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# Complex Backward–Forward Derivative Operator in Non-local-In-Time Lagrangians Mechanics

**Rami Ahmad El-Nabulsi**[1]

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**Abstract** In this paper we introduce non-local-in-time complexified Lagrangians characterized by an expanded complex backward–forward derivative operator which generalize the classical complex derivative operator. We developed the Euler–Lagrange equations and solved them for some special case. We discuss their implications in Newtonian mechanics where a number of applications were illustrated.

**Keywords** Expanded complex backward–forward derivative operator · Nonlocal-in-time complexified Lagrangians · Complexified Newton's mechanics

**Mathematics Subject Classification** Primary 26A33 · 49S05; Secondary 70S05

## 1 Introduction

The theory of non-local-in-time mechanics is not new and was addressed in literature long time ago since the work of Nelson in his derivation of the Schrödinger equation from classical mechanics [18]. Other related approach concerns the work of Nottale on scale relativity characterized by the fractal spacetime concept which amount to consider both the forward and backward motion simultaneously [19]. This concept leads to relate the Schrödinger equation to the complexified Newtonian mechanics. In reality, considering concurrently forward and backward motion was addressed since 1948 by Richard Feynman [10,11] in his spacetime approach to non-relativistic quantum mechanics where position differences of the particle are shifted with respect to each

✉ Rami Ahmad El-Nabulsi
nabulsiahmadrami@yahoo.fr

1 College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641112, Sichuan, China

other. This approach was used recently in [21] where extended Newtonian mechanics was obtained after replacing the kinetic energy of the particles in motion by a non-local-in-time kinetic energy (in fact this was first proposed by Suykens [21] and then further extended by Li et al. [13]). In [20], this approach was used to deal with nonconservative systems and in [6] it was generalized for the case of deformed or non-standard Lagrangians type. It is notable that for non-differentiable particles curves as in quantum mechanics, the classical Lagrangian formalism fails [11]. This problem was addressed in literature by introducing the scale derivative by means of a complex backward–forward derivative operator (CBFDO) which applies to non-differentiable functions [2,19] and was generalized in [19] for the case of fractional embedding theory [4], fractional actionlike variational approach [7,8] and equations on time scales [5,14–16]. These theories seem to be promising, yet lots of work are still required. In this paper, we would like to extend Newton's law by means of a non-local-in-time complexified Lagrangians characterized by an expanded E-CBFDO which generalizes the work done in [6,13,21]. Our main aim is to explore the main consequences of the E-CBFDO in Newtonian mechanics and not to solve the list of higher-order nonlinear differential equations obtained.

The paper is organized as follows: in Sect. 2, we introduce the basic Lagrangian setups of our model: we derive the modified Euler–Lagrange equation and we discuss some of its main consequences; in Sect. 3, we illustrate by discussing the dynamics of some specific Lagrangian models; in the same section we construct the resulting Newton's law and we discuss some of its main impacts; in Sect. 3, we illustrate by discussing some physical applications; conclusions and perspectives are given in Sect. 4.

## 2 Complexified Nonlocal-In-Time Lagrangian from E-CBFDO: The Complex Euler–Lagrange Equation and the Corresponding Newton's Law

We start by introducing the basic setups of our approach.

**Definition 2.1** Let $\mathsf{X}$ be a coordinate of a given dynamical system, $t$ be the proper time and $\tau$ a positive constant. The backward and forward displacement are given respectively by $\mathsf{X}(t-\tau)$ and $\mathsf{X}(t+\tau)$. We define the CBFDO $\mathsf{D}$ of $\mathsf{X}$ by [19]:

$$\begin{aligned}\frac{\mathsf{dX}}{\mathsf{d}t} &\equiv \mathsf{DX} \triangleq \frac{1}{2}\left(\frac{d\mathsf{X}(t+\tau)}{dt}+\frac{d\mathsf{X}(t-\tau)}{dt}\right)-\frac{i}{2}\left(\frac{d\mathsf{X}(t+\tau)}{dt}-\frac{d\mathsf{X}(t-\tau)}{dt}\right),\\ &\equiv \frac{1}{2}\left(D\mathsf{X}(t+\tau)+D\mathsf{X}(t-\tau)\right)-\frac{i}{2}\left(D\mathsf{X}(t+\tau)-D\mathsf{X}(t-\tau)\right)\\ &\equiv \frac{1-i}{2}D\mathsf{X}(t+\tau)+\frac{1+i}{2}D\mathsf{X}(t-\tau), \end{aligned} \tag{1}$$

where $i=\sqrt{-1}\in\mathbb{C}$ and $D=d/dt$.

*Remark 2.1* In fact we can write the $\mathsf{D}$ as:

$$\mathsf{D} = \frac{1}{2}\left(\left.\frac{d}{dt}\right|_B + \left.\frac{d}{dt}\right|_F\right) - \frac{i}{2}\left(\left.\frac{d}{dt}\right|_B - \left.\frac{d}{dt}\right|_F\right) \equiv \frac{1}{2}\left(\mathsf{D}_B + \mathsf{D}_F\right) - \frac{i}{2}\left(\mathsf{D}_B - \mathsf{D}_F\right),$$

where $\mathsf{D}_B\mathsf{X} \equiv D\mathsf{X}(t+\tau)$ and $\mathsf{D}_F\mathsf{X} \equiv D\mathsf{X}(t-\tau)$ are respectively for "backward" and "forward" temporal derivative. Remark that $\mathsf{D}$ holds for only first-order temporal derivative therefore we denote it by $\mathsf{D}^1$.

