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Qualitative Theory of Dynamical Systems

Abelian Integrals and Limit Cycles

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Abstract This survey paper is devoted to introducing some basic concepts and methods about the application of Abelian integral to study the number of limit cycles, especially to the weak Hilbert's 16th problem. We will introduce some recent results in this field.

Keywords Abelian integral · Limit cycle · Weak Hilbert's 16th problem

Mathematics Subject Classification (2000) 34C07 · 34C08 · 37G15

Consider the planar differential systems

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1}$$

where P_n and Q_n are real polynomials of degree at most n. The second half of the famous Hilbert's 16th problem is asking for the maximum number of limit cycles of system (1), denoted by $H(n)$, for all P_n and Q_n, and asking for possible relative positions of the limit cycles.

A *limit cycle* of system (1) is an isolated closed orbit. In many applications the number and positions of limit cycles are important to understand the dynamical behavior of the system. Note that a linear system may have periodic orbits but have no limit cycles, so we assume $n \geq 2$. For *a given system* (1) the number of limit cycles is finite [14,21,41], but the original Hilbert's 16th problem is still open even for the case

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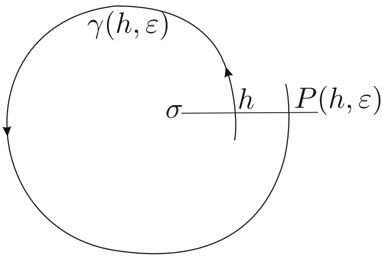

Fig. 1 Construction of displacement function

$n = 2$, and there is no answer if $H(2)$ is finite or not. About the relative positions of limit cycles, Llibre and Rodríguez proved in [61] that any configuration of limit cycles is realizable by a polynomial system of certain degree.

Now we consider a Hamiltonian system X_H:

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x}, \tag{2}$$

where H is a polynomial of degree m, and also consider a related perturbed system

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y} + \varepsilon f(x, y), \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x} + \varepsilon g(x, y), \tag{3}$$

where f and g are polynomials of degrees at most n, and ε is a small parameter.

Suppose that there is a family of ovals, $\gamma_h \subset H^{-1}(h)$, continuously depending on a parameter $h \in (a, b)$. Then we may define the Abelian integral

$$I(h) = \oint_{\gamma_h} f(x, y)dy - g(x, y)dx. \tag{4}$$

It is clear that all γ_h, filling up an annulus for $h \in (a, b)$, are periodic orbits of the Hamiltonian system (2).

A natural question is: How many periodic orbits of X_H keep being unbroken and become the periodic orbits of the perturbed system (3) for small ε? Note that if the number of such orbits is finite, then the orbits are limit cycles of (3).

This question can be proposed in the converse way: Is it possible to find a value $h \in (a, b)$, and some periodic orbits Γ_ε of the perturbed systems (3), such that Γ_ε tends to γ_h (in the sense of Hausdorff distance) as $\varepsilon \to 0$? And how many such Γ_ε for a same h?

To answer this question, we take a segment σ, transversal to each oval γ_h. We choose the values of the function H itself to parameterize σ, and denote by $\gamma(h, \varepsilon)$ a piece of the orbit of the perturbed system (3) between the starting point h on σ and the next intersection point $P(h, \varepsilon)$ with σ, see Fig. 1.

The "next intersection" is possible for sufficiently small ε, since $\gamma(h, \varepsilon)$ is close to γ_h. As usual, the difference $d(h, \varepsilon) = P(h, \varepsilon) - h$ is called the *displacement function.*

Theorem 1 (Poincaré–Pontryagin [72])

$$d(h, \varepsilon) = \varepsilon\,(I(h) + \varepsilon\phi(h, \varepsilon)), \quad \textit{as } \varepsilon \to 0,$$

where $I(h)$ is given in (4), and $\phi(h, \varepsilon)$ is analytic and uniformly bounded for (h, ε) in a compact region near $(h, 0)$, $h \in (a, b)$.

Note that the number of zeros of the displacement function is independent of the choice of the transversal segment σ.

From Theorem 1 by using Implicit Function Theorem and Rolle Theorem it is not hard to prove the following result giving an answer to the above question. We use X_H and $X_{H,\varepsilon}$ to denote the Hamiltonian system (2) and its perturbation (3) respectively, and first give a definition for convenience.

If there exist an $h^* \in (a, b)$ and an $\varepsilon^* > 0$ such that $X_{H,\varepsilon}$ has a limit cycle Γ_ε for $0 < |\varepsilon| < \varepsilon^*$, and Γ_ε tends to γ_{h^*} as $\varepsilon \to 0$, then we will say that Γ_ε *bifurcates from* γ_{h^*}. We say that a limit cycle Γ of $X_{H,\varepsilon}$ *bifurcates from the annulus* $\cup_{h\in(a,b)}\gamma_h$ of X_H, if there is a $h \in (a, b)$ such that Γ bifurcates from γ_h.

