

A Global Regularization for the $(N + 1)$ -Body Problem with the Primaries in a Regular N -gon Central Configuration

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Abstract In the present paper, we consider the restricted planar $(N + 1)$ -body problem where $N \geq 3$ bodies (called primaries) interacting with one another according to Newtonian law which are in the vertices of a regular N -gon with the origin at the center of masses of the coordinate system. We prove that the simultaneous binary collision between the infinitesimal mass and any primary are regularizable, through the implementation of Birkhoff-type transformation.

Keywords Restricted $(N + 1)$ -body problem · Binary collision · Global regularization · Birkhoff's transformation

Mathematics Subject Classification Primary: 70F16, 37N05 · Secondary: 70F15

1 Introduction and Main Result

The classical N -body problem is concerned with the motion of N mass points moving in the space according to the Newtonian law. While the restricted $(N + 1)$ -body problem studies the motion of an infinitesimal mass m_{N+1} under the effects of the Newtonian gravitational force exerted by the N primaries q_1, \dots, q_N of masses m_1, \dots, m_N ,

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respectively. In our approach we will assume that the configuration of the primaries is a regular N -gon central configuration. Let q be the vector position of the infinitesimal mass m_{N+1} which moves on the xy -plane. Consider now $N \geq 3$ particles of mass m_1, \dots, m_N moving on the same plane under the influence of the mutual gravitational attraction and describing circular orbits around their center of mass fixed at the origin of the coordinate system. This system rotates with the same angular velocity $\omega \neq 0$ with respect to the inertial frame. Then, the position vector of the primaries is $q_j(t) = e^{i\omega t} q_j^0$ (i.e., the motion of the primaries consists of circular orbits around their center of gravity) where $i^2 = -1$ for $j = 1, 2, \dots, N$, while the initial configuration (q_1^0, \dots, q_N^0) is of course a central configuration (see details in [8]).

It is a well-known that the 3-gon corresponds to an equilateral triangle, that is, it corresponds to Lagrange's central configuration solution of the 3-body problem, which exist for any value of the masses. Moreover, Perko and Walter [7] and Elmabsout [5] proved that $N \geq 4$ point masses located at the vertices of a regular N -polygon form a central configuration, if and only if, all the masses are equal. In this work we must have in mind only these two possibilities, which will be discussed later.

In an inertial reference frame and choosing suitable units, the equations of motion of the infinitesimal mass m_{N+1} are

$$\ddot{q} = \nabla V(q, t), \quad (1.1)$$

where the Newtonian potential is given by

$$V(q, m_1, \dots, m_N; t) = \sum_{j=1}^N \frac{m_j}{\|q - q_j(t)\|}, \quad (1.2)$$

and m_j are the masses of the primaries. Now we establish the equations of motion in the rotating coordinate system given by (x, y) , (see [6] for details). Thus the equations of motion of body with mass m_{N+1} in (1.1) are expressed as

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \end{aligned} \quad (1.3)$$

where

$$\Omega = \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + W(x, y). \quad (1.4)$$

In order to write the Hamiltonian function associated to (1.3) system, let us introduce the generalized coordinates and the generalized momenta variables

$$\begin{aligned} x &= x, & y &= y, \\ X &= \dot{x} - y, & Y &= \dot{y} + x. \end{aligned}$$

Then the motion of m_{N+1} is governed by the autonomous and two degrees of freedom Hamiltonian function given by

$$H = H(x, y, X, Y) = \frac{1}{2}(X^2 + Y^2) + yX - xY - W(x, y), \quad (1.5)$$

where

$$W = W(x, y) = \sum_{j=1}^N \frac{m_j}{\rho_j}, \quad (1.6)$$

and $\rho_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$, $j = 1, 2, \dots, N$ such that the bodies have positions $q_j = (x_j, y_j)$, $j = 1, 2, \dots, N$ forming a central configuration solution of the N -body problem and their center of mass is fixed at the origin of the coordinate system.

The phase space where the equations of motions (1.3) are well defined is

$$\mathcal{M} = \left\{ (x, y, X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (x, y) \neq (x_j, y_j), j = 1, \dots, N \right\}$$

and the points that have been removed correspond to the binary collisions between the infinitesimal particle m_{N+1} and one of the primaries.

