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On the dynamics of a three-sector growth model

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Abstract Uzawa's two-sector growth model is extended into a three-sector model, where the labor growth rate is variable and bounded over time. The solution of this economic system is determined, as well as its long-run growth and asymptotic stability are investigated.

Keywords Three sector growth models · Variable population growth rate

JEL Classification O41

1 Introduction

The neoclassical growth model theory originated with the work of Solow (1956) and Swan (1956), who independently proposed similar one-sector models. By making particular simple assumptions about the behavior of consumption and labor supply, their model showed how growth in the capital stock, growth in the labor force, and advances in technology interact, and how they affect a nation's total output. However, as simple hypotheses go, the assumption of a constant labor growth rate is not a good approximation to reality. The main problem is that population grows exponentially, and so tends to infinity as time goes to infinity,

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which is clearly unrealistic. Guerrini (2006) investigated the question in the Solow–Swan model of what the impact of changes in the population growth rate would be. Two-sector extensions of the Solow–Swan growth model were introduced by Uzawa (1961, 1963), Meade (1961), Kurz (1963), and then generalized by others into neoclassical multiple sector models (see Morishima 1964). The two sectors corresponded to the production of the consumption good and the production of the investment good, respectively, and the main results focused on the uniqueness and stability of equilibrium. Similarly to what done for the Solow–Swan model, we would like to examine the consequences of relaxing the assumption of a constant population growth rate in neoclassical multi-sector models. This aim is not easy since the dynamical systems corresponding to these models are difficult to be treated, and in fact very little is known of multi-sector models with variable population growth rate. Inspired by the work of Kepenek et al. (2001), in this paper we consider an extension of Uzawa's two sector growth model, where capital goods are heterogeneous, and the labor growth rate is variable and bounded over time. There are two capital goods (productive and unproductive) and one consumption good. The consideration of an unproductive capital good is interesting for the analysis since it leads the model to be represented by two differential equations, one of which is well-known being similar to the differential equation studied by Guerrini (2006). Consequently, the model's dynamic is completely understood, and we are able to write down the solution for this economic system, and determine its long-run growth as well as its asymptotic stability.

2 Assumptions and model

The economy consists of three sectors: productive capital goods, unproductive capital goods, and consumption goods, subindexed by I, H and C, respectively. We assume that productive capital affects the production of productive and unproductive capital goods as well as of consumption goods; unproductive capital affects the production of unproductive capital goods only; consumption capital affects the production of consumption goods only, and may also be consumed. Let us assume Cobb–Douglas production functions of the form

$$Y_i = F_i(K_i, L_i) = K_i^{\alpha_i} L_i^{1-\alpha_i}, \quad i = C, I, H, \tag{1}$$

with $0 < \alpha_i < 1$, $\alpha_I < \alpha_H < \alpha_C$. Y_i is the output of the i-th sector, K_i and L_i the capital and labor used in the i-th sector, respectively. Let P_i be the price in the i-th sector, r be the returns to capital, w be the wage rate, K and L be the aggregate quantities of capital and labor, respectively. We have that factors are paid their marginal product in each sector, i.e.,

$$P_i \frac{\partial F_i}{\partial K_i} = r, \quad P_i \frac{\partial F_i}{\partial L_i} = w, \quad i = C, I, H, \tag{2}$$

and factor employment sums up to the economy's total employment, i.e.