Theorem 2 (see, for example, Theorem 2.4 of part II in [12]) *We suppose that $I(h)$ is not identically zero for $h \in (a, b)$, then the following statements hold.*

(i) *If $X_{H,\varepsilon}$ has a limit cycle bifurcating from γ_{h^*}, then $I(h^*) = 0$.*
(ii) *If there exists an $h^* \in (a, b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$, then $X_{H,\varepsilon}$ has a unique limit cycle bifurcating from γ_{h^*}, moreover, this limit cycle is hyperbolic.*
(iii) *If there exists an $h^* \in (a, b)$ such that $I(h^*) = I'(h^*) = \cdots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$, then $X_{H,\varepsilon}$ has at most k limit cycles bifurcating from the same γ_{h^*}, taking into account the multiplicities of the limit cycles.*
(iv) *The total number (counting the multiplicities) of the limit cycles of $X_{H,\varepsilon}$, bifurcating from the annulus $\cup_{h\in(a,b)}\gamma_h$ of X_H, is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of the Abelian integral $I(h)$ for $h \in (a, b)$.*

From this theorem we see clearly that the number of limit cycles for a perturbation of a Hamiltonian system is closely related to the number of isolated zeros of the corresponding Abelian integral.

We remark here that since the zeros of $I(h)$ also depend on the parameters, appearing in perturbations, they may tend to the endpoints of (a, b), corresponding to critical values of H. At these special values the Implicit Function Theorem can not, in general, be applied to the displacement function, so it is difficulty to give a uniform estimate of the number of zeros for $h \in [a, b]$. It is well know that if one of the endpoints, say a, corresponds to a non-degenerate center of X_H, then $I(h)$ can be extended to the value a analytically (see Theorem 3.9 in Chapter 4 of [56], or Lemma 20 of [4]), and nontrivial $I(h)$ has at most finite number of zeros near a uniformly with respect to parameters, hence the statement (iv) of Theorem 2 can be extended to $[a, b)$. On the other hand, if an endpoint, say b, corresponds to a polycycle (homoclinic or heteroclinic orbit) of X_H, the conclusions are the following. Statement (iv) can be extended to $[a, b]$ if b corresponds to a homoclinic loop, see Roussarie [73] and Mardesic [62]; and in general it surely can not be extended to $[a, b]$ if b corresponds to a heteroclinic loop, as it has been shown by a counter-example with two-saddles loop in [5,20]. At last, if the annulus tends to infinity (globally or partially), then we could make conclusion about the number of limit cycles only in a compact region of the annulus, closer to the boundary if ε is smaller. On the other hand, the system $X_{H,\varepsilon}$ might have limit cycles escaping to infinity as $\varepsilon \to 0$, see the example given by Iliev in Remark 1 of [38].

In general, Arnold [1,2] repeatedly proposed the following problem:

For fixed integers m and n find the maximum $Z(m,n)$ of the numbers of isolated zeros of the Abelian integrals (4).

If we take $m = n + 1$, then system (3) is a special form of system (1), close to the Hamiltonian system (2). In this sense the above problem usually is called the *weak* (*or tangential, infinitesimal*) Hilbert's 16th problem, and the number $\tilde{Z}(n) = Z(n+1,n)$ can be chosen as a lower bound of the Hilbert number $H(n)$.

Recall that an *Abelian integral* is the integral of a rational 1-form along an algebraic oval. If the unperturbed system is integrable but non-Hamiltonian, one has to use a integrating factor, say $\mu(x,y) = 1/R(x,y)$, and the perturbed system can be written in the form

$$\dot{x} = -\frac{\partial F(x,y)}{\partial y} R(x,y) + \varepsilon f(x,y), \quad \dot{y} = \frac{\partial F(x,y)}{\partial x} R(x,y) + \varepsilon g(x,y), \tag{5}$$

and associated to it we define the (generalized, or pseudo) Abelian integral

$$I(h) = \oint_{\gamma_h} \frac{f(x,y)dy - g(x,y)dx}{R(x,y)}, \tag{6}$$

where $\{\gamma_h\}$ are the family of ovals contained in the level curves $\{F(x,y) = h\}$. By the same mechanisms the integral $I(h)$ gives the first approximation of the displacement function. Since $R(x,y)$ and/or $F(x,y)$, in general, are not polynomials, the study of the number of zeros of (6) is more difficult than the study for (4), and most traditional methods fail for this generalized form.

As we discussed above that the Abelian integral $I(h)$, shown in (4), gives the first order approximation of the displacement function of the perturbed system $X_{H,\varepsilon}$, hence, if $I(h)$ is not identically zero then the number of isolated zeros of $I(h)$ gives an upper bound of the number of limit cycles of $X_{H,\varepsilon}$. However, if $I(h) \equiv 0$ for $h \in (a,b)$, then it is natural to express the displacement function in the form

$$d(h,\varepsilon) = \varepsilon I_1(h) + \varepsilon^2 I_2(h) + \cdots + \varepsilon^j I_j(h) + O(\varepsilon^{j+1}), \tag{7}$$

where $I_1(h) \equiv I(h)$, ε small. The question is that if $I_1(h) \equiv 0$, then how to compute the second order approximation $I_2(h)$ and so on ?

The following algorithm to compute $I_{k+1}(h)$, if $I_j(h) \equiv 0$ for $j = 1, 2, \ldots k$, was given by Françoise in [22], see also [81].