Since the motion of all bodies is carried out in a plane, we introduce the complex variables by $z = x + iy$ ($i = \sqrt{-1}$), then the equations (1.3) can be written as

$$\ddot{z} + 2i\dot{z} = 2\frac{\partial\Omega}{\partial\bar{z}}, \quad (1.7)$$

where

$$\Omega(z, \bar{z}) = \frac{1}{2}|z|^2 + V(z, \bar{z}), \quad (1.8)$$

with

$$V(z, \bar{z}) = \sum_{j=1}^N \frac{m_j}{r_j}, \quad (1.9)$$

and

$$r_j = |z - z_j| = \sqrt{(x - x_j)^2 + (y - y_j)^2}, \quad (1.10)$$

representing the distances from the infinitesimal mass to the primaries. The conjugate momenta X and Y associated with the coordinate x and y , respectively, will be denoted by $Z = X + iY \equiv iz + \dot{z}$. So, the Hamiltonian function associated to (1.7) is given by

$$H = \frac{1}{2}|Z|^2 + \text{Im}(z\bar{Z}) + \frac{1}{2}|z|^2 - \Omega(z, \bar{z}) \quad (1.11)$$

or equivalently

$$H = \frac{1}{2}|Z|^2 + \text{Im}(z\bar{Z}) - V(z, \bar{z}). \quad (1.12)$$

Let us define the set on the phase space where the singularities the equation of motion (1.7) due to binary collision does not occur, given by

$$\mathcal{M} = \{(z, Z) \in \mathbb{C} \times \mathbb{C} \mid z \neq z_j \text{ where } j = 1, 2, \dots, N\}.$$

Since we will introduce a coordinate transformation such that the binary collision singularities can be removed simultaneously, we introduce new coordinates such that the barycenter of the primaries is the origin of this new system of coordinates.

Let us remark that in the equilateral triangle configuration the barycenter does not always coincide with its center of mass, but in the regular N -gon (equal masses) configuration, for $N \geq 4$, its barycenter coincides with the center of mass.

Let $z_b = x_b + iy_b$ be the barycenter, so the translation $u = z - z_b = \xi + i\eta$ shifts the origin into the barycenter. Therefore the equation (1.7) in coordinates relative to the barycenter is given by

$$\ddot{u} + 2i\dot{u} = 2\frac{\partial\Omega}{\partial\bar{u}}(u, \bar{u}) \quad (1.13)$$

with

$$\Omega(u, \bar{u}) = \frac{1}{2}|u + z_b|^2 + \sum_{j=1}^N \frac{m_j}{\rho_j}, \quad (1.14)$$

where $\rho_j = |u - u_j|$, $u_j = z_j - z_b$, $j = 1, 2, \dots, N$. So, the Hamiltonian function (1.12) becomes

$$H = \frac{1}{2}|U|^2 + \text{Im}((u + z_b)\bar{U}) - V(u, \bar{u}) \quad (1.15)$$

where $U = Z$ is the (complex) conjugate momenta of u and

$$V(u, \bar{u}) = \sum_{j=1}^N \frac{m_j}{\rho_j} = \sum_{j=1}^N \frac{m_j}{|u - u_j|}.$$

We remark that the Hamiltonian (1.15) has exactly N -points of singularities due the binary collisions between the infinitesimal particle and any of the primaries. These singularities cause that the differential equations describing the motion are undefined because collision singularities occurs. In order to remove the singularities due to collisions, we can introduce new coordinates, in such a way the orbits which approach to the collision can be continued analytically through the collision, in a smooth manner with respect to time, then we say that the collision orbits have been regularized. Several

regularizing techniques are found in the literature, one of these was given by Birkhoff [4] which can be represented by a conformal mapping and a change of the physical time. We now state our main result.

Theorem 1.1 *Consider the planar restricted $(N + 1)$ -body problem in which the $N \geq 3$ bodies (primaries) are at the vertices of a regular N -gon. Then, the N singularities due to binary collisions between the infinitesimal body and any of the primaries are simultaneous regularizable by a canonical change of coordinates and a time-rescaling.*

The proof of this theorem will be given in Sect. 3. A technical preliminary result is given in Sect. 2.

