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Qualitative Theory of Dynamical Systems

Dulac Functions of Planar Vector Fields

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Abstract In this work we give sufficient conditions for the existence of a Dulac function for an arbitrary differential system. This Dulac function allows to discard the existence of limit cycles in its domain of definition if this domain is a simply connected region. If the domain of definition is ℓ-multiple connected then the Dulac function can estimate the number of limit cycles inside the domain.

Keywords Analytic differential systems · Dulac functions · Limit cycles · Planar analytic vector fields

AMS Classification Primary 34C05; Secondary 34C23 · 37G15

1 Introduction

A fundamental problem in the qualitative theory of planar differential systems is the determination of the number and distribution of limit cycles. We recall that a limit cycle is a periodic solution which has an annulus-like neighborhood free of other periodic solutions, see [11]. In fact the 16th Hilbert problem (part b) deals with the maximum number and distribution of limit cycles for a polynomial vector field of degree n, see [6]. This is one of the last two open problems in the Hilbert's list of the first international congress of mathematicians celebrate in Paris in 1900. Even now we cannot answer in general whether, given an arbitrary differential system, it has periodic solution or not and specifically limit cycles. Limit cycles have been

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studied extensively by mathematicians and physicists, in particular the nonexistence, existence, uniqueness and some other properties, see for instance [1,9,11].

The classical method for proving the nonexistence of limit cycles in a simplyconnected region is the Bendixson–Dulac method, see for instance [11] where we can also found some variations of it. For the implementation of this method we need to find what is called a Dulac function.In this work we are interested in determine sufficient conditions for the existence of a Dulac function for an arbitrary differential system.

We consider $\mathcal{C}^1$ two dimensional autonomous systems of differential equations

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

defined on an open subset U of $\mathbb{R}^2$ and their corresponding vector fields $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$ on U and the divergence of $\mathcal{X}$ is $\text{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y$.

First we recall the classical theorems about nonexistence of limit cycles.

Theorem 1 (Bendixson) *If the divergence $\partial P/\partial x + \partial Q/\partial y$ of system* (1) *has constant sign in a simply connected region U, and is not identically zero on any subregion of U, then system* (1)*does not possess any limit cycle*(*in fact a closed trajectory*) *which lies entirely in U.*

The proof Theorem 1 is by contradiction assuming the existence of a limit cycle and applying the Green's formula.

Theorem 2 (Bendixon-Dulac) *If there exists a continuously differentiable function $B(x, y)$ in a simply connected region U such that $\partial(BP)/\partial x+\partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion, then system* (1) *does not possess any limit cycle* (*in fact a closed trajectory*) *which lies entirely in U.*

The proof of Theorem 2 follows the proof of Theorem 1, but use BP and BQ to replace P and Q respectively. The function $B(x, y)$ is called *Dulac function*, and the method of proving nonexistence of closed trajectory is called the *method of Dulac functions*. The method of Dulac functions also gives upper bounds for the number of closed trajectories in a multiply connected region, see also [11]. Initially Dulac functions were used by many authors to prove the absence of limit cycles in a simply connected domain and the uniqueness of a limit cycle in a doubly-connected domain, see for instance [2,10,11].

In fact Theorems 1 and 2 can be extended to multiply connected regions, see [11]. The next important result in order to study the nonexistence and existence of limit cycles was obtained in [3] were it was established the following.

Theorem 3 (Cherkas) *Suppose that in a simply connected domain $U \subset \mathbb{R}^2$, there exists a function $\Psi(x, y)$ of class $\mathcal{C}^1$ and a number $k > 0$ such that*

$$k\,\Psi\,\text{div}\mathcal{X} + \mathcal{X}\Psi > 0,$$

then the domain U contains no limit cycles of system (1).

In this paper we are interested in determine sufficient conditions for the existence of a Dulac function for an arbitrary planar differential system or vector field.

2 Definitions and Preliminary Results

First we establish the definition of Dulac function.

Definition 4 Let $\mathcal{X} = (P, Q)$ be a $\mathcal{C}^1$ vector field defined in the open subset U of $\mathbb{R}^2$. A function $B \in \mathcal{C}^1(U)$ is called a Dulac function of $\mathcal{X}$ in U if

$$\operatorname{div}(B\mathcal{X}) = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}, \tag{2}$$

does not change sign in U and vanishes only on a set Σ of Lebesgue measure zero, where no oval (closed curve homeomorphic to a circle) in Σ is a limit cycle.

