

On the Darboux Integrability of Polynomial Differential Systems

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Abstract This is a survey on recent results on the Darboux integrability of polynomial vector fields in \mathbb{R}^n or \mathbb{C}^n with $n \geq 2$. We also provide an open question and some applications based on the existence of such first integrals.

Keywords Polynomial differential systems · Polynomial vector fields · Darboux integrability

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1 Introduction

In many branches of applied mathematics, physics and, in general, in applied sciences appear nonlinear ordinary differential equations. If a differential equation or vector field defined on a real or complex manifold has a first integral, then its study can be reduced by one dimension. Therefore a natural question is: *Given a vector field on a manifold, how to recognize if this vector field has a first integral defined on an open and dense subset of the manifold?* In general this question has no a good answer up to now.

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In this survey we provide sufficient conditions for the existence of a first integral for polynomial vector fields in \mathbb{R}^n or \mathbb{C}^n with $n \geq 2$. An open question which essentially goes back to Poincaré is presented. Finally some applications of the existence of this kind of first integrals to physical problems, centers, foci, limit cycles and invariant hyperplanes are mentioned.

2 Darboux Theory of Integrability

Since any polynomial differential system in \mathbb{R}^n can be thought as a polynomial differential system inside \mathbb{C}^n we shall work only in \mathbb{C}^n . If our initial differential system is in \mathbb{R}^n , once we get a complex first integral of this system inside \mathbb{C}^n the real and the imaginary parts of it are real first integrals. Moreover if that complex first integral is rational, the same occur for its real and imaginary parts. In short in the rest of the paper we shall work in \mathbb{C}^n .

In this section we study the existence of first integrals for polynomial vector fields in \mathbb{C}^n through the Darboux theory of integrability. The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert's 16th problem [19]. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have (see Darboux [13] and Poincaré [36,37]). This theory shows the fascinating relationships between integrability (a topological phenomenon) and the existence of exact algebraic invariant hypersurfaces formed by solutions for the polynomial vector field. This theory is now known as the *Darboux theory of integrability*, see for more details the Chapter 8 of [16].

2.1 Polynomial Vector Fields in \mathbb{C}^n

As usual $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ denotes the ring of all complex polynomials in the variables x_1, \dots, x_n . We consider the *polynomial vector field* in \mathbb{C}^n

$$\mathcal{X} = \sum_{i=1}^n P_i(x) \frac{\partial}{\partial x_i}, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n,$$

where $P_i = P_i(x) \in \mathbb{C}[x]$ have no common factor for $i = 1, \dots, n$. The integer $d = \max\{\deg P_1, \dots, \deg P_n\}$ is the *degree* of the vector field \mathcal{X} . Usually for simplicity the vector field \mathcal{X} will be denoted by (P_1, \dots, P_n) .

2.2 Invariant Algebraic Hypersurfaces and Darboux Polynomials

Let $f = f(x) \in \mathbb{C}[x]$. We say that $\{f = 0\} \subset \mathbb{C}^n$ is an *invariant algebraic hypersurface* or f is a *Darboux polynomial* of the vector field \mathcal{X} if there exists a polynomial

$K_f \in \mathbb{C}[x]$ such that

$$\mathcal{X}(f) = \sum_{i=1}^n P_i \frac{\partial f}{\partial x_i} = f K_f.$$

The polynomial K_f is called the *cofactor* of f . Note that from this definition the degree of K_f is at most $d-1$, and also that if a solution orbit of the vector field \mathcal{X} has a point on $f=0$, then the whole orbit is contained in $f=0$. This justifies the name of invariant algebraic hypersurface, because it is invariant by the flow of the vector field \mathcal{X} .

Of course if the dimension n is equal to 2, then an invariant algebraic hypersurface is an *invariant algebraic curve*.

If the polynomial f is irreducible in $\mathbb{C}[x]$, then we say that $f=0$ is an *irreducible* invariant algebraic hypersurface, or that f is an *irreducible* Darboux polynomial.

We remark that in the definition of invariant algebraic hypersurface $f=0$ we always allow this curve to be complex; that is $f \in \mathbb{C}[x]$ even in the case of a real polynomial vector fields. As we will see this is due to the fact that sometimes for real polynomial vector fields the existence of a real first integral can be forced by the existence of complex invariant algebraic hypersurfaces, see for more details the chapter 8 of [16]. Of course when we look for a complex invariant algebraic hypersurface of a real polynomial system we are thinking of the real polynomial system as a complex one.

The next result shows that it is sufficient to look for the irreducible invariant algebraic hypersurfaces or irreducible Darboux polynomials.

