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STABILITY AND BIFURCATIONS IN *IS-LM* ECONOMIC MODELS

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Abstract: *In this note, we analyze the local dynamics of a general non-linear fixed-price disequilibrium IS-LM model. We assume investment behavior as a general non-linear function avoiding any Kaldor type assumption. By proving the existence of a family of periodic solutions bifurcating from a steady state, we confirm and extend some results in the literature for IS-LM models reducible to Leinard's equation. We use bifurcation theory and study the effect of a change of the adjustment parameter in the money market upon the solutions of the model as the steady state loses stability.*

We establish analytically that the values of the adjustment parameter in the money market may affect the equilibrium relative to the product market and the government budget constraint. (JEL: *C62, E32*)

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1. *Introduction*

In the past two decades economists devoted particular attention to study phenomena like cycles and complex dynamics related with the existence of indeterminate equilibria in financial markets and global economics (see inter al.: Lorenz, 1989; Benhabib, 1992; Jarsulic, 1993; Benhabib and Perli, 1994;

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Mattana and Venturi, 1999; Fiaschi and Sordi, 2002; Cai, 2005).

In this paper, we analyze the dynamics of a general non-linear fixed price disequilibrium *IS-LM* model with standard markets of goods and money including also a government budget constraint (see Torre, 1977; Schinasi, 1982; Cai, 2005) and with investment behavior as a non-linear function avoiding any Kaldor assumption. We establish analytically that the values of the adjustment parameter in the money market may affect the equilibrium relative to the product market and the government budget constraint. We study the effect of a change of the adjustment parameter in the money market upon the solutions of the model as the steady state loses stability.

By using the Hopf bifurcation theorem (see Guckenheimer and Holmes, 1983) we prove the occurrence of a limit cycle bifurcation.

From a mathematic point of view, we consider a system of three first order non-linear differential equations. In a system whose dimensions are higher than two since the Poincaré-Bendixson is not applicable, the Hopf bifurcation theorem may constitute the only tool for the analysis of cyclical equilibria. We confirm and extend some results usually given in the literature for the disequilibrium systems reducible to forced oscillators or recently formalized as time delay models (see inter al.: Torre, 1977; Schinasi, 1982; Lorenz, 1989; Benhabib, 1992; Medio, 1992; Jarsulic, 1993; De Cesare and Sportelli, 2005; Cai, 2005).

We discuss some economic implications of these oscillating solutions.

The plan of the paper is as follows. Section 2 starts with a very brief presentation of *IS-LM* arguments and discusses some of Schinasi and Torre results. In Section 3 we elaborate a dynamical analysis of the model around the steady state. In Section 4 we choose the adjustment parameter in the money market as a bifurcation parameter[1] and we are able to show the existence of a limit cycle through a Hopf Bifurcation avoiding any Kaldor assumption for the investment function. In the last section, we present some remarks and conclusions.

2. *The General Model*

We shall consider a generalization of a continuous, fixed-price disequilibrium *IS-LM* model (see Schinasi, 1982; Torre,1977; Lorenz, 1989; Jarsulic,1993; Amendola and Gaffard, 1998).

[1] The use of the externality as a bifurcation parameter represents an interesting shift with respect to the classical literature on bifurcations in capital accumulation models where the discount rate has been traditionally privileged.

Specifically, our "generalized model" is given by the non-linear system:

$$\dot{y} = \alpha\left[I(r,y) - S(y^D,w) + G - T(y+B)\right] \tag{2.1}$$

$$\dot{r} = \mu\left[L(r,y,w) - m\right] \tag{2.2}$$

$$\dot{m} = G + B - T(y) \tag{2.3}$$

in the independent state variables *r*, *y* and *m*.

Here, equation (2.1) describes the traditional disequilibrium of dynamic adjustment in the product market; (2.2) describes the corresponding disequilibrium in the money market, and (2.3) represents the governmental budget constraint. The standard *IS-LM* model is described by the markets of goods and money (equations (2.1) and (2.2), see Torre, 1977). The crucial features that distinguishes this specification from most specifications of *IS-LM* models are the general form of the dynamic macro-model (different from the Leinard's equation) and the nonlinear investment functions.

The quantities *I*, *S*, *T*, *G*, *r*, *y*, *B* and *w* represent (respectively): investment, savings, tax collections, government expenditure, interest rate, output (income), interests payment on perpetuities, and wealth. In (2.2), *L* represents the liquidity preference function and *m* is the constant real money supply.

For simplicity, prices are fixed at unity.