$$K_C + K_I + K_H = K, \quad L_C + L_I + L_H = L. \tag{3}$$

Let $y_i = Y_i/L_i$, $k_i = K_i/L_i$, $f_i(k_i) = F(k_i,1)$, $l_i = L_i/L$. Moreover let $k = K/L$ and $\omega = w/r$. Since $\partial F_i/\partial L_i = f_i(k_i) - k_i f_i'(k_i)$ and $\partial F_i/\partial K_i = f_i'(k_i)$, Eqs. (1), (2) and (3) rewrite as

$$y_i = f_i(k_i) = k_i^{\alpha_i}, \quad i = C, I, H, \tag{4}$$

$$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i, \quad i = C, I, H, \tag{5}$$

$$l_C k_C + l_I k_I + l_H k_H = k, \quad l_C + l_I + l_H = 1. \tag{6}$$

The accounting identities of output are $Y = Y_C + (P_I/P_C)\, Y_I + (P_H/P_C)\, Y_H$, with P_I/P_C the price of the productive investment good in terms of the consumption good, and P_H/P_C the price of the unproductive investment good in terms of the consumption good. At any instant of time we suppose that

$$s_I Y = \frac{P_I}{P_C}\, Y_I, \quad s_H Y = \frac{P_H}{P_C}\, Y_H, \tag{7}$$

where s_I and s_H are exogenous savings rates for productive investment goods and unproductive investment goods. The demand for consumption goods is given from what is left, i.e., $(1 - s_I - s_H)Y$. Let the two stock variables grow according to the following rules

$$\dot{K} = Y_I - \delta_K K, \quad \dot{H} = Y_H - \delta_H H, \tag{8}$$

where δ_K and δ_H are the rates of depreciation of K and H, respectively. Physical capital stock, i.e., productive capital, grows only by production of investment goods. Unproductive capital, though it accumulates, behaves like a consumption good. The per capita version of Eq. (8) gives the system of differential equations which characterizes the model, i.e.,

$$\begin{cases} \dot{k} = l_I k_I^{\alpha_I} - [\delta_K + n(t)]k, \\ \dot{h} = l_H k_H^{\alpha_H} - [\delta_H + n(t)]h, \end{cases} \tag{9}$$

where $n(t) = \dot{L}/L$, which denotes the population growth rate, is assumed to be non-constant but variable and bounded over time. More precisely, let $n(t)$ be between prescribed upper and lower limits, i.e., $0 \le n(t) \le M$ for all t, and such that there exists $\lim_{t\to\infty} n(t) = n_\infty$. Moreover, we assume that today's population is given, $L(0) = 1$. In particular, $1 \le L(t) \le e^{Mt}$, for all t.

3 Some remarks

We are going to show some results which will allow expressing k_I and k_H as functions of k.

Lemma 1

$$\omega = \frac{1-\alpha_C}{\alpha_C}\, k_C = \frac{1-\alpha_I}{\alpha_I}\, k_I = \frac{1-\alpha_H}{\alpha_H}\, k_H. \tag{10}$$

Proof Immediate from (4) and (5).

Lemma 2

$$\frac{P_I}{P_C} = \frac{1-\alpha_C}{1-\alpha_I}\frac{y_C}{y_I}, \quad \frac{P_H}{P_C} = \frac{1-\alpha_C}{1-\alpha_H}\frac{y_C}{y_H}. \tag{11}$$

Proof It follows from (2), and from being $\partial F_i / \partial L_i = f_i(k_i) - k_i f_i'(k_i)$.

Proposition 1

$$l_I = \frac{(1-\alpha_I)s_I}{(1-\alpha_I)s_I + (1-\alpha_H)s_H + (1-\alpha_C)(1-s_I-s_H)} > 0, \tag{12}$$

$$l_H = \frac{(1-\alpha_H)s_H}{(1-\alpha_I)s_I + (1-\alpha_H)s_H + (1-\alpha_C)(1-s_I-s_H)} > 0. \tag{13}$$

Proof Equations (7), (11), and $Y = Y_C/(1 - s_I - s_H)$ give

$$\frac{s_I}{1-s_I-s_H} = \frac{P_I}{P_C}\frac{Y_I}{Y_C} = \frac{1-\alpha_C}{1-\alpha_I}\frac{y_C}{y_I}\frac{Y_I}{Y_C} = \frac{1-\alpha_C}{1-\alpha_I}\frac{l_I}{l_C}.$$