Proposition 1 *Suppose $f \in \mathbb{C}[x]$ and let $f = f_1^{n_1} \dots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x]$. Then for a polynomial vector field \mathcal{X} , $f=0$ is an invariant algebraic hypersurface with cofactor K_f if and only if $f_i=0$ is an invariant algebraic hypersurface for each $i=1, \dots, r$ with cofactor K_{f_i} . Moreover $K_f = n_1 K_{f_1} + \dots + n_r K_{f_r}$.*

For a proof of Proposition 1 see for instance [22].

2.3 Exponential Factors

If $f, g \in \mathbb{C}[x]$ are coprime, we write $(f, g) = 1$. Suppose that $(f, g) = 1$, we say that $\exp(g/f)$ is an *exponential factor* of the vector field \mathcal{X} if there exists a polynomial $L_e \in \mathbb{C}[x]$ of degree at most $d-1$ such that

$$\mathcal{X}(\exp(g/f)) = \exp(g/f)L_e.$$

The polynomial L_e is called the *cofactor* of the exponential factor. It is easy to prove that if $\exp(g/f)$ is an exponential factor, then $f=0$ is an invariant algebraic hypersurface. For a proof see again see [22].

2.4 Multiplicity of an Invariant Algebraic Hypersurface or of a Darboux Polynomial

Let $\mathbb{C}_m[x]$ be the \mathbb{C} -vector space of polynomials in $\mathbb{C}[x]$ of degree at most m . Then it has dimension $R = \binom{n+m}{n}$. Let v_1, \dots, v_R be a basis of $\mathbb{C}_m[x]$. Denote by M_R the $R \times R$ matrix

$$\begin{pmatrix} v_1 & v_2 & \dots & v_R \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \dots & \mathcal{X}(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{R-1}(v_1) & \mathcal{X}^{R-1}(v_2) & \dots & \mathcal{X}^{R-1}(v_R) \end{pmatrix}, \quad (1)$$

where $\mathcal{X}^{k+1}(v_i) = \mathcal{X}(\mathcal{X}^k(v_i))$. Then $\det M_R$ is called the m -th *extactic polynomial* of \mathcal{X} . From the properties of the determinant we note that the extactic polynomial is independent of the choice of the basis of $\mathbb{C}_m[x]$ up to a nonzero constant factor. Observe that if $f = 0$ is an invariant algebraic hypersurface of degree m of \mathcal{X} , then f divides the polynomial $\det M_R$. This is due to the fact that if f is a member of a basis of $\mathbb{C}_m[x]$, then f divides the whole column in which f is located.

We say that an irreducible invariant algebraic hypersurface $f = 0$ of degree m has *defined algebraic multiplicity* k or simply *algebraic multiplicity* k if $\det M_R \not\equiv 0$ and k is the maximum positive integer such that f^k divides $\det M_R$; and it has *no defined algebraic multiplicity* if $\det M_R \equiv 0$.

We remark that the matrix (1) already appears in the work of Lagutinskii (see also Dobrovol'skii et al. [15]). For a modern definition of the m -th extactic hypersurface and a clear geometric explanation of its meaning, the readers can consult Pereira [34]. Christopher et al. [11] assuming the irreducibility of the invariant algebraic curves used the extactic curve to study the algebraic multiplicity of invariant algebraic curves of planar polynomial vector fields, and prove the equivalence of the algebraic multiplicity with other three ones: the infinitesimal multiplicity, the integrable multiplicity and the geometric multiplicity.

2.5 Multiplicity and Exponential Factors

The next result presents a characterization under suitable assumptions of the algebraic multiplicity of an invariant algebraic hypersurface using the number of exponential factors of \mathcal{X} associated with the invariant algebraic hypersurface. This characterization due to Llibre and Zhang [28] extends the algebraic multiplicity introduced by Christopher, Llibre and Pereira in [11] for invariant algebraic curves of \mathbb{C}^2 to invariant algebraic hypersurfaces of \mathbb{C}^n . This result is a key point in the Darboux theory of integrability in arbitrary dimension.

Theorem 2 *Let \mathcal{X} be a polynomial vector field. For a given irreducible invariant algebraic hypersurface $f = 0$ of \mathcal{X} with f of degree m , assume that \mathcal{X} restricted to $f = 0$ has no rational first integral. Then f has a defined algebraic multiplicity k if*

and only if the vector field \mathcal{X} has $k - 1$ exponential factors $\exp(g_i/f^i)$, where g_i is a polynomial of degree at most $i \cdot m$ and $(g_i, f) = 1$, for $i = 1, \dots, k - 1$.

