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Qualitative Theory of Dynamical Systems

On the Equivalence of the Melnikov Functions Method and the Averaging Method

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Abstract We prove the equivalence between the Melnikov functions method and the averaging method as tools for finding limit cycles of analytic planar differential systems which are perturbations of a period annulus. We consider any possible change of variables to transform the planar system into a scalar periodic equation which perturbs a continuum of constant solutions. We prove that the Poincaré return map of the planar system and the Poincaré translation map of the scalar equation coincide. For distinct specific changes of variables this was stated before in 2004 by Buică–Llibre and proved in 2015 by Han–Romanovski–Zhang.

Keywords Perturbation of a period annulus · Melnikov functions · Averaging method · Polar-like coordinates

Mathematics Subject Classification 34C07 · 34C29 · 34C25

1 Introduction

The Melnikov functions method and the averaging method are two widely used tools for proving the existence of limit cycles of planar differential systems which are perturbations of a period annulus. As far as we know, arguments to support the idea of their equivalence appeared for the first time in [1], and recently, Han–Romanovski–Zhang in [7] discussed new aspects of this idea.

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Remind that the application of the averaging method for a planar system depends on the change of variables used to transform the planar system into a scalar periodic equation which perturbs a continuum of constant solutions. The ones used in [1] and [7], respectively, are essentially different. So it remained to clarify whether there are other such changes of variables and, in case of positive answer, whether the equivalence of the methods holds in each case. In this paper we give a positive answer to each of these two questions. More exactly, we prove that the Poincaré translation map of the corresponding scalar equation is *some* Poincaré return map of the planar system. So Theorem 6 in Sect. 2 of this paper generalizes and provides a unified proof of Theorems 5.1 and 5.2 in [1] and Lemmas 2.2 and 2.3 in [7]. Note also that the proof of [1, Theorem 5.2] (that states the equivalence between the two methods) is omitted saying that "is a direct consequence of Theorem 5.1 and the definition of displacement and Melnikov functions". (In [1] the displacement map is equivalently used instead of the Poincaré return map.) But [7, Lemma 2.3] is presented with a detailed proof.

Remind also that the Melnikov functions are defined via some Poincaré return map which depends on a transversal section and its parametrization. We discuss the relations between the Melnikov functions obtained for distinct transversal sections and parameterizations (Theorem 7 in Sect. 2), and relate these with the application of the averaging method.

We end this paper with some conclusions presented in Sect. 3.

In the sequel the averaging method for scalar periodic equations, the Melnikov functions method for planar systems and, respectively, the averaging method for planar systems are presented.

In this article we work only with real analytic differential equations and we will not mention this each time. $I \subset \mathbb{R}$ will denote a nonempty and open interval, and $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

The averaging method for scalar periodic equations. For a scalar 2π-periodic equation

$$\frac{dh}{d\theta} = \varepsilon F(\theta, h, \varepsilon), \tag{1}$$

with $F : S^1 \times I \times \mathbb{R} \to \mathbb{R}$, we denote by $\tilde{h}(\theta, h, \varepsilon)$ its solution satisfying $\tilde{h}(0, h, \varepsilon) = h$.

A solution $\tilde{h}(\theta, h^*(\varepsilon), \varepsilon)$ is 2π-periodic if and only if $h^*(\varepsilon)$ is a fixed point of the Poincaré translation map $\tilde{P}(\cdot, \varepsilon)$ defined by

$$\tilde{P}(h, \varepsilon) = \tilde{h}(2\pi, h, \varepsilon), \quad h \in I, \ |\varepsilon| \ll 1.$$

There exist some $\tilde{k} \geq 1$ and some non-null real analytic function $f_{\tilde{k}} : I \to \mathbb{R}$ such that

$$\tilde{P}(h, \varepsilon) - h = \varepsilon^{\tilde{k}} f_{\tilde{k}}(h) + O(\varepsilon^{\tilde{k}+1}).$$

Various tools like the Implicit Function Theorem, Brouwer Topological Degree, Preparation Theorems can be employed to study the fixed points of $\tilde{P}(\cdot, \varepsilon)$ via the zeros of $f_{\tilde{k}}$.

The averaging method of order $\tilde{k}$ for proving existence of 2π-periodic solutions of (1) consists in finding a representation for the function $f_{\tilde{k}}$, finding the number of its zeros and their respective multiplicities [1,5,6]. The function $f_{\tilde{k}}$ is called *averaging/bifurcation function of order $\tilde{k}$*.

Note that we are not interested in transforming (1) in the so-called averaged equation. This aspect and its relation with the above ideas were studied in [7].

The Melnikov functions method for planar systems. We consider now a family of planar analytic vector fields of the form

$$\dot{x} = X_0(x) + \varepsilon X(x, \varepsilon) \tag{2}$$

which also depends on the parameter ε. We assume that the unperturbed system

$$\dot{x} = X_0(x) \tag{3}$$

has a period annulus $\mathcal{P} \subset \mathbb{R}^2$ without equilibria. More exactly, $\mathcal{P}$ is a nonempty, open and connected subset of $\mathbb{R}^2$, filled with closed orbits of system (3). Note that

$$X_0(x) \neq 0 \quad \text{for all} \quad x \in \mathcal{P}.$$

The problem we consider is to find branches of limit cycles of the family (2) that bifurcate from the closed orbits in $\mathcal{P}$ of (3).