2 Preliminary

The convention used here is that a dot always indicates a derivative with respect to the time t , while a star $(\cdot)^*$ indicates a derivative with respect to the complex variable w .

We are going to look for canonical transformations which help us to simplify the regularization. Let $u, w \in \mathbb{C}$, and $u = g(w)$ a complex function. Now, we give the following lemma which shows how the complex change of variables $u = g(w)$ can be extended in order to obtain a canonical (symplectic) transformation.

Lemma 2.1 *Let the point transformation be given by $u = g(w)$ such that $g^*(w) \neq 0$, then the transformation of the conjugate momenta $U = W/\overline{g^*(w)}$ yields a canonical transformation whenever $g^*(w) \neq 0$, namely, $(u, U) \mapsto (w, W)$.*

Proof Observe that the mapping $(u, U) \mapsto (w, W)$ is canonical, if and only if,

$$\operatorname{Re}(\overline{U}du) = \operatorname{Re}(\overline{W}dw)$$

Under the hypotheses we have

$$\overline{U}du = \frac{\overline{W}}{g^*(w)} du = \frac{\overline{W}}{g^*(w)} g^*(w) dw = \overline{W}dw,$$

from which the result follows. \square

3 Proof of the Theorem 1.1

The proof will be performed by a suitably Birkhoff's transformation. Since in the new variable u , the N bodies (primaries) are in a same circle, in fact, they are disposed at the vertices of a regular N -gon. Also, they have the additional property that the center of the circle coincides with their barycenter configuration. With this in mind, easily it follows the following relation among the bodies

$$q_{j+1} = e^{i2\pi/N} q_j, \quad j = 1, \dots, N - 1.$$

Now, we proceed to construct the regularizing transformation as an extension or as a Birkhoff-type, defined by

$$u = g(w) = \alpha w + \frac{\beta}{w^{N-1}}, \quad w \in \mathbb{C}, \quad (3.1)$$

where α and β are constants to be determined later on. It is clear that $g(w)$ in (3.1) is a conformal map satisfying the hypothesis of Lemma 2.1. Next, we introduce the time transformation

$$\frac{dt}{ds} = \frac{1}{|g^*(w)|^2}, \quad (3.2)$$

where s is the new time.

We derive that the Hamiltonian (1.15) in terms of the new variables is

$$H = \frac{1}{2} \frac{|W|^2}{|g^*(w)|^2} + \operatorname{Im} \left((g(w) + z_b) \frac{\overline{W}}{g^*(w)} \right) - V(w, \overline{w}). \quad (3.3)$$

In order to regularize the singularities, we apply a reparametrization of the solutions by fixing the level energy and replacing the Hamiltonian (3.3) by one whose flow is smoothly equivalent. Let us fix the energy level at $h = -C/2$ and the new Hamiltonian function assumes the form

$$\tilde{H} = |g^*(w)|^2 (H + C/2),$$

or explicitly,

$$\begin{aligned} \tilde{H} &= |g^*(w)|^2 \left[\frac{1}{2} \frac{|W|^2}{|g^*(w)|^2} + \operatorname{Im} \left((g(w) + z_b) \frac{\overline{W}}{g^*(w)} \right) - V(w, \overline{w}) + C/2 \right], \\ &= \frac{1}{2} |W|^2 + \operatorname{Im} \left((g(w) + z_b) \overline{g^*(w) W} \right) - |g^*(w)|^2 V(w, \overline{w}) + (C/2) |g^*(w)|^2, \end{aligned} \quad (3.4)$$

where

$$V(w, \overline{w}) = \sum_{j=1}^N \frac{m_j}{|g(w) - u_j|}.$$

We observe that $\tilde{H} = 0$ corresponds to $H = -C/2$.

The next step to achieve the desired regularization is to exhibit the convenient values of the parameters α and β in (3.1). Firstly, we will require that the transformation involving g must eliminate all the singularities simultaneously, and secondly, $g(w)$ must fix all the primaries m_1, m_2, \dots, m_N , in the sense that $w_j = u_j$, for $j = 1, 2, \dots, N$.