We remark that if \mathcal{X} is a planar vector field, then clearly Theorem 2 always holds without the assumption on the non-existence of rational first integrals on $f = 0$. For higher dimensional systems the assumption is necessary as the following example shows.

The real polynomial differential system

$$\dot{x} = 1, \quad \dot{y} = y(y - 2z), \quad \dot{z} = -z(y - z),$$

has $z = 0$ as an invariant plane of multiplicity 2 and its restriction to $z = 0$ has the rational first integral $x + 1/y$. But this system has no exponential factor associated with $z = 0$ as it is proved in the appendix.

This example shows that the additional assumption on the non-existence of the rational first integral on the invariant algebraic hypersurface for polynomial vector fields of dimension larger than 2 is necessary.

2.6 Darboux First Integral

Let M be an open and dense subset of \mathbb{C}^n . A non-constant function $H : M \rightarrow \mathbb{C}$ is a *first integral* of the polynomial vector field \mathcal{X} on M if it is constant on all orbits of \mathcal{X} contained in M ; i.e. $H(x(t)) = \text{constant}$ for all values of t for which the solution $x(t)$ is defined and contained in M . Clearly H is a first integral of \mathcal{X} on M if and only if $\mathcal{X}(H) = 0$ on M . Of course a *rational first integral* is a first integral given by a rational function, defined in the open subset of \mathbb{C}^n where its denominator does not vanish. A *Darboux first integral* is a first integral of the form

$$\left(\prod_{i=1}^r f_i^{l_i} \right) \exp(g/h),$$

where f_i, g and h are polynomials, and the l_i 's are complex numbers.

2.7 Classical Darboux Theory of Integrability in \mathbb{C}^n

The classical Darboux theory of integrability in \mathbb{C}^n with $n \geq 2$ is summarized in the next theorem.

Theorem 3 *Assume that the polynomial vector field \mathcal{X} in \mathbb{C}^n of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \dots, p$ such that f_i are pairwise relatively prime. Then the following statements hold.*

- (a) *If $p \geq N + 1$, then the vector field \mathcal{X} has a Darboux first integral, where $N = \binom{n+d-1}{n}$.*
- (b) *If $p \geq N + n$, then the vector field \mathcal{X} has a rational first integral.*

Statement (a) of Theorem 3 is due to Darboux [13, 14], and statement (b) of Theorem 3 was proved by Jouanolou [20]. For a short proof of statement (b) see Christopher and Llibre [8, 9] for $n = 2$ and Llibre and Zhang [30] for $n \geq 2$.

2.8 Darboux Theory of Integrability in \mathbb{C}^n

The following theorem improves the classical Darboux theory of integrability taking into account not only the invariant algebraic hypersurfaces but also their algebraic multiplicities.

Theorem 4 *Assume that the polynomial vector field \mathcal{X} in \mathbb{C}^n of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ such that the f_i are pairwise relatively prime.*

- (i) *If one of these irreducible invariant algebraic hypersurfaces has no defined algebraic multiplicity, then the vector field \mathcal{X} has a rational first integral.*
- (ii) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have defined algebraic multiplicity q_i for $i = 1, \dots, p$. If \mathcal{X} restricted to each hypersurface $f_i = 0$ with multiplicity larger than 1 has no rational first integral, then the following statements hold.*
 - (a) *If $\sum_{i=1}^p q_i \geq N + 1$, then the vector field \mathcal{X} has a Darboux first integral, where N is the number defined in Theorem 3.*
 - (b) *If $\sum_{i=1}^p q_i \geq N + n$, then the vector field \mathcal{X} has a rational first integral.*

Statement (i) follows from the second part of Theorem 3 of Pereira [34] (see also Theorem 5.3 of [11] for dimension 2). Statement (ii) for dimension 2 (i.e. $n = 2$) follows from [11] and for $n > 2$ it is proved in [28].

Under the assumption (b) of Theorem 4 any orbit of the vector field \mathcal{X} is contained in an invariant algebraic hypersurface. We remark that if the vector field \mathcal{X} is 2-dimensional, then the assumption on the non-existence of rational first integral of \mathcal{X} restricted to the invariant algebraic curves is not necessary.

2.9 Darboux Theory of Integrability in \mathbb{R}^n Taking Into Account the Multiplicity of the Hyperplane at Infinity

In this subsection we show that if the hyperplane at infinity has multiplicity greater than 1, then we can go further improving the Darboux theory of integrability taking into account the multiplicity of the hyperplane at infinity.

