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Qualitative Theory of Dynamical Systems

On the Exponential Stability of Discrete Semigroups

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Abstract Let q be a positive integer and let X be a complex Banach space. We denote by $\mathbb{Z}_+$ the set of all nonnegative integers. Let $P_q(\mathbb{Z}_+, X)$ is the set of all X-valued, q-periodic sequences. Then $P_1(\mathbb{Z}_+, X)$ is the set of all X-valued constant sequences. When $q \geq 2$, we denote by $P_q^0(\mathbb{Z}_+, X)$, the subspace of $P_q(\mathbb{Z}_+, X)$ consisting of all sequences $z(.)$ with $z(0) = 0$. Let T be a bounded linear operator acting on X. It is known, that the discrete semigroup generated (from the algebraic point of view) of T, i.e. the operator valued sequence $T = (T^n)$, is uniformly exponentially stable (i.e. $\lim_{n\to\infty} \frac{\ln \|T^n\|}{n} < 0$), if and only if for each real number μ and each sequences $z(.)$ in $P_1(\mathbb{Z}_+, X)$ the sequences (y_n) given by

$$\begin{cases} y_{n+1} = T(1)y_n + e^{i\mu(n+1)}z(n+1), \\ y(0) = 0 \end{cases}$$

is bounded. In this paper we prove a complementary result taking $P_q^0(\mathbb{Z}_+, X)$ with some integer $q \geq 2$ instead of $P_1(\mathbb{Z}_+, X)$.

Keywords Exponential stability · Discrete semigroups · Periodic sequences

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1 Introduction

Let A be a bounded linear operator acting on a complex Banach space X. A well known theorem of Krein [8,10] says that the system $\dot{x}(t) = Ax(t)$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $y_0 \in X$ the solution of the Cauchy problem

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t} y_0, \\ y(0) = 0 \end{cases}$$

is bounded. The proof of this classic result can be found in [1]. This result can also be extended for strongly continuous bounded semigroups, see [3–5,11].

Under a slightly different assumption the result on stability is also preserved for any strongly continuous semigroups acting on complex Hilbert spaces, see for example [12,13] and references therein. See also [2,9], for counter-examples. In [7,14] the same result were extended for square size matrices in both continuous and discrete cases.

In [6], similar results were obtained in the following manner. *Let $P_1^0(\mathbb{R}_+, X)$ denotes the set of all continuous X-valued functions such that $f(t+1) = f(t)$ for all $t \geq 0$ with $f(0) = 0$ and let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on the Banach space X. If the condition*

$$\sup_{t>0} \left\| \int_0^t e^{i\mu\xi} T(t-\xi) f(\xi) d\xi \right\| < \infty,$$

holds for all $\mu \in \mathbb{R}$ and every $f \in P_1^0(\mathbb{R}_+, X)$ then *T is uniformly exponentially stable.*

In this article we follow a similar approach of the last quoted paper and study the system $x_{n+1} = T(1)x_n$, where $T(1)$ is the algebraic generator of the discrete semi group $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$.

2 Notations and Preliminaries

Let $\mathcal{L}(X)$ be the Banach algebra of all linear and bounded operators acting on a Banach space X. The norm on X and $\mathcal{L}(X)$ is denoted by $\|\|$. By $\mathbb{R}$ we denote the set of all real numbers and $\mathbb{Z}_+$ the set of all non-negative integers.

Let $\mathcal{S}(\mathbb{Z}_+, X)$ be the space of all X-valued bounded sequences endowed with the supremum norm denoted by $\|.\|_\infty$, and $P_0^q(\mathbb{Z}_+, X)$ be the space of q-periodic bounded sequences $z(n)$ with $z(0) = 0$ and let $q \geq 2$ be a given integer. Then clearly $P_0^q(\mathbb{Z}_+, X)$ is a closed subspace of $\mathcal{S}(\mathbb{Z}_+, X)$.

Let A is a bounded linear operator on X and $\sigma(A)$ is its spectrum. By $r(A)$ we denote the spectral radius of A, and is defined as

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

It is well known that $r(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. The resolvent set of A is defined as $\rho(A) := \mathbb{C}\backslash\sigma(A)$, i.e the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is an invertible operator in $\mathcal{L}(X)$.

We recall that A is power bounded if there exists a positive constant M such that $\|A^n\| \le M$ for all $n \in \mathbb{Z}_+$.

3 Exponential Stability of Discrete Semigroups

We recall that a discrete semigroup is a family $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ of bounded linear operators acting on X which satisfies the following conditions

(1) $T(0) = I$, the identity operator on X,
(2) $T(n + m) = T(n)T(m)$ for all $n, m \in \mathbb{Z}_+$.

