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Qualitative Theory of Dynamical Systems

On Sufficient Conditions of Stability of the Permanent Rotations of a Heavy Triaxial Gyrostat

M. Teresa de Bustos Muñoz · Juan Luis García Guirao · Juan Antonio Vera López · Antonio Vigueras Campuzano

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Abstract In this work using as a main tool geometric–mechanics methods we study the permanent rotations of a particular type of heavy triaxial gyrostat. Also, we obtain sufficient conditions of stability of the permanent rotations found previously through the Energy–Casimir method.

Keywords Heavy gyrostat · Permanent rotations · Energy–Casimir method · Lyapunov stability

1 Introduction

The general study of the dynamics of rigid bodies and gyrostats has been treated extensively in the classic literature. Eulerian, Lagrangian and Hamiltonian formulations of

M. T. de Bustos Muñoz
Departmento de Matemática Aplicada, Universidad de Salamanca,
Cases del Parque, 2, 37008 Salamanca, Spain
e-mail: tbustos@usal.es

J. L. G. Guirao (✉) · A. V. Campuzano
Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena,
Hospital de Marina, 30203 Cartagena, Región de Murcia, Spain
e-mail: juan.garcia@upct.es

A. V. Campuzano
e-mail: antonio.vigueras@upct.es

J. A. Vera López
Centro Universitario de la Defensa. Academia General del Aire, Universidad Politécnica de Cartagena,
30720 Santiago de la Ribera, Región de Murcia, Spain
e-mail: juanantonio.vera@cud.upct.es

such dynamics have been the main tools used in the formulation of these problems (see for instance [7,8] or [20]).

It is known that a gyrostat is a mechanical system S made of a rigid body S_1 to which other bodies S_2 are connected; these other bodies may be variable or rigid, but must not be rigidly connected to S_1, so that the movements of S_2 with respect to S_1 do not modify the distribution of mass within the compound system S.

For instance, we can envision a rigid main body S_1, designated as the *platform*, supporting additional bodies S_2, which possess axial symmetry and are designated as *rotors*. These rotors may rotate with respect to the platform in such a way that the mass distribution within the system as a whole is not altered; this will produce an internal angular momentum, designated as *gyrostatic momentum*, which will be normally regarded as a constant. Note that when this constant vector is zero, the motion of the system is reduced to the motion of a rigid body.

Vito Volterra was the first to introduce the concept of a gyrostat in [22], in order to study the motion of the Earth's polar axis and explaining variations in the Earth's latitude by means of internal movements that do not alter the planets's distribution of mass.

Among the various aspects related to these problems that are discussed in the literature, we can highlight the following:

1. Equilibria and stabilities in rigid bodies and gyrostats, either with fixed point or in orbit (see [1,2,5,10,14–16,18,21]).
2. Periodic solutions, bifurcations, or chaos, in various gyrostat motion problems ([6,9,19]).
3. Integrability and first integrals for the problem (see [3,4,11]).

Moving to the study of the motion of rigid bodies and gyrostats, we can consider problems that are simpler, yet not less important, related to the motion of a gyrostat with a fixed point. The study of its equilibria and stabilities in such gyrostats under the gravity potential is interesting as a first approximation towards addressing complex problems.

In this work new approaches, both for classical and modern problems, can arise through the introduction of new mathematical tools. Using mechanics–geometry is one way for such problems to be addressed, as well as new perspectives in their study to be developed. In this paper, we will use such methods to describe the relative equilibria and sufficient conditions of stability of a heavy triaxial gyrostat with a fixed point. We are going to use as tools a variational characterization of relative equilibria and the Energy–Casimir method, that provides sufficient conditions for the Lyapunov stability. The permanent rotations of a heavy rigid body are called Staude rotations in honour to O. Staude (see [17]). The Staude rotations are equilibrium solutions of the Lie-Poisson equations of a heavy rigid body. A very detailed treatment of the Staude rotations of a heavy rigid body are exposed in [11]. The results obtained in this work generalize the results obtained for the heavy rigid body and recover, by these new methods, some classical results for gyrostats in [1,2,5,18] y [14]. In fact, in [5] in a classical way, similar to Mlozeevskii-Staude, the permanent axes of motion for a heavy gyrostat near an inmovable point are studied; in [1] the stability of permanent rotations of a gyrostat, by constructing a Lyapunov function as a combination of the first integrals of motion, is studied and sufficient conditions of stability are given, in the case in which the vector of gyrostatic momentum passes through the center of mass of the gyrostat and coincides with a principal axis of inertia, his results coincides with our results; in [18] the permanent axes of rotation of a gyrostat, under the action of forces resulting of a force function U that depends only on the position of the gyrostat, approximating U por U^2 are given and sufficient conditions of stability for the case in which the gyrostatic momentum is collinear with the vector angular velocity are obtained; in [2] as was done by Rumiantsev, using the second method of Lyapunov, the cone of permanent axes of a gyrostat with a fixed point and the domain stability is investigated; finally in [14] the Routh–Lyapunov theorem and its inverse is used to investigate the satability and bifurcationn of the steady state motions of a heavy gyrostat with a freely rotation rotor.

