Qual. Theory Dyn. Syst. (2010) 9:101–113
DOI 10.1007/s12346-010-0022-9

**Qualitative Theory
of Dynamical Systems**

# Cyclicity versus Center Problem

**Armengol Gasull · Jaume Giné**

Received: 9 January 2009 / Accepted: 26 October 2009 / Published online: 11 August 2010


**Abstract** We prove that there are one-parameter families of planar differential equations for which the center problem has a trivial solution and on the other hand the cyclicity of the weak focus is arbitrarily high. We illustrate this phenomenon in several examples for which this cyclicity is computed.

**Keywords** Polynomial differential system · Focal quantities · Lyapunov constants · Cyclicity · Limit cycles

**Mathematics Subject Classification (2000)** Primary 34C07;
Secondary 34C23 · 34C25 · 37C27

## 1 Introduction and Main Results

Three of the main problems in the qualitative theory of planar polynomial differential systems are: the determination of the number of limit cycles and their distribution in

In memory of Javier Chavarriga.

A. Gasull is partially supported by a MCYT/FEDER grant number MTM2008-03437 and by a CIRIT grant number 2009SGR 410. J. Giné is partially supported by a MCYT/FEDER grant number MTM2008-00694 and by a CIRIT grant number 2009SGR 381.

A. Gasull
Departament de Matemàtiques, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Spain
e-mail: gasull@mat.uab.cat

J. Giné (✉)
Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69, 25001 Lleida, Spain
e-mail: gine@matematica.udl.cat

the plane, called *Hilbert's sixteenth problem*, see [17,18]; the distinction between a center and a focus, called the *center or the center-focus problem*, see [6,13,15,19]; and the determination of the integrable cases, see [9]. Clearly these three problems are strongly related. For instance, the study of the center conditions, gives rise to the Lyapunov constants that are often used to produce limit cycles as well as to find integrable systems and on the other hand, the integrable cases are perturbed to generate limit cycles.

This paper deals with the first two problems and it presents a curious phenomenon which relate them: an one-parameter family of planar vector fields having the cyclicity at the origin arbitrarily high, for which the solution of the center problem is trivial. We prove:

**Theorem 1.1** *Consider an one-parameter family of differential systems of the form*

$$\begin{aligned} \dot{x} &= -y + a^k x(x^2+y^2) + aP(x,y,a),\\ \dot{y} &= x + a^k y(x^2+y^2) + aQ(x,y,a), \end{aligned} \tag{1.1}$$

*where $P$ and $Q$ are analytic functions, starting at least with terms of degree $4$ in $x$ and $y$, and $k \geq 1$ is an integer number. Then:*

(a) *The first Lyapunov constant is $V_3 = 2\pi a^k$ and the origin is a center if and only if $a = 0$.*
(b) *The cyclicity of the origin is at most $k-1$ and there are analytic functions, $P$ and $Q$, for which this upper bound is sharp.*

As we will see, the proof of Theorem 1.1 is not difficult and it is mainly based on the Weierstrass preparation Theorem. Nevertheless, the result is somehow surprising because usually, to obtain many limit cycles, people need to study differential equations having high order weak focus inside the family. On the contrary, in system (1.1) the highest order non-zero Lyapunov constant is the first one, $V_3$. The definition for Lyapunov constants is recalled in next section.

Note that in the above case the high cyclicity of the critical point is caused by the fact that the only significant Lyapunov constant for family (1.1), which is $V_3 = 2\pi a^k$, is such that the ideal generated by it is "far" from being radical. See for instance [14,20,22] for a discussion about the cyclicity of the weak foci in given families and the role of the radicality of the ideal formed by the Lyapunov constants. Indeed an example of a similar situation when $k = 2$ already appears in the last section of [4].

In [4] it is proved that in $m$-parameter families of autonomous planar differential equations the maximum cyclicity of a periodic orbit also can be attained through one-parameter analytic families, see also [21]. Later on, in [4,7], it is shown that this maximum cyclicity can not be necessarily reached through one-parameter algebraic curves. These results are related with Theorem 1.1. From this point of view, our theorem can also be interpreted as a way of giving one-parameter algebraic curves for which the cyclicity of the origin is arbitrarily high.

Although Theorem 1.1 gives an upper bound for the cyclicity of the origin of system (1.1), in general, it is not easy to obtain the actual cyclicity. In the next results we show an effective way for computing it for two concrete families. Before stating them we introduce some notations.

Given the homogeneous polynomials of degree $n$, $P_n(x, y)$ and $Q_n(x, y)$, we write

$$P_n(x, y) = \sum_{k=0}^{n} a_{n-k,k} x^{n-k} y^k, \quad Q_n(x, y) = \sum_{k=0}^{n} b_{n-k,k} x^{n-k} y^k.$$

For sake of simplicity, if there is no ambiguity, we omit the comma in the subindexes. Moreover we denote by $V_{2\ell+1}^{\{n\}}$ the $\ell$th Lyapunov constant of the differential system

$$\begin{aligned} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= x + Q_n(x, y), \end{aligned} \tag{1.2}$$

and by $V_{2\ell+1}^{\{n,m\}}$ the $\ell$th Lyapunov constant of the differential system

$$\begin{aligned} \dot{x} &= -y + P_n(x, y) + P_m(x, y), \\ \dot{y} &= x + Q_n(x, y) + Q_m(x, y). \end{aligned} \tag{1.3}$$

**Theorem 1.2** *Consider the one-parameter family of differential equations*

$$\begin{aligned} \dot{x} &= -y + a^4 x(x^2 + y^2) + a P_4(x, y) + a^3 P_5(x, y), \\ \dot{y} &= x + a^4 y(x^2 + y^2) + a Q_4(x, y) + a^3 Q_5(x, y), \end{aligned} \tag{1.4}$$

*where $P_i$ and $Q_i$ are homogeneous polynomials of degree $i$. Then the cyclicity of the origin is at most* 2.