We consider an analytic transversal section $\Sigma = \{\gamma(h) \ : \ h \in I\}$ to the flow of (3) in $\mathcal{P}$. Remind that this means that $\Sigma \subset \mathcal{P}$, $\gamma : I \to \Sigma$ is analytic and $\gamma'(h) \wedge X_0(\gamma(h)) \neq 0$ for any $h \in I$. The wedge product $\wedge$ between two planar vectors is the determinant of the matrix which have them as columns. We have that, for $|\varepsilon| \ll 1$, Σ is also a transversal section to the flow of (2) in $\mathcal{P}$. Take some $\Sigma' \subset \Sigma$ such that the orbit of (2) that starts in Σ' returns to Σ. We denote by

$$P^{\gamma}(h, \varepsilon), \quad h \in I' \subset I, \ |\varepsilon| \ll 1,$$

the Poincaré first return map to Σ of the flow of (2) in $\mathcal{P}$. In the sequel, the interval I', which may be smaller than I, will be denoted also by I. We have that $h^*(\varepsilon)$ is a fixed point of $P^{\gamma}(\cdot, \varepsilon)$ if and only if the orbit of (2) passing through the point of Σ corresponding to the parameter $h^*(\varepsilon)$ is a closed orbit. There exist some $k \geq 1$ and some analytic function $M_k : I \to \mathbb{R}$ (called *Melnikov function of order k*) such that

$$P^{\gamma}(h, \varepsilon) - h = \varepsilon^k M_k(h) + O(\varepsilon^{k+1}).$$

The Melnikov functions method of order k for proving existence of limit cycles of (2) consists in finding a representation for the function M_k, finding the number of its zeros and their respective multiplicities. Like we mentioned in the discussion on the averaging method, knowing all these about M_k one can provide information on the number of limit cycles of (2) that bifurcates from $\mathcal{P}$.

Note that M_k may depend on the *transversal section* Σ and its *parametrization* $\gamma : I \to \Sigma$.

The averaging method for planar systems. We consider again the planar system (2) such that the unperturbed system (3) has a period annulus $\mathcal{P} \subset \mathbb{R}^2$ without equilibria. *The averaging method to study the existence of limit cycles of system* (2) consists in two steps. The first step is to find a change of variables

$$(h, \theta) = \Phi(x), \quad h \in I, \ \theta \in S^1, \ x \in \mathcal{P}$$

to transform the planar system (2) into some 2π-periodic scalar equation (1) such that, for any $|\varepsilon| \ll 1$, the orbits of (2) are

$$\Gamma_h = \{\Phi^{-1}(\tilde{h}(\theta, h, \varepsilon), \theta) \ : \ \theta \in \mathbb{R}\}, \quad h \in I,$$

which assures that $\tilde{h}(\theta, h^*(\varepsilon), \varepsilon)$ is a 2π-periodic solution of (1) if and only if $\Gamma_{h^*(\varepsilon)}$ is a closed orbit of (2).

The second step is to apply the averaging method to (1) in order to find its 2π-periodic solutions.

2 Main results

Like in the Introduction, we consider a family of analytic planar vector fields of the form

$$\dot{x} = X_0(x) + \varepsilon X(x, \varepsilon) \tag{2}$$

which depends on the small parameter ε. We assume that the unperturbed system

$$\dot{x} = X_0(x) \tag{3}$$

has a period annulus $\mathcal{P} \subset \mathbb{R}^2$ without equilibria. Denote by $\varphi_0(t, q)$, $t \in \mathbb{R}$, $q \in \mathcal{P}$, the flow of (3) in $\mathcal{P}$.

2.1 Polar-like coordinates in a period annulus

We start with some explanations. Saying that some vectorial function $\Psi : I \times S^1 \to \mathbb{R}^2$ is analytic means that $\Psi : I \times \mathbb{R} \to \mathbb{R}^2$ is 2π-periodic in the second variable and analytic.

Saying that some map $A : \mathcal{P} \to S^1$ is analytic means that for each $x^* \in \mathcal{P}$ there exists a nonempty open neighborhood $V \subset \mathcal{P}$ of x^* and an analytic function $a : V \to \mathbb{R}$ such that $A(x) = \{a(x) + 2k\pi \ : \ k \in \mathbb{Z}\}$, the equivalence class of $a(x)$ in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Note that, in this case, taking $a_1 : V_1 \to \mathbb{R}$ and $a_2 : V_2 \to \mathbb{R}$ such that $V_1 \cap V_2 \neq \emptyset$, we have that the function $a_1 - a_2 : V_1 \cap V_2 \to \mathbb{R}$ is constant. Thus, defining $\nabla A(x) = \nabla a(x)$ for all $x \in V$ and for all x^* we have that, for some analytic map $A : \mathcal{P} \to S^1$, $\nabla A : \mathcal{P} \to \mathbb{R}^2$ is an analytic vectorial function.

In this paper we will work with variables (h, θ) as in the next definition.

Definition 1 Let $\Phi = (H, A) : \mathcal{P} \to I \times S^1$ be a real analytic map. We say that $(h, \theta) = \Phi(x)$ are polar-like coordinates (or action-angle like variables) of system (3) in $\mathcal{P}$ if

(i) $\Phi : \mathcal{P} \to I \times S^1$ is a diffeomorphism and
(ii) $H : \mathcal{P} \to \mathbb{R}$ is a first integral of (3).

These new variables (h, θ) are called in this paper *action-angle like variables* since, in particular, the action-angle variables as defined in [2, Section 7.3] satisfy the conditions in the above definition. Moreover, they are also called *polar-like* since when (3) is, in particular, the linear harmonic oscillator $\dot{x}_1 = -x_2, \dot{x}_2 = x_1$, the usual polar coordinates are polar-like.

Note that condition (i) in Definition 1 is imposed just to have a genuine change of variables that maintain the analyticity, and condition (ii) reflects the fact that, for $\varepsilon = 0$, Eq. (1) implies that h must be constant along each orbit of (3).

In practice it is useful to work with the next equivalent definition.