**Definition 2.2** Using the Taylor-series expansions:

$$\begin{aligned}\mathsf{X}(t+\tau) &\approx \mathsf{X}(t) + \tau D\mathsf{X}(t) + \frac{1}{2!}\tau^2 D^2\mathsf{X}(t) + \cdots + \frac{1}{n!}\tau^n D^{(n)}\mathsf{X}(t)\\ &= \mathsf{X}(t) + \sum_{k=1}^{n}\frac{1}{k!}\tau^k D^{(k)}\mathsf{X}(t), \qquad (2)\\ \mathsf{X}(t-\tau) &\approx \mathsf{X}(t) - \tau D\mathsf{X}(t) + \frac{1}{2!}\tau^2 D^2\mathsf{X}(t) + \cdots + \frac{(-1)^n}{n!}\tau^n D^{(n)}\mathsf{X}(t)\\ &= \mathsf{X}(t) + \sum_{k=1}^{n}\frac{(-1)^k}{k!}\tau^k D^{(k)}\mathsf{X}(t), \qquad (3)\end{aligned}$$

we define the E-CBFDO acting on $\mathsf{X}(t)$ by:

$$\begin{aligned}\mathsf{D}^n\mathsf{X}(t) &= \frac{1-i}{2}\left(D\mathsf{X}(t) + \sum_{k=1}^{n}\frac{1}{k!}\tau^k D^{(k+1)}\mathsf{X}(t)\right)\\ &\quad + \frac{1+i}{2}\left(D\mathsf{X}(t) + \sum_{k=1}^{n}\frac{(-1)^k}{k!}\tau^k D^{(k+1)}\mathsf{X}(t)\right),\\ &\equiv D\mathsf{X}(t) + \frac{1-i}{2}\sum_{k=1}^{n}\frac{1}{k!}\tau^k D^{(k+1)}\mathsf{X}(t)\\ &\quad + \frac{1+i}{2}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\tau^k D^{(k+1)}\mathsf{X}(t),\\ &= D\mathsf{X}(t) + \frac{1}{2}\sum_{k=1}^{n}(1+(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)}\mathsf{X}(t)\\ &\quad - \frac{i}{2}\sum_{k=1}^{n}(1-(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)}\mathsf{X}(t). \qquad (4)\end{aligned}$$

**Definition 2.3** We define the non-local-in-time generalized coordinates by:

$$\mathsf{Y}_\tau^n = \frac{\mathsf{X}(t+\tau)+\mathsf{X}(t-\tau)}{2} - i\frac{\mathsf{X}(t+\tau)-\mathsf{X}(t-\tau)}{2},$$
$$\equiv \mathsf{X}(t)+\frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{k!}(1+(-1)^k)\tau^k D^{(k)}\mathsf{X}(t) - i\sum_{k=1}^{n}\frac{1}{k!}\tau^k(1-(-1)^k)D^{(k)}\mathsf{X}(t)\right), \tag{5}$$

and its corresponding E-CBFDO by:

$$\mathsf{D}^n = D + \frac{1}{2}\sum_{k=1}^{n}(1+(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)} - \frac{i}{2}\sum_{k=1}^{n}(1-(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)} \equiv D + \mathcal{D}_R^n - i\mathcal{D}_I^n, \tag{6}$$

where

$$\mathcal{D}_R^n \triangleq \frac{1}{2}\sum_{k=1}^{n}(1+(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)}, \tag{7}$$

$$\mathcal{D}_I^n = \frac{1}{2}\sum_{k=1}^{n}(1-(-1)^k)\frac{1}{k!}\tau^k D^{(k+1)}, \tag{8}$$

Clearly, for $n = 1$, $\mathsf{Y}_\tau^1 = \mathsf{X}(t) - i\tau\dot{\mathsf{X}}(t)$, $\mathsf{D}^1 = D - i\tau D^{(2)}$ and therefore:

$$\mathsf{D}^1\mathsf{Y}_\tau^1 = (D - i\tau D^{(2)})(\mathsf{X}(t) - i\tau\dot{\mathsf{X}}(t)) = \dot{\mathsf{X}}(t) - 2i\tau\ddot{\mathsf{X}}(t) - \tau^2\dddot{\mathsf{X}}(t),$$

whereas for $n = 2$, $\mathsf{Y}_\tau^2 = \mathsf{X}(t) + \frac{1}{2}(\tau^2\ddot{\mathsf{X}}(t) - 2i\tau\dot{\mathsf{X}}(t))$, $\mathsf{D}^2 = D + \frac{1}{2}(\tau^2 D^{(3)} - \frac{i}{2}\tau D^{(2)})$ and accordingly:

$$\mathsf{D}^2\mathsf{Y}_\tau^2 = \dot{\mathsf{X}}(t) - \frac{9}{4}\tau\ddot{\mathsf{X}}(t) + \frac{3}{4}\tau^2\mathsf{X}^{(3)}(t) - \frac{9}{8}i\tau^3\mathsf{X}^{(4)}(t) + \frac{1}{4}\tau^4\mathsf{X}^{(5)}(t),$$

and so on.