Consequently,

$$l_I = \frac{s_I}{1-s_I-s_H}\frac{1-\alpha_I}{1-\alpha_C}\, l_C. \tag{14}$$

Similarly, (7) and (11) imply

$$l_H = \frac{s_H}{1-s_I-s_H}\frac{1-\alpha_H}{1-\alpha_C}\, l_C. \tag{15}$$

Finally, from (6), (14), and (15), we have

$$l_C = \frac{(1-\alpha_C)(1-s_I-s_H)}{(1-\alpha_I)s_I + (1-\alpha_H)s_H + (1-\alpha_C)(1-s_I-s_H)}. \tag{16}$$

The statement now follows substituting (16) in (14), and in (15).

Proposition 2 Set

$$\overline{M}_1 = \frac{\alpha_I}{1-\alpha_I}\frac{(1-\alpha_I)s_I + (1-\alpha_H)s_H + (1-\alpha_C)(1-s_I-s_H)}{\alpha_C(1-s_I-s_H) + (\alpha_I+\alpha_H)s_H},$$

$$\overline{M}_2 = \frac{\alpha_H}{1-\alpha_H}\frac{(1-\alpha_I)s_I + (1-\alpha_H)s_H + (1-\alpha_C)(1-s_I-s_H)}{\alpha_C(1-s_I-s_H) + (\alpha_I+\alpha_H)s_H}.$$

Then

$$k_I = \overline{M}_1\, k, \quad k_H = \overline{M}_2\, k.$$

Proof Equation (10) yields

$$k_H = \frac{\alpha_H}{1-\alpha_H}\,\frac{1-\alpha_I}{\alpha_I}\,k_I, \quad k_C = \frac{\alpha_C}{1-\alpha_C}\,\frac{1-\alpha_I}{\alpha_I}\,k_I.$$

Therefore, it follows from (6) that

$$\left(\frac{\alpha_C}{1-\alpha_C}\,\frac{1-\alpha_I}{\alpha_I}\,l_C + l_I + \frac{\alpha_H}{1-\alpha_H}\,\frac{1-\alpha_I}{\alpha_I}\,l_H\right)k_I = k,$$

Now, by using (12), (13), and (16), we will be able to express k_I in term of k. Similarly the proof for k_H.

Remark 1 $\overline{M}_1$ and $\overline{M}_2$ are both positive. Moreover, $\overline{M}_1 \neq \overline{M}_2$ being $\alpha_I \neq \alpha_H$.

4 Dynamic of the system

Propositions 1 and 2 allow us rewriting the dynamical system (9) as

$$\begin{cases} \dot{k} = M_1 k^{\alpha_I} - [\delta_K + n(t)]k, \\ \dot{h} = M_2 k^{\alpha_H} - [\delta_H + n(t)]h, \\ k(0) = k_0 > 0,\ h(0) = h_0 > 0. \end{cases} \tag{17}$$

where $M_1 = l_I \overline{M}_1^{\alpha_I}$, and $M_2 = l_H \overline{M}_2^{\alpha_H}$. This Cauchy problem has a unique solution $(k(t),\ h(t))$ defined on $[0, +\infty)$ (see Hartman 1982).

Theorem 1 The differential equation $\dot{k} = M_1 k^{\alpha_I} - [\delta_K + n(t)]k$ is solved by

$$k(t) = \mathrm{e}^{-\delta_K t} L(t)^{-1}\left(k_0^{1-\alpha_I} + (1-\alpha_I)M_1\int_0^t \mathrm{e}^{(1-\alpha_I)\delta_K t} L(t)^{1-\alpha_I}\,\mathrm{d}t\right)^{1/(1-\alpha_I)}.$$