The existence of a Dulac function can be used in order to estimate the number of limit cycles of system (1) in its domain of definition. The following result is an extension of Theorem 2 to multiply connected regions, see [11].

Theorem 5 *Let $\mathcal{X} = (P, Q)$ be a $\mathcal{C}^1$ vector field defined in U of $\mathbb{R}^2$. Let U be a ℓ-multiple connected region in $\mathbb{R}^2$. If B is a Dulac function defined in U, then system* (1) *has at most $\ell - 1$ limit cycles which lie entirely in U.*

The method of Dulac function was generalized by Cherkas in [3]. The corresponding generalized Dulac functions are called *Dulac-Cherkas functions* (see [4]), and are defined as follows.

Definition 6 Let $\mathcal{X} = (P, Q)$ be a $\mathcal{C}^1$ vector field defined in the open subset U of $\mathbb{R}^2$. A function $\Psi \in \mathcal{C}^1(U)$ is called a Dulac-Cherkas function of $\mathcal{X}$ in U if there exists a real number $k \neq 0$ such that

$$\Phi := k\,\Psi\,\operatorname{div}\mathcal{X} + \mathcal{X}\Psi > 0, \quad (<0) \quad \text{in } U. \tag{3}$$

The method of Dulac-Cherkas functions not only permits to get an upper bound for the number of limit cycles but also provides information about their stability, see [3,4]. Condition (3) can be relaxed by assuming that Φ may vanish in U on a set of measure zero $\Sigma = \{(x, y) \in U : \Phi = 0\}$, and that no oval of this set is a limit cycle of system (1).

In [3] are considered the Liénard system of the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{4}$$

For such systems Cherkas has contructed a Dulac-Cherkas function of the form $\Psi = y^2/2 + G(x) - A$, where A is a constant and $G(x) = \int_0^x g(s)ds$. In this case $\Phi = -(k/2+1)f(x)y^2 - k(G(x) - A)f(x)$. Hence fixing $k = -2$, if there exist A such that $k(G(x) - A)f(x) \geq 0$ then using Theorem 5 we have that system (4) has at most one limit cycle. In the particular case of the van der Pol equation $g(x) = x$ and $f(x) = \mu(x^2-1)$ we get $\Psi = y^2/2+x^2/2-A$, which gives $\Phi = \mu(x^2/2-A)(x^2-1)$. Taking $A = 1/2$ we have $\Phi = \mu(x^2 - 1)^2/2$ and the corresponding system (4) has at most one limit cycle outside the circle $\Psi = 0$ and no limit cycle in the disk $\Psi < 0$.

In [5] the method is extended to differential systems of the form

$$\dot{x} = p_0(x) + p_1(x)y, \quad \dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2.$$

The generalized Liénard systems

$$\dot{x} = y, \quad \dot{y} = \sum_{j=0}^{\ell} h_j(x)y^j$$

with $\ell \geq 1$ and $h_\ell \not\equiv 0$ are studied in [4].

3 Statement of the Main Results

The main result of the present paper is impose conditions to system (1) in order to have a Dulac function of certain form.

Theorem 7 *The vector field $\mathcal{X} = (P, Q)$ defined in an open subset U of $\mathbb{R}^2$ admits a Dulac function of the form $B = B(z)$ where $z = f(x, y)$ if and only if*

$$\frac{\mathrm{div}\mathcal{X}}{(Pf_x + Qf_y)} = \alpha(z), \qquad \frac{C(x,y)}{(Pf_x + Qf_y)} = \beta(z) \tag{5}$$

where $C(x, y)$ is a function that satisfies that the product $B^k C$, with $k \in \mathbb{R}$, does not change sign in U and only vanishes on a set $\mathcal{N}$ of measure zero where no oval in $\mathcal{N}$ is a limit cycle, and $\alpha(z)$ and $\beta(z)$ are functions exclusively of z. Moreover we can estimate the number of limit cycles of the vector field $\mathcal{X}$ analyzing the domain of definition U and B using Theorems 2 and 5.

The following corollary of Theorem 7 impose certain conditions to system (1) in order to exclude existence of limit cycles.