According (3.4), it follows that the first statement must be satisfied, if we require that

$$g^*(w_j) = 0, \quad j = 1, 2, \dots, N. \tag{3.5}$$

The second constraints require that

$$u_j = g(w_j) = w_j, \quad j = 1, 2, \dots, N. \tag{3.6}$$

Let us remark that $w_{j+1} = e^{i2\pi/N} w_j = e^{i2\pi j/N} w_1$ for $j = 1, 2, 3, \dots, N$. Now, let us go to check that (3.6) conditions are satisfied, for which is enough to verify that $g(w_1) = w_1$, where we get that

$$\beta = (1 - \alpha)w_1^N. \tag{3.7}$$

By definition of g in (3.1), we observe that conditions (3.5) is equivalent to have

$$\begin{aligned} g^*(w) &= \frac{\alpha}{w^N} \left(w^N - \frac{\beta}{\alpha} (N - 1) \right) \\ &= \frac{\alpha}{w^N} (w - u_1)(w - u_2) \cdots (w - u_N). \end{aligned} \tag{3.8}$$

Using the equation (3.8), we can verify that it must satisfy the following N conditions:

$$\begin{aligned} w_1 + w_2 + w_3 + \cdots + w_N &= 0, \\ w_1w_2 + w_1w_3 + \cdots + w_1w_N + w_2w_3 + w_2w_4 + \cdots + w_2w_N + \cdots + \\ &w_{N-2}w_{N-1} + w_{N-2}w_N + w_{N-1}w_N = 0, \\ w_1w_2w_3 + w_1w_2w_4 + \cdots + w_1w_2w_N + w_1w_3w_4 + w_1w_3w_5 + \cdots + \\ &w_1w_3w_N + \cdots + w_2w_3w_4 + w_2w_3w_5 + \cdots + w_2w_3w_N + w_2w_4w_5 + \\ &w_2w_4w_6 + \cdots + w_2w_4w_N + \cdots + w_{N-3}w_{N-2}w_{N-1} + \\ &w_{N-3}w_{N-2}w_N + w_{N-2}w_{N-1}w_N = 0, \\ &\vdots \quad \vdots \quad \vdots \\ w_1w_2 \cdots w_{N-2}w_{N-1} + w_1w_2 \cdots w_{N-2}w_N + w_1w_2 \cdots w_{N-3}w_{N-1}w_N \\ &+ \cdots + w_2w_3 \cdots w_{N-1}w_N = 0, \\ (-1)^N w_1w_2w_3 \cdots w_N &= -\frac{\beta}{\alpha} (N - 1). \end{aligned} \tag{3.9}$$

In order to verify that the first $N - 1$ conditions of (3.9) are satisfied, it is enough to check the first of them. Since $w_{j+1} = e^{i2\pi j/N} w_1$, $j = 1, \dots, N - 1$ are N th roots of the unity, their sum is zero, namely:

$$1 + e^{i2\pi/N} + e^{i4\pi/N} + \cdots + e^{i2\pi(N-1)/N} = 0. \tag{3.10}$$

It yields the first condition of (3.9) which can be written as following

$$\begin{aligned} w_1 + w_2 + w_3 + \cdots + w_N &= w_1 + e^{i2\pi/N} w_1 + e^{i3\pi/N} w_1 + \cdots + e^{i(N-1)\pi/N} w_1 \\ &= (1 + e^{i2\pi/N} + e^{i4\pi/N} + \cdots + e^{i2\pi(N-1)/N}) w_1 \\ &= 0. \end{aligned}$$

Again, as $w_{j+1} = e^{i2\pi j/N} w_1$ from the last equation in (3.9) we have that

$$\frac{w_1^N}{N-1} = \frac{\beta}{\alpha}. \quad (3.11)$$

Combining equations (3.7) and (3.11), we obtain that

$$\alpha = \frac{N-1}{N}, \quad \beta = \frac{w_1^N}{N}. \quad (3.12)$$

From the above considerations, it follows that

$$g(w) - w_j = \frac{(w - w_j)^2}{w^{N-1}} \tilde{g}(w), \quad \text{for } j = 1, 2, \dots, N \quad (3.13)$$

where

$$\begin{aligned} \tilde{g}(w) &= \frac{N-1}{N} w^{N-2} + \frac{N-2}{N} w_j w^{N-3} + \frac{N-3}{N} w_j^2 w^{N-4} + \frac{N-4}{N} w_j^3 w^{N-5} \\ &\quad + \cdots + \frac{2}{N} w_j^{N-3} w + \frac{1}{N} w_j^{N-2}. \end{aligned} \quad (3.14)$$