*Moreover, if we define the function*

$$\phi(c) := V_3^{\{3\}} + V_5^{\{5\}} c^2 + V_7^{\{4\}} c^4,$$

*where $V_3^{\{3\}} = 2\pi$, $V_5^{\{5\}}$ and $V_7^{\{4\}}$ are the Lyapunov constants with expressions given in the Sect. 6, and if we fix the polynomials $P_4$, $P_5$, $Q_4$ and $Q_5$ in such a way that the function $\phi$ has exactly $\ell \leq 2$ simple positive zeros, then $\ell$ limit cycles simultaneously bifurcate from the origin, when the parameter $a > 0$ is small enough.*

Notice that in Theorem 1.2 the coefficients of the bifurcating function $\phi$ are precisely Lyapunov constants of some particular systems of the form (1.2). In general the situation is a little more complicated, as next result shows.

**Theorem 1.3** *Consider the one-parameter family of differential equations*

$$\begin{aligned} \dot{x} &= -y + a^6 x(x^2 + y^2) + a^2 P_4(x, y) + a^5 P_5(x, y) + a P_6(x, y), \\ \dot{y} &= x + a^6 y(x^2 + y^2) + a^2 Q_4(x, y) + a^5 Q_5(x, y) + a Q_6(x, y), \end{aligned} \tag{1.5}$$

*where $P_i$ and $Q_i$ are homogeneous polynomials of degree $i$. Then the cyclicity of the origin is at most 4. Moreover, define the function*

$$\psi(c) := V_3^{\{3\}} + V_5^{\{5\}} c^2 + V_7^{\{4\}} c^4 + V_9^{\{4,6\}} c^6 + V_{11}^{\{6\}} c^8,$$

*where $V_3^{\{3\}} = 2\pi$, $V_5^{\{5\}}$, $V_7^{\{4\}}$, $V_9^{\{4,6\}}$ and $V_{11}^{\{6\}}$ are the Lyapunov constants given in the Sect. 6, and fix the polynomials $P_4$, $P_5$, $P_6$, $Q_4$, $Q_5$ and $Q_6$ in such a way that the function $\psi$ has exactly $\ell \leq 4$ simple positive zeros. Then, $\ell$ limit cycles simultaneously bifurcate from the origin, when the parameter $a > 0$ is small enough.*

Note that in the above results the notation, $V_3^{\{3\}} = 2\pi$, is coherent because $2\pi$ is precisely the first Lyapunov constant of $\dot{x} = -y + x(x^2 + y^2)$, $\dot{y} = x + y(x^2 + y^2)$.

## 2 Preliminary Definitions

In this section we recall the well-known notions of center, focus, weak focus, focal values, Lyapunov constants and cyclicity of a point, see for instance [1,20], adapted to the case we are dealing with, the one-parameter families of planar differential equations.

Consider the differential equation,

$$\begin{aligned} \dot{x} &= -y + P(x, y), \\ \dot{y} &= x + Q(x, y), \end{aligned} \tag{2.1}$$

where $P$ and $Q$ are analytic functions starting at least with terms of degree 2. Its origin is usually called a *weak focus* and it is well-known that it is either a center or a focus. To distinguish which of the possibilities occur one of the standard methods is based on the computation of the derivatives of the return map at the origin. These derivatives are obtained by solving a system of recursive differential equations. This technique is described for instance in the book of Andronov et al. [1] and it is used in almost all classical works, see [2,3,10,12]. For further use, we quickly recall it: Take polar coordinates $(r, \theta)$, defined by $x = r\cos\theta$, $y = r\sin\theta$ in system (2.1). In these coordinates it becomes

$$\frac{dr}{d\theta} = g(r, \theta) = g_2(\theta) r^2 + g_3(\theta) r^3 + \cdots, \tag{2.2}$$

where $g$ is analytic and for all $k \geq 2$, the functions $g_k$ are trigonometrical polynomials. Let

$$r = r(\theta, r_0) := r_0 + u_2(\theta) r_0^2 + u_3(\theta) r_0^3 + \cdots, \tag{2.3}$$

be the solution of (2.2) satisfying $r(0, r_0) = r_0$. Taking $\theta = 0$ we obtain the initial conditions $u_k(0) = 0$ for all $k \geq 2$. Substituting expression (2.3) in (2.2) we get a recursive differential system to compute $u_k(\theta)$ and the displacement function is given by

$$d(r_0) = r(2\pi, r_0) - r_0 = u_2(2\pi)r_0^2 + u_3(2\pi)r_0^3 + \cdots$$
$$:= v_2 r_0^2 + v_3 r_0^3 + \cdots$$

The values $v_k$, $k \geq 2$, are called *the focal values* of the origin. It is well known that either all the values $v_k$ vanish for $k \geq 2$, and in this case the origin is a center, or otherwise the first non zero focal value corresponds to some $k = 2m + 1$ odd and in this case the origin is a focus. In this situation it is said that the origin is a *weak focus of order m* and the value $V_{2m+1} := v_{2m+1}$ is called the *mth Lyapunov constant* of the system. Notice that the sign of this Lyapunov constant gives the stability of the origin.