Proof The evolution of capital k is described by a non-linear differential equation, which is of Bernoulli type. The substitution $z = k^{1-\alpha_I}$ yields a linear differential equation in z, i.e. $\dot{z} = (1-\alpha_I)M_1 - (1-\alpha_I)[\delta_K + n(t)]z$, which is known to be solved by

$$z(t) = \mathrm{e}^{-\int_0^t (1-\alpha_I)[\delta_K + n(t)]\mathrm{d}t}\left(z_0 + \int_0^t (1-\alpha_I)M_1\,\mathrm{e}^{\int_0^t (1-\alpha_I)[\delta_K + n(t)]\mathrm{d}t}\,\mathrm{d}t\right).$$

Replacing $n(t) = \dot{L}(t)/L(t)$ yields

$$\int_0^t (1-\alpha_I)[\delta_K + n(t)]\mathrm{d}t = (1-\alpha_I)[\delta_K t + lnL(t)].$$

Therefore

$$z(t) = \left[\mathrm{e}^{\delta_K t}L(t)\right]^{-(1-\alpha_I)}\left(z_0 + (1-\alpha_I)M_1\int_0^t \mathrm{e}^{(1-\alpha_I)\delta_K t}L(t)^{1-\alpha_I}\mathrm{d}t\right).$$

The statement now follows expressing z in terms of k.

Theorem 2 The differential equation $\dot{h} = M_2 k^{\alpha_H} - [\delta_H + n(t)]h$ is solved by

$$h(t) = \mathrm{e}^{-\delta_H t}L(t)^{-1}\left(h_0 + M_2\int_0^t k(t)^{\alpha_H}\mathrm{e}^{\delta_H t}L(t)\mathrm{d}t\right).$$

Proof The evolution of capital h is described by a linear differential equation and so its solution is

$$h(t) = \mathrm{e}^{-\int_0^t [\delta_H + n(t)]\mathrm{d}t}\left(h_0 + \int_0^t M_2 k(t)^{\alpha_H}\,\mathrm{e}^{\int_0^t [\delta_H + n(t)]\mathrm{d}t}\mathrm{d}t\right).$$

The statement now follows by proceeding as in the previous proof.

Theorem 3 Let ($k(t)$, $h(t)$) be the solution of the Cauchy problem (17). Then

$$\lim_{t\to\infty}(k(t), h(t)) = \left(\left[\frac{M_1}{\delta_K + n_\infty}\right]^{\frac{1}{1-\alpha_I}}, \frac{M_2}{\delta_H + n_\infty}\left[\frac{M_1}{\delta_K + n_\infty}\right]^{\frac{\alpha_H}{1-\alpha_I}}\right). \tag{18}$$

Proof First, note that

$$\int_0^t \mathrm{e}^{(1-\alpha_I)\delta_K t}L(t)^{1-\alpha_I}\mathrm{d}t \geq \int_0^t \mathrm{e}^{(1-\alpha_I)\delta_K t}\mathrm{d}t = [\mathrm{e}^{(1-\alpha_I)\delta_K t} - 1]/(1-\alpha_I)\delta_K.$$

Thus, as t tends to infinity, the integral on the left of the above inequality diverges being divergent that on the right hand side. Next, Hopital's rule allows to evaluate the following limit

$$\lim_{t\to\infty} k(t)^{1-\alpha_I} = \lim_{t\to\infty}\frac{k_0^{1-\alpha_I} + (1-\alpha_I)M_1\int_0^t \mathrm{e}^{(1-\alpha_I)\delta_K t}L(t)^{1-\alpha_I}\mathrm{d}t}{[\mathrm{e}^{\delta_K t}L(t)]^{1-\alpha_I}} = \lim_{t\to\infty}\frac{M_1}{\delta_K + n(t)}.$$

Consequently,

$$\lim_{t\to\infty} k(t) = \left[\frac{M_1}{\delta_K + n_\infty}\right]^{\frac{1}{1-\alpha_I}}.$$