Corollary 8 *Under the assumptions of Theorem 7 the following statements hold for the case $k = 1$:*

- (i) *Taking $z = f(x, y) = x$ if $\beta(z) - \alpha(z)$ is a function only of the variable x (or taking $z = f(x, y) = y$ if $\beta(z) - \alpha(z)$ is a function only of the variable y) then system (1) has not limit cycles in U if U is a simply connected region.*
- (ii) *Taking $z = f(x, y) = xy$ if $\beta(z) - \alpha(z)$ is a function only of xy then system (1) has not limit cycles in U if U is a simply connected region.*

Statement (i) of the previous corollary was already given in [8]. The authors of [8] impose the condition that C do not change sign and vanish only in a measure zero set because for the case $k = 1$ the Dulac function B do not change sign. Theorem 7 is a generalization of the results given in [7,8] with some consequences about the existence of limit cycles for system (1) in the domain of definition of B.

The possible applications of Theorem 7 are based on the fact that if we impose that $\mathrm{div}(B\mathcal{X}) = B^k C$, with $k \in \mathbb{R}$, we obtain the linear partial differential equation

$$P\frac{\partial B}{\partial x} + Q\frac{\partial B}{\partial y} + B\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = B^k(x, y)C(x, y). \tag{6}$$

Using the characteristics method we can try to solve Eq. (6) choosing the function C in order to satisfy the conditions of Theorem 7.

4 Proof of Theorem 7 and Corollary 8

Proof of Theorem 7 We assume that $B = B(z)$ where $z = f(x, y)$, then applying the chain rule, expression (2) is transformed to

$$P\frac{dB}{dz}\frac{\partial f}{\partial x} + Q\frac{dB}{dz}\frac{\partial f}{\partial y} + B\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right). \tag{7}$$

The expression (7) must have constant sign in U. Now we impose that

$$P\frac{dB}{dz}\frac{\partial f}{\partial x} + Q\frac{dB}{dz}\frac{\partial f}{\partial y} + B\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = B^k C, \tag{8}$$

where $B^k C$ must be a function of defined sign. For the case $k \neq 1$ we have

$$\frac{dB}{dz} + B\,\frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{Pf_x + Qf_y} = B^k\,\frac{C}{Pf_x + Qf_y}, \tag{9}$$

where f_x and f_y are the partial derivatives of f respect to x and y. Under the conditions of the theorem the differential Eq. (9) becomes

$$\frac{dB}{dz} + B\,\alpha(z) = B^k\,\beta(z), \tag{10}$$

whose solution is given by

$$B(z) = \left[e^{(k-1)\int_0^z \alpha(s)ds}\left(C_1(1-k)\int_0^z e^{(1-k)\int_0^s \alpha(\tau)d\tau}\beta(s)ds\right)\right]^{\frac{1}{1-k}}, \tag{11}$$

where C_1 is an arbitrary constant. Moreover, for the case $k = 1$, we can isolate dB/dz from Eq. (8) and we have

$$\frac{\frac{dB}{dz}}{B} = \frac{C - \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}{Pf_x + Qf_y}, \tag{12}$$

The left hand-side of Eq. (12) is a function of z, hence the right hand-side must be also a function of z and we obtain

$$\frac{\frac{dB}{dz}}{B} = \frac{C - \mathrm{div}\mathcal{X}}{Pf_x + Qf_y} = \beta(z) - \alpha(z). \tag{13}$$

If equality (13) is satisfied we get that a possible Dulac function of system (1) takes the form

$$B = C_1 e^{\int_0^z (\beta(s) - \alpha(s))ds},$$

where C_1 is an arbitrary constant. Now if the product $B^k C$ does not change sign and only vanishes in a measure zero set for $k \neq 1$, or C does not change sign and only vanishes in a measure zero set for $k = 1$, we can estimate the number of limit cycles of the vector field $\mathcal{X}$ analyzing the domain of definition of U and B using Theorems 2 and 5 and the proof of Theorem 7 follows. □