To study the cyclicity, instead of considering a fixed system like (2.1), we will consider an one-parameter family of differential equations,

$$\begin{aligned} \dot{x} &= -y + P(x, y, a), \\ \dot{y} &= x + Q(x, y, a), \end{aligned} \tag{2.4}$$

where again $P$ and $Q$ are analytic functions starting at least with terms of degree 2 in $x$ and $y$, and $a \in \mathbb{R}$. First observe that for each value of $a$ we have different focal values that will be denoted by $v_k(a)$, $k \geq 2$. We already know that $v_2(a) \equiv 0$. In this setting, the *m-th Lyapunov constant* is the function $V_{2m+1}(a) := v_{2m+1}(a)$, and is defined for the values of $a$ for which $V_3(a) = V_5(a) = \cdots = V_{2m-1}(a) = 0$.

Recall that a *limit cycle* for a planar autonomous differential equation is a periodic orbit of the system which is isolated in the set of all its periodic orbits. Given Eq. (2.4) we will say that the cyclicity of the origin for $a = a^*$ is $k$ if the following conditions hold:

(i) There exist constants $\varepsilon > 0$ and $\delta > 0$ such that for $|a - a^*| < \varepsilon$ the maximum number of limit cycles of system (2.4) in $\{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 < \delta^2\}$ is at most $k$.

(ii) For $|a - a^*|$ small enough there are $k$ limit cycles, say $\Gamma_a^1, \Gamma_a^2, \ldots, \Gamma_a^k$, that tend to $(0, 0)$ when $a$ tends to $a^*$ and such that their cuts with a transversal section through the origin depend smoothly on $a$.

For short we will denote this cyclicity by $\text{Cyc}(a^*) = k$.

We must comment that item (ii) of the above definition is not the usual way of saying that $k$ is the actual cyclicity. In many works the existence of these $k$ limit cycles is only ensured for a sequence of values of $a$ tending to $a^*$, see for instance [4,20]. We have chosen the above one because it is also very natural and is more convenient for our point of view. In any case, it can be seen that both definitions coincide under analycity assumptions, see [4,5].

As a consequence of the above definition, for each value of $a$ the origin of (2.4) can have a different cyclicity. The maximum cyclicity, when $a$ varies in $\mathbb{R}$, among all these cyclicities will be called the *cyclicity at the origin for the family* (2.4) and is $\sup_{a \in \mathbb{R}} (\text{Cyc}(a))$.

In this paper sometimes the one-parameter families will be viewed as

$$\begin{aligned} \dot{x} &= -y + P(x, y, a, \mathbf{b}), \\ \dot{y} &= x + Q(x, y, a, \mathbf{b}), \end{aligned} \tag{2.5}$$

where again $P$ and $Q$ are analytic functions starting at least with terms of degree 2 in $x$ and $y$, $a \in \mathbb{R}$ is the parameter and $\mathbf{b} \in \mathbb{R}^m$ are some fixed numbers. In this situation, clearly the cyclicity of the origin, as an one-parameter family, depends on $a$ and $\mathbf{b}$ and we can write it as $\mathrm{Cyc}(a, \mathbf{b})$. When we say that the *cyclicity at the origin for the one-parameter families given by system* (2.5) *is at most* $k$ we mean that

$$\sup_{\mathbf{b} \in \mathbb{R}^m} \Big( \sup_{a \in \mathbb{R}} (\mathrm{Cyc}(a, \mathbf{b})) \Big) \leq k.$$

Notice that the above cyclicity is smaller or equal (usually smaller) than the one obtained by considering system (2.5) as a $(m+1)$-parameter family with parameters $(a, \mathbf{b})$.

## 3 Proof of Theorem 1.1

We state and prove a theorem slightly stronger that Theorem 1.1.

**Theorem 3.1** *Consider an one-parameter family of differential systems of the form*

$$\begin{aligned} \dot{x} &= -y + a^k x(x^2 + y^2) + P(x, y, a), \\ \dot{y} &= x + a^k y(x^2 + y^2) + Q(x, y, a), \end{aligned} \tag{3.1}$$

*where $P$ and $Q$ are analytic functions, starting at least with terms of degree* 4 *in $x$ and $y$, and $k \geq 1$ is an integer number. Then:*

(a) *The cyclicity of the origin is at most $k$ and there are functions $P$ and $Q$ for which this upper bound is sharp.*
(b) *If we assume that there exists some integer $m$, $1 \leq m \leq k$, such that the focal values $v_i(a) = a^m W_i(a)$ for $i \geq 3$, where $W_i$ are analytic functions at $a = 0$, then the cyclicity of the origin is at most $k - m$. Furthermore, there are functions $P$ and $Q$ for which this upper bound is reached.*