2 Lie–Poisson Equations of a Heavy Gyrostat with a Fixed Point

Let the rigid part of a gyrostat S be fixed in one of its points O, which will be taken as the origin of two systems of reference; one fixed $OX_1X_2X_3$ and another mobile $Ox_1x_2x_3$, fixed in the body, and whose axes are directed according to the principal inertia directions of the gyrostat at O. We will suppose that the only external forces come from the Newtonian attraction that a fixed point P (or a rigid body with a spherical distribution of masses) of mass M exerts over the gyrostat S, of total mass m. Besides, if the relative motion of its mobile part with respect to its rigid part is supposedly known, the system of axes $OX_1X_2X_3$ is chosen in such a way that the point P is over the axis OX_3, in its negative part, at a constant distance $r = |\mathbf{OP}|$, so that $\mathbf{r} = -r\mathbf{k}$, being $\mathbf{k} = (k_1, k_2, k_3)$ the unitary vector of the axis OX_3 expressed in the mobile system and the mutual potential between the point P and the gyrostat S is given by the function U, then according to the angular momentum theorem and the kinematic equations of Poisson, the equations of motion of this problem in the mobile system (see [11]) takes the form:

$$
\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 + \frac{l_2\pi_3}{I_3} - \frac{l_3\pi_2}{I_2} + k_2\frac{\partial U}{\partial k_3} - k_3\frac{\partial U}{\partial k_2}\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_3 I_1}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} + k_3\frac{\partial U}{\partial k_1} - k_1\frac{\partial U}{\partial k_3}\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_2 I_1}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2} - \frac{l_2\pi_1}{I_1} + k_1\frac{\partial U}{\partial k_2} - k_2\frac{\partial U}{\partial k_1}\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2}\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3}\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}
\end{aligned}
\tag{1}
$$

where

- $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the diagonal inertia tensor;
- $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the angular momentum of the gyrostat considered as a rigid body;
- $\boldsymbol{\Pi} = \boldsymbol{\pi} + \mathbf{l}$ is the total angular momentum vector for the gyrostat, where $\mathbf{l} = (l_1, l_2, l_3)$ is the gyrostatic momentum, assumed to be constant;
- $\mathbf{k} = (k_1, k_2, k_3)$ is the Poisson vector;
- $\mathbf{v}_1 \cdot \mathbf{v}_2$ is the scalar (vector) product in $\mathbb{R}^3$;
- $|\mathbf{v}|$ is the Euclidean norm of $\mathbf{v} \in \mathbb{R}^3$;
- $\times$ is the cross (vector) product in $\mathbb{R}^3$;
- $\mathbf{z} = (\boldsymbol{\pi}, \mathbf{k}) = (\pi_1, \pi_2, \pi_3, k_1, k_2, k_3)$;
- $\nabla_{\mathbf{z}} f$ is the gradient of f with respect to the vector $\mathbf{z} \in \mathbb{R}^6$.

In order to apply the Energy–Casimir method, we must declare the mechanical system in question to be a Lie–Poisson system with a certain Hamiltonian function $\mathcal{H}$, in this case it is given by the formula

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + U(k_1, k_2, k_3)$$

Then, it is easy to see that the Eq. (1) can be written in vector form using the following relations

$$\begin{aligned} \frac{d\pi}{dt} &= (\pi + \mathbf{l}) \times \nabla_\pi \mathcal{H} + \mathbf{k} \times \nabla_{\mathbf{k}} \mathcal{H} \\ \frac{d\mathbf{k}}{dt} &= \mathbf{k} \times \nabla_\pi \mathcal{H} \end{aligned} \tag{2}$$

Finally, we provide (using [12]) the following result that completely describes the equations as a Lie–Poisson system.

Proposition 1 (Lie–Poisson brackets of a gyrostat with a fixed point). *The geometric structure associated to the motion of a gyrostat with a fixed point O and constant gyrostatic momentum, is given by the following Lie–Poisson brackets:*

$$\{F, G\}(\boldsymbol{\pi}, \mathbf{k}) = -(\boldsymbol{\pi} + \mathbf{l}) \cdot (\nabla_{\boldsymbol{\pi}} F \times \nabla_{\boldsymbol{\pi}} G) - \mathbf{k} \cdot (\nabla_{\boldsymbol{\pi}} F \times \nabla_{\mathbf{k}} G + \nabla_{\mathbf{k}} F \times \nabla_{\boldsymbol{\pi}} G)$$

defined in $\mathbb{R}^3 \times \mathbb{R}^3$*, where* $F, G \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$*,* $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ *is the angular momentum of the gyrostat S, considered as a rigid body,* $\mathbf{k} = (k_1, k_2, k_3)$ *is the Poisson vector and* $\mathbf{l} = (l_1, l_2, l_3)$ *is the gyrostatic momentum.*