Proposition 2 *Let $H : \mathcal{P} \to \mathbb{R}$ be a first integral of (3). We have that $(h, \theta) = \Phi(x) = (H(x), A(x))$ are polar-like coordinates of system (3) in $\mathcal{P}$ if and only if there exists an invertible analytic map $\Psi : I \times S^1 \to \mathcal{P}$ such that $\Phi = \Psi^{-1}$, $H(\Psi(h, \theta)) = h$ and the Jacobian determinant $J\Psi(h, \theta) \neq 0$ for all $(h, \theta) \in I \times S^1$.*

Proof When $(h, \theta) = \Phi(x)$ are polar-like coordinates as in Definition 1 it is clear that $\Psi = \Phi^{-1}$ satisfies the required properties. We would like to discuss in more detail the reversed implication. So, we have an invertible analytic map $\Psi : I \times S^1 \to \mathcal{P}$ with the inverse $\Phi : \mathcal{P} \to I \times S^1$. The condition $H(\Psi(h, \theta)) = h$ for all $(h, \theta) \in I \times S^1$ clearly implies that the first component of Φ is $H : \mathcal{P} \to I$, which, by hypothesis, is a first integral of (3). It remained to prove only that $\Phi = (H, A) : \mathcal{P} \to I \times S^1$ is analytic. From now on we see Ψ as $\Psi : I \times \mathbb{R} \to \mathcal{P}$, which, by hypothesis, is a local analytic diffeomorphism and is 2π-periodic with respect to the second variable. Let $x^* \in \mathcal{P}$ and $(h^*, \theta^*) \in I \times \mathbb{R}$ be such that $h^* = H(x^*)$ and $\theta^* \in A(x^*)$. We have that there exist some neighborhoods $U \subset I \times \mathbb{R}$ of (h^*, θ^*) and $V \subset \mathcal{P}$ of x^* and a local analytic inverse $\phi : V \to U$ of Ψ. Clearly the first component of ϕ is H, hence H is analytic. Denoting its second component by $a : V \to U$ we have that $a(x) \in A(x)$ for all $x \in V$, which proves the analyticity of $A : \mathcal{P} \to S^1$, too. The proof is finished. □

The following result holds.

Lemma 3 *Assume that $(h, \theta) = \Phi(x)$ are polar-like variables of (3), and let $\Psi = \Phi^{-1}$. Let $T(h)$ denote the main period of $\varphi_0(\cdot, q)$ where $H(q) = h$. Then*

(i) the first integral $H : \mathcal{P} \to \mathbb{R}$ has no critical points and, for each $\theta^* \in S^1$, $\Sigma = \{\Psi(h, \theta^*) \; : \; h \in I\}$ is a transversal section to the flow of (3) in $\mathcal{P}$;
(ii) the change to the polar-like variables (h, θ) transforms (3) in the analytic system

$$\dot{h} = 0, \quad \dot{\theta} = \Theta_0(h, \theta), \tag{4}$$

where $\Theta_0 : I \times \mathbb{R} \to \mathbb{R}$ is 2π-periodic in the second variable and of definite sign;

(iii) *there exists a real analytic function* $\tau : I \times \mathbb{R} \to \mathbb{R}$, $\tau_h(t) = \tau(h,t)$, *with the properties* $\tau_h(0) = 0$, $\tau_h : \mathbb{R} \to \mathbb{R}$ *is a diffeomorphism,* $\tau_h(t + T(h)) = \tau_h(t) + 2\pi$, *for each* $(h,t) \in I \times \mathbb{R}$, *such that*

$$\Psi(h,\theta) = \varphi_0(\tau_h^{-1}(\theta), \Psi(h,0)), \quad \textit{for all} \ (h,\theta) \in I \times \mathbb{R}.$$

Proof (i) The gradient vector $\nabla H(x)$ is not null for any $x \in \mathcal{P}$ due to the fact that is the first line in the Jacobian matrix $D\Phi(x)$ which is invertible for any $x \in \mathcal{P}$. In addition, this fact also assures that $\nabla H \wedge \nabla A \neq 0$ in $\mathcal{P}$. Since

$$\nabla H \cdot X_0 = 0 \ \text{ in } \ \mathcal{P}$$

the following relation, which will be useful in the sequel, holds true

$$\nabla A \cdot X_0 \neq 0 \ \text{ in } \ \mathcal{P}.$$

We remind that the relation $\nabla H \cdot X_0 = 0$ is valid due to the fact that, by Definition 1, H is a first integral of X_0. In order to show that $\Sigma = \{\Psi(h,\theta^*) \ : \ h \in I\}$ is a transversal section, it is sufficient to prove that

$$\frac{\partial \Psi}{\partial h}(h,\theta) \wedge X_0(\Psi(h,\theta)) \neq 0 \quad \text{for all} \ (h,\theta) \in I \times S^1. \tag{5}$$

For this, we use that the product of the Jacobian matrices $D\Phi$ and $D\Psi$ is the identity matrix. Since $D\Phi = \begin{pmatrix} \nabla H \\ \nabla A \end{pmatrix}$ and $D\Psi = \begin{pmatrix} \frac{\partial \Psi}{\partial h} & \frac{\partial \Psi}{\partial \theta} \end{pmatrix}$ we have that $\nabla A(\Psi(h,\theta)) \cdot \frac{\partial \Psi}{\partial h}(h,\theta) = 0$ for all $(h,\theta) \in I \times S^1$. This and $\nabla A \cdot X_0 \neq 0$ in $\mathcal{P}$ prove (5).

