Qual. Theory Dyn. Syst. (2018) 17:67–80
https://doi.org/10.1007/s12346-017-0223-6

Qualitative Theory of Dynamical Systems

Existence of Positive Periodic Solutions for a Neutral Delay Predator–Prey Model with Hassell–Varley Type Functional Response and Impulse

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Received: 7 December 2016 / Accepted: 7 January 2017 / Published online: 13 January 2017

Abstract In this paper, we discuss a neutral delay predator–prey model with Hassel–Varley type functional response and impulse is investigated. By using Mawhin coincidence degree theory, we obtain some sufficient conditions for the existence of positive periodic solutions. We extend some known work.

Keywords Neutral predator–prey model · Periodic solutions · Impulse · Coincidence degree theory

1 Introduction

The predator–prey model was first introduced by Lotka and Volterra, and then the traditional predator–prey models have been extensively studied [1–16]. Many models have been established to describe the relationships between species and the outer environment and the connections between species. In 1969, Hassell and Varley proposed a predator–prey model (HV, for short), in which the functional response dependents on the predator density in different way [1]. In Ref. [2], Wang considered the periodicity to a non-autonomous predator–prey model with HV functional response and delay in the prey specific growth term

This work is supported by the Natural Science Foundation of China (Grant No. 11471146), and partially supported by PAPD of Jiangsu Province.

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$$\begin{cases} N_1'(t) = N_1(t)\left[a(t) - b(t)N_1(t-\tau(t)) - \frac{c(t)N_2(t)}{mN_2^{\gamma}(t)+N_1(t)}\right], \\ N_2'(t) = N_2(t)\left[-d(t) + \frac{r(t)N_1(t)}{mN_2^{\gamma}(t)+N_1(t)}\right]. \end{cases} \quad 0 < \gamma < 1.$$

Recently, the neutral differential equation with delay has been widely studied and the applications in mathematical ecology will continue to be one of the dominant themes due to its universal existence and importance [3,4]. Many excellent results have been done for the predator–prey model with neutral and delays. For example, Zhang and Zheng [5] considered the following neutral delay predator–prey model with Holling type II functional response

$$\begin{cases} x'(t) = x(t)\Big[a(t) - bx(t-\sigma_1) - \rho x'(t-\sigma_2) - \frac{c(t)x(t)y(t)}{my(t)+x(t)}\Big], \\ y'(t) = y(t)\Big[-d(t) + \frac{f(t)x(t-\tau)}{my(t-\tau)+x(t-\tau)}\Big]. \end{cases}$$

On the other side, in population dynamics, perturbations occur in a more-or-less fashion for many reasons. For example, mating habits, hunting, harvesting, birth, etc. This perturbation bring sudden change to the model. To describe the mathematical ecology systems more realistically, it need to consider the impulse term. In the recent years, impulsive differential equations have been extensively studied [6,7]. By applying impulsive differential equations theory, many authors investigated the mathematical ecology systems with impulse [8–11]. However there are few papers discussing the impulse neutral differential systems.

Motivated by the above work, in this paper, we study the following neutral delay predator–prey model with HV type functional response and impulse

$$\begin{cases} \left.\begin{array}{l} x' = x(t)\left[r(t) - b(t)x(t-\tau(t)) - \rho x'(t-\sigma_1(t)) - \frac{c(t)y(t)}{my^{\gamma}(t)+x(t)}\right], \\ y' = y(t)\left[-d(t) + \frac{a(t)x(t-\sigma_2)}{my^{\gamma}(t-\sigma_2)+x(t-\sigma_2)}\right], \end{array}\right\} t \neq t_k, k=1,2,\ldots, 0<\gamma\leq 1, \\ \left.\begin{array}{l} x(t_k^+) - x(t_k) = \theta_{1k}x(t_k), \\ y(t_k^+) - y(t_k) = \theta_{2k}y(t_k), \end{array}\right\} t = t_k, k = 1, 2, \ldots, \end{cases} \tag{1.1}$$

with initial condition

$$\begin{aligned} & x(t) = \varphi(t),\ x'(t) = \varphi'(t), \\ & \qquad \varphi \in C([-\sigma, 0], [0, +\infty)) \cap C^1([-\sigma, 0], [0, +\infty)),\ \varphi(0) > 0, \\ & y(t) = \psi(t),\ y'(t) = \psi'(t), \\ & \qquad \psi \in C([-\sigma, 0], [0, +\infty)) \cap C^1([-\sigma, 0], [0, +\infty)),\ \psi(0) > 0, \end{aligned} \tag{1.2}$$

where x and y represent prey and predator densities at time t, respectively. $a(t)$, $b(t)$, $c(t)$, $d(t)$, $r(t)$, $\tau(t)$, $\sigma_1(t)$ are continuous nonnegative T-periodic functions. m, ρ and γ are positive constants, σ_2 is a small positive constant. $\theta_{ik} > -1$, $i = 1, 2$, $k \in N^+ = 1, 2, \ldots$. Furthermore, $\sigma := \max_{t\in[0,T]}\{\tau(t), \sigma_1(t), \sigma_2\}$.