In order to construct the complexified Lagrangian formalism, we let $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t) \in C^2([a, b] \times \mathbb{C}^n \times \mathbb{C}^n; \mathbb{C})$ be an admissible smooth complexified Lagrangian function with $(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t) \to \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)$ assumed to be a $C^2$ function with respect to all its arguments, i.e. continuously differentiable with respect to all of its arguments. The corresponding complexified action function is given in our arguments by: $S = \int_a^b \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)dt$ where $\mathsf{Y}_\tau^n \in C^1[a, b]$ is sufficient in our arguments. We can now find using the standard variational approach the function $\mathsf{Y}_\tau^n$ for which the action functional subject to given boundary conditions $\mathsf{Y}_\tau^n(a) = \mathsf{Y}_{\tau,a}^n$ and $\mathsf{Y}_\tau^n(b) = \mathsf{Y}_{\tau,b}^n$ has an extremum.

*Remark 2.2* Before we proceed to explore the complexified Lagrangian dynamics, it is notable to mention that classical mechanics is extended in our formalism into the complex domain. Though in our paper we deal with the Lagrangian approach, we expect naturally that the Hamiltonian formalism is complexified and its corresponding analytic Hamilton function satisfies a family of Cauchy–Riemann conditions depending on the value of $n$. In reality, complexified Hamiltonian systems are obtained by complexifying the canonical variables whereas in our approach the complexification is obtained by means of non-local coordinates. The phase space structure of certain complexified dynamical systems described by higher-order Lagrangians is under construction. We expect that the new complexified Hamiltonian formalism will lead to a generalized structure of the phase space. In physics and mathematics, many methods have been constructed to jump from the real space to the complex space. To the best of our knowledge, the formalism addressed in this paper was not attacked before in literature and it represents a novel way to encode higher-order derivatives to the usual first-order classical mechanics. This may have interesting features in complexified quantum mechanics.

**Theorem 2.1** *Given the action functional* $S = \int_a^b \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)dt$. *If* $\mathsf{Y}_\tau^n$ *are solutions, then the following complexified Euler–Lagrange equation (CELE) holds:*

$$\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{Y}_\tau^n} - \mathsf{D}^n\left(\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{D}^n\mathsf{Y}_\tau^n}\right) = 0. \tag{9}$$

**Lemma 2.1** *In terms of the derivative operators* $D$ *and* $\mathcal{D}^n$, *the CELE takes the form:*

$$\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{Y}_\tau^n} - D\left(\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{D}^n\mathsf{Y}_\tau^n}\right) - \mathcal{D}_R^n\left(\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{D}^n\mathsf{Y}_\tau^n}\right)$$
$$+ i\mathcal{D}_I^n\left(\frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)}{\partial \mathsf{D}^n\mathsf{Y}_\tau^n}\right) = 0. \tag{10}$$

*Remark 2.3* It is obvious that the complexified Lagrangian $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n, \mathsf{Y}_\tau^n, t)$ contains higher-order derivatives terms and accordingly, one can always write Eq. (9) as a higher-order complexified Euler–Lagrange equations depending on the value of $n$.

To clarify, for $n = 1$, we have $\mathsf{L}_1(\mathsf{D}^1\mathsf{Y}_\tau^1, \mathsf{Y}_\tau^1, t) = \mathsf{L}_1(\dot{\mathsf{X}}(t) - 2i\tau\ddot{\mathsf{X}}(t) - \tau^2\dddot{\mathsf{X}}(t), \mathsf{X}(t) - i\tau\dot{\mathsf{X}}(t), t) \equiv \mathsf{L}_1$ and therefore the corresponding CELE (10) takes the form:

$$\frac{\partial \mathsf{L}_1}{\partial \mathsf{Y}_\tau^1} - D\left(\frac{\partial \mathsf{L}_1}{\partial \mathsf{D}^1\mathsf{Y}_\tau^1}\right) - \mathcal{D}_R^1\left(\frac{\partial \mathsf{L}_1}{\partial \mathsf{D}^1\mathsf{Y}_\tau^1}\right) + i\mathcal{D}_I^1\left(\frac{\partial \mathsf{L}_1}{\partial \mathsf{D}^1\mathsf{Y}_\tau^1}\right) = 0,$$

which is also written using Eqs. (7) and (8) as:

$$\frac{\partial \mathsf{L}_1}{\partial \mathsf{Y}_\tau^1} - \frac{d}{dt}\left(\frac{\partial \mathsf{L}_1}{\partial \mathsf{D}^1\mathsf{Y}_\tau^1}\right) + i\tau\frac{d^2}{dt^2}\left(\frac{\partial \mathsf{L}_1}{\partial \mathsf{D}^1\mathsf{Y}_\tau^1}\right) = 0.$$