(ii) Note first that the change to the polar-like variables (h,θ) transforms (3) in the analytic system

$$\dot{h} = \nabla H \cdot X_0|_{x=\Psi(h,\theta)}, \quad \dot{\theta} = \Theta_0(h,\theta) = \nabla A \cdot X_0|_{x=\Psi(h,\theta)}. \tag{6}$$

The relations proved at (i) show that system (6) has the form (4), where $\Theta_0 : I \times \mathbb{R} \to \mathbb{R}$ is 2π-periodic in the second variable and satisfies $\Theta_0(h,\theta) \neq 0$ for all $(h,\theta) \in I \times \mathbb{R}$. This assures that Θ_0 has definite sign in $I \times \mathbb{R}$.

(iii) In the sequel we denote, for each $h \in I$, by $\tau_h(t) = \tau(h,t)$ the unique solution of the IVP $\dot{\theta} = \Theta_0(h,\theta)$, $\theta(0) = 0$. It is clear that $\tau_h(0) = 0$, and, since $\Theta_0(h,\cdot)$ is bounded and of definite sign, $\tau_h : \mathbb{R} \to \mathbb{R}$ is a real analytic diffeomorphism. Let $Q(h) > 0$ be such that $\tau_h(Q(h)) = 2\pi$. Then, the 2π-periodicity of $\Theta_0(h,\cdot)$ assures that $\tau_h(t + Q(h)) = \tau_h(t) + 2\pi$ for all $t \in \mathbb{R}$.
The following relation holds between the flows of systems (3) and, respectively, (4),

$$\Psi(h,\tau_h(t)) = \varphi_0(t, \Psi(h,0)), \quad \text{for all} \ t \in \mathbb{R}.$$

If we put here $t = \tau_h^{-1}(\theta)$ the relation from the conclusion is obtained. Also, from this relation one can get that $Q(h) = T(h)$. □

Next we present examples of polar-like coordinates (h, θ) in the period annulus $\mathcal{P}$ of some planar system $\dot{x} = X_0(x)$.

• The first example presents the classical polar coordinates. We consider

$$X_0(x_1, x_2) = (-x_2, x_1),\ \mathcal{P} = \mathbb{R}^2\backslash\{0\},\ H(x_1, x_2) = \sqrt{x_1^2 + x_2^2},\ I = (0, \infty)$$

and

$$\Psi : (0, \infty) \times S^1 \to \mathbb{R}^2\backslash\{0\}, \quad \Psi(h, \theta) = (h \cos\theta, h \sin\theta).$$

It is easy to prove using Proposition 2 that $(h, \theta) = \Phi(x)$, with $\Phi = \Psi^{-1}$, are polar-like coordinates in the sense of Definition 1. Denote $\Phi = (H, \mathcal{A})$. We consider that it worth to make some more comments with respect to *the polar angle map* $\mathcal{A} : \mathbb{R}^2\backslash\{0\} \to S^1$. For this we consider the following four analytic functions

$$\begin{aligned}
&a_1 : \{(x_1, x_2)\ :\ x_1 > 0\} \to (-\frac{\pi}{2}, \frac{\pi}{2}), \quad a_1(x) = \arctan\frac{x_2}{x_1}\\
&a_2 : \{(x_1, x_2)\ :\ x_1 < 0\} \to (\frac{\pi}{2}, \frac{3\pi}{2}), \quad a_2(x) = \pi + \arctan\frac{x_2}{x_1}\\
&a_3 : \{(x_1, x_2)\ :\ x_2 > 0\} \to (0, \pi), \quad a_3(x) = \operatorname{arccot}\frac{x_1}{x_2}\\
&a_4 : \{(x_1, x_2)\ :\ x_2 < 0\} \to (\pi, 2\pi), \quad a_4(x) = \pi + \operatorname{arccot}\frac{x_1}{x_2}\,.
\end{aligned}$$

Note that the union of their domains of definition is $\mathbb{R}^2\backslash\{0\}$ and that, for any $i \in \{1, 2, 3, 4\}$, $a_i(x) \in \mathcal{A}(x)$ for all x for which $a_i(x)$ is defined. This proves that $\mathcal{A} : \mathbb{R}^2\backslash\{0\} \to S^1$ is analytic. In addition, one can easily see that

$$\nabla\mathcal{A}(x) = \frac{1}{|x|^2}\, x^{\perp} \quad \text{for all } x \in \mathbb{R}^2\backslash\{0\},$$

where $|x|^2 = x_1^2 + x_2^2$ and $x^{\perp} = (-x_2, x_1)$.

Another remark is that the function $\arctan(x_2/x_1)$ is defined almost everywhere in $\mathbb{R}^2\backslash\{(0, 0)\}$ and is the expression of both $a_1(x)$ and $a_2(x) - \pi$. This explains why the polar angle is many times used as having the expression $\theta = \arctan(x_2/x_1)$.

• The second example presents the quasi homogeneous polar coordinates of Lyapunov. We consider

$$X_0(x_1, x_2) = (-x_2^3, x_1),\ \mathcal{P} = \mathbb{R}^2 \setminus \{0\},\ H(x_1, x_2) = \sqrt[4]{2x_1^2 + x_2^4},\ I = (0, \infty)$$

and

$$\Psi : (0, \infty) \times S^1 \to \mathbb{R}^2\backslash\{0\}, \quad \Psi(h, \theta) = (h^2 \operatorname{Cs}\theta, h \operatorname{Sn}\theta),$$

where $(\mathrm{Cs}\, t, \mathrm{Sn}\, t) = \varphi_0(t, (1/\sqrt{2}, 0))$ for all $t \in \mathbb{R}$ and are periodic functions with the main period $T > 0$. The expression of T is given, for example, in [4]. This time S^1 denotes $\mathbb{R}/T\mathbb{Z}$. Using the classical properties $2\mathrm{Cs}^2\,\theta + \mathrm{Sn}^4\,\theta = 1$, $(\mathrm{Cs}\,\theta)' = -\mathrm{Sn}^3\,\theta$, $(\mathrm{Sn}\theta)' = \mathrm{Cs}\theta$ (which follows immediately by the definition of Cs and Sn), one can easily prove using Proposition 2 that (h, θ) are polar-like coordinates for $\dot{x} = X_0(x)$.