By using the Mawhin coincidence theory, we establish some criteria to guarantee the existence of positive periodic solutions of systems (1.1) and (1.2). Our results show that the impulsive neutral delay model (1.1) and (1.2) preserve the original periodicity without the neutral term and impulse under appropriate periodic impulse perturbations. From Theorem 3.1 of this paper, we can easily see that the condition $[A_3]$ of Theorem 3.1 in paper [2] could be removed.

2 Preliminaries

Definition 2.1 $(x(t), y(t))^T \in C([-\sigma, +\infty), (0, +\infty), (0, +\infty))$ is said to be a solution of the initial value problem (1.1) and (1.2) on $[-\sigma, +\infty)$ if

(i) $x(t),\ y(t)$ are absolutely continuous on each interval $(0, t_1]$ and $(t_k, t_{k+1}]$, $k \in N^+$;
(ii) for any $t_k,\ k \in N^+,\ (x(t_k^+), y(t_k^+))^T$ and $(x(t_k^-), y(t_k^-))^T$ exist and $(x(t_k^-), y(t_k^-))^T = (x(t_k), y(t_k))^T$;
(iii) $(x(t), y(t))^T$ satisfies (1.1) and (1.2) for almost everywhere (a.e.) in $[0, \infty)\backslash\{t_k\}$ and satisfies $x(t_k^+) - x(t_k) = \theta_{1k} x(t_k),\ y(t_k^+) - y(t_k) = \theta_{2k} y(t_k)$, for $t = t_k,\ k \in N^+$.

We make the following assumptions

$[H_1]$ $0 < t_1 < t_2 < \cdots < t_k < \cdots$ are fixed points and $\lim_{k\to\infty} t_k = +\infty$;
$[H_2]$ $\{\theta_{ik}\}$ are real sequences such that $\theta_{ik} > -1$ and $\prod_{0<t_k<t}(1 + \theta_{ik}),\ i = 1, 2$ are T-periodic functions.

Under the assumptions $[H_1]$ and $[H_2]$, we consider the following system

$$\begin{cases} N_1'(t) = N_1(t)\left[r(t) - B(t)N_1(t-\tau(t)) - \delta N_1'(t-\sigma_1(t)) - \frac{C(t)N_2(t)}{MN_2^{\gamma}(t)+\theta N_1(t)}\right], \\ N_2'(t) = N_2(t)\left[-d(t) + \frac{A(t)N_1(t-\sigma_2)}{MN_2^{\gamma}(t\sigma_2)+\theta N_1(t-\sigma_2)}\right], \end{cases} \quad 0 < \gamma \le 1, \tag{2.1}$$

with initial condition

$$\begin{aligned} &p(t) = \varphi(t), \quad p'(t) = \varphi'(t), \\ &\qquad \varphi \in C([-\sigma, 0], [0, \infty)) \cap C^1([-\sigma, 0], [0, \infty)), \quad \varphi(0) > 0, \\ &q(t) = \psi(t), \quad q'(t) = \psi'(t), \\ &\qquad \psi \in C([-\sigma, 0], [0, \infty)) \cap C^1([-\sigma, 0], [0, \infty)), \quad \psi(0) > 0, \end{aligned} \tag{2.2}$$

where

$$A(t)=a(t)\prod_{0<t_k<t}(1+\theta_{1k}),\quad \delta=\rho\prod_{0<t_k<t-\sigma(t)}(1+\theta_{1k}),$$
$$B(t)=b(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k}),\quad C(t)=c(t)\prod_{0<t_k<t}(1+\theta_{2k}),$$
$$M=m\left(\prod_{0<t_k<t}(1+\theta_{2k})\right)^{\gamma},\quad \theta=\prod_{0<t_k<t}(1+\theta_{1k}). \tag{2.3}$$

Lemma 2.1 *Suppose that* $[H_1]$ *and* $[H_2]$ *hold, then*

(i) if $(N_1(t),N_2(t))^T$ *is a solution of* (2.1) *and* (2.2)*, then* $(x(t),y(t))^T$ *is a solution of* (1.1) *and* (1.2)*, where*

$$x(t)=\prod_{0<t_k<t}(1+\theta_{1k})N_1(t),\quad y(t)=\prod_{0<t_k<t}(1+\theta_{2k})N_2(t);$$

(ii) if $(x(t),y(t))^T$ *is a solution of* (1.1) *and* (1.2)*, then* $(N_1(t),N_2(t))^T$ *is a solution of* (2.1) *and* (2.2)*, where*

$$N_1(t)=\prod_{0<t_k<t}(1+\theta_{1k})^{-1}x(t),\quad N_2(t)=\prod_{0<t_k<t}(1+\theta_{2k})^{-1}y(t).$$