The Poisson tensor associated to the brackets, is given by the matrix:

$$B(\boldsymbol{\pi},\mathbf{k})=\begin{pmatrix}0 & -\pi_3-l_3 & \pi_2+l_2 & 0 & -k_3 & k_2\\ \pi_3+l_3 & 0 & -\pi_1-l_1 & k_3 & 0 & -k_1\\ -\pi_2-l_2 & \pi_1+l_1 & 0 & -k_2 & k_1 & 0\\ 0 & -k_3 & k_2 & 0 & 0 & 0\\ k_3 & 0 & -k_1 & 0 & 0 & 0\\ -k_2 & k_1 & 0 & 0 & 0 & 0\end{pmatrix}$$

And the Lie–Poisson equations associated to the Hamiltonian

$$\mathcal{H}=\frac{1}{2}\left(\frac{\pi_1^2}{I_1}+\frac{\pi_2^2}{I_2}+\frac{\pi_3^2}{I_3}\right)+U(k_1,k_2,k_3)$$

are expressed by the formulas (2).

Remark 2 Note that, if the gyrostatic momentum is zero then these equations are reduced to the equations of the rigid body with a fixed point.

The problem has two Casimir functions given by

$$(\boldsymbol{\pi}+\mathbf{l})\cdot\mathbf{k}=(\pi_1+l_1)k_1+(\pi_2+l_2)k_2+(\pi_3+l_3)k_3=p_\psi$$

with $p_\psi\in\mathbb{R}$ and

$$\mid\mathbf{k}\mid^2=k_1^2+k_2^2+k_3^2=1.$$

This system has, in general, three integrals of motion in involution, the Hamiltonian $\mathcal{H}$ and the two previous casimir functions.

2.1 The Gravitational Potential

We consider the Newtonian attraction of a point $\mathbf{P}(x,y,z)$, of mass M, on a body S, of total mass m. The elementary force $\mathbf{df}$ with the point $\mathbf{P}$ attracts the point $\mathbf{P}'$ of mass dm, of the body S, is given by

$$\mathbf{df}=-GM\frac{\Lambda}{\mid\Lambda\mid^3}$$

being $\Lambda=\mathbf{r}'-\mathbf{r}$, the vector of $\mathbf{P}'$ related to $\mathbf{P}$, and G the universal gravitational constant.

The total force $\mathbf{f}$ of the point $\mathbf{P}$ on the body S, is

$$\mathbf{f}=-GM\int_S\frac{\Lambda}{\mid\Lambda\mid^3}dm$$

If we keep in mind that, with regard to the coordinates (x', y', z') of $\mathbf{P}'$ and (x, y, z) of $\mathbf{P}$ the following relationship is verified

$$\nabla_{\mathbf{r}'}(\frac{1}{\mid \Lambda \mid}) = -\nabla_{\mathbf{r}}(\frac{1}{\mid \Lambda \mid}) = -\frac{\Lambda}{\mid \Lambda \mid^3}$$

we can introduce the potential function V given by

$$U = -GM\int_S \frac{1}{\mid \Lambda \mid} dm$$

obtaining

$$\mathbf{f} = \nabla_{\mathbf{r}}(U).$$

Let us see that the function U can be expressed by means of a series. We consider

$$r = \mid \mathbf{r} \mid, \; r' = \mid \mathbf{r}' \mid, \; h = \frac{r'}{r}$$

with θ is the angle formed for $\mathbf{P}$, $\mathbf{O}$ and $\mathbf{P}'$ and

$$\begin{aligned} \mid \Lambda \mid^2 &= r^2 + r'2 - 2rr' \cos\theta \\ \cos\theta &= \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'}. \end{aligned}$$

For $r' < r$ we obtain the following power series

$$\frac{1}{\mid \Lambda \mid} = \frac{1}{r\sqrt{1 - 2h\cos\theta + h^2}} = \frac{1}{r}\sum_{n=0}^{\infty} h^n P_n(\cos\theta)$$

with $P_n(x)$ the Legendre polynomials given by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n[(x^2-1)^n]}{dx^n}.$$

The potential function U is given by the following power series

$$U = \sum_{n=0}^{\infty} U_n = -\frac{GM}{r}\sum_{n=0}^{\infty}\int_S h^n P_n(\cos\theta) dm$$

where the series converges absolutely for $r > R$, being R the maximum of r' corresponding to points of S.