For $n=2$, we have: $\mathsf{L}_2(\dot{\mathsf{X}}(t)-\frac{9}{4}\tau\ddot{\mathsf{X}}(t)+\frac{3}{4}\tau^2X^{(3)}(t)-\frac{9}{8}i\tau^3\mathsf{X}^{(4)}(t)+\frac{1}{4}\tau^4\mathsf{X}^{(5)}(t),$ $\mathsf{X}(t)+\frac{1}{2}(\tau^2\ddot{\mathsf{X}}(t)-2i\tau\dot{\mathsf{X}}(t)),t)\equiv L_2$ and the corresponding CELE is:

$$\frac{\partial \mathsf{L}_2}{\partial \mathsf{Y}_\tau^2}-D\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)-\mathcal{D}_R^2\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)+i\mathcal{D}_I^2\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)=0,$$

which also may be written as:

$$\frac{\partial \mathsf{L}_2}{\partial \mathsf{Y}_\tau^2}-\frac{d}{dt}\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)-\frac{\tau^2}{2}\frac{d^3}{dt^3}\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)+i\tau\frac{d^2}{dt^2}\left(\frac{\partial \mathsf{L}_2}{\partial \mathsf{D}^2\mathsf{Y}_\tau^2}\right)=0,$$

and so on for larger values of $n$.

One quick consequence of the CELE concerns the Lagrangian $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}_\tau^n,\mathsf{Y}_\tau^n,t)=\frac{1}{2}\mathsf{D}^n\mathsf{Y}_\tau^n\mathsf{D}^n\mathsf{Y}_\tau^n-\frac{1}{2}\mathsf{Y}_\tau^n\mathsf{Y}_\tau^n$ which is similar in form to the harmonic oscillator Lagrangian. However, this is a complexified Lagrangian of higher-order of $n$. The equation of motion as derived from Eq. (10) takes the form:

$$-\mathsf{Y}_\tau^n-D(\mathsf{D}^n\mathsf{Y}_\tau^n)-\mathcal{D}_R^n(\mathsf{D}^n\mathsf{Y}_\tau^n)+i\mathcal{D}_I^n(\mathsf{D}^n\mathsf{Y}_\tau^n)=0. \tag{11}$$

For $n=1$, we have $\mathsf{Y}_\tau^1=\mathsf{X}(t)-i\tau\dot{\mathsf{X}}(t)$, $\mathsf{D}^1=D-i\tau D^{(2)}$, $\mathsf{D}^1\mathsf{Y}_\tau^1=\dot{\mathsf{X}}(t)-2i\tau\ddot{\mathsf{X}}(t)-\tau^2\dddot{\mathsf{X}}(t)$ and the following relations hold accordingly:

$$D(\mathsf{D}^1\mathsf{Y}_\tau^1)=\ddot{\mathsf{X}}(t)-2i\tau\mathsf{X}^{(3)}(t)-\tau^2\mathsf{X}^{(4)}(t), \tag{12}$$

$$\mathcal{D}_R^1(\mathsf{D}^1\mathsf{Y}_\tau^1)=0, \tag{13}$$

$$\mathcal{D}_I^1(\mathsf{D}^1\mathsf{Y}_\tau^1)=\tau D^{(2)}(\mathsf{D}^1\mathsf{Y}_\tau^1)=\tau\mathsf{X}^{(3)}(t)-2i\tau^2\mathsf{X}^{(4)}(t)-\tau^3\mathsf{X}^{(5)}(t), \tag{14}$$

and the equation of motion takes the form:

$$i\tau^3\mathsf{X}^{(5)}(t)-3\tau^2\mathsf{X}^{(4)}(t)-3i\tau\mathsf{X}^{(3)}(t)+\ddot{\mathsf{X}}(t)-i\tau\dot{\mathsf{X}}(t)+\mathsf{X}(t)=0, \tag{15}$$

which is a higher-order nonlinear complexified differential equation. Notice that for $\tau=0$, Eq. (15) is reduced to $\ddot{\mathsf{X}}(t)+\mathsf{X}(t)=0$ which corresponds for the second-order differential equation for a classical oscillator.

For $n=2$, we have $\mathsf{Y}_\tau^2=\mathsf{X}(t)+\frac{1}{2}(\tau^2\ddot{\mathsf{X}}(t)-2i\tau\dot{\mathsf{X}}(t))$, $\mathsf{D}^2=D+\frac{1}{2}(\tau^2D^{(3)}-\frac{i}{2}\tau D^{(2)})$ and the corresponding derivative $\mathsf{D}^2\mathsf{Y}_\tau^2=\dot{\mathsf{X}}(t)-\frac{9}{4}\tau\ddot{\mathsf{X}}(t)+\frac{3}{4}\tau^2\mathsf{X}^{(3)}(t)-\frac{9}{8}i\tau^3\mathsf{X}^{(4)}(t)+\frac{1}{4}\tau^4\mathsf{X}^{(5)}(t)$. The following relations hold consequently:

$$D(\mathsf{D}^2\mathsf{Y}_\tau^2)=\ddot{\mathsf{X}}(t)-\frac{9}{4}\tau\mathsf{X}^{(3)}(t)+\frac{3}{4}\tau^2\mathsf{X}^{(4)}(t)-\frac{9}{8}i\tau^3\mathsf{X}^{(5)}(t)+\frac{1}{4}\tau^4\mathsf{X}^{(6)}(t), \tag{16}$$

$$\mathcal{D}_R^2(\mathsf{D}^2\mathsf{Y}_\tau^2)=\frac{1}{2}\tau^2\left(\mathsf{X}^{(4)}(t)-\frac{9}{4}\tau\mathsf{X}^{(5)}(t)+\frac{3}{4}\tau^2\mathsf{X}^{(6)}(t)-\frac{9}{8}i\tau^3\mathsf{X}^{(7)}(t)+\frac{1}{4}\tau^4\mathsf{X}^{(8)}(t)\right), \tag{17}$$

$$\mathcal{D}_I^2(\mathsf{D}^2\mathsf{Y}_\tau^2)=\tau\left(\mathsf{X}^{(3)}(t)-\frac{9}{4}\tau\mathsf{X}^{(4)}(t)+\frac{3}{4}\tau^2\mathsf{X}^{(5)}(t)-\frac{9}{8}i\tau^3\mathsf{X}^{(6)}(t)+\frac{1}{4}\tau^4\mathsf{X}^{(7)}(t)\right), \tag{18}$$

and the equation of motion takes the form:

$$
\begin{aligned}
&-\frac{1}{8}\tau^6 \mathsf{X}^{(8)}(t) + \frac{13}{16} i\tau^5 \mathsf{X}^{(7)}(t) + \frac{1}{2}\tau^4 \mathsf{X}^{(6)}(t) + \frac{15i-9}{8}\tau^3 \mathsf{X}^{(5)}(t) \\
&\quad -\frac{5+9i}{4}\tau^2 \mathsf{X}^{(4)}(t) + \frac{9+4i}{4}\tau \mathsf{X}^{(3)}(t) + \frac{1}{2}(\tau^2-2)\ddot{\mathsf{X}}(t) + i\tau\dot{\mathsf{X}}(t) - \mathsf{X}(t) = 0.
\end{aligned}
\tag{19}
$$

Once again for $\tau = 0$, Eq. (19) is reduced to the second-order differential equation for a classical oscillator $\ddot{\mathsf{X}}(t) + \mathsf{X}(t) = 0$. It is obvious that this approach generates complex ordinary differential equations for the dynamical variables. One naturally expects that the present complexified Lagrangians formalisms will generate complex classical Hamiltonians which will satisfy complex ordinary differential equations and therefore the study of complex ordinary differential equations is crucial. These classes of Hamiltonians were discussed largely in literature in classical and quantum theories [1,2]. It will be of interest to study in a future work the matching Noether operators in the complex domain.

It is noteworthy that if one pick a deformed Lagrangian of the form $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t) = \mathsf{Y}^n_\tau\mathsf{Y}^n_\tau + \mathsf{Y}^n_\tau\mathsf{D}^n\mathsf{Y}^n_\tau$, its equation of motion is $\mathsf{Y}^n_\tau = 0$. For $n = 1$, $\mathsf{Y}^1_\tau = \mathsf{X}(t) - i\tau\dot{\mathsf{X}}(t) = 0$ which gives $\mathsf{X}(t) = c_1 t^{-it/\tau}$ whereas for $n = 2$, we have $\tau^2\ddot{\mathsf{X}}(t) - 2i\tau\dot{\mathsf{X}}(t) + 2\mathsf{X}(t) = 0$ and the solution is given by $\mathsf{X}(t) = c_2 e^{-i(\sqrt{3}-1)t/\tau} + c_3 e^{i(\sqrt{3}+1)t/\tau}$ and so on. Here $c_j$, $j = 1, 2, \ldots$ are constants of integration. It is striking that in the standard formalism, the solution is simply: $\mathsf{X}(t) = 0$.

In order at this stage to discuss the implications of the E-CBFDO on Newton's law, we consider the Lagrangian $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t) = \frac{1}{2}\mathsf{D}^n\mathsf{Y}^n_\tau\mathsf{D}^n\mathsf{Y}^n_\tau - \boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau)$, $\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau)$ being the potential of the system in motion. From Eq. (9) we find effortlessly:

$$
\mathsf{D}^n\mathsf{D}^n\mathsf{Y}^n_\tau = -\frac{\partial\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau)}{\partial\mathsf{Y}^n_\tau}, \tag{20}
$$