- For another example we consider $X_0(x_1, x_2) = (-x_2(1+x_1),\ x_1(1+x_1))$, $\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2\backslash\{0\} : x_1^2+x_2^2 < 1\}$, $H(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, $I = (0, 1)$ and

$$\Psi : (0, 1) \times S^1 \to \mathcal{P}, \quad \Psi(h, \theta) = (h\cos\theta, h\sin\theta).$$

Again Proposition 2 assures that (h, θ) are polar-like coordinates for $\dot{x} = X_0(x)$.

- An example from [1] considers $X_0(x_1, x_2) = (-x_2+x_1^2,\ x_1+x_1x_2)$, $\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2\backslash\{0\} : x_1^2-2x_2-1 < 0\}$, $H(x_1, x_2) = \dfrac{\sqrt{x_1^2 + x_2^2}}{x_2 + 1}$, $I = (0, 1)$ and

$$\Psi : (0, 1) \times S^1 \to \mathcal{P}, \quad \Psi(h, \theta) = \left(\frac{h\cos\theta}{1 - h\sin\theta}, \frac{h\sin\theta}{1 - h\sin\theta}\right).$$

Again, using Proposition 2 one can prove that (h, θ) are polar-like coordinates in the sense of Definition 1 for the system $\dot{x} = X_0(x)$. Note that

$$\Psi(h, \theta) = \frac{h}{1 - h\sin\theta}(\cos\theta, \sin\theta),$$

thus, θ is also the polar angle coordinate (like in the previous examples, except the Lyapunov coordinates).

The next result characterizes the vector fields X_0 for which the polar angle map can be used to define polar-like coordinates.

Proposition 4 *Let $H : \mathcal{P} \to \mathbb{R}$ be an analytic first integral of system* (3) *without critical points. Assume that 0 belongs to the bounded component of $\mathbb{R}^2\backslash\mathcal{P}$.*

If $x^\perp \cdot X_0(x) \neq 0$ for all $x \in \mathcal{P}$ then $(h, \theta) = (H(x), \mathcal{A}(x))$ (where $\mathcal{A}$ is the polar angle map) are polar-like coordinates of (3).

The reversed implication is also valid.

Proof Since H has no critical points in $\mathcal{P}$ (which is an open and connected subset of the plane), we have that $I = H(\mathcal{P})$ is an open interval of real numbers and that two distinct closed orbits have two distinct energy levels. Also, the fact that the origin of the plane belongs to the bounded component of $\mathbb{R}^2\backslash\mathcal{P}$, implies that any closed orbit in $\mathcal{P}$ encircles the origin, which, in turn, gives that the polar angle map takes any value in S^1 along each closed orbit. All these assure that the map $\Phi = (H, \mathcal{A}) : \mathcal{P} \to I \times S^1$ is surjective. Moreover, Φ is one-to-one if and only if for each $h \in I$ the map $\mathcal{A}|_{\Gamma_h}$ is one-to-one. Here Γ_h is the closed orbit of energy level h. We claim that $\mathcal{A}|_{\Gamma_h}$ is one-to-one for each $h \in I$. Then Φ is invertible.

On the other hand, since $\nabla\mathcal{A}(x) = \frac{1}{|x|^2}\, x^{\perp}$ we have that the condition $x^{\perp}\cdot X_0(x) \neq 0$ implies that $\nabla\mathcal{A}\cdot X_0 \neq 0$, which, in turn, since $\nabla H \cdot X_0 = 0$, implies that $\nabla H \wedge \nabla\mathcal{A} \neq 0$. Note that the Jacobian determinant $J\Phi = \nabla H \wedge \nabla\mathcal{A}$. Hence, Φ is a local analytic diffeomorphism. Remind that we proved before that $\Phi = (H, \mathcal{A})$ is invertible and that, by hypothesis, H is a first integral of X_0. This is sufficient to conclude that Φ satisfies all the conditions in Definition 1.

Now we prove the claim. For fixed $h \in I$ denote by $\varphi(t)$ the solution corresponding to the orbit Γ_h. Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be an analytic function such that $\vartheta(t) \in A(\varphi(t))$. Then $\dot{\vartheta} = (\nabla A \cdot X_0) \circ \varphi = (\frac{1}{|x|^2}\, x^{\perp} \cdot X_0) \circ \varphi \neq 0$. Then the function ϑ is one-to-one. This implies that $\mathcal{A}|_{\Gamma_h}$ is one-to-one. The claim is proved.

We skip the proof of the reversed implication since it is easy. □

The next result is a reciprocal of Lemma 3 (iii), thus another characterization of the notion of polar-like coordinates is obtained. It also shows that given a transversal section $\Sigma = \{\gamma(h) : h \in I\}$, there exists polar-like variables such that $\Psi(h, 0) = \gamma(h)$.