In order to use the infinity of \mathbb{R}^n as an additional invariant hyperplane for studying the integrability of the vector field \mathcal{X} , we need the Poincaré compactification for the vector field \mathcal{X} , see for more details Cima and Llibre [12]. In the chart U_1 using the change of variables

$$x_1 = \frac{1}{z}, \quad x_2 = \frac{y_2}{z}, \quad \dots, \quad x_n = \frac{y_n}{z}. \quad (2)$$

the vector field \mathcal{X} is transformed to

$$\bar{\mathcal{X}} = -z\bar{P}_1(y)\frac{\partial}{\partial z} + (\bar{P}_2(y) - y_2\bar{P}_1(y))\frac{\partial}{\partial y_2} + \dots + (\bar{P}_n(y) - y_n\bar{P}_1(y))\frac{\partial}{\partial y_n},$$

where $\bar{P}_i = z^d P_i(1/z, y_2/z, \dots, y_n/z)$ for $i = 1, \dots, n$ and $\bar{y} = (z, y_2, \dots, y_n)$. We note that $z = 0$ is an invariant hyperplane of the vector field $\bar{\mathcal{X}}$ and that the infinity of \mathbb{R}^n may be identified with $z = 0$ of the vector field $\bar{\mathcal{X}}$. So we can define the algebraic multiplicity of $z = 0$ for the vector field $\bar{\mathcal{X}}$.

We say that the infinity of \mathcal{X} has *defined algebraic multiplicity* k or simply *algebraic multiplicity* k if $z = 0$ has defined algebraic multiplicity k for the vector field $\bar{\mathcal{X}}$; and that it has *no defined algebraic multiplicity* if $z = 0$ has no defined algebraic multiplicity for $\bar{\mathcal{X}}$.

In [40] Schlomiuk and Vulpe gave a definition for the algebraic multiplicity of the line at infinity for a planar vector field using a limit inside the definition. In fact the two definitions are equivalent. But ours is easier to apply for computing the algebraic multiplicity of the line at infinity for a given planar vector field.

Similar to Theorem 2 we have the following result characterizing the existence of exponential factors associated with the hyperplane at infinity.

Theorem 5 *Assume that $\bar{\mathcal{X}}$ restricted to $z = 0$ has no rational first integral. Then $z = 0$ has algebraic multiplicity k for $\bar{\mathcal{X}}$ if and only if $\bar{\mathcal{X}}$ has $k - 1$ exponential factors $\exp(\bar{g}_i/z^i)$, $i = 1, \dots, k - 1$ with $\bar{g}_i \in \mathbb{C}_i[y]$ having no factor z .*

The following result improves the Darboux theory of integrability in \mathbb{R}^n taking into account the algebraic multiplicity of the hyperplane at infinity, for a proof see [29].

Theorem 6 *Assume that the polynomial vector field \mathcal{X} in \mathbb{R}^n of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \dots, p$ such that the f_i are pairwise relatively prime.*

- (i) *If one of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field \mathcal{X} has a rational first integral.*
- (ii) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have defined algebraic multiplicity q_i for $i = 1, \dots, p$ and that the invariant hyperplane at infinity has defined algebraic multiplicity k . If the vector field restricted to each invariant hypersurface including the hyperplane at infinity having algebraic multiplicity larger than 1 has no rational first integral, then the following hold.*
 - (a) *If $\sum_{i=1}^p q_i + k \geq N + 2$, then the vector field \mathcal{X} has a Darboux first integral, where $N = \binom{n+d-1}{n}$.*
 - (b) *If $\sum_{i=1}^p q_i + k \geq N + n + 1$, then the vector field \mathcal{X} has a rational first integral.*

We note that if the hyperplane at infinity is not taken into account, then Theorem 6 is exactly Theorem 4. Also if the hyperplane at infinity has algebraic multiplicity 1, then it does not contribute to integrability by comparing Theorem 6 with Theorem 4.

We remark that for the moment we do not have an analogous to Theorem 6 for polynomial vector fields in \mathbb{C}^n which takes into account the multiplicity of the infinity. In [29] are shown some difficulties for obtaining the extension of Theorem 6 from polynomial vector fields in \mathbb{R}^n to polynomial vector fields in \mathbb{C}^n .