Denote $dH = H_x dx + H_y dy$, $\omega = f dy - g dx$, where H, f and g are polynomials in x and y, $\deg(H) = n + 1$, $\max(\deg(f), \deg(g)) = n$. Then Eqs. (2) and (3) can be written as the Pfaffian forms $dH = 0$ and $dH - \varepsilon\omega = 0$ respectively. As before, we use γ_h to denote the family of ovals contained in the level curves $H^{-1}(h)$, σ a segment transversal to γ_h and parameterized by H, and $\gamma(h,\varepsilon)$ a piece of the orbit of $dH - \varepsilon\omega = 0$ between the starting point h on σ and the next intersection point $P(h,\varepsilon)$ with σ. By using these notations Theorem 1 (Poincaré–Pontryagin) can be shown shortly as follows.

The integration of $dH - \varepsilon\omega = 0$ over $\gamma(h,\varepsilon)$ gives

$$d(h,\varepsilon) = \int\limits_{\gamma(h,\varepsilon)} dH = \varepsilon \int\limits_{\gamma(h,\varepsilon)} \omega = \varepsilon \int\limits_{\gamma_h} \omega + O(\varepsilon^2).$$

Following [22], we say that the polynomial H satisfies the condition $(*)$ if and only if for all polynomial 1-forms ω:

$$\int_{\gamma_h} \omega = 0 \;\Leftrightarrow \text{there are polynomials } g \text{ and } R \text{ such that } \omega = g dH + dR. \qquad (*)$$

Theorem 3 [22] *Assume that H satisfies the condition* $(*)$ *and* $I_j(h) \equiv 0$ *in (7) for* $j = 1, 2, \ldots, k$. *Then there are* $g_1, \ldots, g_k$; $R_1, \ldots, R_k$ *such that*

$$\omega = g_1\, dH + dR_1,\;\; g_1\omega = g_2\, dH + dR_2, \ldots, g_{k-1}\omega = g_k\, dH + dR_k$$

and

$$I_{k+1}(h) = \int\limits_{\gamma_h} g_k\, \omega.$$

Note that Gavrilov [26] shows that for "generic" polynomial Hamiltonian H, the condition $(*)$ holds. Some further discussions concerning Theorem 3 can be found in [30].

Varchenko [78] and Khovanskii [43] proved in 1984 that for given m and n the number $Z(m,n)$ is uniformly bounded, i.e. $Z(m,n) < \infty$. This result certainly is important. However, it is a purely existential statement, giving no information on the number $Z(m,n)$. We will introduce a recent result in [3] which gives an explicit bound to $Z(m,n)$. There are many works dealing with restricted versions of the problem (restriction on H or on the class of f and g). We list some of them below.

- It is natural to think about a possibility to find $\tilde{Z}(n) = Z(n+1,n)$ exactly for smaller n, and this succeed by several authors only for $n = 2$ (1993 to 2002). For generic cases $\tilde{Z}(2) = 2$, see Gavrilov [27], Horozov and Iliev [33], Li and Zhang [86] or Markov [63] and Li and Zhang [52]. A unified proof appears in [6]. For degenerate cases $I(h)$ has at most one zero, but this gives no information about the cyclicity of the period annulus, higher order approximations must be considered. Iliev in [37] gives formulas (called second- or third-order Melnikov function) to determine the cyclicity for all degenerate cases. The cyclicity of the period annulus (or annuli) is 3 for the Hamiltonian triangle case [35], and is 2 for all other seven cases (see [11,28,34,88,90], and a unified proof in [48]).
- For elliptic Hamiltonian $H = y^2 + P_k(x)$, where P_k is a polynomial of degree k, there is a series works to study the number of zeros of Abelian integrals for

different classes of perturbations, see for example [71] and [15–18,38,44,65,79]. More citations can be found in the second part of [12].
- Takens [77] and Arnold [1] proposed the $1:q$ resonance problem. Except for the case $q = 4$, the codimension two cases have been completed solved, and codimension ≥ 3 cases are partially solved. The study is related to Abelian integrals. A systematic introduction about this problem can be found in [10].
- The study of quadratic perturbations of quadratic integrable and non-Hamiltonian systems was done for some special classes, we will introduce it for reversible case in more details later on.
- Many authors studied this problem for certain H under perturbations f and g, belonging to some function classes. Here H, f and g are not necessarily polynomials.

There are several methods to study the number of zeros of Abelian integrals: the method based on Picard–Fuchs equation and related Riccati equation; the method based on the Argument Principle (see [71]), the averaging method (see [60]), the method by using Chebyshev property (see [24,31]), and the method based on complexification of the Abelian integrals (see [41]).

For more details about the results and methods listed above, see the second part of [12,46], sections 6–8 of [53,82], and references therein.

In the second part of this survey paper we briefly introduce some recent results about (or using) Abelian integrals.

1. In 2010 G. Binyamini, D. Novikov and S. Yakovenko obtain an uniform upper bound for $\tilde{Z}(n) = Z(n+1, n)$.

   Theorem 4 [3]

$$\tilde{Z}(n) \leq 2^{2^{Poly(n)}},$$

   where $Poly(n) = O(n^p)$ stands for an explicit polynomially growing term with the exponent p not exceeding 61.

   This is the first explicit uniform bound for the number of isolated zeros of Abelian integrals $I(h)$ in (4) for $m = n+1$. To prove this result, the authors of [3] make complexification of the Abelian integrals and reduce the weak Hilbert 16th problem to a question about zeros of solutions to an integrable Pfaffian system subject to a condition on its monodromy. They use the fact that Abelian integrals of a given degree are horizontal sections of a regular flat meromorphic connection defined over $\mathbb{Q}$ (the Gauss–Manin connection) with a quasiunipotent monodromy group.
   Based on above result, the authors also made the following conjecture in [3]:

$$Z(m,n) \leq 2^{2^{\mathrm{Poly}(n)}} + O(m), \quad \text{as } n, m \to +\infty.$$

2. By improving the method and results of [13] and Propositions 8.5–8.6 and Theorem 8.7 of [53], recently Han and Li give a lower bound for $\tilde{Z}(n) = Z(n+1, n)$, hence give lower bound for the Hilbert number $H(n)$.