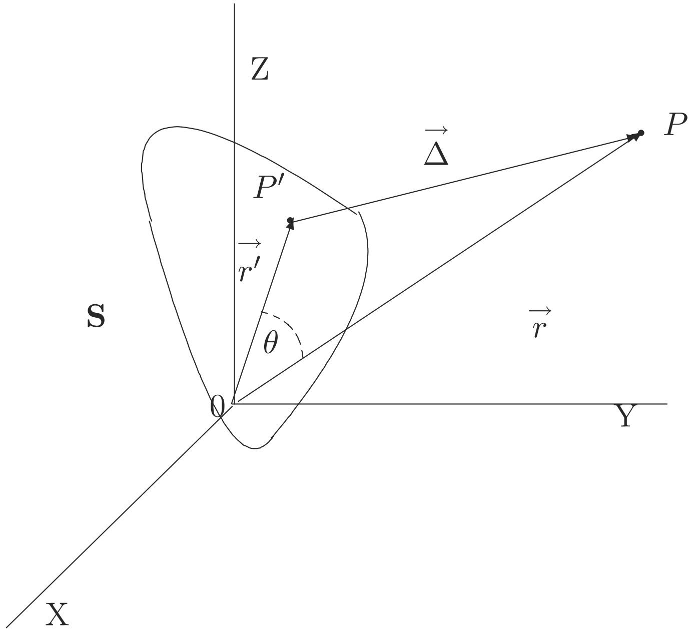

Fig. 1 The multipolar series of the potential

In this work we consider the first terms of the previous series which are

$$U_0 = -\frac{GM}{r}\int_S dm = -\frac{GM}{r},$$

$$U_1 = -\frac{GM}{r}\int_S \frac{r'}{r}\cos\theta dm = -\frac{GM}{r^3}\int_S rr'\cos\theta dm = -\frac{GM}{r^3}\mathbf{r}\cdot\int_S \mathbf{r}'dm$$
$$= -\frac{GMm}{r^3}\mathbf{r}\cdot\mathbf{r}_0 = mg(x_0k_1 + y_0k_2 + z_0k_3)$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the position of the mass center of S in the mobile frame $Ox_1x_2x_3$. Note that, the first term of the multipolar development of the potential U_0 is a numerical constant (Fig. 1).

2.2 Equations of Motion of a Heavy Gyrostat with a Fixed Point

In the Eq. (1) if the gravitational potential U considered is $U_0 + U_1$ then

$$\frac{d\pi_1}{dt} = \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 + \frac{l_2\pi_3}{I_3} - \frac{l_3\pi_2}{I_2} + mg\,(z_0k_2 - y_0k_3)$$
$$\frac{d\pi_2}{dt} = \left(\frac{I_3 - I_1}{I_3 I_1}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} + mg\,(x_0k_3 - z_0k_1)$$

$$\begin{aligned}
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_2 I_1}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2} - \frac{l_2\pi_1}{I_1} + mg\,(y_0k_1 - x_0k_2)\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2}\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3}\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}
\end{aligned} \tag{3}$$

In the following sections we study the equilibrium solutions and sufficient conditions for the stability of the same ones in the case of a heavy gyrostat with this particular mass geometry and gyrostatic momentum:

- *The mass center of the gyrostat lies in one of the principal axis of inertia at the point O, the axis Ox_1.*
- *The gyrostatic momentum is $\mathbf{l} = (l_1, 0, 0)$.*

3 Staude Rotations

Recall that the Staude rotations are the permanent rotations of the gyrostat i. e. the equilibrium solutions of the Lie–Poisson equations of a heavy gyrostat. In this section we characterize the equilibrium solutions of (3) under the two previous mentioned hypotesis. In [11], pages 65–93, we obtain the following result for a heavy rigid body in absence of gyrostatic momentum $\mathbf{l}$.

Proposition 3 *For a heavy rigid body with mass center in the principal inertia axis Ox_1 the only Staude rotations are the following*

$$\begin{aligned}
\mathbf{E}_1 &= (\pi_1 = I_1\omega\sin\varphi,\ \pi_2 = I_2\omega\cos\varphi,\ \pi_3 = 0,\ k_1 = \sin\varphi,\ k_2 = \cos\varphi,\ k_3 = 0)\\
\mathbf{E}_2 &= (\pi_1 = 0,\ \pi_2 = I_2\omega\sin\varphi,\ \pi_3 = I_3\omega\cos\varphi,\ k_1 = 0,\ k_2 = \sin\varphi,\ k_3 = \cos\varphi)\\
\mathbf{E}_3 &= (\pi_1 = I_1\omega\sin\varphi,\ \pi_2 = 0,\ \pi_3 = I_3\omega\cos\varphi,\ k_1 = \sin\varphi,\ k_2 = 0,\ k_3 = \cos\varphi)\\
\mathbf{E}_4 &= (\pi_1 = I_1\omega,\ \pi_2 = 0,\ \pi_3 = 0,\ k_1 = \pm 1,\ k_2 = 0,\ k_3 = 0)\\
\mathbf{E}_5 &= (\pi_1 = 0,\ \pi_2 = I_2\omega,\ \pi_3 = 0,\ k_1 = 0,\ k_2 = \pm 1,\ k_3 = 0)\\
\mathbf{E}_6 &= (\pi_1 = 0,\ \pi_2 = 0,\ \pi_3 = I_3\omega,\ k_1 = 0,\ k_2 = 0,\ k_3 = \pm 1)
\end{aligned}$$

with $\omega^2 = \frac{mgx_0}{2(I_1-I_2)\sin\varphi}$ for $\mathbf{E}_1$, $\omega^2 = \frac{mgx_0}{2(I_2-I_3)\sin\varphi}$ for $\mathbf{E}_2$ and $\omega^2 = \frac{mgx_0}{2(I_1-I_3)\sin\varphi}$ for $\mathbf{E}_3$. On the other hand, $\omega \in \mathbb{R}$ is a free parameter for $\mathbf{E}_4$, $\mathbf{E}_5$ and $\mathbf{E}_6$. Also for $\mathbf{E}_1$

Variation of φ	*Geometry of Mass*
$(0, \pi)$	$x_0 < 0, I_2 < I_1$
	$x_0 > 0, I_2 > I_1$
$(\pi, 2\pi)$	$x_0 < 0, I_2 > I_1$
	$x_0 > 0, I_2 < I_1$

(4)

Similar results are valid for $\mathbf{E}_2$ *and* $\mathbf{E}_3$.