Now, again an application of Hopital's rule yields

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{h_0 + M_2 \int_0^t k(t)^{\alpha_H} e^{\delta_H t} L(t) dt}{e^{\delta_H t} L(t)} = \frac{M_2}{\delta_H + n_\infty}\left[\frac{M_1}{\delta_K + n_\infty}\right]^{\frac{\alpha_H}{1-\alpha_I}}.$$

Remark 2 If $n(t) = n$, i.e. the population growth rate is constant, then the dynamical system (17) has a unique non-trivial steady state, say (k_*^n, h_*^n), to which the economy converges in the long run, as it can be seen from (18) being now $n_\infty = n$. If the population growth rate is variable over time, then there are no steady states, but we still have convergence to a point in the long run. Recalling that $0 \le n(t) \le M$ for all t, we deduce that $0 \le n_\infty \le M$. In particular, for $n_\infty = 0$, $n_\infty = M$, we see that $k_*^M < k_*^0$, and $h_*^M < h_*^0$. Consequently, $\|(k_*^M, h_*^M)\| < \|(k_*^0, h_*^0)\|$. Thus, the lower is n_∞ the farther is the point where the economy stabilizes in the long run.

Remark 3 In case of a logistic population growth law, $\dot{L}(t) = aL(t) - bL(t)^2, a > b > 0$, where $a = (1 - \alpha_I)\delta_K/(1 + \alpha_I)$, and $\alpha_H = 1/2$, it is possible to show that the solution $(k(t), h(t))$ writes in closed form through the Hypergeometric function ${}_2F_1$ (see Whittaker and Watson 1927 for the definition of ${}_2F_1$).

5 Asymptotic stability of the solution

Let $(k(t), h(t); k_0, h_0)$ denote the solution of problem (17). We are going to prove that this solution is asymptotically stable. This means first that the solution is Lyapunov stable, i.e., all solutions which start close to (k_0, h_0) remain near the curve $(k, h) = (k(t), h(t); k_0, h_0)$ for all $t > 0$. Mathematically, this means, first, that for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$, such that, for every $t > 0$, the inequality $||(\hat{k}_0, \hat{h}_0) - (k_0, h_0)|| < \eta(\varepsilon)$ implies

$$||(k(t), h(t); \hat{k}_0, \hat{h}_0) - (k(t), h(t); k_0, h_0)|| < \varepsilon.$$

Second, there exists $\mu > 0$ such that if $||(\hat{k}_0, \hat{h}_0) - (k_0, h_0)|| < \mu$, then

$$\lim_{t\to\infty} \{(k(t), h(t); \hat{k}_0, \hat{h}_0) - (k(t), h(t); k_0, h_0)\} = 0.$$

Now, let us rewrite the system of differential equations (17) in matrix form. Setting

$$Z = \begin{bmatrix} k \\ h \end{bmatrix}, \quad B(t) = \begin{bmatrix} -\delta_K - n(t) & 0 \\ 0 & -\delta_H - n(t) \end{bmatrix}, \quad C = \begin{bmatrix} M_1 k^{\alpha_I} & 0 \\ 0 & M_2 k^{\alpha_H} \end{bmatrix},$$

we have

$$\dot{Z} = B(t)Z + C.$$

Theorem 4 The solution of problem (17) is asymptotically stable.

Proof Let $Z(t) = Z(t; k_0, h_0)$, and $\hat{Z}(t) = \hat{Z}(t; \hat{k}_0, \hat{h}_0)$. From Theorem 3, $\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \hat{Z}(t)$, and so $\lim_{t\to\infty}[\hat{Z}(t) - Z(t)] = 0$. This means that $\forall\varepsilon > 0$ there exists $T > 0$ such that $||\hat{Z}(t) - Z(t)|| < \varepsilon$ for $t > T$. In particular, $\forall\varepsilon > 0$ there exists $\eta_1 > 0$ such that $||\hat{Z}_0 - Z_0|| < \eta_1$ implies the inequality $||\hat{Z}(t) - Z(t)|| < \varepsilon$ when $t > T$. The purpose is now to show that this inequality is also true when $0 < t \leq T$. In order to do so, we start considering the following inequality

$$||\hat{Z}(t) - Z(t)|| \leq ||\hat{Z}_0 - Z_0|| + \int_0^t \left[||B(t)||||\hat{Z}(t) - Z(t)|| + ||\hat{C} - C||\right] \mathrm{d}t. \tag{19}$$