Let $T(1)$ denote the algebraic generator of the semigroup $\mathbb{T}$. Then it is clear that $T(n) = T^n(1)$ for all $n \in \mathbb{Z}_+$. The growth bound of $\mathbb{T}$ is denoted by $\omega_0(\mathbb{T})$ and is defined as

$$\omega_0(\mathbb{T}) := \inf\{\omega \in \mathbb{R} : \text{ there exists } M_\omega > 0 \\ \text{such that } \|T(n)\| \le M_\omega e^{\omega n} \text{ for all } n \in \mathbb{Z}_+\}.$$

The family $\mathbb{T}$ is uniformly exponentially stable if $\omega_0(\mathbb{T})$ is negative, or equivalently, if there exist two positive constants M and ω such that $\|T(n)\| \le Me^{-\omega n}$ for all $n \in \mathbb{Z}_+$.

We recall the following lemma without proof from [6], so that the paper will be self contained

Lemma 3.1 *Let $A \in \mathcal{L}(X)$. If the inequality*

$$\sup_{n\in\mathbb{Z}_+} \Big|\Big| \sum_{k=0}^{n} e^{i\mu k} A^k \Big|\Big| = M_\mu < \infty, \text{ for all } \mu \in \mathbb{R}, \tag{3.1}$$

then $r(A) < 1$.

We denote by $\mathcal{A}$ the set of all X-valued, q-periodic sequences z such that

$$z(n) = n(n - q)T(n)x, \quad n \in \{0, 1, \dots, q - 1\}, \quad x \in X.$$

Now we state and prove our main result.

Theorem 3.2 *Let $T(1)$ is the algebraic generator of the discrete semigroup $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ on X and $\mu \in \mathbb{R}$. Then the following holds true.*

(1) *If the system $x_{n+1} = T(1)x_n$ is uniformly exponentially stable then for each real number μ and each q-periodic sequence $(z(n))$ with $z(0) = 0$ the solution of the Cauchy Problem*

$$\begin{cases} y_{n+1} = T(1)y_n + e^{i\mu(n+1)}z(n+1), \\ y(0) = 0 \end{cases} \qquad (T(1), \mu, 0)$$

is bounded.

(2) *If for each real number* μ *and each* q*-periodic sequence* $(z(n))$ *in* $\mathcal{A}$ *the solution of the Cauchy Problem* $(T(1), \mu, 0)$ *is bounded, with the assumption that the operator* $e^{i\mu n}\sum_{\nu=1}^{q-1} e^{i\mu\nu}\nu(q-\nu)x$ *is non zero for each non zero* x *in* X. *Then* $\mathbb{T}$ *is uniformly exponentially stable.*

Proof **(1)** Here we show that if $\mathbb{T}$ is uniformly exponentially stable then the solution (y_n) of $(T(1), \mu, 0)$ is bounded.

Let M and ν be positive constants such that $\|T(n)\| \leq Me^{-\nu n}$, for all $n \in \mathbb{Z}_+$. The solution of the Cauchy Problem $(T(1), \mu, 0)$ is given by

$$y_n = \sum_{k=0}^{n} e^{iuk} T(n-k)z(k). \tag{3.2}$$

Taking norm of both sides

$$\begin{aligned}
\|y_n\| &= \|\sum_{k=0}^{n} e^{iuk} T(n-k)z(k)\| \\
&\leq \sum_{k=0}^{n} \|e^{iuk} T(n-k)z(k)\| \\
&= \sum_{k=0}^{n} \|e^{iuk}\| \|T(n-k)\| \|z(k)\| \\
&\leq \sum_{k=0}^{n} \|T(n-k)\| \|z(k)\| \\
&\leq \sum_{k=0}^{n} Me^{-\nu(n-k)}M', \text{ where } M' = \sup_{k\in\mathbb{Z}_+} \|z(k)\| \\
&= M''e^{-\nu n}\sum_{k=0}^{n} e^{\nu k}, \text{ where } M'' = MM' \\
&= M''e^{-\nu n}\left(\frac{1-e^{(n+1)\nu}}{1-e^{\nu}}\right), \text{where } \nu > 0 \\
&\leq M''e^{-\nu n} \\
&\leq \frac{M''}{e^{\nu}}.
\end{aligned}$$

Thus the solution of the Cauchy Problem $(T(1), \mu, 0)$ is bounded.

(2) The proof of the second part is not so easy. Let us divide n by q i.e. $n = Nq + r$ for some $N \in \mathbb{Z}_+$, where $r \in \{0, 1, \ldots, q-1\}$.