Proposition 5 *Let $\Sigma = \{\gamma(h) : h \in I\}$ be an analytic transversal section to the flow of (3) in $\mathcal{P}$. Denote $T(h) > 0$ the main period of $\varphi_0(\cdot, \gamma(h))$.*

Let $\tau : I \times \mathbb{R} \to \mathbb{R}$ $\tau_h(t) = \tau(h, t)$, be a real analytic function with the properties $\tau_h(0) = 0$, $\tau_h : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, $\tau_h(t + T(h)) = \tau_h(t) + 2\pi$, for each $(h, t) \in I \times \mathbb{R}$. Consider the map

$$\Psi : I \times \mathbb{R} \to \mathcal{P}, \quad \Psi(h, \theta) = \varphi_0(\tau_h^{-1}(\theta), \gamma(h)), \qquad \text{for all } (h, \theta) \in I \times \mathbb{R}.$$

Then $(h, \theta) = \Psi^{-1}(x)$ are polar-like coordinates of (3).

Proof First we remind that, given the analytic transversal section to the flow of (3) in $\mathcal{P}$, $\Sigma = \{\gamma(h) : h \in I\}$, there exists a unique analytic first integral $H : \mathcal{P} \to \mathbb{R}$ without critical points such that $H(\gamma(h)) = h$ for all $h \in I$.

Now note that Ψ is 2π-periodic in the second variable. Indeed, for any $(h, \theta) \in I \times \mathbb{R}$ we have

$$\Psi(h, \theta + 2\pi) = \varphi_0(\tau_h^{-1}(\theta + 2\pi), \gamma(h)) = \varphi_0(\tau_h^{-1}(\theta) + T(h), \gamma(h)) = \Psi(h, \theta).$$

Hence $\Psi : I \times S^1 \to \mathcal{P}$ is well-defined and using its expression, it is not difficult to see that Ψ is an invertible analytic map and that $H(\Psi(h, \theta)) = h$ for all $(h, \theta) \in I \times S^1$. Using Proposition 2 it remained to prove that the Jacobian determinant $J\Psi(h, \theta) \neq 0$ for all $(h, \theta) \in I \times S^1$.

We have that

$$\begin{aligned}
J\Psi &= \frac{\partial \Psi}{\partial h} \wedge \frac{\partial \Psi}{\partial \theta}\,,\\
\frac{\partial \Psi}{\partial h} &= X_0(\Psi(h, \theta)) \frac{d}{dh}(\tau_h^{-1}(\theta)) + D_q\varphi_0(\tau_h^{-1}(\theta), \gamma(h))\gamma'(h),\\
\frac{\partial \Psi}{\partial \theta} &= X_0(\Psi(h, \theta)) \frac{d}{d\theta}(\tau_h^{-1}(\theta)).
\end{aligned}$$

We claim that for any $(h, t) \in I \times \mathbb{R}$, the vectors $y_1(t)$ and $y_2(t)$ are linearly independent, where

$$y_1(t) = D_q\varphi_0(t, \gamma(h))\gamma'(h) \text{ and } y_2(t) = X_0(\varphi_0(t, \gamma(h))).$$

It is not difficult to see that this implies that the vectors $\dfrac{\partial\Psi}{\partial h}$ and $\dfrac{\partial\Psi}{\partial\theta}$ are linearly independent, thus $J\Psi(h, \theta) \neq 0$, for all $(h, \theta) \in I \times \mathbb{R}$.

In order to prove the claim note that the functions $y_1(t)$ and $y_2(t)$ satisfy the linear variational system

$$\dot{y} = DX_0(\varphi_0(t, \gamma(h)))y.$$

Then the linear independence of $y_1(0)$ and $y_2(0)$ implies the linear independence of $y_1(t)$ and $y_2(t)$ for all $t \in \mathbb{R}$. Using that $\varphi_0(0, q) = q$ and the Jacobian matrix $D_q\varphi_0(0, q)$ is the identity, we get

$$y_1(0) = \gamma'(h) \text{ and } y_2(0) = X_0(\gamma(h)).$$

These vectors are linearly independent due to the fact that Σ is transversal to X_0. Thus the claim is proved and also the proposition. □

Remark that polar-like coordinates for Hamiltonian systems were defined in [7] like in the above proposition, with H being the Hamiltonian and the angle as function of time given by $\tau_h(t) = \frac{2\pi}{T(h)}t$. Of course, if one prefers to work with the angle variable as function of time, this is the most simple formula to consider.

2.2 Poincaré maps and the averaging method

Theorem 6 *Let $\Phi = (H, A) : \mathcal{P} \to I \times S^1$ be such that $(h, \theta) = \Phi(x)$ are polar-like coordinates of system (3) in $\mathcal{P}$. Let $\Psi = \Phi^{-1}$. Then*

(i) this change of variables transforms system (2) into an analytic system of the form

$$\dot{h} = \varepsilon F(h, \theta, \varepsilon), \quad \dot{\theta} = \Theta(h, \theta) + \varepsilon R(h, \theta, \varepsilon), \quad h \in I, \quad \theta \in S^1, \tag{7}$$

where $\Theta : I \times S^1 \to \mathbb{R}$ has definite sign, and for which $\{\theta = 0\}$ is a transversal section;

(ii) the orbits of (2) are

$$\Gamma_h = \{\Psi(\tilde{h}(\theta, h, \varepsilon), \theta) \ : \ \theta \in S^1\}, \quad h \in J,$$

where $\tilde{h}(\cdot, h, \varepsilon)$ satisfies $\tilde{h}(0, h, \varepsilon) = h$ and is a solution of the scalar equation

$$\frac{dh}{d\theta} = \varepsilon\frac{F(h, \theta, \varepsilon)}{\Theta(h, \theta) + \varepsilon R(h, \theta, \varepsilon)}, \quad |\varepsilon| \ll 1, \quad h \in J, \quad \theta \in S^1; \tag{8}$$

where J is an open and bounded interval such that its closure $\overline{J} \subset I$;

(iii) the Poincaré 2π-translation map of (8) is the Poincaré first return map of (2) associated to the transversal section $\Sigma = \{\Psi(h, 0) \,:\, h \in J\}$.

