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) - N(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right)\delta_{1\varepsilon}.$$

Suppose that $|\delta_0| + |\varepsilon|$ is sufficiently small and $h_1(\delta_0, \varepsilon) = 0$, then

$$(\alpha(\varepsilon)\omega(\delta_0, \varepsilon) + 1)\left(1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right) = N(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right)\delta_{1\varepsilon},$$

note that

$$N(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right)\delta_{1\varepsilon} \to 0, \quad 1 + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)}) \to 1, \text{ as } |\delta| + |\varepsilon| \to 0,$$

then for sufficiently small $|\delta_0| + |\varepsilon|$, there exists a constant μ such that

$$0 > \mu > \alpha(\varepsilon)\omega(\delta_0, \varepsilon) \geq -1. \tag{47}$$

We define the function

$$\begin{aligned} h_2(\delta, \varepsilon) &:= d_{1\varepsilon}^{2\lambda(\varepsilon)-1} \frac{\partial}{\partial \delta} h_1(\delta, \varepsilon) \\ &= \alpha(\varepsilon) + o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)}) - N(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right) d_{1\varepsilon}^{2\lambda(\varepsilon)-1}, \end{aligned}$$

then the zeros of $\frac{\partial}{\partial \delta} h_1(\delta, \varepsilon)$ are equivalent to those of $h_2(\delta, \varepsilon)$. Since for any $\nu > 0$,

$$d_{1\varepsilon}^{\nu}\omega \to 0, \text{ as } |\delta| + |\varepsilon| \to 0,$$

and

$$\begin{aligned} h_2(\delta, \varepsilon)\omega = \alpha(\varepsilon)\omega &+ \left(o(d_{1\varepsilon}^{k_0}) + O(d_{1\varepsilon}^{\lambda(\varepsilon)})\right)\omega \\ &- N(\varepsilon)\left(1 + O(\delta_{1\varepsilon}^{\frac{1}{2}})\right) d_{1\varepsilon}^{2\lambda(\varepsilon)-1}\omega, \end{aligned} \tag{48}$$

then for a small ε fixed, from (47) and (48) we have that $h_2(\delta_0, \varepsilon)$ has the same sign as $\alpha(\varepsilon)$, where δ_0 is a zero of $h_1(\delta, \varepsilon)$. Therefore, $h_1(\delta, \varepsilon)$ has at most one zeros for sufficiently small $|\delta| + |\varepsilon|$. Consequently, using Rolle's Theorem yields that $h(\delta, \varepsilon)$ has at most two zeros. Therefore, the proof is now complete. □

5 An Example

Consider a planar PWS system in the form

$$\begin{cases} \dot{x}_1 = -1, \\ \dot{x}_2 = -3x_1^2 - 4x_1 + 2\varepsilon_1(x_1 + 1), \end{cases} \qquad (x_1, x_2)^T \in \Omega_+, \tag{49}$$

$$\begin{cases} \dot{x}_1 = x_1 + 4x_2 + 5 + \varepsilon_2(x_1 + 1), \\ \dot{x}_2 = 4x_1 + x_2 + 5 + 2\varepsilon_2(x_2 + 1), \end{cases} \qquad (x_1, x_2)^T \in \Omega_-, \tag{50}$$

where ε_1 and ε_2 are small parameters, $\Omega_\pm$ and Ω_0 are the same as defined in the general system (1).

When $\varepsilon_1 = \varepsilon_2 = 0$, we can check that the unperturbed system has a generalized heteroclinic loop Γ, which has a generalized singular point P at the origin, a hyperbolic saddle point at $S = (-1, -1)^T$ and the other intersection between Γ and x_1-axis is point $Q = (-2, 0)^T$. The branches of Γ are in the following form

$$\begin{aligned} \Gamma_u &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 = x_1 \, , x_1 \in (-1, 0)\}, \\ \Gamma_0 &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 = x_1^3 + 2x_2^2 \, , x_1 \in [-2, 0)\}, \\ \Gamma_s &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 = -x_1 - 2 \, , x_1 \in [-2, -1)\}. \end{aligned}$$

It is clear that $\lambda^- = -3$, $\lambda^+ = 5$ and $\lambda_0 = 3/5 > 1/2$. Then from Theorem 1 it follows that Γ is asymptomatically stable.

When $0 < \varepsilon_1 \ll 1$ and $\varepsilon_2 = 0$, there are no changes in subsystem (50). The flow of subsystem (49) starting from $P = (0, 0)^T$ exactly crosses the point Q. Substituting $(x_1, x_2) = (0, 0)$ into (50) yields $(\dot{x}_1, \dot{x}_2) = (-1, 2\varepsilon_1)$. Therefore, a new homoclinic loop Γ_{ε_1} appears. Since $\lambda^+ + \lambda^- = 2 > 0$, then by [5, Theorem 1], we can obtain that the homoclinic loop Γ_{ε_1} is unstable. Thus, a stable limit cycle appears if $0 < \varepsilon_1 \ll 1$ and $\varepsilon_2 = 0$.

When $0 < \varepsilon_2 \ll \varepsilon_1 \ll 1$, S is also the hyperbolic saddle point of subsystem (50). But Γ_s and Γ_u change to be Γ_s^ε and Γ_u^ε respectively, which are in the form

$$\begin{aligned} \Gamma_s^\varepsilon &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 + 1 = k_-(x_1 + 1) \, , x_2 \in (-1, 0)\}, \\ \Gamma_u^\varepsilon &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 + 1 = k_+(x_1 + 1) \, , x_2 \in (-1, 0)\}, \end{aligned}$$

where $k_\pm = (\varepsilon_2 \pm \sqrt{64 + \varepsilon_2^2})/8$. We can check that Γ_u^ε and Γ_s^ε intersect Ω_0 at $P_\varepsilon^- = (-1 - k_-, 0)^T$ and $Q_\varepsilon^- = (-1 - k_+, 0)^T$ respectively, where $-1 - k_- < 0$

and $-1-k_+ < -2$. Thus, the homoclinic loop Γ_{ε_1} breaks and from [5, Theorem 2] it follows that another one limit cycle appears. As known in Theorem 2, the perturbed system has at most two limit cycles near the generalized heteroclinic loop. Therefore, there are only two limit cycles in the perturbed system under the condition $0 < \varepsilon_2 \ll \varepsilon_1 \ll 1$.

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