For each $j \in \mathbb{Z}_+$, we consider the set $A_j := \{1 + jq, 2 + jq, \dots, q - 1 + jq\}$, also let $B_N := \{Nq + 1, \dots, Nq + r\}$ if $r \geq 1$ and $B := \{0, q, 2q, \dots, Nq\}$ then clearly

$$\cup_{j=0}^{N-1} A_j \cup B_N \cup B = \{0, 1, 2, \dots, n\}.$$

From (3.2) we know that the solution of the Cauchy Problem $(T(1), \mu, 0)$ is

$$y_n = \sum_{k=0}^{n} e^{i\mu k} T(n-k) z(k).$$

The sequence $z(.)$ belongs to the set $\mathcal{A}$ if and only if there exists $x \in X$ such that

$$z(k) = \begin{cases} (k - jq)[(1 + j)q - k]T(k - jq)x, & \text{if } \; k \in A_j, \\ k(q - k)T(k)x, & \text{if } \; k \in B_N, \\ 0, & \text{if } \; k \in C. \end{cases}$$

Then clearly $\big(z(k)\big) \in \mathcal{A}$. Thus

$$\begin{aligned} y_n &= \sum_{k=0}^{n} e^{i\mu k} T(n-k) z(k) \\ &= \sum_{k \in \cup_{j=0}^{N-1} A_j} e^{i\mu k} T(Nq + r - k) z(k) \\ &\quad + \sum_{k \in B_N} e^{i\mu k} T(Nq + r - k) z(k) \\ &\quad + \sum_{k \in C} e^{i\mu k} T(Nq + r - k) z(k) \\ &= \sum_{j=0}^{N-1} \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k} T(Nq + r - k) z(k) \\ &\quad + \sum_{k=Nq+1}^{Nq+r} e^{i\mu k} T(Nq + r - k) z(k) \\ &= J_1 + J_2. \end{aligned}$$

where

$$\begin{aligned} J_1 &= \sum_{j=0}^{N-1} \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k} T(Nq + r - k)(k - jq)[q - (k - jq)]T(k - jq)x \\ &= \sum_{j=0}^{N-1} T(Nq + r - jq) \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k}(k - jq)[q - (k - jq)]x \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{N-1} T(Nq + r - jq) e^{i\mu jq} \sum_{\nu=1}^{q-1} e^{i\mu\nu} \nu(q - \nu) x \\
&= \sum_{j=0}^{N-1} e^{-i\mu(Nq+r-jq)} T(Nq + r - jq) e^{i\mu(Nq+r)} \sum_{\nu=1}^{q-1} e^{i\mu\nu} \nu(q - \nu) x \\
&= \sum_{\omega=r+q}^{r+Nq} e^{-i\mu\omega} T^{\omega}(1) e^{i\mu n} \sum_{\nu=1}^{q-1} e^{i\mu\nu} \nu(q - \nu) x \\
&= \sum_{\omega=r+q}^{n} e^{-i\mu\omega} T^{\omega}(1) S(x)
\end{aligned}
$$

with $S(x) = e^{i\mu n} \sum_{\nu=1}^{q-1} e^{i\mu\nu} \nu(q - \nu) x$ with the assumption that $S(x) \neq 0$ for each non zero x in X.

And

$$
\begin{aligned}
J_2 &= \sum_{\rho=0}^{r-1} e^{i\mu(Nq+r-\rho)} T(\rho) z(Nq + r - \rho) x \\
&= \sum_{\rho=0}^{r-1} e^{i\mu(Nq+r-\rho)} T(\rho) z(r - \rho) x.
\end{aligned}
$$

Taking norm of both sides

$$
\begin{aligned}
\|J_2\| &= \| \sum_{\rho=0}^{r-1} e^{i\mu(Nq+r-\rho)} T(\rho) z(r - \rho) x \| \\
&\leq \| \sum_{\rho=0}^{r-1} T(\rho) x \| \|z(r - \cdot)\|_{\infty} \\
&\leq M, \text{ where } M \text{ is some constant}
\end{aligned}
$$

i.e. J_2 is bounded.

Hence,

$$
\sum_{k=0}^{n} e^{i\mu k} T(n - k) z(k) = \sum_{\omega=r+q}^{n} e^{-i\mu\omega} T^{\omega}(1) S(x) + \sum_{\rho=0}^{r-1} e^{i\mu(n-\rho)} T(\rho) z(r - \rho) x.
$$

Now by our assumption (y_n) is bounded i.e.

$$
\sup_{n\geq 0} \|y_n\| = \sup_{n\geq 0} \| \sum_{k=0}^{n} e^{i\mu k} T(n - k) z(k) \| < \infty.
$$

Thus

$$\sup_{n\geq 0} \| \sum_{\omega=r+q}^{n} e^{-i\mu\omega} T^{\omega}(1)S(x) + \sum_{\rho=0}^{r-1} e^{i\mu(n-\rho)} T(\rho)z(r-\rho)x \| < \infty.$$

Which implies that

$$\sup_{n\geq 0} \| \sum_{\omega=r+q}^{n} e^{-i\mu\omega} T^{\omega}(1)S(x) \| < \infty.$$

i.e.

$$\sup_{n\geq 0} \| \sum_{\omega=r+q}^{n} e^{-i\mu\omega} T^{\omega}(1) \| < \infty.$$

Thus by Lemma 3.4, $\mathbb{T}$ is uniformly exponentially stable. □

Corollary 3.3 *The system $x_{n+1} = T(1)x_n$ is uniformly exponentially stable if and only if for each real number μ and each q-periodic bounded sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy Problem $(T(1), \mu, 0)$ is bounded.*

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