*Proof* To compute the focal values we follow the steps described in the previous section. In polar coordinates system (3.1) becomes

$$\frac{dr}{d\theta} = r^3(a^k + a g(r, \theta, a)), \tag{3.2}$$

where $g$ is an analytic function. Writing

$$r = r(\theta, a, r_0) := r_0 + u_3(\theta, a) r_0^3 + u_4(\theta, a) r_0^4 + \cdots, \tag{3.3}$$

we get

$$\begin{aligned} d(a, r_0) = r(2\pi, a, r_0) - r_0 &= u_3(2\pi, a) r_0^3 + u_4(2\pi, a) r_0^4 + \cdots \\ &:= v_3(a) r_0^3 + v_4(a) r_0^4 + \cdots \end{aligned}$$

where the $v_k(a)$ are the focal values and concretely $v_3(a) = 2\pi a^k$. The limit cycles of the system near the origin correspond to initial conditions that are positive values of $r_0$ which are solutions of the analytic equation

$$F(a, r_0) := \frac{d(a, r_0)}{r_0^3} = 2\pi a^k + O(r_0) = 0, \tag{3.4}$$

where

$$F(0,0) = 0, \quad \frac{\partial F}{\partial a}(0,0) = 0, \quad \ldots, \quad \frac{\partial^{k-1} F}{\partial a^{k-1}}(0,0) = 0, \quad \frac{\partial^k F}{\partial a^k}(0,0) \neq 0.$$

When $a \neq 0$ it is clear that $\mathrm{Cyc}(a) = 0$. So, in order to get the maximum cyclicity of the origin we have to compute $\mathrm{Cyc}(0)$.

Using the Weierstrass preparation Theorem, see [1, p. 388], we know that Eq. (3.4) can be written as

$$F(a, r_0) = (a^k + \varphi_{k-1}(r_0)a^{k-1} + \cdots + \varphi_1(r_0)a + \varphi_0(r_0))\Phi(a, r_0) = 0,$$

where $\Phi(0, 0) \neq 0$ and the involved functions are analytic. Clearly, near $(0, 0)$, to study the cyclicity of the origin it suffices to study the equation

$$G(a, r_0) = a^k + \varphi_{k-1}(r_0)a^{k-1} + \cdots + \varphi_1(r_0)a + \varphi_0(r_0) = 0.$$

Fixed $r_0$, the above equation has at most $k$ solutions. On the contrary, to prove that the cyclicity of the origin is at most $k$, we have to prove that, fixed $a$, the maximum number of positive values of $r_0$ satisfying the above equation is also $k$. Note that if we do not impose that the solutions $r_0$ are positive this upper bound is not true as the following example having $2k$ solutions for $a > 0$ shows,

$$(a - r_0^2)(a - 2r_0^2)(a - 3r_0^2) \cdots (a - (k-1)r_0^2)(a - kr_0^2) = 0.$$

Assume that the cyclicity is greater than $k$. Then, there exist at least $k + 1$ limit cycles $\Gamma_a^i$, $i = 1, \ldots, k+1$, tending to $(0, 0)$ when $a$ goes to zero. Associated to these limit cycles and to the transversal section $\{\theta = 0\}$ there are $k + 1$ smooth functions $r_0 = R_i(a) > 0$, $i = 1, \ldots, k + 1$ defined in a neighborhood of $a = 0$, and such that $G(a, R_i(a)) \equiv 0$. Therefore given $r_0^* > 0$ small enough there are at least $k + 1$ values of $a$ belonging to the set $Z := \cup_{i=1}^{k+1}\{R_i^{-1}(r_0^*)\}$, such that if $a \in Z$ then $G(a, r_0) = 0$. This is in contradiction with the fact that $G(a, r_0^*)$ is a non-zero polynomial in $a$ of degree $k$. Thus the upper bound is proved.

Now we present an example showing that this upper bound is reached. Consider the following family of differential equations

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2)(a^k + \alpha_1 a^{k-1} r^2 + \cdots + \alpha_{k-1} a r^{2(k-1)} + \alpha_k r^{2k}), \\ \dot{y} &= x + y(x^2 + y^2)(a^k + \alpha_1 a^{k-1} r^2 + \cdots + \alpha_{k-1} a r^{2(k-1)} + \alpha_k r^{2k}), \end{aligned} \tag{3.5}$$

where $r^2 = x^2 + y^2$ and $\alpha_i, i = 1, 2, \ldots, k$ are real constants to be fixed. In polar coordinates system (3.5) writes as

$$\frac{dr}{d\theta} = r^3(a^k + \alpha_1 a^{k-1} r^2 + \alpha_2 a^{k-2} r^4 + \cdots + \alpha_{k-1} a r^{2(k-1)} + \alpha_k r^{2k}).$$

The limit cycles of the above equation are the circles whose radii correspond to positive isolated solutions of

$$a^k + \alpha_1 a^{k-1} r^2 + \alpha_2 a^{k-2} r^4 + \cdots + \alpha_{k-1} a r^{2(k-1)} + \alpha_k r^{2k} = 0. \tag{3.6}$$

If we perform the change $r^2 = a\rho$, then Eq. (3.6) for $a > 0$ transforms in the following equivalent equation

$$1 + \alpha_1 \rho + \alpha_2 \rho^2 + \cdots + \alpha_{k-1} \rho^{k-1} + \alpha_k \rho^k = 0.$$

By choosing suitable $\alpha_i$ for $i = 1, 2, \ldots, k$ the above equation has $k$ positive roots and the result follows.

