Qual. Theory Dyn. Syst. (2011) 10:43–49
DOI 10.1007/s12346-011-0036-y

Qualitative Theory of Dynamical Systems

On the Dulac Functions

Osvaldo Osuna · Gabriel Villaseñor

Received: 7 January 2010 / Accepted: 12 November 2010 / Published online: 4 February 2011

Abstract In this work we present an alternative method to build Dulac functions that allow one to discard the existence of periodic solutions for differential equations in the plane using partial differential equations. We give some examples to illustrate applications of these results.

Keywords Bendixson–Dulac criterion · Dulac functions · Quasilinear partial differential equations

Mathematics Subject Classification (2000) 34A34 · 34C25

1 Introduction

There exist two important goals in the study of differential equations, the first one is to describe the dynamics around fixed points, and the second is the analysis of the periodic trajectories that are there in a given system. There are criteria that allow us to know if a system of equations does or does not contain periodic solutions. In the particular case of a system defined on the plane, one has, for example, criterion like Poincaré–Bendixson, Bendixson–Dulac, the index theory and special systems as the

O. Osuna
Instituto de Física y Matemáticas, Universidad Michoacana, Edif. C-3,
Cd. Universitaria, Morelia, Michoacán, 58040, México
e-mail: osvaldo@ifm.umich.mx

G. Villaseñor (✉)
Departamento de Ciencias Básicas, Instituto Tecnológico de Morelia, Edif. Z-3,
Av. Morelos Norte S/N, Morelia, Michoacán, México
e-mail: gabriel@ifm.umich.mx

system gradient where it is known that periodic orbits cannot exist, among others see [1,4,7].

In this paper we are interested in constructing Dulac functions for a system, in order to aid the reader, we remember the criterion of Bendixson–Dulac see [3].

Theorem 1.1 (Bendixson–Dulac criterion) *Let* $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ *and* $h(x_1, x_2)$ *be functions* C^1 *in a simply connected domain* $D \subset \mathbb{R}^2$ *such that* $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ *does not change sign in* D *and vanishes at most on a set of measure zero. Then the system*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \quad (x_1, x_2) \in D, \end{cases} \tag{1.1}$$

does not have periodic orbits in D.

According to this criterion, to rule out the existence of periodic orbits of the system (1.1) in a simply connected region D, we need to find a function $h(x_1, x_2)$ that satisfies the conditions of the theorem of Bendixson–Dulac, such function h is called a *Dulac function*. It is usually not easy to determine such a function, however it is possible to propose some candidates of the form $h = 1, x_1^s, x_2^s, e^{x_1 x_2}, x_1^s x_2^t$, for example:

Example Consider the system

$$\begin{cases} \dot{x_1} = (x_2 - 2)x_1^2 \\ \dot{x_2} = ax_1 + 4x_2 \end{cases}$$

and let $h = x_1^{-2}$ then

$$\frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} = \frac{\partial(x_2 - 2)x_1^2 x_1^{-2}}{\partial x_1} + \frac{\partial(ax_1 + 4x_2)x_1^{-2}}{\partial x_2} = 4x_1^{-2} \geq 0,$$

and the system does not contain periodic orbits in the subset $x_1 > 0$.

In Sáez and Szánto [6] gave a construction of certain Dulac functions using some methods for the construction of Lyapunov functions.

Our goal is to find another method to construct Dulac functions, and for this we will use partial differential equations.

2 Methods to Obtain Dulac functions

The list of functions h given above only serves a limited number of systems; therefore we discuss some more general situations.

(1) First propose another positive or negative function $k(x_1, x_2)$ that only vanishes on a set of measure zero, such that it satisfies

$$\frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} = k(x_1, x_2).$$

(2) Now we will take advantage of the freedom of k to obtain some results. Then take $k(x_1, x_2) = c(x_1, x_2)h(x_1, x_2)$, with c positive or negative that only vanishes on a set of measure zero, and substitute this relation into the previous equation to obtain the following expression

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} + h \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = c(x_1, x_2)h(x_1, x_2), \tag{2.1}$$

we can rewrite this equation

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right). \tag{2.2}$$

(3) Find the solution of this first order quasilinear partial differential equation.
(4) Finally, test h to see if it is a Dulac function for the system (1.1).