The main result of this section is the following.

Theorem 4 *For a heavy gyrostat with mass center in the principal inertia axis* Ox_1 *and gyrostatic momentum* $\mathbf{l} = (l_1, 0, 0)$*, the only Staude rotations of* (3) *are the following points of* $\mathbb{R}^3 \times S^2$

$$\begin{aligned}
\mathbf{E}_1 &= (\pi_1 = I_1\omega\sin\varphi,\ \pi_2 = I_2\omega\cos\varphi,\ \pi_3 = 0,\ k_1 = \sin\varphi,\ k_2 = \cos\varphi,\ k_3 = 0)\\
\mathbf{E}_2 &= (\pi_1 = I_1\omega\sin\varphi,\ \pi_2 = 0,\ \pi_3 = I_3\omega\cos\varphi,\ k_1 = \sin\varphi,\ k_2 = 0,\ k_3 = \cos\varphi)\\
\mathbf{E}_3 &= (\pi_1 = I_1\omega,\ \pi_2 = 0,\ \pi_3 = 0,\ k_1 = \pm 1,\ k_2 = 0,\ k_3 = 0)\\
\mathbf{E}_4 &= (\pi_1 = 0,\ \pi_2 = I_2\omega,\ \pi_3 = 0,\ k_1 = 0,\ k_2 = \pm 1,\ k_3 = 0)\\
\mathbf{E}_5 &= \left(\pi_1 = 0,\ \pi_2 = 0,\ \pi_3 = \frac{mgx_0}{l},\ k_1 = 0,\ k_2 = 0,\ k_3 = \pm 1\right)
\end{aligned}$$

with

$$\left(\omega + \frac{l_1}{2(I_1 - I_2)\sin\varphi}\right)^2 = \frac{mgx_0}{2(I_1 - I_2)\sin\varphi} + \left(\frac{l_1}{2(I_1 - I_2)\sin\varphi}\right)^2$$

for $\mathbf{E}_1$ *and*

$$\left(\omega + \frac{l_1}{2(I_1 - I_3)\sin\varphi}\right)^2 = \frac{mgx_0}{2(I_1 - I_3)\sin\varphi} + \left(\frac{l_1}{2(I_1 - I_3)\sin\varphi}\right)^2$$

for $\mathbf{E}_2$. *For* $\mathbf{E}_3$ *and* $\mathbf{E}_4$*,* $\omega \in \mathbb{R}$ *is a free parameter. The conditions* (4) *are the same for* $\mathbf{E}_1$ *and* $\mathbf{E}_2$.

Proof From the Eq. (3) we obtain in an immediate way that the points $\mathbf{E}_i$ for $i = 3, 4, 5$ are stationary solutions. The other Staude rotations of (3) are critical points of the following function

$$\mathcal{H}_{\lambda,\mu} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgx_0k_1 + \lambda(\boldsymbol{\pi} + \mathbf{l}) \cdot \mathbf{k} + \mu\ \mid \mathbf{k} \mid^2$$

with λ and μ real parameters to determine.

According to the equations of motion, we can deduce that for all Staude rotations must verify $\Omega = \omega\mathbf{k}$, being $\Omega = \left(\frac{\pi_1}{I_1}, \frac{\pi_2}{I_2}, \frac{\pi_3}{I_3}\right)$ the angular velocity of S and $\omega \in \mathbb{R}$. Moreover, all equilibria are also tied by $(\pi_1 + l_1)k_1 + \pi_2k_2 + \pi_3k_3 = p_\psi$ with $p_\psi \in \mathbb{R}$ and $k_1^2 + k_2^2 + k_3^2 = 1$. We can assume that these have the expression

$$\mathbf{E} = (I_1\omega u,\ I_2\omega v,\ I_3\omega w,\ u,\ v,\ w)$$

with $u^2 + v^2 + w^2 = 1$. Using

$$\left\{ \left.\frac{\partial \mathcal{H}_{\lambda,\mu}}{\partial \pi_i}\right|_{\mathbf{E}} = 0,\ \left.\frac{\partial \mathcal{H}_{\lambda,\mu}}{\partial k_i}\right|_{\mathbf{E}} = 0,\ i = 1, 2, 3 \right\}$$

we obtain $v = 0$ or $w = 0$. Considering $w = 0$ using $u^2 + v^2 = 1$ we propose for possible solution of the previous system

$$\mathbf{E}_1 = (\pi_1 = I_1\omega \sin\varphi,\ \pi_2 = I_2\omega\cos\varphi,\ \pi_3 = 0,\ k_1 = \sin\varphi,\ k_2 = \cos\varphi,\ k_3 = 0)$$

and if $v = 0$ using $u^2 + w^2 = 1$ the following

$$\mathbf{E}_2 = (\pi_1 = I_1\omega \sin\varphi,\ \pi_2 = 0,\ \pi_3 = I_3\omega\cos\varphi,\ k_1 = \sin\varphi,\ k_2 = 0,\ k_3 = \cos\varphi).$$