Proof (i) It is easy to see that $x(t)=\prod_{0<t_k<t}(1+\theta_{1k})N_1(t)$, $y(t)=\prod_{0<t_k<t}(1+\theta_{2k})N_2(t)$ are absolutely continuous on every interval $(t_k,t_{k+1}]$. For any $t\neq t_k$, $k\in N^+$, one has

$$x'(t)-x(t)\left[r(t)-b(t)x(t-\tau(t))-\rho x'(t-\sigma_1(t))-\frac{c(t)y(t)}{my^{\gamma}(t)+x(t)}\right]$$
$$=\prod_{0<t_k<t}(1+\theta_{1k})N_1'(t)-\prod_{0<t_k<t}(1+\theta_{1k})N_1(t)$$
$$\times\left[r(t)-b(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k})N_1(t-\tau(t))\right.$$
$$=-\rho\prod_{0<t_k<t-\sigma_1(t)}(1+\theta_{1k})N_1(t-\sigma(t))$$
$$\left.-\frac{c(t)\prod_{0<t_k<t}(1+\theta_{2k})N_2(t)}{m\left(\prod_{0<t_k<t}(1+\theta_{2k})\right)^{\gamma}N_2^{\gamma}(t)+\prod_{0<t_k<t}(1+\theta_{1k})N_1(t)}\right]$$
$$=\prod_{0<t_k<t}(1+\theta_{1k})\left\{N_1'(t)-N_1(t)\left[r(t)-b(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k})N_1(t-\tau(t))\right.\right.$$
$$-\rho\prod_{0<t_k<t-\sigma_1(t)}(1+\theta_{1k})N_1(t-\sigma(t))$$

$$
\begin{aligned}
&\left.\left.-\frac{c(t)\prod_{0<t_k<t}(1+\theta_{2k})N_2(t)}{m\left(\prod_{0<t_k<t}(1+\theta_{2k})\right)^{\gamma}N_2^{\gamma}(t)+\prod_{0<t_k<t}(1+\theta_{1k})N_1(t)}\right]\right\}\\
&=\prod_{0<t_k<t}(1+\theta_{1k})\left\{N_1'(t)-N_1(t)\Big[r(t)-B(t)N_1(t-\tau(t))-\delta N_1'(t-\sigma_1(t))\right.\\
&\left.\left.-\frac{C(t)N_2(t)}{MN_2^{\gamma}(t)+\theta N_1(t)}\right]\right\}\\
&=0. \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
&y'(t)-y(t)\left[-d(t)+\frac{a(t)x(t-\sigma_2)}{my^{\gamma}(t-\sigma_2)+x(t-\sigma_2)}\right]\\
&\prod_{0<t_k<t}(1+\theta_{2k})\left\{N_2'(t)-N_2(t)\Big[-d(t)+\frac{A(t)N_1(t-\sigma_2)}{MN_2^{\gamma}(t-\sigma_2)+\theta N_1(t-\sigma_2)}\Big]\right\}\\
&=0. \qquad (2.5)
\end{aligned}
$$

On the other hand, for any $t=t_k,\ k\in N^+$, we have

$$
\begin{aligned}
x(t_k^+)&=\lim_{t\to t_k^+}\prod_{0<t_j<t}(1+\theta_{1j})N_1(t)=\prod_{0<t_j\le t_k}(1+\theta_{1j})N_1(t_k),\\
y(t_k^+)&=\lim_{t\to t_k^+}\prod_{0<t_j<t}(1+\theta_{2j})N_2(t)=\prod_{0<t_j\le t_k}(1+\theta_{2j})N_2(t_k),
\end{aligned}
$$

and

$$
x(t_k)=\prod_{0<t_j<t_k}(1+\theta_{1j})N_1(t_k),\quad y(t_k)=\prod_{0<t_j<t_k}(1+\theta_{2j})N_2(t_k).
$$

Thus we obtain

$$
x(t_k^+)=(1+\theta_{1k})x(t_k),\quad y(t_k^+)=(1+\theta_{2k})y(t_k). \qquad (2.6)
$$

From (2.2)–(2.6), we know that $(x(t),\ y(t))^T$ is a solution of (1.1) and (1.2).