Proof of Corollary 8
(i) In the case $z = f(x, y) = x$ and $\beta(z) - \alpha(z)$ is a function only of x we have that Eq. (13) takes the form $\frac{d}{dx}(\log B) = (C(x, y) - \mathrm{div}\mathcal{X})/P = h(x)$. Hence $B = e^{\int_0^x h(s)ds}$ and applying Theorem 7 the result follows. The proof is analogous for the case $z = f(x, y) = y$ and when $\beta(z) - \alpha(z)$ is only a function only of y.
(ii) In the case $z = f(x, y) = xy$ and $\beta(z) - \alpha(z)$ is a function only of xy we have that Eq. (12) takes the form $\frac{d}{dz}(\log B) = (C(x, y) - \mathrm{div}\mathcal{X})/(Py + Qx) = \beta(z) - \alpha(z)$. Hence $B = e^{\int_0^z (\beta(s) - \alpha(s))dz}$ and applying Theorem 7 we obtain the result.

□

5 Examples

In this section we give two examples where we apply the results developed in this work.

The Lotka-Volterra differential systems model biological systems in which two species interact, and therefore the systems are defined in the first quadrant for $x, y > 0$. For these systems we have the following result.

Proposition 9 *Consider the Lotka-Volterra differential system*

$$\dot{x} = x\,P(x, y), \qquad \dot{y} = y\,Q(x, y), \quad \textit{for} \quad x, y > 0, \tag{14}$$

where P and Q are $\mathcal{C}^1$ functions. System (14) has not limit cycles in the domain where $xP_x + yQ_y > 0$ or $xP_x + yQ_y < 0$.

Proof Assume that system (14) admits a Dulac function $B \in \mathcal{C}^1(U)$, then this Dulac function must satisfies that expression (2) does not change sign in the first quadrant and vanishes only on a measure zero set. Now we impose that, in fact, $\operatorname{div}(B\mathcal{X}) = BC$ and we obtain the linear partial differential Eq. (6) which in this case is

$$xP\frac{\partial B}{\partial x} + yQ\frac{\partial B}{\partial y} + B(P + Q + xP_x + yQ_y) = BC. \tag{15}$$

We choose $C = xP_x + yQ_y$ and we have

$$xP\frac{\partial B}{\partial x} + yQ\frac{\partial B}{\partial y} = -B(P + Q). \tag{16}$$

A solution of this partial differential equation is $B = c_1 e^{-\log(xy)} = c_1/(xy)$ which has definite sign positive or negative depending of the arbitrary constant c_1. Now in order to apply Theorem 7 we must to impose that C does not change sign. In this case we have $C = xP_x + yQ_y > 0$ or $C = xP_x + yQ_y < 0$. It is clear that the z of Theorem 7 is in this example $z = 1/(xy)$. □

Proposition 10 *Consider the differential system*

$$\dot{x} = f_1(x) + f_2(y), \qquad \dot{y} = xg(y), \tag{17}$$

where f_1, f_2 and g are $\mathcal{C}^1$ functions. If $\frac{df_1}{dx}(x) > 0$ or $\frac{df_1}{dx}(x) < 0$ and the domain of definition of $g(y)$ is simply connected, system (17) has not limit cycles in that domain.

Proof of Corollary 8 In this case we assume that the system has a Dulac function $B(z)$ which is a unique function of $z = y$ and consequently we apply the statement (i) of Corollary 8. Therefore we compute the Dulac function from

$$\frac{d}{dx}(\log B) = \frac{\frac{dB}{dz}}{B} = \frac{(C(x, y) - \operatorname{div}\mathcal{X})}{P} = \alpha(y). \tag{18}$$

For system (17) Eq. (18) takes the form

$$\frac{\frac{dB}{dz}}{B} = \frac{(C(x, y) - \frac{df_1}{dx}(x) - x\frac{dg}{dy}(y))}{xg(y)}. \tag{19}$$

Now we take $C(x, y) = \frac{df_1}{dx}(x)$ and Eq. (20) becomes

$$\frac{\frac{dB}{dz}}{B} = \frac{\frac{dg}{dy}(y)}{g(y)}, \tag{20}$$

which implies that the possible Dulac function $B = c_2 e^{\log(g(y))} = c_2 g(y)$ which has definite sign positive or negative depending of the arbitrary constant c_2. In order to apply Theorem 7 we must to impose that C does not change sign. In this case we have $C = \frac{df_1}{dx}(x) > 0$ or $C = \frac{df_1}{dx}(x) < 0$. □

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