Theorem 5 [32] *For any integer $k \geq 1$, there exists a constant B_k satisfying*

$$\lim_{k\to\infty} \frac{B_k}{\ln(k+1)} = \frac{1}{2\ln 2},$$

such that for $n = 2^i(k+1) - 1, i \geq 1$

$$\tilde{Z}(n) \geq \frac{1}{2\ln 2}(n+1)^2 \ln(n+1) - B_k(n+1)^2 + 3n + \frac{4}{3}.$$

Moreover, one can take $B_1 = \frac{5}{6}$, $B_2 = \frac{19}{108} + \frac{\ln 3}{2\ln 2}$.

Theorem 6 [32]

(i) $\tilde{Z}(n) \geq n^2$, *for $n \geq 23$.*
(ii) *For $k \geq 1$, $\tilde{Z}(2k+1) \geq (2k+1)^2$ for $k \neq 4$ and $\tilde{Z}(9) \geq 80$.*

Theorem 7 [32] *For any sufficiently small $\varepsilon > 0$ there exists a positive number n^*, depending on ε, such that*

$$\tilde{Z}(n) > \left(\frac{1}{2\ln 2} - \varepsilon\right)(n+2)^2 \ln(n+2), \ \textit{for}\ n > n^*$$

Hence,

$$\lim_{n\to\infty} \inf \frac{\tilde{Z}(n)}{(n+2)^2 \ln(n+2)} \geq \frac{1}{2\ln 2}.$$

That is to say, $\tilde{Z}(n)$ grows at least as rapidly as $\frac{1}{2\ln 2}(n+2)^2 \ln(n+2)$.

3. In 2011 Grau et al. [31] gave a Chebyshev criterion for Abelian integrals, as a generalization of [51] from two-dimension to higher dimension. The advantage of this method is that to study the number of zeros of an Abelian integral one only needs to make some purely algebraic computations, unlike the usual way to make complicated differential and integral computations. But this method can be used, up to now, only for restricted forms of the first integrals, like $H(x, y) = \Phi(x) + \Psi(y)$ or $H(x, y) = A(x) + B(x)y^a$, where H is an analytic function in some open subset of the plane that has a local minimum at the origin. Then there exists a punctured neighborhood of the origin foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$. If the Abelian integral (4) or (6) can be expressed as

$$I(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_{n-1} I_{n-1}(h),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are arbitrary constants, then the number of zeros of $I(h)$ is related to the Chebyshev property of the functions $I_0, I_1, \ldots, I_{n-1}$.
Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval L of R.

(a) $\{f_0, f_1, \ldots, f_{n-1}\}$ is a *Chebyshev system (in short, T-system)* on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on L.

(b) An ordered set of n functions $(f_0, f_1, \ldots, f_{n-1})$ is a *complete Chebyshev system (in short, CT-system)* on L if $\{f_0, f_1, \ldots, f_{k-1}\}$ is a T-system for all $k = 1, 2, \ldots, n$.

(c) An ordered set of n functions $(f_0, f_1, \ldots, f_{n-1})$ is an *extended complete system (in short, ECT-system)* on L if, for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on L counted with multiplicities.

We assume that Φ and Ψ are analytic and $x\Phi'(x) > 0$ for any $x \in (x_\ell, x_r) \setminus \{0\}$ and $y\Psi'(y) > 0$ for any $y \in (y_\ell, y_r) \setminus \{0\}$. Then Φ and Ψ must have even multiplicity at 0. Thus, there exist two analytic involutions σ_1 and σ_2 such that

$$\Phi(x) = \Phi(\sigma_1(x)) \quad \text{for all } x \in (x_\ell, x_r)$$

and

$$\Psi(y) = \Psi(\sigma_2(y)) \quad \text{for all } y \in (y_\ell, y_r).$$

Note that $\sigma_i(0) = 0$. For a given function κ, we define its balance with respect to σ as

$$\mathscr{B}_\sigma(\kappa)(x) = \kappa(x) - \kappa(\sigma(x)).$$

Theorem 8 [31] *Consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x)g(y)dx, \quad i = 0, 1, \ldots, n-1,$$

where $\{f_i\}$ are analytic in $x \in (x_\ell, x_r)$ and g is analytic in $y \in (y_\ell, y_r)$, for each $h \in (0, h_0)$, γ_h is the oval surrounding the origin inside the level curve $\{\Phi(x) + \Psi(y) = h\}$. Let σ_1 and σ_2 be the involutions associated to Φ and Ψ, respectively. Setting $g_0 = g$, define $g_{i+1} = \frac{g_i'}{\Psi'}$. Then $(I_0, I_1, \ldots, I_{n-1})$ is an ECT-system on $(0, h_0)$ if the following hypotheses are satisfied:

(a) $\left(\mathscr{B}_{\sigma_1}\left(\frac{f_0}{\Phi'}\right), \mathscr{B}_{\sigma_1}\left(\frac{f_1}{\Phi'}\right), \ldots, \mathscr{B}_{\sigma_1}\left(\frac{f_{n-1}}{\Phi'}\right)\right)$ *is a CT-system on $(0, x_r)$, and*

(b) $(\mathscr{B}_{\sigma_2}(g_0), \ldots, \mathscr{B}_{\sigma_2}(g_{n-1}))$ *is a CT-system on $(0, y_r)$ and $\mathscr{B}_{\sigma_2}(g_0)(y) = o(y^{2m(n-2)})$.*

If the Hamiltonian function H and the function $g(y)$ have the following forms

$$H(x, y) = A(x) + B(x)y^{2m}, \quad g(y) = y^{2s-1},$$

where $s \in \mathbb{N}$, H has a local minimum at the origin by assumption, $B(0) > 0$, and A has a local minimum at $x = 0$. Thus, as before, there exists an involution σ satisfying $A(x) = A(\sigma(x))$ for all $x \in (x_\ell, x_r)$.

Theorem 9 [31] *Consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x)y^{2s-1}dx, \quad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$*,* γ_h *is the oval surrounding the origin inside the level curve* $\{A(x) + B(x)y^{2m} = h\}$*. Let* σ *be the involution associated to* A *and define*

$$\ell_i = \mathscr{B}_\sigma\left(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\right).$$

Then $(I_0, I_1, \ldots, I_{n-1})$ *is an ECT-system on* $(0, h_0)$ *if* $s > m(n-2)$ *and* $(\ell_0, \ell_1, \ldots, \ell_{n-1})$ *is a CT-system on* $(0, x_r)$*.*

In application problems, if the condition $s > m(n-2)$ is not satisfied, then the following Lemma is useful.

Lemma ([31]) *Let* γ_h *be an oval inside the level curve* $\{A(x) + B(x)y^2 = h\}$ *and we consider a function* F *such that* F/A' *is analytic at* $x = 0$*. Then, for any* $k \in \mathbb{N}$*,*

$$\int_{\gamma_h} F(x)y^{k-2}dx = \int_{\gamma_h} G(x)y^k dx,$$

where $G(x) = \frac{2}{k}\left(\frac{BF}{A'}\right)'(x) - \left(\frac{B'F}{A'}\right)(x)$*.*

By using Theorem 9 the authors of [31] also gave simpler proofs for some know results, including the results in [66, 88], as well as some new results about cyclicity problem of quadratic integrable systems, which we will introduce later on. Besides, by using Theorem 8 [79] studied perturbations of a class of hyperelliptic Hamiltonian systems with one nilpotent saddle.
Recently Mañosas and Villadelprat [64] generalized the result of Theorem 9 to the case that for any nontrivial linear combination of $n-1$ Abelian integrals to have at most $n-1+k$ zeros counted with multiplicities, i.e. $(I_0, I_1, \ldots, I_{n-1})$ is a *Chebyshev system with accuracy* k.

Theorem 10 [64] *Consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x)y^{2s-1}dx, \quad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$*,* γ_h *is the oval inside the level curve* $\{A(x)+B(x)y^{2m} = h\}$*. Let* σ *be the involution associated to* A *and define*

$$\ell_i = \mathscr{B}_\sigma\left(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\right).$$

If the following conditions are verified (where $W[\ell_0, \cdots, \ell_i]$ *stands for the Wronskian determinant of the functions* $\ell_0, \ldots, \ell_i$*):*

(a) $W[\ell_0, \ldots, \ell_i]$ *is non-vanishing on* $(0, x_r)$ *for* $i = 0, 1, \ldots, n-2$*,*
(b) $W[\ell_0, \ldots, \ell_{n-1}]$ *has* k *zeros on* $(0, x_r)$ *counted with multiplicities, and*
(c) $s > m(n+k-2)$*, then any nontrivial linear combination of* $I_0, I_1, \ldots, I_{n-1}$ *has at most* $n-1+k$ *zeros on* $(0, h_0)$ *counted with multiplicities.*

As an application of Theorem 10, the authors of [64] gave a simple proof about the number of zeros of Abelian integral corresponding to one period annulus in [19].

4. In a recent paper Gasull et al. [23] use Chebyshev property to study some perturbed Abel equations. Consider the family of analytic functions

$$I_{k,\alpha}(y) := \int_a^b \frac{g^k(t)}{(1-yg(t))^\alpha}dt, \quad k = 0, 1, \ldots, n,$$

where $\alpha, a, b \in \mathbb{R}$ and $g(t)$ is a continuous non identically vanishing function on $[a, b]$. $I_{k,\alpha}(y)$ is defined on the open interval J given by the connected component of the set $\{y \in \mathbb{R} : 1 - yg(t) > 0 \text{ for all } t \in [a, b]\}$ which contains the origin.