First, we note that

$$||B(t)|| = \delta_K + \delta_H + 2n(t) \leq \delta_K + \delta_H + 2M = N. \tag{20}$$

Second, we see by the Mean Value Theorem that

$$\begin{aligned} ||\hat{C} - C|| &= M_1|\hat{k}(t)^{\alpha_I} - k(t)^{\alpha_I}| + M_2|\hat{k}(t)^{\alpha_H} - k(t)^{\alpha_H}|, \\ &\leq (M_1\alpha_I\xi_1^{\alpha_I - 1} + M_2\alpha_H\xi_2^{\alpha_H - 1})|\hat{k}(t) - k(t)|, \end{aligned}$$

for some $\xi_1, \xi_2 \in (k(t), \hat{k}(t))$ if $k(t) < \hat{k}(t)$, or some $\xi_1, \xi_2 \in (\hat{k}(t), k(t))$ if $\hat{k}(t) < k(t)$. Consequently, known results on the dynamic of $k(t)$ (see Guerrini 2006) imply

$$||\hat{C} - C|| < \beta|\hat{k}(t) - k(t)| \leq \beta||\hat{Z}(t) - Z(t)||, \tag{21}$$

where β is some positive constant. Next, replacing (20) and (21) in (19) yields

$$||\hat{Z}(t) - Z(t)|| \leq ||\hat{Z}_0 - Z_0|| + \int_0^t \left[(N + \beta)||\hat{Z}(t) - Z(t)||\right] dt.$$

Now, Gronwall's inequality, and the fact that $t \leq T$ yield

$$||\hat{Z}(t) - Z(t)|| \leq ||\hat{Z}_0 - Z_0||\mathrm{e}^{\int_0^t (N+\beta)\mathrm{d}t} \leq ||\hat{Z}_0 - Z_0||\mathrm{e}^{(N+\beta)T}.$$

Therefore, if we set $\eta_2 = \varepsilon/\mathrm{e}^{(N+\beta)T}$, then $\forall\varepsilon > 0$ there exists $\eta_2 > 0$ such that $||\hat{Z}_0 - Z_0|| < \eta_2$ implies $||\hat{Z}(t) - Z(t)|| < \varepsilon$ when $t \leq T$. So, we can conclude that $\forall\varepsilon > 0$ there exists $\eta > 0$, $\eta = \min\{\eta_1, \eta_2\}$, such that $||\hat{Z}_0 - Z_0|| < \eta$ implies $||\hat{Z}(t) - Z(t)|| < \varepsilon$ for all $t > 0$, i.e. the statement.

6 Conclusions

In this paper, we have considered an extension of Uzawa's two sector growth model, where capital goods are heterogeneous, and the labor growth rate is non-constant but variable over time. This set-up led the model to be represented by a two-dimensional dynamical system in which one of its two equations can be explicitly solved. In this way, we could get some insight into the dynamics of the model like for example to determine the model's solution, investigate the long-run behavior of the economic system, and examine its asymptotic stability. In particular, we derived that countries with same initial per capita values grow toward the same point in the long run as well as small variations of the initial per capita values do not change very much the economic growth process of a country. Moreover, countries whose population growth rates converge to values n_∞ closer to zero will have a more efficient economic growth process than those who do not, and this because the lower the n_∞ the farther the point where the economy stabilizes in the long run. Thus, it is important for a country to have an efficient population control policy.

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