Proof (i) follows directly from Definition 1 and Lemma 3.

(ii) Denote by $(\chi(t, h, \varepsilon), \tau(t, h, \varepsilon))$ the solution of system (7) satisfying $h(0) = h$, $\theta(0) = 0$. Since $\{\theta = 0\}$ is a transversal section to (7), the orbits of (2) are

$$\Gamma_h = \{\Psi(\chi(t, h, \varepsilon), \tau(t, h, \varepsilon)) \,:\, t \in \mathbb{R}\}, \quad h \in I. \tag{9}$$

The fact that $\Theta : J \times S^1 \to \mathbb{R}$ has definite sign and is bounded assures that, for $|\varepsilon| \ll 1$, the right-hand side of the second equation in (7) also has definite sign and is bounded on $J \times S^1$. Then the function $\tau(\cdot, h, \varepsilon) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism. Moreover, this assures that Eq. (8) is well-defined in $J \times \mathbb{R}$. Then

$$\tilde{h}(\theta, h, \varepsilon) = \chi(t, h, \varepsilon) \quad \text{when } \theta = \tau(t, h, \varepsilon).$$

The conclusion follows after we replace this relation in (9).

(iii) We use the description of the orbits of (2) as at (ii). When starting from the point $\Psi(h, 0) \in \Sigma$, the first return of Γ_h to Σ is at the point $\Psi(\tilde{h}(2\pi, h, \varepsilon), 2\pi) \in \Sigma$. Then, by definition, the Poincaré return map to Σ of (2) is

$$P(h, \varepsilon) = \tilde{h}(2\pi, h, \varepsilon)$$

and the proof is finished. □

When $(h, \theta) = (H(x), \mathcal{A}(x))$ (where $\mathcal{A}$ is the polar angle coordinate, the (partial) proof of this result has been given in [1, Theorems 5.1 and 5.2], while when the angle as function of time is $\theta = 2\pi t / T(h)$ (where $T(h)$ is the main period of the closed orbit of level h), the proof was given in [7, Lemmas 2.2 and 2.3].

2.3 Melnikov functions

In the Introduction we defined the Poincaré return map $P^\gamma(h, \varepsilon)$ of (2) associated to an analytic transversal section $\Sigma = \{\gamma(h) \,:\, h \in I\}$. Given some analytic first integral of (3), $H : \mathcal{P} \to \mathbb{R}$, without critical points we say that Σ is parameterized with the values of H when $H(\gamma(h)) = h$ for all $h \in I$. Note that, when considering another parametrization of the same transversal section, $\tilde{\gamma} : J \to \Sigma$, then Σ is parameterized with the values of the first integral $\tilde{H} = \tilde{\gamma}^{-1} \circ \gamma \circ H$, which is also analytic and without critical points in $\mathcal{P}$.

In the next result we will consider two distinct parameterizations for Σ, $\gamma : I \to \Sigma$ and $\tilde{\gamma} : J \to \Sigma$, and a geometrically distinct to Σ transversal section S, both parameterized with the values of the same first integral H.

Theorem 7 *Let P^γ and $P^{\tilde\gamma}$ be the Poincaré return maps of* (2) *associated to $\gamma : I \to \Sigma$ and, respectively, $\tilde\gamma : J \to \Sigma$. Let $k, \tilde k \geq 1$ be such that*

$$P^\gamma(h,\varepsilon) = h + \varepsilon^k M_k^\gamma(h) + O(\varepsilon^{k+1}) \quad and \quad P^{\tilde\gamma}(s,\varepsilon) = s + \varepsilon^{\tilde k} M_{\tilde k}^{\tilde\gamma}(s) + O(\varepsilon^{\tilde k+1})\,.$$

Then the corresponding Melnikov functions satisfy

$$\tilde k = k \quad and \quad M_k^{\tilde\gamma}(s)\tilde\gamma'(s) = M_k^\gamma(h)\gamma'(h) \;\; for \;\; s = (\tilde\gamma^{-1}\circ\gamma)(h).$$

We consider now two analytic transversal sections Σ and S both parameterized with the values of the same first integral H of (3)*. Denote the corresponding Poincaré maps by $P(h,\varepsilon)$ and $R(h,\varepsilon)$ and let $k, l \geq 1$ be such that*

$$P(h,\varepsilon) = h + \varepsilon^k M_k(h) + O(\varepsilon^{k+1}) \quad and \quad R(h,\varepsilon) = h + \varepsilon^l \mu_l(h,\varepsilon) + O(\varepsilon^{l+1})\,.$$

Then the corresponding Melnikov functions satisfy

$$k = l \quad and \quad M_k(h) = \mu_k(h).$$

Proof Let $h \in I$ and $s = (\tilde\gamma^{-1}\circ\gamma)(h)$. By the definitions of Poincaré map and transversal section, we have

$$\gamma(P^\gamma(h,\varepsilon)) = \tilde\gamma(P^{\tilde\gamma}(s,\varepsilon)).$$

Hence

$$\gamma(h + \varepsilon^k M_k^\gamma(h) + O(\varepsilon^{k+1})) = \tilde\gamma(s + \varepsilon^{\tilde k} M_{\tilde k}^{\tilde\gamma}(s) + O(\varepsilon^{\tilde k+1})).$$

After taking the derivative with respect to ε in the above relation we get

$$\begin{aligned}&\gamma'(h + \varepsilon^k M_k^\gamma(h) + O(\varepsilon^{k+1}))(k\varepsilon^{k-1} M_k^\gamma(h) + O(\varepsilon^{k+1}))\\ &\quad = \tilde\gamma'(s + \varepsilon^{\tilde k} M_{\tilde k}^{\tilde\gamma}(s) + O(\varepsilon^{k+1}))(\tilde k\varepsilon^{\tilde k-1} M_{\tilde k}^{\tilde\gamma}(s) + O(\varepsilon^{k+1})).\end{aligned}$$

Assuming that $\tilde k < k$, we obtain that the Melnikov function $M_{\tilde k}^{\tilde\gamma}$ is identically 0, which contradicts the definition of $\tilde k$. Because of the symmetry we get $\tilde k = k$. The relation from the conclusion is obtained after dividing by $k\varepsilon^{k-1}$ and then taking $\varepsilon = 0$.