For $n = 1$, Eq. (20) is reduced to:

$$
i\tau^3\mathsf{X}^{(5)}(t) - 3\tau^2\mathsf{X}^{(4)}(t) - (2i+1)\tau\mathsf{X}^{(3)}(t) + \ddot{\mathsf{X}}(t) = -\frac{\partial\boldsymbol{\mathcal{V}}(\mathsf{Y}^1_\tau)}{\partial\mathsf{Y}^1_\tau}, \tag{21}
$$

whereas for $n = 2$, we find:

$$
\begin{aligned}
&-\frac{1}{2}\tau^6\mathsf{X}^{(8)}(t) + \frac{13}{16}i\tau^5\mathsf{X}^{(7)}(t) + \frac{1}{2}\tau^4\mathsf{X}^{(6)}(t) \\
&\quad + \frac{9}{8}(1+2i)\tau^3\mathsf{X}^{(5)}(t) - \frac{5+9i}{4}\tau^2\mathsf{X}^{(4)}(t) \\
&\quad + \frac{9+4i}{4}\tau X^{(3)}(t) - \ddot{\mathsf{X}}(t) = -\frac{\partial\boldsymbol{\mathcal{V}}(\mathsf{Y}^2_\tau)}{\partial\mathsf{Y}^2_\tau}.
\end{aligned}
\tag{22}
$$

and so on. These give an extended complexified Newton's law holding higher-order derivative terms.

**Lemma 2.2** *For the case of deformed Lagrangians of the form* $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t) = \mathsf{Y}^n_\tau \mathsf{D}^n \mathsf{Y}^n_\tau + k\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau), k \in \mathbb{R}$, *the equation of motion is* $\partial\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau)/\partial\mathsf{Y}^n_\tau = 0$ *and a dynamics still occurs for power-law potential* $\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau) \propto (\mathsf{Y}^n_\tau)^N$, $N = 2, 3, 4, \ldots$

*Proof* The proof is straightforward and is obtained from Eq. (9). For $N = 2, 3, 4 \ldots$ we find $\mathsf{Y}^n_\tau = 0$ and therefore for $n = 1, 2, 3, \ldots$ a dynamics still occur as shown in the previous section. □

*Remark 2.4* For the case of the deformed Lagrangian $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t) = \mathsf{Y}^n_\tau \mathsf{D}^n \mathsf{Y}^n_\tau + k\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau), k \in \mathbb{R}$, we can pick a periodic potential of the form $\boldsymbol{\mathcal{V}}(\mathsf{Y}^n_\tau) = \cos(\mathsf{Y}^n_\tau)$ which gives $\sin(\mathsf{Y}^n_\tau) = 0$ or $\mathsf{Y}^n_\tau = \mathsf{n}\pi, \mathsf{n} \in \mathbb{Z}$. For $n = 1$ the solution is given by $\mathsf{X}(t) = \mathsf{n}\pi + c_4 e^{-it/\tau}$.

**Definition 2.4** Let $\mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t)$ be an admissible smooth Lagrangian function. We define the higher-order force by

$$\mathsf{F}_n = \frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t)}{\partial \mathsf{Y}^n_\tau}, \tag{23}$$

and the higher-order momentum by:

$$\mathsf{P}_n = \frac{\partial \mathsf{L}_n(\mathsf{D}^n\mathsf{Y}^n_\tau, \mathsf{Y}^n_\tau, t)}{\partial \mathsf{D}^n\mathsf{Y}^n_\tau}. \tag{24}$$

**Lemma 2.3** *The second complexified higher-order Newton's law is expressed by:*

$$\begin{aligned}\mathsf{F}_n = \mathsf{D}^n\mathsf{P}_n &= (D + \boldsymbol{\mathcal{D}}^n_R - i\boldsymbol{\mathcal{D}}^n_I)\mathsf{P}_n \equiv (D + \boldsymbol{\mathcal{D}}^n_R)\mathsf{P}_n - i\boldsymbol{\mathcal{D}}^n_I\mathsf{P}_n \\ &= (\mathsf{F}_n)_R + i(\mathsf{F}_n)_I,\end{aligned} \tag{25}$$

*where* $(\mathsf{F}_n)_R \triangleq (D + \boldsymbol{\mathcal{D}}^n_R)\mathsf{P}_n$ *and* $(\mathsf{F}_n)_I \triangleq -\boldsymbol{\mathcal{D}}^n_I\mathsf{P}_n$ *are respectively the real and imaginary part of the complexified force.*

*For* $n = 1$, $(\mathsf{F}_1)_R \triangleq (D + \boldsymbol{\mathcal{D}}^1_R)\mathsf{P}_1 = \dot{\mathsf{P}}_1$, $(\mathsf{F}_1)_I \triangleq -\boldsymbol{\mathcal{D}}^1_I\mathsf{P}_1 = -\tau\mathsf{D}^{(2)}\mathsf{P}_1 = -\tau\ddot{\mathsf{P}}_1$ *and therefore the Newton's law is* $\mathsf{F}_1 = \dot{\mathsf{P}}_1 - i\tau\ddot{\mathsf{P}}_1$.