The proof of statement (b) is similar, but considering the equation

$$F_m(a, r_0) := \frac{d(a, r_0)}{a^m r_0^3} = 2\pi a^{k-m} + O(r_0) = 0,$$

instead of (3.4) and taking system (3.5) with $\alpha_k = \alpha_{k-1} = \cdots = \alpha_{k-m+1} = 0$. Notice that the hypotheses on the focal values imply that $F_m$ is an analytic function. □

*Proof of Theorem* 1.1. In the proof of (a) of the above theorem it is seen that $V_3 = 2\pi a^k$. By using this fact the proof of (a) follows. To prove (b), we notice that we can apply Theorem 3.1.(b), with $m = 1$, because in (1.1) there is the parameter $a$ in front of both functions $P$ and $Q$. □

As a corollary of Theorem 3.1 we obtain next result that will be used in Sect. 4.

**Corollary 3.2** *Consider the one-parameter family of differential systems*

$$\begin{aligned} \dot{x} &= -y + a^k x(x^2 + y^2) + a^\alpha P_4(x, y) + a^\beta P_5(x, y) + a^\gamma P_6(x, y), \\ \dot{y} &= x + a^k y(x^2 + y^2) + a^\alpha Q_4(x, y) + a^\beta Q_5(x, y) + a^\gamma Q_6(x, y), \end{aligned} \tag{3.7}$$

*where $P_i$ and $Q_i$ are homogeneous polynomials of degree $i$, and $k$, $\alpha$, $\beta$ and $\gamma$ are natural numbers satisfying $k \geq 2$, $\alpha \geq 1$, $\beta \geq 2$ and $\gamma \geq 1$. Then the cyclicity of the origin is at most $k - 2$.*

*Proof* The result is a consequence of Theorem 3.1.(b) with $m = 2$. We prove that the focal values of system (3.7) can be written as $v_i(a) = a^2 W_i(a)$ for $i \geq 3$, for polynomial functions $W_i$. This result follows by controlling the powers of $a$ that appear in the functions involved in the algorithm for the computation of the focal values described in Sect. 2.

We remark that $v_3(a) = 2\pi a^k$, $v_4 = 0$ and $v_5(a) = S_5 a^\beta + 6\pi^2 a^{2k}$ where $S_5$ is a polynomial in the coefficients of $P_i$ and $Q_i$, and does not depend on $a$. Therefore to ensure that $v_i(a) = a^2 W_i(a)$, for some polynomials $W_i$ and all $i \geq 3$, we need to impose that $k \geq 2$ and $\beta \geq 2$. If either $k = 1$ or $\beta = 1$ we only can ensure that the cyclicity of the origin is at most $k - 1$. □

## 4 Proof of Theorems 1.2 and 1.3

For sake of shortness we only give the details of the proof for Theorem 1.2. Instead of working with system (1.4), we consider the more general system (3.7) with $P_6 = Q_6 = 0$,

$$\begin{aligned} \dot{x} &= -y + a^k x(x^2 + y^2) + a^\alpha P_4(x, y) + a^\beta P_5(x, y), \\ \dot{y} &= x + a^k y(x^2 + y^2) + a^\alpha Q_4(x, y) + a^\beta Q_5(x, y). \end{aligned} \tag{4.1}$$

Arguing in the same way as in the proof of Theorem 3.1, we get that the displacement function of system (4.1) is

$$d(a, r_0) = v_3(a) r_0^3 + v_4(a) r_0^4 + \cdots$$

where the $v_k(a)$ are the focal values. From long, but straightforward computations, we obtain:

$$\begin{aligned} &v_3(a) = 2\pi a^k, \quad v_4(a) = 0, \quad v_5(a) = V_5^{\{5\}} a^\beta + 6\,\pi^2 a^{2k}, \\ &v_6(a) = A_6\, a^{\beta+k}, \quad v_7(a) = V_7^{\{4\}} a^{2\alpha} + A_7\, a^{\beta+k} + 20\,\pi^3 a^{3k}, \\ &v_8(a) = A_8\, a^{\alpha+2k} + B_8\, a^{\alpha+\beta}, \end{aligned}$$

where $A_6$, $A_7$, $A_8$ and $B_8$ are polynomial expressions in the coefficients of $P_4$, $Q_4$, $P_5$ and $Q_5$ and do not depend on $a$. The Newton's diagram of $d(a, r_0)$, see [1, p. 392], depends on the parameters $\alpha$, $\beta$ and $k$. The choice $\alpha = 1$, $\beta = 3$ and $k = 4$ provides a side of the boundary of the Newton's diagram containing exactly the points (4, 3), (3, 5) and (2, 7). Hence, writing $a = \lambda^2$ and $r_0 = c\lambda$ we obtain

$$d(\lambda^2, c\lambda) = \lambda^{11} c^3 \left(\phi(c) + O(\lambda)\right). \tag{4.2}$$