In the previous subsection we showed by an example that the assumption on the non-existence of rational first integral of \mathcal{X} restricted to an invariant algebraic hypersurface with multiplicity larger than 1 is necessary for the vector field in \mathbb{R}^n with $n > 2$. There are also examples showing that if $n > 2$, the additional assumption is also necessary for the hyperplane at infinity with multiplicity larger than 1, see [29]. If \mathcal{X} is a planar vector field, then this additional assumption about the rational first integral is not necessary, it would imply that such a curve consists of singular points only. We have the following

Corollary 7 *Assume that the polynomial vector field \mathcal{X} in \mathbb{R}^2 of degree $d > 0$ has irreducible invariant algebraic curves $f_i = 0$ with defined algebraic multiplicity q_i for $i = 1, \dots, p$ and that the invariant straight line at infinity has defined algebraic multiplicity k . Then the following hold.*

- (a) *If $\sum_{i=1}^p q_i + k \geq \binom{d+1}{2} + 2$, then the vector field \mathcal{X} has a Darboux first integral.*
- (b) *If $\sum_{i=1}^p q_i + k \geq \binom{d+1}{2} + 3$, then the vector field \mathcal{X} has a rational first integral.*

2.10 Construction of the First Integrals

From the previous main results of this section it follows easily statements (ii) and (iii) of the next theorem. Statements (i) and (iv) without taking into account the cofactors of the exponential factors are essentially due to Darboux [13]. The rest of the statements come from Christopher and Llibre [8,9].

Theorem 8 *Suppose that a polynomial vector field \mathcal{X} of degree d in \mathbb{C}^n admits p irreducible invariant algebraic hypersurfaces $f_i = 0$ such that the f_i are pairwise relatively prime with cofactors K_i for $i = 1, \dots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$.*

- (i) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$, if and only if the (multi-valued) function*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} \left(\exp \left(\frac{g_1}{h_1} \right) \right)^{\mu_1} \dots \left(\exp \left(\frac{g_q}{h_q} \right) \right)^{\mu_q} \tag{3}$$

is a first integral of \mathcal{X} .

- (ii) *If $p+q \geq N+1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$.*
- (iii) *If $p+q \geq N+n$, then \mathcal{X} has a rational first integral.*
In the particular case that $n = 2$ the following statements also hold.

- (iv) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -\operatorname{div}(P, Q)$, if and only if function (3) is an integrating factor of \mathcal{X} .
- (v) If $p+q = N$ then function (3) is a first integral if $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$, or an integrating factor if $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -\operatorname{div}(P, Q)$, under the condition that not all $\lambda_i, \mu_j \in \mathbb{C}$ are zero.

3 First Integrals Obtained by the Darboux Theory of Integrability in Dimension Two

In this section we summarize the effectiveness of the Darboux theory of integrability in dimension two, i.e. which sort of integrals does it capture. For additional details on this section see [16], Prolle and Singer [38], and Singer [41].

The idea of determining what type of functions can arise as the result of evaluating an indefinite integral or solving a differential equation goes back to Liouville. The modern formulation of these ideas is usually done through differential algebra. Some of the advantages over an analytic approach are first that the messy details of branch points etc., are hidden completely, and second the Darboux theory of integrability can be studied using symbolic computation.

3.1 Elementary and Liouvillian Functions

We assume that the set of functions we are interested in forms a field together with a number of *derivations*. We call such an object a *differential field*. The process of adjoining more functions to a given set of functions is described by a tower of such fields:

$$F_0 \subset F_1 \subset \dots \subset F_n.$$

Of course, we must also specify how the derivations of F_0 are extended to derivations on each F_i .

The fields we are interested in arise by adding exponentials, logarithms or the solutions of algebraic equations based on the previous set of functions. That is we take

$$F_i = F_0(\theta_1, \dots, \theta_i),$$

where one of the following holds:

- (i) $\delta\theta_i = \theta_i \delta g$, for some $g \in F_{i-1}$ and for each derivation δ ;
- (ii) $\delta\theta_i = g^{-1} \delta g$, for some $g \in F_{i-1}$ and for each derivation δ ;
- (iii) θ_i is algebraic over F_{i-1} .

If we have such a tower of fields, F_n is called an *elementary* extension of F_0 .

This is essentially what we mean by a function being expressible in closed form. We call the set of all elements of a differential field which are annihilated by all the derivations the *field of constants*. We shall always assume that the field of constants is algebraically closed.

We say that our system has an *elementary* first integral if there is an element u in an elementary extension field of the field of rational functions $\mathbb{C}(x, y)$ with the same field of constants such that $Du = 0$. The derivations on $\mathbb{C}(x, y)$ are of course d/dx and d/dy .