Theorem 11 [23] *For any* $n \in \mathbb{N}$ *and any* $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$*, the ordered set of functions* $(I_{0,\alpha}, I_{1,\alpha}, \ldots, I_{n,\alpha})$ *is an ECT-system on* J*. When* $\alpha \in \mathbb{Z}^-$ *it is an ECT-system on* J *if and only if* $n \le -\alpha$*. In particular, the case where the set of functions is an ECT-system, any non-trivial function of the form*

$$\Phi_\alpha(y) := \sum_{k=0}^{n} a_k I_{k,\alpha}(y),$$

with $a_k \in \mathbb{R}$*, has at most* n *zeros in* J *counting multiplicities.*

As an application of this result it is possible to determine upper bounds for the number of isolated 2π-periodic solutions which appear when one performs a first order analysis in ε of generalized Abel equations

$$\frac{dx}{dt} = \frac{\cos(t)}{q-1}\, x^q + \varepsilon P_n(\cos(t), \sin(t))\, x^p, \tag{AE}$$

where $q, p \in \mathbb{N}\backslash\{0, 1\}$, $q \neq p$, and P_n being a polynomial of degree n. If $x = \varphi(t, \rho, \varepsilon)$ is the solution of equation (AE) starting at $x = \rho$, then:

$$\varphi(2\pi, \rho, \varepsilon) = \rho + \varepsilon\rho^p \Phi_\alpha(\rho^{q-1}) + O(\varepsilon^2),$$

where Φ_α is the function introduced in Theorem 11 for $g(t) = \sin(t)$, $\alpha = (p-q)/(q-1)$ and suitable real constants $a_0, a_1, \ldots, a_n$. It is well-known that simple zeros in $(-1, 1)\backslash\{0\}$ of $\Phi_\alpha(\rho^{q-1})$, give rise to initial conditions for isolated 2π-periodic solutions which tend to these zeros as ε goes to 0. We call these 2π-periodic solutions, *periodic solutions obtained by a first order analysis*. Thus, from Theorem 11 we have:

Theorem 12 [23] *The maximum number of 2π-periodic solutions of the generalized Abel equation (AE), obtained by a first order analysis, is n when q is even and $2n$ when q is odd. Moreover in both cases these upper bounds are sharp.*

This Theorem improves several known results, without heavy computations. This method can be used for a wide class of functions $g(t)$.

5. By studying the corresponding Abelian integrals, Li et al. [47] constructed a cubic system with at least 13 limit cycles in 2009. The system has form (3), where H, f and g are given by

$$H(x, y) = \int_0^x x(x+1)(x-\lambda)dx + \frac{y^4}{4} - \frac{k^2y^2}{2} = F(x) + (\frac{y^2-k^2}{2})^2 - \frac{k^4}{4},$$

$$f(x, y) = 0, \quad g(x, y) = y(\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2),$$

where $\lambda, k, \alpha_1, \alpha_2, \alpha_3$ and α_4 are real numbers, and

$$F(x) = \frac{x^4}{4} + \frac{1-\lambda}{3}x^3 - \frac{\lambda}{2}x^2.$$

The phase portrait of the unperturbed system X_H is shown in Fig. 2.
Using the result in IV of [15] by a suitable perturbation, related to $\alpha_1, \alpha_2, \alpha_3$ and some value of $\lambda \in (0, 1)$, the system (3), i.e. $X_{H,\varepsilon}$ for small ε, has at least 5 limit cycles in the half plane $y > 0$, 1 of them bifurcated from the region $\{\gamma_2\}$ and 4 from the region $\{\gamma_3\}$ in Fig. 2. It is proved in [47] that by the perturbation, related to a suitable α_4 and big $k > 0$, one more limit cycle appears from the region $\{\gamma_3\}$. By symmetry, system $X_{H,\varepsilon}$ has at least 12 limit cycles, with one more big limit cycle, for big $k > 0$ appearing from the region $\{\gamma_7\}$. So the total number of limit cycles is at least 13.

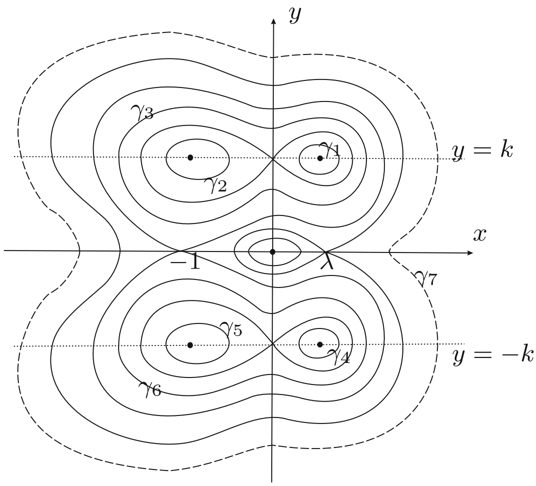

Fig. 2 The phase portrait of the unperturbed system X_H.

We remark that another example of cubic system with at least 13 limit cycles was constructed by Li and Liu [54]. The system has at least 12 limit cycles, surrounding two foci respectively in (6,6)-distribution, appearing by the Hopf bifurcation of order 6 and by symmetry. Moreover, the system has at least one more big limit cycle surrounding the 12 limit cycles. The (6,6)-distribution of limit cycles for cubic system, constructed by Hopf bifurcation of order 6, was found earlier by Yu and Han in [84].

6. The cyclicity problem of quadratic reversible systems under quadratic perturbations is a interesting and difficult problem. It is well known, see [74] or [91] for example, that any quadratic reversible system with a center at the origin can be written in the form

$$\dot{x} = -y + \alpha x^2 + \beta y^2, \quad \dot{y} = x(1 + \gamma y). \tag{8}$$

If $\gamma = 0$, then the first integral contains an exponential function, the study of its cyclicity problem under quadratic perturbations, in general, is difficult, Li [55] proved that the cyclicity is two if $\alpha = \beta \neq 0$, and there is a simpler proof recently in [58].