Until here we have constructed a method that allows us to discard periodic orbits associated with a system of differential equations in the plane as summarized in the following:

Theorem 2.1 *For the system of differential equations* (1.1) *a solution h of the associated system* (2.2) *(for some function c which does not change sign and vanishes only on a subset of measure zero) is a Dulac function for* (1.1) *in any simply connected region A contained in $D \setminus \{h^{-1}(0)\}$.*

Proof The validity of the result follows from the steps (1)–(4). □

Corollary 2.2 *For the system of differential equations* (1.1)*, if* (2.2) *(for some function c which does not change of sign and it vanishes only on a subset of measure zero) has a solution h on D such that h does not change sign and vanishes only on a subset of measure zero, then h is a Dulac function for* (1.1) *on D.*

Proof Given a function $l : D \to \mathbb{R}$ we denote by $Z_l := \{(x_1, x_2) \in D / l(x_1, x_2) = 0\}$, we have $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ does not change sign and vanishes at most in $Z_c \cup Z_h$ which have measure zero. □

Remark 2.3 With the choice of k in step 2 we obtain a linear equation (1.1), which is simpler to work, nevertheless other choices are possible.

Remark 2.4 The choice of c is used to simplify the equation (2.1), notice that always we can take c to be a constant nonzero.

Example Let us consider the system

$$\begin{aligned} \dot{x_1} &= x_1^2 x_2^2 (1 - \cos x_2) \\ \dot{x_2} &= 3x_1^2 x_2 \end{aligned}$$

Replacing in the equation (1.1) we have the expression

$$x_1^2x_2^2(1-\cos x_2)\frac{\partial h}{\partial x_1}+3x_1^2x_2\frac{\partial h}{\partial x_2}=h\left[c(x_1,x_2)-\left(2x_1x_2^2(1-\cos x_2)+3x_1^2\right)\right]$$

that it is a quasilinear partial differential equation, applying the method of the characteristic we obtain the associate system

$$\begin{aligned} dx_1 &= x_1^2x_2^2(1-\cos x_2)dt \\ dx_2 &= 3x_1^2x_2dt \\ dh &= h\left[c-\left(2x_1x_2^2(1-\cos x_2)+3x_1^2\right)\right]dt \end{aligned}$$

of the two first equations eliminating the parameter t we obtain

$$3x_1^2x_2dx_1=x_1^2x_2^2(1-\cos x_2)dx_2,$$

solving this equation we obtain the first characteristic

$$3x_1+x_2\sin x_2+\cos x_2-\frac{1}{2}x_2^2=C_1$$

taking $c := 3x_1^2$ in the last equation and multiplying the equation by x_1 we have

$$x_1dh=-h\left(2x_1^2x_2^2(1-\cos x_2)\right)dt$$

which, by replacing in the first equation of the associate system, we can rewrite as

$$x_1dh=-2hdx_1$$

and solving we have $h=\frac{C_2}{x_1^2}$, note that h is a Dulac function for the given system in $x_1>0$ or $x_1<0$, in particular, if there is a periodic orbit it must cut $x_1=0$, which is not possible.

Example Let us consider the system

$$\begin{aligned} \dot{x_1} &= 2x_1\cos x_1 \\ \dot{x_2} &= e^{x_2}+3x_2 \end{aligned}$$

Replacing in the equation (2.2) and taking $H(x_1,x_2)=\ln h(x_1,x_2)$ we have the expression

$$2x_1\cos x_1\frac{\partial H}{\partial x_1}+(e^{x_2}+3x_2)\frac{\partial H}{\partial x_2}=c(x_1,x_2)-(2\cos x_1-2x_1\cos x_1+e^{x_2}+3)$$

if we take $c := 3 + e^{x_2} + 2\cos x_1$, then the equation is reduced to

$$2x_1 \cos x_1 \frac{\partial H}{\partial x_1} + (e^{x_2} + 3x_2)\frac{\partial H}{\partial x_2} = 2x_1 \cos x_1,$$

that it is a partial differential equation of the form

$$f(x_1)H_{x_1} + g(x_2)H_{x_2} = G(x_1) + G(x_2),$$

the solution of this system (see [5]) is given by

$$H = x_1 + \Phi\left(\int \frac{dx_1}{2x_1 \cos x_1} - \int \frac{dx_2}{e^{x_2} + 3x_2}\right)$$

and therefore

$$h = exp\left[x_1 + \Phi\left(\int \frac{dx_1}{2x_1 \cos x_1} - \int \frac{dx_2}{e^{x_2} + 3x_2}\right)\right]$$

where Φ is an arbitrary function.