Thus we readily obtain that $\tilde{g}(w_j) = (N-1)w_j^{N-1} \neq 0$, $j = 1, 2, \dots, N$, in particular we have

$$\begin{aligned} \frac{|g^*(w)|^2}{|g(w) - w_j|} &= \frac{(N-1)^2 |w - w_1|^2 |w - w_2|^2 \cdots |w - w_N|^2}{N^2 |w|^N |g(w) - w_j|} \\ &= \frac{(N-1)^2 |w - w_1|^2 |w - w_2|^2 \cdots |w - w_N|^2}{N^2 |w|^N |w - w_j|^2 |\tilde{g}(w)|^2} \\ &= \frac{(N-1)^2}{N^2} |w|^{N-2} \\ &\quad \frac{|w - w_1|^2 |w - w_2|^2 \cdots |w - w_{j-1}|^2 |w - w_{j+1}|^2 \cdots |w - w_N|^2}{|\tilde{g}(w)|^2}. \end{aligned} \quad (3.15)$$

Therefore, the singularities due to binary collisions have been simultaneously removed, and the theorem is proved. \square

We finalize this section proving the following result.

Proposition 3.1 *Let $u \in \mathbb{C}$ be a complex number. For the number of pre-images of $u = g(w)$ we have two cases:*

- (i) $u = w_j$ for $j = 1, 2, \dots, N$ has at the maximum $N - 1$ distinct pre-images.
- (ii) $u \neq w_j$ for $j = 1, 2, \dots, N$ has N pre-images, each of them different of w_j .

Proof The proof of item (i) follows using Eqs. (3.13) and (3.14), and observing that $\tilde{g}(w_j) \neq 0$ and \tilde{g} is a complex polynomial of degree $N - 2$.

To prove (ii), we will use (3.1) and find the roots of the complex polynomial given by

$$(N - 1)w^N - Nuw^{N-1} + w_1^N = 0. \tag{3.16}$$

If $u \neq w_j$, we define the auxiliary function $\zeta(w) = (N - 1)w^N - Nuw^{N-1} + w_1^N$. Then, we calculate its complex derivative

$$\begin{aligned} \zeta^*(w) &= N(N - 1)w^{N-1} - N(N - 1)uw^{N-2} \\ &= N(N - 1)w^{N-2}(w - u). \end{aligned}$$

From here $\zeta^*(w) = 0$, if and only if, $w = 0$ or $w = u$. But, $\zeta(0) = w_1^N \neq 0$ and $\zeta(u) = w_1^N - u^N$, which is different from zero since $u \neq w_j, j = 1, 2, \dots, N$. Therefore, it follows that $\zeta(w)$ has only simple zeroes. Thus, we concluded the proof. \square

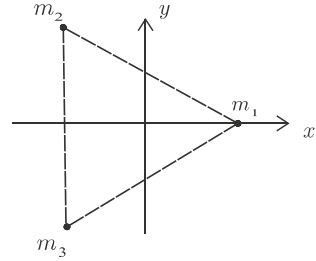
4 Applications

In this section, we illustrate the usefulness of Theorem 1.1 with some examples of the N -body problem.

Example In [1] was considered the problem of global regularization in the equilateral triangle restricted four-body problem. This restricted problem is obtained assuming that the primaries are in an equilateral or Lagrangian configuration, where the authors assume that the primaries have masses $m_1 = 1 - 2\mu, m_2 = m_3 = \mu$ (with $\mu \in [0, 1/2]$), respectively. Since this configuration there exists for all the values of the masses, in [3] was considered the study of the restricted equilateral problem for different masses. Thus, here we are going to regularize the restricted equilateral problem for arbitrary masses. We choose units of length and orientation of the axes such that the primaries forming an equilateral triangle (see Fig. 1), and the primaries are located at $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$ where

$$\begin{aligned} x_1 &= \frac{|M_1|}{M_1} \sqrt{M_2}, \\ y_1 &= 0, \\ x_2 &= -\frac{|M_1| [(m_2 - m_3)m_3 + m_1(2m_2 + m_3)]}{2M_1 \sqrt{M_2}}, \end{aligned}$$