The multipliers are $\lambda = -\omega$ and $\mu = B\omega^2/2$ for $\mathbf{E}_1$ and $\mu = C\omega^2/2$ for $\mathbf{E}_2$. After some computations, the angular velocity of rotation ω satisfies the following equation

$$\left(\omega + \frac{l_1}{2(I_1 - I_2)\sin\varphi}\right)^2 = \frac{mgx_0}{2(I_1 - I_2)\sin\varphi} + \left(\frac{l_1}{2(I_1 - I_2)\sin\varphi}\right)^2$$

for $\mathbf{E}_1$ and

$$\left(\omega + \frac{l_1}{2(I_1 - I_3)\sin\varphi}\right)^2 = \frac{mgx_0}{2(I_1 - I_3)\sin\varphi} + \left(\frac{l_1}{2(I_1 - I_3)\sin\varphi}\right)^2$$

for $\mathbf{E}_2$. □

4 Sufficient Conditions of Stability of the Staude Rotations

We are going to use the Energy–Casimir method, see [13] for details, to obtain the sufficient conditions for the Lyapunov stability of the Staude rotations $\mathbf{E}_i$, $i = 1, 2, 3, 4$.

Theorem 5 (Energy–Casimir) *Be* $(\mathbf{M}, \mathbf{B}, \mathcal{H})$ *a Lie–Poisson system. We consider* $\mathbf{z}_e$ *a equilibrium of the equations*

$$\frac{d\mathbf{z}}{dt} = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}(\mathbf{z}).$$

Be $C_1, C_2, \ldots, C_r \in \mathfrak{F}(M)$ *integrals of the system and*

$$\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_r} = \mathcal{H} + \sum_{i=1}^{r} \phi_i(C_i)$$

where $\phi_i \in C^\infty(\mathbb{R})$, $i = 1, \ldots, r$ *takes so that* $\mathbf{d}\mathcal{H}_{\phi_1,\ldots,\phi_r}(\mathbf{z}_e) = 0$. *Then, if* $\mathbf{d}^2\mathcal{H}_{\phi_1,\ldots,\phi_r}(\mathbf{z}_e)\,|\,_{W\times W}$ *being*

$$W = \ker \mathbf{d}C_1(\mathbf{z}_e) \cap \ker \mathbf{d}C_2(\mathbf{z}_e) \cap \ldots \cap \ker \mathbf{d}C_r(\mathbf{z}_e)$$

is defined positive or negative, then $\mathbf{z}_e$ *is stable in the sense of Lyapunov. In particular, if* $W = 0$, *then* $\mathbf{z}_e$ *is stable in the sense of Lyapunov.*

4.1 Sufficient Conditions of Stability of E_1 and E_2

The main result is the following:

Theorem 6 *A sufficient condition for the Lyapunov stability of* $\boldsymbol{E}_1$ *is*

$$\begin{aligned} I_2 &> I_1, \\ I_2 &> I_3. \end{aligned} \tag{5}$$

A sufficient condition for the Lyapunov stability of $\boldsymbol{E}_2$ *is*

$$\begin{aligned} I_3 &> I_1, \\ I_3 &> I_2. \end{aligned} \tag{6}$$

Proof Similarly, we consider the function

$$\mathcal{H}_{\phi_1,\phi_2} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgx_0k_1 + \phi_1((\boldsymbol{\pi} + \mathbf{l}) \cdot \mathbf{k}) + \phi_2(|\,\mathbf{k}\,|^2)$$

with ϕ_i smooth real functions with $\phi_1''(\mathbf{E}_i) = 0$ and $\phi_2''(\mathbf{E}_i) = 0$ for $i = 1, 2$. The multipliers are $\lambda = \phi_1'(\mathbf{E}_i) = -\omega$ and $\mu = \phi_1'(\mathbf{E}_i) = I_2\omega^2/2$ for $\mathbf{E}_1$ or $\mu = \phi_1'(\mathbf{E}_i) = I_3\omega^2/2$ for $\mathbf{E}_2$ respectively. The Hessian matrix $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i)$ for $i = 1, 2$ are

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i) = \begin{pmatrix} \frac{1}{I_1} & 0 & 0 & \lambda & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 & 0 & \lambda & 0 \\ 0 & 0 & \frac{1}{I_3} & 0 & 0 & \lambda \\ \lambda & 0 & 0 & 2\mu & 0 & 0 \\ 0 & \lambda & 0 & 0 & 2\mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & 2\mu \end{pmatrix}.$$