Another class of integrals we are interested in are the *Liouvillian* ones. Here we say that an extension F_n is a *Liouvillian* extension of F_0 if there is a tower of differential fields as above which satisfies conditions (i), (iii) and

$$(ii)' \quad \delta_\alpha \theta_i = h_\alpha \text{ for some elements } h_\alpha \in F_{i-1} \text{ such that } \delta_\alpha h_\beta = \delta_\beta h_\alpha.$$

This last condition, mimics the introduction of line integrals into the class of functions. Clearly (ii) is included in (ii)'.

This class of functions represents those functions which are obtainable “by quadratures”. An element u of a Liouvillian extension field of $\mathbb{C}(x, y)$ with the same field of constants is said to be a *Liouvillian* first integral.

A function of the form $e^{w_0 + \sum c_i \ln(w_i)}$, where c_i are constants and w_i are rational functions is called *Darboux* function.

As we shall see the Darboux theory of integrability finds all Liouvillian first integrals of the planar polynomial vector fields.

3.2 The Relation Between a First Integral and Its Associated Integrating Factor

We consider the following classes of functions, polynomial, rational, Darboux, elementary and Liouvillian. We note that each of these classes of functions is contained in the following one of the previous list. The simplest functions which can be first integrals of polynomial differential systems are functions of one of the mentioned classes. The Darboux theory of integrability allows to compute all the first integrals belonging to one of these classes, see [16, 38, 41]. In the next result and for polynomial differential systems in \mathbb{R}^2 or \mathbb{C}^2 we summarize the explicit relationships between the functions defining the first integrals and their integrating factors.

Theorem 9 *Let \mathcal{X} be a planar polynomial vector field whose components are relatively prime.*

- (a) *If \mathcal{X} has a Liouvillian first integral, then it has a Darboux integrating factor.*
- (b) *If \mathcal{X} has an elementary first integral, then it has an integrating factor of the form a rational function to power $1/n$ for positive integer n .*
- (c) *If \mathcal{X} has a Darboux first integral, then it has a rational integrating factor.*
- (d) *If \mathcal{X} has a rational first integral, then it has a rational integrating factor.*
- (e) *If \mathcal{X} has a polynomial first integral, then it has a polynomial integrating factor.*

Statement (a) is due to Singer [41], see also Christopher [7] and Pereira [35]. Statement (b) was proved by Prelle and Singer [38]. Statement (c) was shown by Chavarriga, Giacomini, Giné and Llibre in [4]. The proof of statement (d) follows easily. Finally the proof of statement (e) follows from [4] and [17], this last paper is due to Ferragut, Llibre and Mahdi.

4 An Open Question for Planar Polynomial Vector Fields

In all this section $n = 2$.

From Jouanolou's result [see Theorem 4 (ii.b)] it follows that for a given planar polynomial differential system of degree d the maximum degree of its irreducible invariant algebraic curves is bounded, since either it has a finite number $p < [d(d + 1)/2] + 2$ of invariant algebraic curves, or all its trajectories are contained in invariant algebraic curves and the system admits a rational first integral. Thus for each polynomial differential system there is a natural number N which bounds the degree of all its *irreducible* invariant algebraic curves. A natural question which goes back to Poincaré [36,37] is: *to give an effective procedure to find N* . Partial answers to this question were given by Cerveau and Lins Neto [3], Carnicer [2], Campillo and Carnicer [1], and Walcher [43]. These results depend on either restricting the nature of the polynomial differential system, or more specifically on the singularities of its invariant algebraic curves.

Of course, given such a bound for N , it is then easy to compute the invariant algebraic curves of the system and also describe its elementary or Liouvillian first integrals (modulo any exponential factors) see for instance Man and Maccallum [31], Christopher [6], and Pearson, Lloyd and Christopher [33].

Unfortunately for the class of polynomial differential systems with fixed degree d , there does not exist a uniform upper bound $N(d)$ for N as shown by the polynomial differential system of degree 1:

$$\dot{x} = rx, \quad \dot{y} = sy,$$

with r and s be positive integers. This system has a rational first integral

$$H = \frac{y^r}{x^s}.$$

and hence invariant algebraic curves $x^s - hy^r = 0$ for all $h \in \mathbb{C} \setminus \{0\}$ and r and s relatively prime.

A conjecture (coming from Poincaré) was that the following question would have a positive answer:

For a given $d \geq 2$ is there a positive integer $M(d)$ such that if a polynomial vector field of degree d has an irreducible invariant algebraic curve of degree $\geq M(d)$, then it has a rational first integral.