Fig. 1 The equilateral restricted four-body problem



$$\begin{aligned} y_2 &= \frac{\sqrt{3}}{2} \frac{m_3}{\sqrt{M_2}}, \\ x_3 &= -\frac{|M_1|}{2} \frac{1}{\sqrt{M_2}}, \\ y_3 &= -\frac{\sqrt{3}}{2} \frac{m_2}{\sqrt{M_2}} \end{aligned} \quad (4.1)$$

where $M = m_1 + m_2 + m_3 = 1$, $M_1 = m_2(m_3 - m_2) + m_1(m_2 + 2m_3)$ and $M_2 = m_2^2 + m_2m_3 + m_3^2$.

In this model the potential W in (1.9) is given by

$$\begin{aligned} W &= W(x, y) = \sum_{j=1}^3 \frac{m_j}{\rho_j} \\ &= \frac{m_1}{\sqrt{(x-x_1)^2 + y^2}} + \frac{m_2}{\sqrt{(x-x_2)^2 + (y-y_2)^2}} + \frac{m_3}{\sqrt{(x-x_3)^2 + (y-y_3)^2}}. \end{aligned} \quad (4.2)$$

In order to apply Theorem 1.1, firstly we need to change the origin of the coordinate system to the barycenter of the triangle. Note that the barycenter has coordinates

$$(x_b, y_b) := \left(\frac{|M_1|}{\sqrt{2}M_1} \left[\frac{m_2^2 - m_1m_2 - m_1m_3 + m_3^2}{\sqrt{M_2}} \right], \frac{m_3 - m_2}{2\sqrt{3}\sqrt{M_2}} \right).$$

Following the previous notation, let $z = u + (x_b, y_b)$, where $u = \xi + i\eta$. We have that the transformation locates the primaries at

$$\begin{aligned} \xi_1 &= \frac{m_2 + m_3}{2\sqrt{M_2} \operatorname{sgn}(M_1) + m_2(m_3 - m_2)}, \\ \eta_1 &= \frac{m_2 - m_3}{2\sqrt{3}\sqrt{M_2}}, \\ \xi_2 &= -\frac{\tilde{m}_2}{2\sqrt{M_2} \operatorname{sgn}(M_1)}, \\ \eta_2 &= \frac{m_2 + 2m_3}{2\sqrt{3}\sqrt{M_2}}, \\ \xi_3 &= -\frac{\tilde{m}_3}{2\sqrt{M_1} \operatorname{sgn}(M_1)}, \\ \eta_3 &= -\frac{2m_2 + m_3}{2\sqrt{3}\sqrt{M_2}} \end{aligned} \quad (4.3)$$

where $u_j = \xi_j + i\eta_j$, $j = 1, 2, 3$.

Therefore, according (3.12) the global regularization of the restricted equilateral problem with arbitrary masses is given by

$$g(w) = \alpha w + \frac{\beta}{w^2},$$

where

$$\alpha = \frac{2}{3}, \quad \beta = \frac{w_1^3}{3} = \frac{1}{3} \left(\frac{(m_2 + m_3)}{2\sqrt{M_2}\text{sgn}(M_1) + m_2(m_3 - m_2)} + i \frac{m_2 - m_3}{2\sqrt{3}\sqrt{M_2}} \right)^3.$$

We observe the transformation for binary collisions given in [1] can be recovered by taking masses $m_1 = 1 - 2\mu, m_2 = m_3 = \mu$.

Now, we proceed to use Proposition 3.1. The pre-images in the case $u = w_j$ are given by w_j and the zeroes of the function \tilde{g} given by (3.14) which in this case is

$$\tilde{g}(w) = \frac{2}{3} \left(w + \frac{1}{2}w_1 \right).$$

Therefore, in the case $u = w_j$, there are exactly two pre-images, $w = w_j$ and $w = -\frac{w_1}{2}$.