Computing $W = \ker \mathbf{d}C_1(\mathbf{E}_i) \cap \ker \mathbf{d}C_2(\mathbf{E}_i)$ with $C_1 = \phi_1((\pi + \mathbf{l}) \cdot \mathbf{k}))$ and $C_2 = \phi_2(|\,\mathbf{k}\,|^2)$ we obtain

$$W = span\,(\{\cos\varphi\; \mathbf{e}_1 - \sin\varphi\; \mathbf{e}_2, \mathbf{e}_3, \cos\varphi\; \mathbf{e_4} - \sin\varphi\; \mathbf{e}_5, \mathbf{e}_6\})$$

for $\mathbf{E}_1$ and

$$W = span\,(\{\cos\varphi\; \mathbf{e}_1 - \sin\varphi\; \mathbf{e}_3, \mathbf{e}_2, \cos\varphi\; \mathbf{e_4} - \sin\varphi\; \mathbf{e}_6, \mathbf{e}_5\})$$

for $\mathbf{E}_2$ with $\mathcal{B}$=$\{\mathbf{e}_i\}_{i=1..6}$ the canonical basis of $\mathbb{R}^6$. Then $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i)\Big|_{W\times W}$ are

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_1)\Big|_{W\times W} = \begin{pmatrix} \frac{\cos^2\varphi(I_2 - I_1) + I_1}{I_2 I_1} & 0 & \lambda & 0 \\ 0 & \frac{1}{I_3} & 0 & 0 \\ \lambda & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \end{pmatrix}$$

and

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_2)\Big|_{W\times W} = \begin{pmatrix} \frac{\cos^2\varphi(I_3-I_1)+I_1}{I_3I_1} & 0 & \lambda & 0 \\ 0 & \frac{1}{I_2} & 0 & 0 \\ \lambda & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \end{pmatrix}.$$

The Sylvester criterion for positive definiteness of the matrix $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_1)\big|_{W\times W}$ are the following

$$\Delta_1 = \frac{\cos^2\varphi(I_2-I_1)+I_1}{I_2I_1} > 0, \quad \Delta_2 = \frac{2\mu-\lambda^2I_2}{I_2I_3} > 0,$$
$$\Delta_3 = \left|\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_1)\Big|_{W\times W}\right| = \frac{2\mu\left(2\mu-\lambda^2I_2\right)}{I_2I_3} > 0. \tag{7}$$

Analogously, the Sylvester criterion for positive definiteness of the matrix $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_2)|_{W\times W}$ are

$$\Delta_1 = \frac{\cos^2\varphi(I_3-I_1)+I_1}{I_3I_1} > 0, \quad \Delta_2 = \frac{2\mu-\lambda^2I_1}{I_1I_3} > 0,$$
$$\Delta_3 = \left|\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_2)\Big|_{W\times W}\right| = \frac{2\mu\left(2\mu-\lambda^2I_1\right)}{I_1I_3} > 0. \tag{8}$$

Using the values of $\lambda = -\omega$ and $\mu = I_2\omega^2/2$ in () we obtain the sufficient conditions

$$\Delta_1 = \frac{\cos^2\varphi(I_2-I_1)+I_1}{I_2I_1} > 0, \quad \Delta_2 = \frac{\omega^2\cos^2\varphi(I_2-I_1)}{I_3I_1} > 0,$$
$$\Delta_3 = \frac{\omega^4\cos^2\varphi(I_2-I_1)(I_2-I_3)}{I_3I_1} > 0.$$

with are equivalent to (5). Using $\lambda = -\omega$ and $\mu = I_3\omega^2/2$ in () we obtain the sufficient conditions

$$\Delta_1 = \frac{\cos^2\varphi(I_3-I_1)+I_1}{I_2I_1} > 0, \quad \Delta_2 = \frac{\omega^2\cos^2\varphi(I_3-I_1)}{I_3I_1} > 0,$$
$$\Delta_3 = \frac{\omega^4\cos^2\varphi(I_3-I_1)(I_3-I_2)}{I_3I_1} > 0.$$

with are equivalent to (10). □

4.2 Sufficient Conditions of Stability of E_3 and E_4

The Staude rotation $\boldsymbol{E}_3$ and $\boldsymbol{E}_4$ correspond physically to the motion of the gyrostat around the principal inertia axis Ox_1 and Ox_2 respectively. The main result is the following:

Theorem 7 *A sufficient condition for the Lyapunov stability of* $\boldsymbol{E}_3$ *is*

$$(I_1 - I_2)\omega^2 + l_1\omega > \pm mgx_0,$$
$$(I_1 - I_3)\omega^2 + l_1\omega > \pm mgx_0. \tag{9}$$

A sufficient condition for the Lyapunov stability of $\boldsymbol{E}_4$ *is*

$$(I_2 - I_3)\omega^2 + l_1\omega > \pm mgx_0,$$
$$(I_2 - I_1)\omega^2 + l_1\omega > \pm mgx_0. \tag{10}$$