(ii) Since $x(t)=\prod_{0<t_k<t}(1+\theta_{1k})N_1(t),\ \ y(t)=\prod_{0<t_k<t}(1+\theta_{2k})N_2(t)$ are absolutely continuous on every interval $(t_k,t_{k+1}],\ k\in N^+$. According to definition 2.1, $\forall\, k\in N^+$, we have

$$
\begin{aligned}
N_1(t_k^+)&=\prod_{0<t_j\le t_k}(1+\theta_{1j})^{-1}x(t_k^+)=\prod_{0<t_j<t_k}(1+\theta_{1j})^{-1}x(t_k)=N_1(t_k),\\
N_2(t_k^+)&=\prod_{0<t_j\le t_k}(1+\theta_{2j})^{-1}y(t_k^+)=\prod_{0<t_j<t_k}(1+\theta_{2j})^{-1}y(t_k)=N_2(t_k),
\end{aligned}
$$

and

$$N_1(t_k^-) = \prod_{0<t_j\le t_{k-1}} (1+\theta_{1j})^{-1} x(t_k^-) = N_1(t_k),$$
$$N_2(t_k^-) = \prod_{0<t_j\le t_{k-1}} (1+\theta_{2j})^{-1} y(t_k^-) = N_2(t_k),$$

which implies that $N_1(t),\ N_2(t)$ are continuous on $[-\sigma, +\infty)$. It is easy to prove that $N_1(t),\ N_2(t)$ are absolutely continuous on $[-\sigma, +\infty)$.

Similar to the proof of the case (i), we could show that

$$N_1(t) = \prod_{0<t_k<t} (1+\theta_{1k})^{-1} x(t), \quad N_2(t) = \prod_{0<t_k<t} (1+\theta_{2k})^{-1} y(t)$$

are the solutions of systems (2.1) and (2.2). The proof of Lemma 2.1 is completed.

From Lemma 2.1, we only study the existence of positive periodic solutions of systems (2.1) and (2.2) and obtain the existence of positive periodic solutions of systems (1.1) and (1.2).

In order to present sufficient conditions for guaranteeing the existence of positive periodic solutions for the systems (2.1) and (2.2), we introduce the coincidence degree theorem.

Let X and Y be two Banach spaces, $L : \mathrm{D}omL \subset X \to Y$ is a linear map, and $N : X \to Y$ is a continuous map. If dim Ker$L$ = codim Im$L < +\infty$ and I$mL \in Y$ is closed, then we call the operator L is a Fredholm operator with index zero. And if L is a Fredholm operator with index zero and there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that ImP = Ker$L$, ImL = Ker$Q$ = Im$(I - Q)$, then $L|_{DomL\cap KerP} : (I - P)X \to$ ImL has an inverse function, we set it as K_p. Assume $\Omega \in X$ is any open set, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N(\overline{\Omega}) \in X$ is relative compact, then we say $N \in \overline{\Omega}$ is L-compact.

Lemma 2.2 [17] *Let X and Y be both Banach spaces, $L : DomL \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \in Y$ be an open bounded set, and $N : \overline{\Omega} \to X$ be L-compact on $\overline{\Omega}$. If all the following conditions hold*

[C_1] *$Lx \ne \lambda Nx$, for $x \in \partial\Omega \cap DomL$, $\lambda \in (0, 1)$;*
[C_2] *$Nx \notin ImL$, for $x \in \partial\Omega \cap KerL$;*
[C_3] *$deg\{JQN, \Omega \cap KerL, 0\} \ne 0$, where $J : ImQ \to KerL$ is an isomorphism;*

then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap \mathrm{Dom}L$.

Lemma 2.3 [18,19] *If $\tau \in C^1(R, R)$ with $\tau(t + T) = \tau(t)$ and $\tau'(t) < 1$ for $t \in [0, T]$, then the function $\delta(t) = t - \tau(t)$ has a unique inverse $\delta^{-1}(t)$ satisfying $\delta \in C(R, R)$ with $\delta^{-1}(s + T) = \delta^{-1}(s) + T$ for $s \in [0, T]$.*

For convenience, we denote

$$\bar{f} = \frac{1}{T}\int_0^T f(t)dt, \quad f^L = \min_{t\in[0,T]} f(t), \quad f^M = \max_{t\in[0,T]} f(t),$$

where f is a nonnegative T-periodic continuous function.

3 Main Results

Theorem 3.1 *Assume that* $[H_1]$, $[H_2]$ *and the following conditions hold*

$[H_3]$ $\tau'(t) < 1$;
$[H_4]$ $1 > \delta e^R$, *where* R *is defined in the proof;*
$[H_5]$ $M\overline{r} > \overline{C}$, $\frac{e^R A^M}{\overline{d}M} > \theta$, $\frac{\overline{A}}{\overline{d}} > \theta$.