We consider now the case of two transversal sections. Denote by $\pi_\varepsilon(h) = \pi(\varepsilon, h)$ the transition map between Σ and S along the flow of (2). Note that $\pi_0(h) = h$, hence $\pi_0'(h) = 1$. Then

$$\pi_\varepsilon \circ P_\varepsilon = R_\varepsilon \circ \pi_\varepsilon,$$

which further gives

$$\pi(h + \varepsilon^k M_k(h) + O(\varepsilon^{k+1}), \varepsilon) = \pi(h,\varepsilon) + \varepsilon^l \mu_l(\pi(h,\varepsilon)) + O(\varepsilon^{l+1})\,. \qquad (10)$$

It can be proved that

$$\lim_{\varepsilon \to 0} \frac{\pi(h + \varepsilon^k M_k(h) + O(\varepsilon^{k+1}), \varepsilon) - \pi(h, \varepsilon)}{\varepsilon^l} = 0 \quad \text{when} \ \ l < k.$$

Assuming that $l < k$, after dividing by ε^l relation (10) and taking the limit as $\varepsilon \to 0$, we would obtain that the Melnikov function $\mu_l(\pi_0(h)) = \mu_l(h)$ is identically 0. This contradicts the definition of l. Because of the symmetry it is not necessary to consider the case $l > k$. Hence $l = k$. It can be proved that

$$\lim_{\varepsilon \to 0} \frac{\pi(h + \varepsilon^k M_k(h) + O(\varepsilon^{k+1}), \varepsilon) - \pi(h, \varepsilon)}{\varepsilon^k} = M_k(h).$$

Now, the relation between the Melnikov functions is obtained after dividing by ε^k relation (10) and taking the limit as $\varepsilon \to 0$. □

3 Conclusions

We conclude that, from theoretical point of view, the averaging method for finding limit cycles of planar systems, applied using any possible polar-like coordinates, is nothing else than the study of *some* Poincaré return map. If, for a given system studied with both methods, M_k denotes the Melnikov function and $f_{\tilde{k}}$ denotes the averaging function, then $k = \tilde{k}$ and either $M_k(h) = f_k(h)$ (when the first integral H used to parameterize the transversal section associated to M_k is also the first integral associated to the polar-like coordinates), or $M_k(h)\gamma'(h) = f_k(s)\tilde{\gamma}'(s)$ for $s = (\tilde{\gamma}^{-1} \circ \gamma)(h)$ and $\tilde{\gamma}(s) = \Psi(s, 0)$.

From practical point of view, the computations necessary to find a representation (or even the expression in some concrete examples) of f_k and, respectively, of M_k, are totally different. We do not discuss here the complexity of these computations in neither of these two methods. Hence, we do not conclude on the efficiency of these methods. The application of one method or another (or the choosing of the polar-like variables when applying the averaging method) is a matter of personal taste and skills, as it is mentioned also by Li in [3, Section 3.4]. Just as a simple example, remind that the first order Melnikov function for a near-Hamiltonian system with the period annulus $\mathcal{P} = \{\Gamma_h \ : \ h \in I\}$, where h is the level of the Hamiltonian H, has the representation

$$M_1(h) = \oint_{\Gamma_h} Q(x)dx_1 - P(x)dx_2,$$

where (P, Q) are the components of $X_0(x)$ from (3), and (x_1, x_2) are the components of x. It can be proved that, performing the polar-like change of variables in the above contour integral one arrives to the bifurcation/averaging function f_1,

$$f_1(h) = \int_0^{2\pi} \frac{F(h, \theta, 0)}{\Theta(h, \theta)} d\theta \, ,$$

where F and Θ are as in Eq. (8). It is known that the computation of higher order Melnikov functions and, respectively, bifurcation functions, is more involved [5,8].

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References

1. Buică, A., Llibre, J.: Averaging methods for finding periodic orbits via Brouwer degree. Bull. Sci. Math. **128**, 7–22 (2004)
2. Chicone, C.: Ordinary Differential Equations with Applications. Springer, New York (2006)
3. Christopher, C., Li, C.: Limit Cycles of Differential Equations. Advanced Courses in Mathematics - CRM Barcelona. Birkhaüser, Boston (2007)
4. Cima, A., Gasull, A., Mañosas, F.: Cyclicity of a family of vector fields. J. Math. Anal. Appl. **196**, 921–937 (1995)
5. Giné, J., Grau, M., Llibre, J.: Averaging theory at any order for computing periodic orbits. Phys. D. **250**, 58–65 (2013)
6. Giné, J., Llibre, J., Wu, K., Zhang, X.: Averaging methods of arbitrary order, periodic solutions and integrability. J. Differ. Equ. **260**, 4130–4156 (2016)
7. Han, M., Romanovski V. G., Zhang X.: Equivalence of the Melnikov function method and the averaging method. Qual. Theory Dyn. Syst. **15**, 471–479 (2016)
8. Ilyashenko, Y., Yakovenko, S.: Lectures on analytic differential equations. Graduate Studies in Mathematics, vol. 86, p 625. American Mathematical Society, Providence, RI (2008)