## 3 Physical Implications of the Complexified Nonlocal-In-Time Lagrangian from E-CBFDO

*A*: One interesting consequence of our approach concerns the E-CBFDO of a given function $f(\mathsf{Y}^n_\tau, t)$. For illustration purpose, we choose $n = 1$. Its derivative is then given by:

$$\mathsf{D}^1 f(\mathsf{Y}^1_\tau, t) = (D - i\tau D^{(2)}) f(\mathsf{Y}^1_\tau, t) \equiv \left(\frac{d}{dt} - i\tau\frac{d^2}{dt^2}\right) f(\mathsf{Y}^1_\tau, t). \tag{26}$$

Expanding its total differential to the first order gives for $D$:

$$\begin{aligned}\frac{df(\mathsf{Y}_\tau^1,t)}{dt} &= \frac{\partial f(\mathsf{Y}_\tau^1,t)}{\partial t} + \nabla f(\mathsf{Y}_\tau^1,t)\cdot\frac{d(\mathsf{X}(t)-i\tau\dot{\mathsf{X}}(t))}{dt}\\ &= \frac{\partial f(\mathsf{Y}_\tau^1,t)}{\partial t} + (v-i\tau\gamma)\cdot\nabla f(\mathsf{Y}_\tau^1,t),\end{aligned} \tag{27}$$

and for $D^{(2)}$:

$$\begin{aligned}\frac{d^2 f(\mathsf{Y}_\tau^1,t)}{dt^2} = {}& \frac{\partial^2 f(\mathsf{Y}_\tau^1,t)}{\partial t^2} + (\gamma - i\tau\mathsf{J})\cdot\nabla f(\mathsf{Y}_\tau^1,t)\\ &+2\left((v\cdot\nabla)-i\tau\left(\gamma\cdot\nabla\right)\right)\frac{\partial f(\mathsf{Y}_\tau^1,t)}{\partial t}\\ &+(v^2-\tau^2\gamma^2)\Delta f(\mathsf{Y}_\tau^1,t) - 2i\tau\left((v\cdot\nabla)\gamma\right.\\ &\left.+(\gamma\cdot\nabla)v\right)\cdot\nabla f(\mathsf{Y}_\tau^1,t),\end{aligned} \tag{28}$$

where $\mathsf{J}$ is the jerk, i.e. the rate of change of the vector acceleration $\gamma$ and $v$ is the velocity vector. Accordingly, Eq. (26) gives:

$$\begin{aligned}\mathsf{D}^1 f(\mathsf{Y}_\tau^1,t) = {}& (1-\tau^2(\gamma\cdot\nabla)-2i\tau\,(v\cdot\nabla))\frac{\partial f(\mathsf{Y}_\tau^1,t)}{\partial t}\\ &-i\tau\frac{\partial^2 f(\mathsf{Y}_\tau^1,t)}{\partial t^2} + (v-2i\tau\gamma)\cdot\nabla f(\mathsf{Y}_\tau^1,t)\\ &-i\tau(v^2-\tau^2\gamma^2)\Delta f(\mathsf{Y}_\tau^1,t) + 2\tau^2\left((v\cdot\nabla)\gamma + (\gamma\cdot\nabla)v - \frac{1}{2}\mathsf{J}\right)\\ &\cdot\nabla f(\mathsf{Y}_\tau^1,t).\end{aligned} \tag{29}$$

For $\tau = 0$, this equation is reduced to $\mathsf{D}^1 = \frac{\partial}{\partial t} - v\cdot\nabla$ as it is expected. At first sight, we observe that the material derivative, i.e. the time rate of change of a function while moving with the particle, is complexified. For the case of a constant velocity, Eq. (29) is reduced to:

$$\begin{aligned}\mathsf{D}^1 f(\mathsf{Y}_\tau^1,t) = {}& (1-2i\tau v\nabla)\,\frac{\partial f(\mathsf{Y}_\tau^1,t)}{\partial t} - i\tau\frac{\partial^2 f(\mathsf{Y}_\tau^1,t)}{\partial t^2}\\ &+v\nabla f(\mathsf{Y}_\tau^1,t) - i\tau v^2\Delta f(\mathsf{Y}_\tau^1,t).\end{aligned} \tag{30}$$

As a simple application, we consider a small volume $V$ whose boundary moves with a fluid of density $\rho$ assumed to be constant inside the volume [22]. Assuming that the mass of the fluid $M = \rho V$ is conserved as the fluid moves with velocity $v$ assumed to be constant without loss of generality, then for $n = 1$, we can write using Eq. (30): $\mathsf{D}^1 M = V\mathsf{D}^1\rho = 0$ which gives $\mathsf{D}^1\rho = 0$. Using Eq. (28) we find:

$$(1-2i\tau v\nabla)\,\frac{\partial\rho}{\partial t} - i\tau\frac{\partial^2\rho}{\partial t^2} - i\tau v^2\Delta\rho = 0, \tag{31}$$

which is the modified continuity equation. If the fluid density is constant in time but varies with distance, e.g. height, than Eq. (31) is reduced to the Laplace equation $\Delta\rho = 0$. For the case of a stationary fluid, the energy density varies as $\rho(t) = c_5 + c_6 e^{-it/\tau}$ which tends to a constant for very large time. A number of implications in fluid mechanics are under construction.