If $\gamma \neq 0$, then by scaling we can change to $\gamma = -2$, and system (8) takes the form

$$\dot{x} = -y + \alpha x^2 + \beta y^2, \quad \dot{y} = x(1 - 2y). \tag{9}$$

By the changes

$$x = \frac{1}{2}\bar{x}, \quad y = -\frac{1}{2}(\bar{y} - 1), \quad t = 2\bar{t},$$

and writing (x, y, t) instead of $(\bar{x}, \bar{y}, \bar{t})$, we obtain

$$\dot{x} = \alpha x^2 + \beta y^2 - 2(\beta - 1)y + (\beta - 2), \quad \dot{y} = -2xy. \tag{10}$$

System (10) has an invariant straight line $\{y = 0\}$, and has a center at (0, 1). The singularity $\left(0, -\frac{2-\beta}{\beta}\right)$ is also a center if $0 < \beta < 2$, and is a saddle if $\beta < 0$ or $\beta > 2$. If $\alpha(2 - \beta) > 0$, then the system has two saddles at $\left(\pm\sqrt{\frac{2-\beta}{\alpha}}, 0\right)$. There is a nilpotent singularity at (0, 0) if $\beta = 2$, and one singularity goes to infinity as $\beta \to 0$.
If $\alpha(\alpha + 1)(\alpha + 2) \neq 0$, then the first integral of (10) is given by

$$H = |y|^{\alpha} \left(x^2 + \frac{\beta\, y^2}{\alpha + 2} + \frac{2(1 - \beta)\, y}{\alpha + 1} + \frac{\beta - 2}{\alpha}\right) = h, \tag{11}$$

with integrating factor $\mu = |y|^{\alpha - 1}$.
We denote the vector field (9) or (10) by $X_{\alpha,\beta}$. As in [91], we use Q_3^H, Q_3^R, Q_3^{LV} and Q_4 for quadratic integrable classes of Hamiltonian, reversible, Lotka–Volterra and codimension 4, respectively. Then all degenerate reversible cases are:

- $X_{\alpha,\beta} \in Q_3^R \cap Q_3^H \setminus \{Q_3^{LV} \cup Q_4\}$ if $\alpha = 1$ and $\beta \neq -1$;
- $X_{\alpha,\beta} \in Q_3^R \cap Q_3^H \cap Q_3^{LV} \setminus \{Q_4\}$ (Hamiltonian triangle) if $(\alpha, \beta) = (1, -1)$.
- $X_{\alpha,\beta} \in Q_3^R \cap Q_3^{LV} \setminus \{Q_3^H \cup Q_4\}$ if $\alpha + \beta = 0$ and $\alpha \neq 1$; and
- $X_{\alpha,\beta} \in Q_3^R \cap Q_4 \setminus \{Q_3^H \cup Q_3^{LV}\}$ if $(\alpha, \beta) = (-4, 2)$ or $(-2/3, 0)$.

In other cases $X_{\alpha,\beta} \in Q_3^R \setminus \{Q_3^H \cup Q_3^{LV} \cup Q_4\}$, and it is called a generic reversible quadratic system. To study its cyclicity under quadratic perturbations we need to estimate the number of zeros of the (generalized) Abelian integral

$$M(h) = \int_{\gamma_h} |y|^{\alpha-2}(c_1 + c_2 y + c_3 y^2)\, x\, dy. \tag{12}$$

It is proved in [8] that, if $\alpha \neq 0, -1, -2$ and $\beta \neq 0, 2$, the associated Picard-Fuchs equation has a finite order $\mathscr{K}$ if α is rational and $\mathscr{K} = \infty$ if α is irrational. This means that the study of cyclicity problem for quadratic reversible systems under quadratic perturbations is very difficult, and the problem is open. It is natural to consider the cases that $\mathscr{K}$ is smaller, and try to study the cyclicity more precisely in these cases. From the proof of Lemma 3.1 in [8] we know that $\mathscr{K}_{\alpha,\beta} = \mathscr{K}_{-(\alpha+2),2-\beta}$, hence it is enough to study $\mathscr{K}$ for $\alpha > -1$. We have that for $\alpha \neq 0, -1, -2$ and $\beta \neq 0, 2$

- If $|\alpha| < 1$, $\alpha = \pm\frac{m}{n}$, $0 < m < n$, $(m, n) = 1$, then $\mathscr{K} = 2n$;
- If $\alpha \geq 1$ is an integer, then $\mathscr{K} = \alpha + 2$;
- If $\alpha > 1$, $\alpha \in \mathbb{Q}$ is not an integer, $\alpha = [\alpha] + \frac{m}{n}$, then $\mathscr{K} = ([\alpha] + 2)n$.

In particular (also use the property $\mathscr{K}_{\alpha,\beta} = \mathscr{K}_{-(\alpha+2),2-\beta}$),

- $\mathscr{K} = 3$ if $\alpha = 1$ or $\alpha = -3$;
- $\mathscr{K} = 4$ if $\alpha = 2, -4, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{5}{2}$;
- $\mathscr{K} \geq 5$, otherwise.