Let $c^*$ be a simple positive solution of $\phi(c) = 0$. Then if we define

$$g(\lambda, c) := \frac{d(\lambda^2, c\lambda)}{\lambda^{11} c^3} = \phi(c) + O(\lambda),$$

notice that

$$g(0, c^*) = 0, \quad \text{and} \quad \frac{\partial g}{\partial c}(0, c^*) = \phi'(c^*) \neq 0.$$

By using the implicit function theorem we know that there exists a unique function $\lambda = \Lambda(c)$ such that $\Lambda(c^*) = 0$ and in a neighborhood of $(0, c^*)$, $g(\Lambda(c), c) \equiv 0$. Therefore the equation $d(a, r_0) = 0$, for $c$ small enough, has the solutions $(a, r_0) = (\Lambda^2(c), c\Lambda(c))$. In other words, for $a > 0$, there exists a function $r_0 = R(a)$ such that

$$d(a, R(a)) \equiv 0, \quad \text{and} \quad \lim_{a \to 0} \frac{R(a)}{\sqrt{a}} = c^*.$$

Hence, $r_0 \simeq c^*\sqrt{a}$ is the initial condition of a limit cycle that bifurcates from the origin. By using the same reasoning for all the simple roots of $\phi$ the theorem follows.

Notice, that a priori, by using Corollary 3.2 we already knew that the cyclicity of the origin of system (1.4) is at most 2. Our proof shows that for this concrete family the cyclicity is exactly 2.

In order to prove Theorem 1.3 we follow similar steps starting from system (3.7). The main difference is that for this case the computations are much longer. In this case we obtain,

$$\begin{aligned}
v_3(a) &= 2\pi a^k, \quad v_4(a) = 0, \quad v_5(a) = V_5^{\{5\}} a^\beta + 6\,\pi^2 a^{2k}, \\
v_6(a) &= A_6\, a^{\beta+k}, \quad v_7(a) = V_7^{\{4\}} a^{2\alpha} + A_7\, a^{\beta+k} + 20\,\pi^3 a^{3k}, \\
v_8(a) &= A_8\, a^{\alpha+2k} + B_8\, a^{\alpha+\beta} + C_8\, a^{k+\gamma}, \\
v_9(a) &= V_9^{\{4,6\}} a^{\alpha+\gamma} + A_9\, a^{2\alpha+k} + B_9\, a^{2k+\beta} + C_9\, a^{2\beta} + 70\,\pi^4\, a^{4k}, \\
v_{10}(a) &= A_{10} a^{3\alpha} + B_{10}\, a^{\alpha+k+\beta} + C_{10}\, a^{\alpha+3k} + D_{10}\, a^{2k+\gamma} + E_{10}\, a^{\beta+\gamma}, \\
v_{11}(a) &= V_{11}^{\{6\}} a^{2\gamma} + A_{11} a^{2\alpha+\beta} + B_{11}\, a^{2\alpha+2k} + C_{11}\, a^{k+2\beta} + D_{11}\, a^{3k+\beta} \\
&\quad + E_{10}\, a^{k+\beta+\gamma} + 252\,\pi^5 a^{5k},
\end{aligned}$$

where $A_i, B_i, C_i, D_i$ and $E_i$ are polynomial expressions in the coefficients of $P_4$, $Q_4$, $P_5$, $Q_5$, $P_6$ and $Q_6$ and do not depend on $a$. Fixing the values $\alpha = 2$, $\beta = 5$, $\gamma = 1$ and $k = 6$, Eq. (4.2) writes as

$$d(\lambda^2, c\lambda) = \lambda^{15} c^3 (\psi(c) + O(\lambda)),$$

and the proof follows as in the previous situation. □

To end this section we give a concrete example to illustrate the use of Theorems 1.2 and 1.3. Consider system

$$\begin{aligned}
\dot{x} &= -y + a^6 x(x^2 + y^2) + a^2 \left( \frac{80}{3}\, x y^3 + y^4 \right) - \frac{100}{3}\, a^5 x^3 y^2 - a \left( \frac{80}{7}\, x y^5 + \frac{14}{45}\, y^6 \right), \\
\dot{y} &= x + a^6 y(x^2 + y^2) + 20\, a x^2 y^4.
\end{aligned}$$

By using the expressions given in the Sect. 6, we get that the function $\psi(c)$ introduced in Theorem 1.3 is $\pi(c^2 - 1)(c^2 - 2)(c^2 - 3)(c^2 - 4)/12$. Then, when the parameter $a > 0$ is small enough, 4 limit cycles simultaneously bifurcate from the origin and their initial conditions are $r_{0,\ell} \simeq \sqrt{\ell a}$, $\ell = 1, 2, 3, 4$. Moreover we know that the cyclicity of the origin for this one-parameter family is 4.