Then systems (1.1) *and* (1.2) *have at least one* T*-periodic solution.*

Proof It is not difficult to see that the solution of system (2.1) remains positive for all $t \in R$. Let

$$N_1(t) = e^{u(t)}, \quad N_2(t) = e^{v(t)}.$$

Then system (2.1) can be reformulated in the following form

$$\begin{cases} u'(t) = r(t) - B(t)e^{u(t-\tau(t))} - \delta e^{u(t-\sigma_1(t))}u'(t-\sigma_1(t)) - \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)}+\theta e^{u(t)}}, \\ v'(t) = -d(t) + \frac{A(t)e^{u(t-\sigma_2)}}{Me^{\gamma v(t-\sigma_2)}+\theta e^{u(t-\sigma_2)}}. \end{cases} \tag{3.1}$$

In order to apply Lemma 2.2 to study the existence of positive periodic solutions to above system, set

$$X = Y = \{z(t) = (u(t), v(t))^T \in C(R, R^2) : z(t+\omega) \equiv z(t)\}$$

and denote

$$|z| = |u| + |v|, \ |z|_\infty = \max_{t\in[0,\omega]} |z| \text{ and } \|z\| = |z|_\infty + |z'|_\infty.$$

Then X and Y are both Banach spaces when they are endowed with the norms $\|\cdot\|$ and $|\cdot|_\infty$, respectively.

Define operators L, P and Q as follows, respectively

$$L : \text{Dom}L \cap X \to Y, \quad Lz = \frac{dz}{dt}; \quad p(z) = \frac{1}{T}\int_0^T z(t)dt; \quad Q(z) = \frac{1}{T}\int_0^T z(t)dt,$$

where $\text{Dom}L = \{z|z \in X : z(t) \in C^1(R, R^2)\}$, and define $N : X \to Y$ by the form

$$Nz = \begin{pmatrix} \Lambda_1(z,t) \\ \Lambda_2(z,t) \end{pmatrix},$$

where

$$\Lambda_1(z,t) = r(t) - B(t)e^{u(t-\tau(t))} - \delta e^{u(t-\sigma_1(t))}u'(t-\sigma_1(t)) - \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}},$$
$$\Lambda_2(z,t) = -d(t) + \frac{A(t)e^{u(t-\sigma_2)}}{Me^{\gamma v(t-\sigma_2)} + \theta e^{u(t-\sigma_2)}}.$$

Then $\text{Ker}L = R^2$, and $\text{Im}L = \{z \in Y : \int_0^T z(t)dt = 0\}$ is closed in Y. Furthermore, $\dim \text{Ker}L = \text{codim Im}L$, and P, Q are both continuous projections satisfying

$$\text{Im}P = \text{Ker}L, \quad \text{Im}L = \text{Ker}Q = \text{Im}(I-Q).$$

So L is a Fredholm operator with index zero, which implies that L has a unique inverse. Define $K_p : \text{Im}L \to \text{Ker}P \cap \text{Dom}L$ being the inverse of L. By simply calculating, one has

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^T \int_0^t z(s)dsdt = \int_T^t z(s)ds + \frac{1}{T}\int_0^T sz(s)ds.$$

Therefore

$$QNz = \begin{pmatrix} \frac{1}{T}\int_0^T \Lambda_1^*(z,s)ds \\ \frac{1}{T}\int_0^T \Lambda_2(z,s)ds \end{pmatrix},$$

where $\Lambda_1^*(z,t) = r(t) - B(t)e^{u(t-\tau(t))} - \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)}+\theta e^{u(t)}}$. Then we obtain

$$K_p(I-Q)Nz = \begin{pmatrix} \int_T^t \Lambda_1(z,s)ds + \frac{1}{T}\int_0^T s\Lambda_1(z,s)ds + \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T \Lambda_1^*(z,s)ds \\ \int_T^t \Lambda_2(z,s)ds + \frac{1}{T}\int_0^T s\Lambda_2(z,s)ds + \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T \Lambda_2(z,s)ds \end{pmatrix}.$$

Obviously, it is not difficult to check by the Lebesgue convergence theorem that QN and $K_p(I-Q)N$ are both continuous. By using Arzela-Ascoli Theorem, we know that operator $K_p(I-Q)N(\overline{\Omega})$ is compact and $QN(\overline{\Omega})$ is bounded for any open set $\Omega \in X$. So $N \in \Omega$ is L-compact on $\overline{\Omega}$.

Corresponding to operator equation $Lz = \lambda Nz$ for $\lambda \in (0,1)$, we have

$$\begin{cases} u'(t) = \lambda\Big[r(t) - B(t)e^{u(t-\tau(t))} - \delta e^{u(t-\sigma_1(t))}u'(t-\sigma_1(t)) - \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)}+\theta e^{u(t)}}\Big], \\ v'(t) = \lambda\Big[-d(t) + \frac{A(t)e^{u(t-\sigma_2)}}{Me^{\gamma v(t-\sigma_2)}+\theta e^{u(t-\sigma_2)}}\Big]. \end{cases} \tag{3.2}$$