On the other hand, in the case $u \neq w_j$, according (3.16) the pre-images are given by roots of

$$2w^3 - 3uw^2 + w_1^3 = 0. \tag{4.4}$$

Clearly, it has the following three different solutions

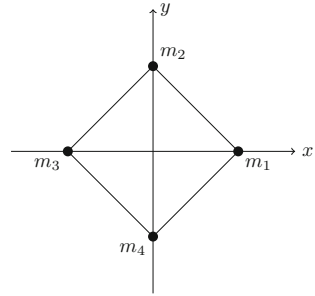
$$\begin{aligned} \tilde{w}_1 &= \frac{1}{2} \left(u + \frac{u^2}{(u^3 - 2w_1^3 + 2\sqrt{-u^3w_1^3 + w_1^6})^{1/3}} \right. \\ &\quad \left. + (u^3 - 2w_1^3 + 2\sqrt{-u^3w_1^3 + w_1^6})^{1/3} \right), \\ \tilde{w}_{2,3} &= \frac{u}{2} - \frac{(1 \pm i\sqrt{3})u^2}{4(u^3 - 2w_1^3 + 2\sqrt{-u^3w_1^3 + w_1^6})^{1/3}} \\ &\quad - \frac{1}{4}(1 \mp i\sqrt{3}) \left(u^3 - 2w_1^3 + 2\sqrt{-u^3w_1^3 + w_1^6} \right)^{1/3}. \end{aligned}$$

Then, we conclude that for $u \neq w_j$ there are exactly three pre-images.

Example Now consider the $(4 + 1)$ -body problem with primaries in a regular 4-gon. It is widely known that the only regular polygon with four sides is a square. Albuoy [2] showed that equal mass must be required, so we may suppose without loss of generality that the primaries masses are $m_1 = m_2 = m_3 = m_4 = 1$.

We will assume that the primaries are fixed at $(x_1, y_1) = (1, 0), (x_2, y_2) = (0, 1), (x_3, y_3) = (-1, 0)$ and $(x_4, y_4) = (0, -1)$, see Fig. 2.

Fig. 2 The square four-body problem



Let us remark that in this problem the barycenter and the center of mass coincide, so it is not necessary to translate the coordinate system as in the last example. Now, to performed the process of regularization, we are going to give the global regularizing transformation, which comes from (3.12) obtaining

$$g(w) = \frac{3}{4}w + \frac{w_1^4}{4w^3}.$$

We shall now apply the Proposition 3.1 to determine the number of pre-images of the function $g(w)$. From (3.14) we have the function

$$\tilde{g}(w) = \frac{3}{4}w^2 + \frac{1}{2}w_1w + \frac{1}{4}w_1^4.$$

Thus in the case $u = w_j$, there are three pre-images, namely $w = w_j$ and the corresponding two roots of the quadratic polynomial $p(w) = 3w^2 + 2w_1w + w_1^4$. We also have that for $u \neq w_j$ the number of pre-images of a point u under the transformation $g(w)$, we need to solve the equation given by (3.16). Explicitly these are

$$\begin{aligned} \tilde{w}_{1(2)} = & \frac{u}{3} \\ & - \frac{1}{3\sqrt{2}} \sqrt{2u^2 + \frac{3w_1^4}{(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} + 3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} \\ & \pm \frac{1}{2}\sqrt{\varrho}, \end{aligned}$$

where

$$\begin{aligned} \varrho = & \frac{8u^2}{9} - \frac{2w_1^4}{3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} - \frac{2}{3}(u^2 - w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3} \\ & - \frac{8\sqrt{2}u^3}{9\sqrt{2u^2 + \frac{3w_1^4}{(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} + 3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}}}, \end{aligned}$$

and

$$\tilde{w}_{3(4)} = \frac{u}{3} + \frac{1}{3\sqrt{2}} \sqrt{2u^2 + \frac{3w_1^4}{(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} + 3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3} \pm \frac{1}{2}\sqrt{\zeta}},$$

where

$$\zeta = \frac{8u^2}{9} - \frac{2w_1^4}{3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} - \frac{2}{3}(u^2 - w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3} + \frac{8\sqrt{2}u^3}{9\sqrt{2u^2 + \frac{3w_1^4}{(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}} + 3(u^2w_1^4 - \sqrt{u^4w_1^8 - w_1^{12}})^{1/3}}}.$$

We arrive thus at the conclusion that $u = g(w)$ has four pre-images.

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