## 5 Final Remarks

**Some considerations on two-parameter families.** When instead of being one-parameter we consider two-parameter families of planar differential equations, similar phenomena to the ones described in this paper appear. We present the following two examples:

$$\begin{aligned} \dot{x} &= -y + a^k b^\ell x(x^2+y^2) + abP(x,y,a,b), \\ \dot{y} &= x + a^k b^\ell y(x^2+y^2) + abQ(x,y,a,b), \end{aligned} \tag{5.1}$$

where $P$ and $Q$ are analytic functions (starting at least with terms of degree 4 in $x$ and $y$) and $k \geq 1$ and $\ell \geq 1$ are integer numbers; and

$$\begin{aligned} \dot{x} &= -y + a^k x(x^2+y^2) + b^\ell x(x^2+y^2)^2 + aP_1(x,y,a,b) + bP_2(x,y,a,b), \\ \dot{y} &= x + a^k y(x^2+y^2) + b^\ell y(x^2+y^2)^2 + aQ_1(x,y,a,b) + bQ_2(x,y,a,b), \end{aligned} \tag{5.2}$$

where $P_1$, $P_2$, $Q_1$ and $Q_2$ are analytic functions (starting at least with terms of degree 6 in $x$ and $y$) and $k \geq 1$ and $\ell \geq 1$ are integer numbers.

For Eq. (5.1), $V_3 = 2\pi a^k b^\ell$ and the origin is a center if and only if either $a = 0$ or $b = 0$. Moreover, by taking the case $a = b$ and applying Theorem 3.1.(b) with $m = 2$, it is clear that there are functions $P$ and $Q$ for which the cyclicity is at least $k + \ell - 2$. Similarly, for Eq. (5.2), $V_3 = 2\pi a^k$, $V_5 = 2\pi b^\ell$, the origin is a center if and only if $a = b = 0$ and there are nonlinearities for which the cyclicity is at least $\max(k, \ell) - 1$.

**Other limit cycles for system (1.1).** Notice that when $a = 0$ system (1.1) is precisely the linear center. From this point of view when $a$ is small it can be seen as the perturbation of a global Hamiltonian center. In this situation the algorithm introduced in [11], see also [14,16], allows to compute the first non-zero Melnikov function associated to the system. The positive simple zeros of this function control the limit cycles of the system that tend to the periodic orbits of the linear system.

**Note added in proof** Motivated by this paper systems (5.1) and (5.2) have been studied in [8].

## 6 Appendix

Some Lyapunov constants $V_k^{\{n\}}$ and $V_9^{\{4,6\}}$, for several particular systems of the forms (1.2) and (1.3).

$$\begin{aligned} V_5^{\{5\}} &= \frac{\pi}{8}(a_{14} + a_{32} + 5a_{50} + 5b_{05} + b_{23} + b_{41}), \\ V_7^{\{4\}} &= \frac{\pi}{64}(-7a_{04}a_{13} - 3a_{13}a_{22} - 3a_{04}a_{31} - 3a_{22}a_{31} - 3a_{13}a_{40} \\ &\quad -7a_{31}a_{40} - 28a_{04}b_{04} - 2a_{22}b_{04} - 2a_{13}b_{13} + 7b_{04}b_{13} - 2a_{04}b_{22} \end{aligned}$$

$$
\begin{aligned}
&+2a_{40}b_{22}+3b_{13}b_{22}+2a_{31}b_{31}+3b_{04}b_{31}+3b_{22}b_{31}+2a_{22}b_{40}\\
&+28a_{40}b_{40}+3b_{13}b_{40}+7b_{31}b_{40}),\\
V_{11}^{\{6\}} = &\frac{\pi}{512}(33a_{06}a_{15}+9a_{15}a_{24}+9a_{06}a_{33}+5a_{24}a_{33}+5a_{15}a_{42}+5a_{33}a_{42}\\
&+5a_{06}a_{51}+5a_{24}a_{51}+9a_{42}a_{51}+5a_{15}a_{60}+9a_{33}a_{60}+33a_{51}a_{60}\\
&+198a_{06}b_{06}+12a_{24}b_{06}+2a_{42}b_{06}+12a_{15}b_{15}+2a_{33}b_{15}-33b_{06}b_{15}\\
&+12a_{06}b_{24}+2a_{24}b_{24}-2a_{60}b_{24}-9b_{15}b_{24}+2a_{15}b_{33}-2a_{51}b_{33}\\
&-9b_{06}b_{33}-5b_{24}b_{33}+2a_{06}b_{42}-2a_{42}b_{42}-12a_{60}b_{42}-5b_{15}b_{42}\\
&-5b_{33}b_{42}-2a_{33}b_{51}-12a_{51}b_{51}-5b_{06}b_{51}-5b_{24}b_{51}-9b_{42}b_{51}\\
&-2a_{24}b_{60}-12a_{42}b_{60}-198a_{60}b_{60}-5b_{15}b_{60}-9b_{33}b_{60}-33b_{51}b_{60}).\\
V_{9}^{\{4,6\}} = &\frac{\pi}{1920}(135a_{06}a_{13}+189a_{04}a_{15}+91a_{15}a_{22}+33a_{13}a_{24}+45a_{06}a_{31}\\
&+27a_{24}a_{31}+63a_{04}a_{33}+57a_{22}a_{33}+229a_{15}a_{40}+183a_{33}a_{40}-a_{13}a_{42}\\
&+21a_{31}a_{42}+45a_{04}a_{51}+75a_{22}a_{51}+405a_{40}a_{51}-231a_{13}a_{60}\\
&-189a_{31}a_{60}+540a_{06}b_{04}-78a_{24}b_{04}-184a_{42}b_{04}-1194a_{60}b_{04}\\
&+1134a_{04}b_{06}+336a_{22}b_{06}+1194a_{40}b_{06}+84a_{15}b_{13}+18a_{33}b_{13}\\
&+189b_{06}b_{13}+30a_{13}b_{15}-405b_{04}b_{15}+30a_{06}b_{22}-12a_{24}b_{22}-46a_{42}b_{22}\\
&-336a_{60}b_{22}-75b_{15}b_{22}+84a_{04}b_{24}+46a_{22}b_{24}+184a_{40}b_{24}-21b_{13}b_{24}\\
&+46a_{15}b_{31}+12a_{33}b_{31}-30a_{51}b_{31}+231b_{06}b_{31}+b_{24}b_{31}-12a_{13}b_{33}\\
&-18a_{31}b_{33}-183b_{04}b_{33}-57b_{22}b_{33}-18a_{24}b_{40}-84a_{42}b_{40}-1134a_{60}b_{40}\\
&-45b_{15}b_{40}-63b_{33}b_{40}+18a_{04}b_{42}+12a_{22}b_{42}+78a_{40}b_{42}-27b_{13}b_{42}\\
&-33b_{31}b_{42}-46a_{13}b_{51}-84a_{31}b_{51}-229b_{04}b_{51}-91b_{22}b_{51}-189b_{40}b_{51}\\
&-30a_{22}b_{60}-540a_{40}b_{60}-45b_{13}b_{60}-135b_{31}b_{60}).
\end{aligned}
$$