See for instance the open question 2 of Christopher and Llibre [9], or the question at the end of the introduction of Lins Neto [21].

The conjecture has a negative answer, two counterexamples appeared at the same time, one due to Moulin Ollagnier [32], and another due to Christopher and Llibre [10]. Later on other counterexamples appeared see for instance Chavarriga and Grau [5]. But all these counterexamples exhibit a Darboux first integral or a Darboux integrating factor. So we conjecture that the following *open question* would have a positive answer (see [22]):

There is some number $D(d)$ for which any polynomial differential system of degree d having some irreducible invariant algebraic curve of degree $\geq D(d)$ has a Darboux first integral or Darboux integrating factor.

5 Some Applications

The Darboux theory of integrability has been successfully applied to the study of some physical models. Thus, for instance, for the classical Bianchi IX system, and for the Einstein–Yang–Mills differential equations, Llibre and Valls in [25] and [26] provided a complete description of its Darboux polynomials, exponential factors, rational first integrals and Darboux first integrals. Similar studies was done by Valls [42] for the Rikitake system, and for the Lorenz system by Llibre and Zhang [27] and Zhang [44].

The proof of the classification of all centers of planar polynomial differential systems of degree 2 can be strongly simplified using the Darboux first integrals, see Schlomiuk [39] and the chapter 8 of Dumortier, Llibre and Artés [16].

Darboux first integrals can be used for obtaining new classes of integrable planar polynomial differential systems having a focus, see Giné and Llibre [18].

Using the Darboux theory of integrability Llibre and Rodríguez in [24] proved that every finite configuration of disjoint simple closed curves of the plane is topologically realizable as the set of limit cycles of a polynomial vector field. Moreover the realization can be made by algebraic limit cycles, and explicit polynomial vector fields exhibiting any given finite configuration of limit cycles are given.

In [23] Llibre and Medrado gave best possible upper bounds of the maximal number of invariant hyperplanes, of the maximal number of parallel invariant hyperplanes, and of the maximal number of invariant hyperplanes that pass through a single point for the polynomial vector fields in \mathbb{C}^n with a given degree.

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6 Appendix

Proposition 10 *The real polynomial differential system*

$$\dot{x} = 1, \quad \dot{y} = y(y - 2z), \quad \dot{z} = -z(y - z), \quad (4)$$

has no exponential factors associated to the invariant plane $z = 0$ of multiplicity 2.

Proof By contrary we assume that system (4) has an exponential factor associated with $z = 0$. Let $E = \exp(g/z^s)$ be the exponential factor with cofactor L , where g is a polynomial of degree at most $s \geq 1$, and L is a polynomial of degree at most 1.

By the definition of exponential factor we get that g should satisfy the equation

$$\frac{\partial g}{\partial x} + y(y - 2z)\frac{\partial g}{\partial y} - z(y - z)\frac{\partial g}{\partial z} = -s(y - z)g + Lz^s. \quad (5)$$

Set $L = L_0 + L_1$, where L_0 is a constant and L_1 is a homogeneous polynomial of degree 1. In what follows we also denote by g_i the homogeneous polynomial of degree i of g .

If $\deg g = s$, equating the homogeneous part of degree $s + 1$ of (5) we get

$$y(y - 2z) \frac{\partial g_s}{\partial y} - z(y - z) \frac{\partial g_s}{\partial z} = -s(y - z)g_s + L_1 z^s.$$

Using the Euler's formula for homogeneous functions this last equation is equivalent to

$$x(y - z) \frac{\partial g_s}{\partial x} + y(2y - 3z) \frac{\partial g_s}{\partial y} = L_1 z^s. \quad (6)$$

We claim that $L_1 \equiv 0$. For proving it, we set $L_1 = ax + by + cz$ and write Eq. (6) in

$$x(y - z) \frac{\partial g_s}{\partial x} = (ax + by + cz)z^s - y(2y - 3z) \frac{\partial g_s}{\partial y}. \quad (7)$$

Clearly we must have $c = 0$. Also we must have $b = 0$, because x divides $by z^s - y(2y - 3z) \frac{\partial g_s}{\partial y}$. Finally we prove $a = 0$. Since x divides $\partial g_s / \partial x$, we set $\partial g_s / \partial x = x h_{s-2}(x, y, z)$ with h_{s-2} a homogeneous polynomial of degree $s - 2$. It follows from (7) that

$$(y - z) \frac{\partial g_s}{\partial x} = az^s - y(2y - 3z)h_{s-2}(x, y, z). \quad (8)$$

If $a \neq 0$, we must have $\frac{\partial g_s}{\partial x} = -az^{s-1}$. But from (8) we are in contradiction. Hence we must have $a = 0$, and consequently $L_1(x, y, z) \equiv 0$. The claim follows.