We remark that in Theorem 1 of [25] Gautier et al. classify all quadratic reversible systems, whose phase curves are algebraic curves of genus one, into 18 cases (r1)–(r18). They use the complex form of the integrable quadratic system with a center at origin as follows

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, \qquad a, b \in \mathbb{R}, \quad z = x + iy.$$

Note that $\gamma \neq 0$ in (8) is equivalent to $a \neq b$ here. Under this condition it is not hard to translate above equation to form (9) or (10) with relations

$$\alpha = -\frac{a+b+2}{a-b}, \quad \beta = \frac{a+b-2}{a-b},$$

and their conditions (r1)–(r18) will be changed to following simpler forms:

- (r1)–(r6) correspond to $\alpha = -3, 1, -\frac{3}{2}\,(\beta \neq 2), -\frac{1}{2}\,(\beta \neq 0), -4$ and 2, respectively;
- (r7), (r10), (r12), (14), (r16) and (r18) correspond to $\beta = 0$ and $\alpha = -\frac{4}{3}, -\frac{2}{3}, -\frac{1}{3}, -\frac{3}{4}, -\frac{1}{4}$ and $\frac{1}{2}$ respectively;
- (r8), (r9), (r11), (r13), (r15) and (r17) correspond to $\beta = 2$ and $\alpha = -\frac{2}{3}, -\frac{4}{3}, -\frac{5}{3}, -\frac{5}{4}, -\frac{7}{4}$ and $-\frac{5}{2}$ respectively.

It is clear that each of cases (r1)–(r6) corresponds to a one-parameter family of systems, and each of other cases corresponds to a single system with a critical value $\beta = 0$ or $\beta = 2$ and some special value of α. We list the results about cyclicity problem of families (r1)–(r6), including many recent works.

- Family (r2) ($\alpha = 1$) is the intersection of reversible and Hamiltonian classes, the cyclicity problem of the period annulus or annuli was completely solved, see the introduction above [11,28,34,35,88,90,48].
- Family (r1) ($\alpha = -3$) was completely studied, see [19] for $\beta = 1$ (also see [64] for a simple proof about the number of zeros in one annulus by using Chebysev property); [66] for $\beta = -1$; [83] for $\beta \in (-\infty, 0) \setminus \{-1\}$; [39] for $\beta \in (0, 2)$; [40] for $\beta \in [2, +\infty) \setminus \{3\}$; [49] for $\beta = 3$ (a reversible and Lotka-Volterra case) and [76] for $\beta = 0$.
- Family (r3) ($\alpha = -\frac{3}{2}$) was also completely studied, see [57] for $\beta \in (0, 2)$; [89] for $\beta \in (2, +\infty)$ and [59] for $\beta \in (-\infty, 0] \cup \{2\}$.
- Family (r4) ($\alpha = -\frac{1}{2}$) was studied in [9] for $\beta \in (0, 2)$.
- Family (r5) ($\alpha = -4$) was studied in Theorem 2.1 of [8] for $\beta \neq 0, 2, 4$.
- Family (r6) ($\alpha = 2$) was studied in Theorem 1.1 of [8] for $\beta \in (0, 2)$.

Note that if $\beta \in (0, 2)$, then system $X_{\alpha,\beta}$ has two centers and two period annuli. The papers listed above studied the maximal number of zeros of Abelian integral for each annulus and also for two annuli in the same time. In fact, the bifurcation diagrams in parameter space were obtained. Besides, [58] proved that the cyclicity under quadratic perturbations is two in case $\beta = \alpha+2, \alpha(\alpha+2) > 0$ and $\alpha \neq 1$.

A quadratic system possessing at least 4 limit cycles (in (3,1)-distribution), constructed from a quadratic reversible system by quadratic perturbations, was found in 1982 in [45]. If $a = 0$ then system (30) of this paper is reversible, see the first footnote on page 1093 of [45]. Recently, [57] studies this problem in family (r3) ($\beta \in (0, 2)$) and [85] studies this problem by using different normal forms.
We also list the results about the cyclicity problem of cases (r7)–(r18), which are solved completely. Note that in each case the unperturbed system has no parameters.

- (r7) in [31,68];
- (r8), (r13) and (r16) in [7];
- (r9) in [69];
- (r10) in [36];
- (r11) in [25,31];
- (r12) in [70];
- (r14) in [31,80];
- (r15) in [31,67];
- (r17) in [31];
- (r18) in [25].

7. Concerning the quadratic perturbations of period annulus of quadratic reversible and Lotka-Volterra systems ($Q_3^R \cap Q_3^{LV}$, i.e. $\alpha + \beta = 0$ in (9) or (10)), several results give cyclicity 3 or 2. If the phase curves are algebraic curves of genus one, then [25] classifies them into 6 cases (rlv1)-(rlv6). Above mentioned [35,49] studied the cases (rlv1) and (rlv2) respectively, and [31,75] studied the cases (rlv3) and (rlv4) respectively. Besides, [50] studied the case $(\alpha, \beta) = (-1, 1)$ by using second order averaging.
8. Concerning the quadratic perturbations of period annulus of quadratic codimension four center ($Q4$), using Abelian integral, based on Picard-Fuchs equations and argument principle, Gavrilov and Iliev [29] proved that the cyclicity is less or equal to eight. Recently Zhao improved this number from eight to five, see [87].

We make a final remark that similar to the Hilbert's 16th problem, its weak form (about the number of zeros of Abelian integrals) is still far from completely solved. Some new methods, new approaches, and new techniques need to be developed.

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