Example Consider the system

$$\begin{cases} \dot{x}_1 = -2x_1x_2, \\ \dot{x}_2 = (x_1x_2)^2\cos(x_1) + (1 + x_1{}^2)x_2 + x_2{}^2. \end{cases}$$

Taking $c := 1 + x_1{}^2$, we obtain

$$\begin{aligned} &-2x_1x_2\frac{\partial h}{\partial x_1} + \left((x_1x_2)^2\cos(x_1) + (1 + x_1{}^2)x_2 + x_2{}^2\right)\frac{\partial h}{\partial x_2} \\ &= h(-2x_1x_2\cos(x_1)), \end{aligned}$$

assume that h depends only on x_1, then the solution is

$$h = \exp\left(\int -2x_1\cos(x_1)\right).$$

This example provides a Dulac function completely different from those listed in the introduction.

Example Consider the Lotka–Volterra equations

$$\begin{cases} \dot{x_1} = x_1(k - ax_1 - bx_2) \\ \dot{x_2} = x_2(m - ex_1 - dx_2) \end{cases}$$

Since these equations model biological systems in which two species interact, we take $x_1 > 0$ and $x_2 > 0$. Replacing in equation (2.2), we obtain

$$x_1(k - ax_1 - bx_2)\frac{\partial h}{\partial x_1} + x_2(m - ex_1 - dx_2)\frac{\partial h}{\partial x_2}$$
$$= h\,[c - (k - ax_1 - bx_2) - (m - ex_1 - dx_2) + ax_1 + dx_2]$$

choosing $ad \geq 0$ where a, d are not both zero, we can take $c := -ax_1 - dx_2$,

$$x_1(k - ax_1 - bx_2)\frac{\partial h}{\partial x_1} + x_2(m - ex_1 - dx_2)\frac{\partial h}{\partial x_2}$$
$$= h\,[-(k - ax_1 - bx_2) - (m - ex_1 - dx_2)]$$

and rewriting with $r_1 := k - ax_1 - bx_2$ and $r_2 := m - ex_1 - dx_2$

$$x_1 r_1 \frac{\partial h}{\partial x_1} + x_2 r_2 \frac{\partial h}{\partial x_2} = h\,(-r_1 - r_2)\,,$$

the solution of this partial differential equation is

$$h = \frac{1}{x_1 x_2}.$$

Example Consider the system of the first example

$$\begin{cases} \dot{x_1} = (x_2 - 2)x_1^2 \\ \dot{x_2} = ax_1 + 4x_2 \end{cases}$$

replacing in equation (2.2) we have

$$(x_2 - 2)x_1^2 \frac{\partial h}{\partial x_1} + (ax_1 + 4x_2)\frac{\partial h}{\partial x_2} = h\,(c - (2x_1(x_2 - 2) + 4))$$

and taking $c = 4$ we obtain

$$(x_2 - 2)x_1^2 \frac{\partial h}{\partial x_1} + (ax_1 + 4x_2)\frac{\partial h}{\partial x_2} = h\,(-(2x_1(x_2 - 2)))$$

and supposing that h only depends on x_1 then

$$(x_2 - 2)x_1^2 \frac{\partial h}{\partial x_1} = h\,(-(2x_1(x_2 - 2)))$$

and solving we have $h = x_1^{-2}$.

Note that we have recovered the function of Dulac used in the first example.

This algorithm also can be applied to find a function h in the following general situation see [2].

Proposition 2.5 *Let D_0 be an $l-$connected subset of D (i.e. D_0 has l gaps or $\Pi_1(D_0) \approx \mathbb{Z} * \mathbb{Z} * \ldots (l) * \mathbb{Z}$), and supposes there exists a function $h \in C^1(D_0, \mathbb{R})$ such that $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ does not change sign and vanishes only on a set of measure zero, then the system has at the most l periodic orbits in D_0.*

Acknowledgments O. Osuna was partially supported by CONACYT, Grant, 50483-F. We thank the referee for various comments that helped improving the paper.

References

1. Bendixson, I.: Sur les curbes définiés par des équations différentielles. Acta Math. **24**, 1–88 (1901)
2. Gassull, A., Giacomini, H.: A new criterion for controlling the number of limit cycles of some generalized Liénard equations. J. Dyn. Diff. Equ. **185**, 54–73 (2002)
3. McCluskey, C.C., Muldowney, J.S.: Bendixson–Dulac criteria for difference equations. J. Dyn. Diff. Equ. **10**, 567–575 (1998)
4. Lawrence Perko: Differential Equations And Dynamical Systems. Springer, Berlin (2006)
5. Polyanin, A.D., Zaitsev, V.F., Moussiaux, A.: Handbook of First Order Partial Differential Equations. Taylor and Francis, London (2002)
6. Saéz, E., Szánto, I.: On the construction of certain Dulac function. Trans. Automat Control **33**, 856 (1988)
7. Strogatz, S.: Nonlinear Dynamics and Chaos. Addison-Wesley, Reading (1994)