Assume that $(u(t),\ v(t))^T \in X$ is a T-period solution of (3.2) for a certain $\lambda \in (0,\ 1)$. Integrating (3.2) over the interval $[0,\ T]$, we obtain

$$\int_0^T \Big[B(t)e^{u(t-\tau(t))} + \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}}\Big]dt = \bar{r}T, \tag{3.3}$$

$$\int_0^T \Big[\frac{A(t)e^{u(t-\sigma_2)}}{Me^{\gamma v(t-\sigma_2)} + \theta e^{u(t-\sigma_2)}}\Big]dt = \bar{d}T. \tag{3.4}$$

In view of Lemma 2.3 and $[H_3]$, one has

$$\int_0^T B(t)e^{u(t-\tau(t))}dt = \int_{-\tau(0)}^{T-\tau(T)} \frac{B(\delta^{-1}(s))e^{u(s)}}{1-\tau'(\delta^{-1}(s))}ds = \int_0^T \frac{B(\delta^{-1}(s))e^{u(s)}}{1-\tau'(\delta^{-1}(s))}ds.$$

It follows from (3.3) that

$$T\bar{r} = \int_0^T \frac{B(\delta^{-1}(s))e^{u(s)}}{1-\tau'(\delta^{-1}(s))}ds + \int_0^T \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}}dt,$$

which implies

$$\int_0^T e^{u(t)}dt \le \frac{T\bar{r}}{P^L} := TU_1, \tag{3.5}$$

where $P = \frac{B(\delta^{-1}(s))}{1-\tau'(\delta^{-1}(s))}$.

Multiplying both sides of the second equation of (3.2) by $e^{\gamma v(t)}$, and integrating them from 0 to T, we have

$$\int_0^T v'(t)e^{\gamma v(t)}dt = \lambda \int_0^T \Big[-d(t)e^{\gamma v(t)} + \frac{A(t)e^{u(t-\sigma_2)+\gamma v(t)}}{Me^{\gamma v(t-\sigma_2)} + \theta e^{u(t-\sigma_2)}}\Big]dt,$$

which leads to

$$\int_0^T d(t)e^{\gamma v(t)}dt = \int_0^T \frac{A(t)e^{u(t-\sigma_2)+\gamma v(t)}}{Me^{\gamma v(t-\sigma_2)} + \theta e^{u(t-\sigma_2)}}dt < \frac{A^M}{M}\int_0^T e^{u(t-\sigma_2)}dt.$$

Then

$$\int_0^T Me^{\gamma v(t)}dt < \frac{A^M}{d^L M}\int_0^T e^{u(t-\sigma_2)}dt \le \frac{A^M TU_1}{d^L M} := TU_2. \tag{3.6}$$

Now we prove by the following two cases: $v(t) \ge 0$ and $v(t) < 0$.

Case 1 If $v(t) \ge 0$, then $e^{\gamma v(t)} \ge 1$. Together with (3.6) yields

$$U_2 \ge \frac{1}{T}\int_0^T e^{\gamma v(t)}dt \ge 1,$$

which implies that there exists $\xi_1 \in [0, T]$ such that $v(\xi_1) \leq \frac{lnU_2}{\gamma}$, therefore

$$U_1 \geq \frac{1}{T}\int_0^T e^{u(t)}dt > \frac{d^L M}{TA^M}\int_0^T e^{\gamma v(t)}dt \geq \frac{Md^L}{A^M}. \tag{3.7}$$

Thus there exists $\eta_1 \in [0, T]$ such that $u(\eta_1) \leq \max\{|lnU_1|, \ |ln\frac{Md^L}{A^M}|\}$.

Case 2 If $v(t) < 0$, then $e^{\gamma v(t)} < 1$. According to (3.3), we have

$$T\overline{r} < B^M \int_0^T e^{u(t)}dt + \frac{T}{M}\overline{C},$$

which implies

$$\int_0^T e^{u(t)}dt > \frac{T}{B^M}\left(\overline{r} - \frac{\overline{C}}{M}\right) := TL_1. \tag{3.8}$$

Again from (3.4), one has $U_1 > \frac{1}{T}\int_0^T e^{u(t)}dt > L_1$. So there exists $\eta_2 \in [0, T]$ such that $u(\eta_2) \leq \max\left\{|lnU_1|, \ |lnL_1|\right\}$. We can choose $\eta \in [0, T]$ such that $u(\eta) \leq \max\{|lnU_1|, \ |lnL_1|, \ |ln\frac{Md^L}{A^M}|\} := W_1$.