*B*: In order to have a simple idea about the implications of the present approach in classical theory, we consider the action $S = \int_a^b \mathsf{L}_1(\mathsf{D}^1\mathsf{Y}_\tau^1, \mathsf{Y}_\tau^1, t)dt$ as a functional of the upper limit of integration, the variation of the action gives the complex momentum $\mathsf{P}_1 = \nabla S$ and the complex energy $\mathsf{E}_1 = -\partial S/\partial t$ [19]. For given system characterized by the Lagrangian $\mathsf{L}_1(\mathsf{D}^1\mathsf{Y}_\tau^1, \mathsf{Y}_\tau^1, t) = \frac{1}{2}m\mathsf{D}^1\mathsf{Y}_\tau^1\mathsf{D}^1\mathsf{Y}_\tau^1 - \Phi$ where $\Phi$ is the classical potential and $m$ is the mass of the particle, Eq. (9) gives $m\mathsf{D}^1\mathsf{D}^1\mathsf{Y}_\tau^1 = -\nabla\Phi$ from we deduce the complex momentum $\mathsf{P}_1 = m\mathsf{D}^1\mathsf{Y}_\tau^1$ which already gives $\mathsf{D}^1\mathsf{Y}_\tau^1 = \nabla S/m$. By introducing a complex function $\Psi = e^{i\alpha S}$ where $\alpha$ is a real constant, we obtain $\mathsf{D}^1\mathsf{Y}_\tau^1 = -(i/m\alpha)\nabla \ln\Psi$ and from $m\mathsf{D}^1\mathsf{D}^1\mathsf{Y}_\tau^1 = -\nabla\Phi$ we find at the end $\nabla\Phi = i\mathsf{D}^1(\nabla \ln\Psi)/\alpha$. Using Eqs. (26)–(28), we find after some algebra:

$$\begin{aligned}\frac{\alpha}{i}\nabla\Phi = &\left(1 - 2i\tau\left((v\cdot\nabla) - i\tau\left(\gamma\cdot\nabla\right)\right)\right)\frac{\partial(\nabla \ln\Psi)}{\partial t} - i\tau\frac{\partial^2(\nabla \ln\Psi)}{\partial t^2}\\ &+(v - 2i\tau\gamma - \tau^2\mathsf{J} + 2\tau^2\left((v\cdot\nabla)\gamma + (\gamma\cdot\nabla)v\right))\\ &\cdot\nabla(\nabla \ln\Psi) - i\tau(v^2 - \tau^2\gamma^2)\Delta(\nabla \ln\Psi).\end{aligned} \tag{32}$$

Assuming for simplicity a constant velocity, i.e. $\mathsf{J} = \gamma = 0$, then Eq. (32) is approximated by:

$$\begin{aligned}\frac{\alpha}{i}\nabla\Phi = &\left(1 - 2i\tau v\nabla\right)\frac{\partial(\nabla \ln\Psi)}{\partial t} - i\tau\frac{\partial^2(\nabla \ln\Psi)}{\partial t^2}\\ &+ v\nabla(\nabla \ln\Psi) - i\tau v^2\Delta(\nabla \ln\Psi).\end{aligned} \tag{33}$$

By letting $\phi = \nabla \ln\Psi$ and $\Lambda = \nabla\Phi$ we can write Eq. (33) as:

$$\frac{\alpha}{i}\Lambda = \left(1 - 2i\tau v\nabla\right)\frac{\partial\phi}{\partial t} - i\tau\frac{\partial^2\phi}{\partial t^2} + v\nabla\phi - i\tau v^2\Delta\phi. \tag{34}$$

This equation may be splitted into two partial differential equations:

$$\frac{\partial\phi}{\partial t} + v\nabla\phi = 0, \tag{35}$$

$$v^2\Delta\phi + \frac{\partial^2\phi}{\partial t^2} + 2v\nabla\frac{\partial\phi}{\partial t} = \frac{\alpha}{\tau}\Lambda. \tag{36}$$

Deriving Eq. (35) with respect to time and replacing into Eq. (36) we find:

$$\Delta\phi - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\alpha}{\tau v^2}\Lambda, \tag{37}$$

which is the wave equation with a source term. In this case one may rewrite the given complex nonlinear differential equations splitting the complex from real parts and then investigate the dynamics of the resulting differential partial equations. One then expects this approach to have interesting consequences in quantum mechanics.

## 4 Conclusions

In this work, we have introduced the notion of a complex backward–forward derivative operator applied to non-local-in-time generalized coordinates of a given Lagrangian system. We have constructed the Lagrangian formalism and derived the corresponding equations of motion in particular the Euler–Lagrange equation and the Newton's equations of motion. It was observed that the present formalism generates dynamical equations of motion holding higher-order derivatives terms and the extended complexified Newton's law holds higher-order derivative terms. This gives hope to construct Newtonian's theory with higher-order corrections terms which is useful in quantum physics. It was observed that the present formalism may have some motivating consequences in fluid mechanics and quantum mechanics. The corresponding Hamiltonian formalism and a list of physical applications are under progress. It should be mentioned that complexified Lagrangian mechanics was discussed in literature through different aspects [3,9,12,17,20] and have many important applications in quantum theory, yet our approach generalizes these approach by the present of the E-CBFDO and non-local-in-time generalized coordinates. The complexified Lagrangians will yield naturally complexified Hamiltonian dynamical systems and complex differential equations which render the study of complex ordinary and partial differential equations crucial and we expect to have several physically interesting applications.

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