Now we will prove that $g_s = rz^s$. Set

$$g_s(x, y, z) = zg_{s-1}(x, y, z) + q_s(x, y),$$

with g_{s-1} and q_s homogenous polynomials of degrees $s - 1$ and s , respectively. Substituting the expression of g_s into (6) and set $z = 0$, we get

$$x \frac{\partial q_s}{\partial x} + 2y \frac{\partial q_s}{\partial y} = 0.$$

This last equation can be written in

$$y \frac{\partial q_s(x, y)}{\partial y} + sq_s(x, y) = 0.$$

Its general solution is $q_s(x, y) = c(x)y^{-s}$. So we must have $q_s(x, y) \equiv 0$. Now set

$$g_{s-1}(x, y, z) = zg_{s-2}(x, y, z) + q_{s-1}(x, y),$$

with g_{s-2} and q_{s-1} homogenous polynomials of degrees $s-2$ and $s-1$, respectively. Then similarly we can prove from (6) that $q_{s-1}(x, y) \equiv 0$, and so

$$g_s(x, y, z) = z^2 g_{s-2}(x, y, z).$$

By induction we can prove that $g_s = rz^s$.

The above proof shows that the exponential factor $E = exp(g/z^s)$ (if exists) has a constant cofactor and $g = h(x, y, z) + rz^s$ with h a polynomial of degree at most $s-1$. But in this case the exponential factor E is essentially the same with g of degree less than s .

If $\deg g = s-1$, equating the homogeneous part of degree s of Eq. (5) we get that $L = L_0$ and g_{s-1} satisfies the following equation

$$y(y-2z)\frac{\partial g_{s-1}}{\partial y} - z(y-z)\frac{\partial g_{s-1}}{\partial z} = -s(y-z)g_{s-1} + L_0z^s. \quad (9)$$

If $s=1$, it is easy to show that $g_{s-1} = 0$. So system (4) has no exponential factor. For $s > 1$, set $g_{s-1} = zp_{s-2}(x, y, z) + q_{s-1}(x, y)$ with p_{s-2} and q_{s-1} homogeneous polynomials of degrees $s-2$ and $s-1$, respectively. Then we obtain from (9) with $z=0$ that $y\partial q_{s-1}/\partial y = -sq_{s-1}$. This last equation has only the solution $q_{s-1}(x, y) = 0$. So we have $g_{s-1} = zp_{s-2}(x, y, z)$.

From Eq. (9) we get that

$$y(y-2z)\frac{\partial p_{s-2}}{\partial y} - (y-z)\left(p_{s-2} + z\frac{\partial p_{s-2}}{\partial z}\right) = -s(y-z)p_{s-2} + L_0z^{s-1}. \quad (10)$$

If $s=2$, we get from (10) that $p_{s-2} = 0$, and so system (4) has no exponential factor. For $s > 2$, set $p_{s-2} = zp_{s-3}(x, y, z) + q_{s-2}(x, y)$ with p_{s-3} and q_{s-2} homogeneous polynomials of degrees $s-3$ and $s-2$, respectively. Then we obtain from (10) with $z=0$ that $y\partial q_{s-2}/\partial y = -(s-1)q_{s-2}$. This equation has only the solution $q_{s-2}(x, y) = 0$. So we have $p_{s-2} = zp_{s-3}(x, y, z)$. Moreover p_{s-3} satisfies the following equation

$$y(y-2z)\frac{\partial p_{s-3}}{\partial y} - (y-z)\left(2p_{s-3} + z\frac{\partial p_{s-3}}{\partial z}\right) = -s(y-z)p_{s-3} + L_0z^{s-2}.$$

This equation has a similar form than (10).

By induction we can prove that $g_{s-1} = bz^{s-1}$ with b a constant. Substituting g_{s-1} into Eq. (9) yields $b=0$. So g has degree at most $s-2$.

If $\deg g = k \leq s-2$, then Eq. (5) implies that $L=0$. Moreover, we have

$$y(y-2z)\frac{\partial g_k}{\partial y} - z(y-z)\frac{\partial g_k}{\partial z} = -s(y-z)g_k. \quad (11)$$

Working in a similar way to the proof of (9) we can prove that (11) has only the solution $g_k = 0$.

The above proof shows that system (4) has no exponential factor associated with the invariant plane $z = 0$. This completes the proof of the proposition. \square

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