By the mean value theorem of differential calculus, we have

$$\overline{r}T > \int_0^T B(t)e^{u(t-\tau(t))}dt = B(\zeta_1)\int_0^T e^{u(t-\tau(t))}dt.$$

In view of (3.3) and Lemma 2.3, one has

$$\overline{r}T > \int_0^T B(t)e^{u(t-\tau(t))}dt = \int_0^T \frac{B(\delta^{-1}(t))}{1-\tau'(\delta^{-1}(t))}e^{u(t)}dt = E(\zeta_2)\int_0^T e^{u(t)}dt,$$

where $E(t) = \frac{B(\delta^{-1}(t))}{1-\tau'(\delta^{-1}(t))}$. Thus

$$B(\zeta_1)\int_0^T e^{u(t-\tau(t))}dt + E(\zeta_2)\int_0^T e^{u(t)}dt < 2\overline{r}T.$$

There exists $\zeta_3 \in [0, T]$ such that

$$B(\zeta_1)e^{u(\zeta_3-\tau(\zeta_3))} + E(\zeta_2)e^{u(\zeta_3)} < 2r(\zeta_3).$$

Since $\int_0^T r(t)dt > 0, \ r^M > 0$, we obtain $u(\zeta_3) < ln\frac{2r^M}{E^L}$ and $u(\zeta_3 - \tau(\zeta_3)) < \frac{2r^M}{B^L}$. Therefore

$$u(t) + \lambda\delta e^{u(t-\tau(t))} \le u(\zeta_3) + \lambda\delta e^{u(\zeta_3-\tau(\zeta_3))} + \int_0^T \left|\frac{d}{dt}u(t) + \lambda\delta e^{u(t-\tau(t))}\right| dt$$
$$< ln\frac{2r^M}{E^L} + \delta\frac{2r^M}{E^L} + 2\bar{r}T := R,$$

which implies that $u(t) < R$.

From (3.2) and (3.3), we have

$$\int_0^T \left|\frac{d}{dt}\left[u(t) + \lambda\delta e^{u(t-\tau(t))}\right]\right| dt$$
$$= \lambda\int_0^T \left|r(t) - B(t)e^{u(t-\tau(t))} - \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}}\right| dt$$
$$\le \int_0^T \left[r(t) + B(t)e^{u(t-\tau(t))} + \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}}\right] dt = 2\bar{r}T.$$

By the mean value theorem of differential calculus, we see that there exists $\xi_1 \in [0, T]$ such that

$$\int_0^T E(t)e^{u(t)}dt = E(\xi_1)\int_0^T e^{u(t)}dt = E(\xi_1)\int_0^T \frac{e^{u(t-\tau(t))}}{1-\tau'(t)}dt < \bar{r}T.$$

Thus

$$\int_0^T |u'(t)|dt \le \int_0^T r(t)dt + \int_0^T \left[B(t)e^{u(t-\tau(t))} + \frac{C(t)e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}}\right] dt$$
$$+ \int_0^T \left|\delta e^{u(t-\tau(t))}u'(t-\tau(t))\right| dt$$
$$\le 2\bar{r}T + \delta e^R \int_0^T |u'(t)|dt,$$

which implies that

$$\int_0^T |u'(t)|dt < \frac{2\bar{r}T}{1-\delta e^R} := R_1.$$

It is easy to see that

$$|u(t)| \le |u(\eta)| + \int_0^T |u'(t)|dt < W_1 + R_1 := M_1.$$

According to the second equation of (3.2), one has

$$\bar{d}T = \int_0^T \frac{A(t)e^{u(t-\sigma_2)}}{Me^{\gamma v(t-\sigma_2)} + \theta e^{u(t-\sigma_2)}}dt < A^M\int_0^T \frac{1}{\theta + \frac{M}{e^R}e^{\gamma v(t-\sigma_2)}}dt.$$

By the mean value theorem of differential calculus, there exists $\zeta \in [0, T]$ such that

$$\frac{\overline{d}T}{A^M} < \frac{T}{\theta + \frac{M}{e^R} e^{\gamma v(\zeta-\sigma_2)}},$$

which yields that

$$e^{v(\zeta-\sigma_2)} < \left(\frac{e^R A^M}{\overline{d}M} - \theta\right)^{\frac{1}{\gamma}} := R_2 \quad \text{i.e.} \quad v(\zeta) < lnR_2.$$

It is not difficult to see that $\int_0^T |v'(t)|dt < 2\overline{d}T$, thus

$$|v(t)| \leq |v(\zeta - \sigma_2)| + \int_0^T |v'(t)|dt < 2\overline{d}T + lnR_2 := M_2.$$

In view of (3.2), we get

$$|u'| \leq \frac{1}{1 + \delta e^{-M_1}} \left(r^M + B^M + \frac{C^M e^{M_2}}{Me^{-\gamma M_2} + \theta e^{-\gamma M_1}}\right) := M_3,$$
$$|v'| \leq d^M + \frac{A^M e^M}{Me^{-\gamma M_2} + \theta e^{-\gamma M_1}}; = M_4.$$

Now it is easy to see that

$$\|z\| = \left\|(u, v)^T\right\| = |z|_\infty + |z'|_\infty \leq M_1 + M_2 + M_3 + M_4 := M_5.$$

M_5 is independent of λ. Set $M^* = M_5 + 1$, and take $\Omega = \{z = (u, v)^T : z < M^*\}$. It is clear that Ω verifies the condition $[C_1]$ in Lemma 2.2.