## References


1. Andronov, A.A., Leontovich, E.A., Gordon, I.I., Maier, A.G.: Theory of Bifurcations of Dynamic Systems on a Plane. Wiley, New York (1967)
2. Bautin, N.N.: On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Mat. Sb. **30**(72), 181–196 (1952)
3. Bautin, N.N.: On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Am. Math. Soc. Transl. **100**, 397–413 (1954)
4. Caubergh, M.: Configurations of zeroes of analytic functions. C. R. Acad. Sci. Paris Sér. I Math. **333**, 307–312 (2001)
5. Caubergh, M.: Limit cycles near vector fields of center type, PhD thesis. Limburgs Universitair Centrum, Belgium (2004)
6. Caubergh, M., Dumortier, F.: Hopf-Takens bifurcations and centres. J. Differ. Equ. **202**, 1–31 (2004)
7. Caubergh, M., Dumortier, F.: Algebraic curves of maximal cyclicity. Math. Proc. Camb. Philos. Soc. **140**, 47–70 (2006)
8. Caubergh, M., Gasull, A.: Absolute cyclicity, Lyapunov quantities and center conditions. J. Math. Anal. Appl. **366**, 297–309 (2010)
9. Chavarriga, J., Giacomini, H., Giné, J., Llibre, J.: On the integrability of two-dimensional flows. J. Differ. Equ. **157**, 163–182 (1999)
10. Dulac, H.: Détermination et intégration d'une certain classe d'équations différentielle ayant pour point singulier un center. Bull. Sci. Math. Sér. (2) **32**, 230–252 (1908)

11. Françoise, J.P.: Successive derivatives of a first return map, application to the study of quadratic vector fields. Ergodic Theory Dyn. Syst. **16**, 87–96 (1996)
12. Frommer, M.: Über das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung rationaler Unbestimmheitsstellen. Math. Ann. **109**, 395–424 (1934)
13. Gasull, A., Torregrosa, J.: A new approach to the computation of the Lyapunov constants. The geometry of differential equations and dynamical systems. Comput. Appl. Math. **20**, 149–177 (2001)
14. Gasull, A., Torregrosa, J.: A relation between small amplitude and big limit cycles. Rocky Mt. J. Math. **31**, 1277–1303 (2001)
15. Giné, J., Santallusia, X.: On the Poincaré–Lyapunov constants and the Poincaré series. Appl. Math. (Warsaw) **28**, 17–30 (2001)
16. Iliev, I.D., Perko, L.M.: Higher order bifurcations of limit cycles. J. Differ. Equ. **154**, 339–363 (1999)
17. Ilyashenko, Yu.: Centennial history of Hilbert's 16th problem. Bull. Am. Math. Soc. (N.S.) **39**, 301–354 (2002)
18. Li, J.: Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Int. J. Bifurc. Chaos Appl. Sci. Eng. **13**, 47–106 (2003)
19. Pearson, J.M., Lloyd, N.G., Christopher, C.J.: Algorithmic derivation of centre conditions. SIAM Rev. **38**, 619–636 (1996)
20. Roussarie, R.: Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progress in Mathematics, vol. 164. Birkhäuser, Basel (1998)
21. Roussarie, R.: Melnikov functions and Bautin ideal. Qual. Theory Dyn. Syst. **2**, 67–78 (2001)
22. Żołądek, H.: Quadratic systems with center and their perturbations. J. Differ. Equ. **109**, 223–273 (1994)