Proof Now, we consider the function

$$\mathcal{H}_{\phi_1,\phi_2} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgx_0k_1 + \phi_1((\boldsymbol{\pi} + \mathbf{l}) \cdot \mathbf{k}) + \phi_2(\mid \mathbf{k} \mid^2)$$

with ϕ_i smooth real functions with $\phi_1''(\mathbf{E}_i) = 0$ and $\phi_2''(\mathbf{E}_i) = 0$. The multipliers are $\lambda = \phi_1'(\mathbf{E}_i) = \mp\omega$ and $\mu = \phi_1'(\mathbf{E}_i) = (I_1\omega^2 + l_1\omega \mp mgx_0)/2$ or $\mu = (I_1\omega^2 + l_1\omega \mp mgx_0)/2$ for $\mathbf{E}_3$ and $\mathbf{E}_4$, respectively. The Hessian matrix $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i)$ for $i = 3, 4$ are

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i) = \begin{pmatrix} \frac{1}{I_1} & 0 & 0 & \lambda & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 & 0 & \lambda & 0 \\ 0 & 0 & \frac{1}{I_3} & 0 & 0 & \lambda \\ \lambda & 0 & 0 & 2\mu & 0 & 0 \\ 0 & \lambda & 0 & 0 & 2\mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & 2\mu \end{pmatrix}.$$

Computing $W = \ker \mathbf{d}C_1(\mathbf{E}_i) \cap \ker \mathbf{d}C_2(\mathbf{E}_i)$ with $C_1 = \phi_1((\pi + \mathbf{l}) \cdot \mathbf{k})$ and $C_2 = \phi_2(\mid \mathbf{k} \mid^2)$ we obtain

$$W = span\,(\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6\})$$

for $\mathbf{E}_3$ and

$$W = span\,(\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_6\})$$

for $\mathbf{E}_4$ with $\mathcal{B}$=$\{\mathbf{e}_i\}_{i=1..6}$ the canonical basis of $\mathbb{R}^6$. Then $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_i)\big|_{W\times W}$ are

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_3)\Big|_{W\times W} = \begin{pmatrix} \frac{1}{I_2} & 0 & \lambda & 0 \\ 0 & \frac{1}{I_3} & 0 & 0 \\ \lambda & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \end{pmatrix}.$$

and

$$\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_4)\Big|_{W\times W} = \begin{pmatrix} \frac{1}{I_1} & 0 & \lambda & 0 \\ 0 & \frac{1}{I_3} & 0 & 0 \\ \lambda & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \end{pmatrix}.$$

The Sylvester criterion for positive definiteness of the matrix $\mathbf{d}^2\mathcal{H}_{\phi_1,\phi_2}(\mathbf{E}_3)\big|_{W\times W}$ are the following

$$\Delta_1 = \begin{vmatrix} \frac{1}{I_2} & 0 & \lambda \\ 0 & \frac{1}{I_3} & 0 \\ \lambda & 0 & 2\mu \end{vmatrix} = \frac{2\mu - \lambda^2 I_2}{I_2 I_3} > 0,$$

$$\Delta_2 = \left|\mathbf{d}^2\mathcal{H}_{\lambda,\mu}(\mathbf{E}_3)\Big|_{W\times W}\right| = \frac{2\mu\left(2\mu - \lambda^2 I_2\right)}{I_2 I_3}. \tag{11}$$

Analogously, the Sylvester criterion for positive definiteness of the matrix $\mathbf{d}^2\mathcal{H}_{\lambda,\mu}$ $(\mathbf{E}_4)|_{W\times W}$ are

$$\Delta_1 = \begin{vmatrix} \frac{1}{I_1} & 0 & \lambda \\ 0 & \frac{1}{I_3} & 0 \\ \lambda & 0 & 2\mu \end{vmatrix} = \frac{2\mu - \lambda^2 I_1}{I_1 I_3} > 0,$$

$$\Delta_2 = \left|\mathbf{d}^2\mathcal{H}_{\lambda,\mu}(\mathbf{E}_4)\Big|_{W\times W}\right| = \frac{2\mu\left(2\mu - \lambda^2 I_1\right)}{I_1 I_3}. \tag{12}$$

Using $\lambda = \mp\omega$ and $\mu = (I_1\omega^2 + l_1\omega \mp mgx_0)/2$ in (11) we obtain the sufficient conditions (9). Using $\lambda = \mp\omega$ and $\mu = (I_2\omega^2 + l_1\omega \mp mgx_0)/2$ in (12) we obtain the sufficient conditions (10). □

Remark 8 For the permanent rotation $\mathbf{E}_5$, the Energy–Casimir Method don't give us information about sufficient conditions of stability because the quadratic form is semidefinite. Local analysis using canonical variables are needed. This investigation is a work in progress.

5 Conclusions

In this paper we consider the non-canonical Hamiltonian dynamics of a heavy gyrostat with a fixed point with the mass center of the gyrostat placed in one of the principal axis of inertia at the point O, the axis Ox, and with gyrostatic momentum $\mathbf{l} = (l_1, 0, 0)$. By means of geometric–mechanics methods we study the Staude rotations of this system. Also, we use the Energy–Casimir method to obtain sufficient conditions of the Staude rotations $\mathbf{E}_i$, $i = 1, 2, 3, 4$.

Derive this conditions applying the classical method of Lyapunov–Chetaev has very tedious and less systematic that the method of the Energy–Casimir. The advantages of the last method in stability problems of gyrostat dynamics is clear.

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