When $z = (u, v)^T \in \partial\Omega \cap \text{KerL} = \partial\Omega \cap R^2,\ z = (u, v)^T$ is a constant vector in R^2 with $||z|| = M^*$, we have

$$QNz = \begin{pmatrix} \bar{r} - \bar{B}e^u - \frac{\overline{C}e^v}{Me^{\gamma v} + \theta e^u} \\ -\bar{d} + \frac{\overline{A}e^u}{Me^{\gamma v} + \theta e^u} \end{pmatrix} \neq 0.$$

This prove that condition $[C_2]$ in Lemma 2.2 is satisfied.

Finally, we will show that condition $[C_3]$ in Lemma 2.2 holds.

Define the homotopy $\phi :\ \text{Dom}L \times [0, 1] \to X$ by

$$\phi(u, v, \mu) = \begin{pmatrix} \bar{r} - \bar{B}e^{u(t)} \\ -\bar{d} + \frac{\overline{A}e^{u(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}} \end{pmatrix} + \mu \begin{pmatrix} -\frac{\overline{C}e^{v(t)}}{Me^{\gamma v(t)} + \theta e^{u(t)}} \\ 0 \end{pmatrix},$$

where $\mu \in [0, 1]$ is a parameter. When $(u, v)^T \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap R^2,\ (u, v)^T$ is a constant vector in R^2 with $\|(u, v)^T\| = W$. We will show that when $(u, v)^T \in \partial\Omega \cap \text{Ker}L$, $\phi((u, v)^T, \mu) \neq 0$. The following algebraic equation

$$\phi(u, v, 0) = 0$$

has a unique solution $(u^*, v^*)^T$, which satisfy $e^{u^*} = \frac{\overline{r}}{\overline{B}}$, $e^{\gamma v^*} = \frac{\overline{r}}{M\overline{B}}\big(\frac{\overline{A}}{\overline{d}} - \theta\big)$. Define the homomorphism $J : \text{Im}Q \to \text{Ker}L$, $Jz \equiv z$, a direct calculation shows that

$$\begin{aligned}\deg\{JQN, \Omega \cap \text{KerL}, 0\} &= \deg\{QN, \Omega \cap \text{KerL}, 0\} = \deg\{\phi(u, v, 1), \Omega \cap \text{KerL}, 0\}\\ &= \deg\{\phi(u, v, 0), \Omega \cap \text{KerL}, 0\} = 1 \neq 0.\end{aligned}$$

Therefore, we have verified all the requirements of Mawhin coincidence theorem in Ω and the system (3.1) has at least one positive T-periodic solution. Then, by Lemma 2.2, we derive that systems (1.1) and (1.2) have at least one positive T-periodic solution. This completes the proof. □

Remark 3.1 In (1.1) and (1.2), if $\rho = \theta_{ik} = 0$, $i = 1, 2$, $k \in N^+$, then the model discussed in [2] is a special case of (1.1) and (1.2). From the proof of our main result, we know that the condition $[A_3]$ of Theorem 3.1 in [2] could be removed. So our main result generalized Theorem 3.1 in [2].

Remark 3.2 In (1.1) and (1.2), if $\tau(t) = \sigma_1$, $\sigma_1(t) = \sigma_2$, $\gamma = 1$, $\theta_{1k} = \theta_{2k} = 0$, we can obtain that the model in [7] is a special case of (1.1) and (1.2).

4 Example

Considering the following neutral delay predator–prey model with HV-type functional response and impulse

$$\begin{cases}\left.\begin{aligned} x' &= x(t)\left[6+cost-(5+cost)x(t-\tfrac{sint}{2})-10^{-10}x'(t-cost)-\tfrac{(2+sint)y(t)}{100y^{\gamma}(t)+x(t)}\right],\\ y' &= y(t)\left[3+sint+\tfrac{(10+cost)x(t-0.1)}{100y^{\gamma}(t-0.1)+x(t-0.1)}\right],\end{aligned}\right\} t \neq 2k\pi,\quad k \in Z, 0<\gamma<1,\\ \left.\begin{aligned} x(t_k^+) - x(t_k) &= -0.1x(t_k),\\ y(t_k^+) - y(t_k) &= 0.1y(t_k).\end{aligned}\right\} t = 2k\pi,\ k \in Z,\end{cases}$$

It is easy to calculate $\tau'(t) = \frac{cost}{2} < 1$, $\theta < 1$, , $R > 24\pi$, $\delta e^R \ll 1$, $\overline{r} = 6$, $\overline{C} = 2\theta$, $M = 4\theta$, $\overline{d} = 3$, $\overline{A} = 10\theta$, $A^M > 11$. We can check that all the conditions of Theorem 3.1 hold, then the system has at least one $2\pi-$periodic solution.

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