Here, $\alpha > 0$ and $\mu > 0$ are the adjustment parameters in their respective markets. Moreover, we assume that *G* and *B* are positive constants. Next, we define the disposable income y^D and the wealth *w* as follows:

$$y^D = y + B - T(y+B) \tag{2.4}$$

$$w = m + \frac{B}{r} \tag{2.5}$$

Clearly, we always have:

$$\frac{\partial w}{\partial r} = w_r = -\frac{B}{r^2} < 0 \tag{2.6}$$

(Throughout the paper, the symbols f_r, g_r etc. denote the first partial derivative in the respective variables).

Finally, for simplicity, we assume that the functions: I, S, T are of class C^2 (twice continuously differentiable) in all their arguments: r, y, w and $\boldsymbol{m}$.

In our formulation, we completely avoid any Kaldor (sigmoid) assumption on the investment function I, (Schinasi,1982; Cai, 2005).

Recall that a stationary (equilibrium) point of our system is any solution of

$$\dot{y} = \dot{r} = \dot{m} = 0$$

Assuming the existence of such a solution at some point $P^* = (y*, r*, m*) \in \Re^3_+$ we want to analyze its local properties (e.g. stability, etc.) around P^*.

3. *Steady State Analysis*

It is well-known that the local dynamical properties of a non-linear system, at a hyperbolic equilibrium point P^*, are described in terms of the Jacobian matrix $J(P^*) \equiv J^*$, for brevity. In fact, the nature of the eigenvalues of J^* plays a key role.

At a generic point $P = (r, y, m)$, given the order of the variables, we shall rewrite our system in the equivalent form

$$\dot{r} = \mu\left[L(r, y, w) - m\right] \equiv \mu f(r, y, m) \tag{3.1}$$

$$\dot{y} = \alpha\left[I(r, y) - S\left(y^D(y), w\right) + G - T(y + B)\right] \equiv \alpha g(r, y, m) \tag{3.2}$$

$$\dot{m} = G + B - T(y) \equiv h(y) \tag{3.3}$$

with h: $R \rightarrow R$.

Since G and B are fixed, a choice of the policies implies

$$h'(y) = -T_y$$

because all deficit must be financed either by creation of money or by creation of new debt.

Remark1. Unlike Schinasi (1982), we allow the functions L and S (liquidity and savings) to depend on wealth w.

The monotonicity of the restrictions:

$$r \to I(.,.,r),\ r \to L(.,.,r),\ w \to L(.,.,w),\ y \to S(.,y,.) \text{ and } w \to S(.,.,w)$$

is assumed and these assumptions imply the (economic) conditions:

$$I_r > 0,\ L_r > 0,\ L_w > 0,\ S_y > 0 \text{ and } S_w > 0$$

Moreover, since $T(y)$ is increasing, we may assume for convenience that

$$0 < T_y < 1$$

The Jacobian matrix J of our new system is

(3.4) $$J(r,y,m) = \begin{pmatrix} \mu f_r & \mu f_y & \mu f_m \\ \alpha g_r & \alpha g_y & \alpha g_m \\ 0 & -T_y & 0 \end{pmatrix}$$

with determinant

(3.5) $$DetJ = T_y \alpha\mu\ (f_r g_m - g_r f_m)$$

and trace

(3.6) $$TrJ = \mu f_r + \alpha g_y$$

Simple computations, together with (2.6) and formula (2.4), yield

(3.7) $$f_r = (L_r + L_w w_r) = L_r - \frac{BL_w}{r^2} < 0 \text{ always}$$

(3.8) $$f_y = L_y \text{ and } f_m = L_w w_m - 1 = L_w - 1$$

(3.9) $$g_r = \left(I_r - S_w w_r\right) = I_r + \frac{BS}{r^2} < 0$$

(3.10) $$g_y = I_y - \left[S_{y^D}\right]\left[\left(y^D\right)_y\right] - T_y = I_y - \left\{T_y + S_{y^D}\left[1 - T_y\right]\right\}$$

lastly $g_m = -S_m w_m = -S_m < 0$ always.

Another invariant to be considered is the sum, B (J), of the principal minors of J:

(3.11) $$B(J) = -\alpha\, S_w T_y + \alpha\mu\left(f_r g_y - g_r f_y\right)$$

Theorem 1. A hyperbolic stationary point $P^* \in R_+^3$ of the system (3.1), (3.2), (3.3) is locally asymptotically stable if the following assumptions hold at P^*:

(3.12) $\mu \geq \left(-f_r\right)^{-1}\left[\alpha\, g_y + N\right]$, with N given by (3.13) at P^*

i) the marginal propensity to invest out of income is greater than unity

$$I_y - S_y\left[1 - T_y\right] > T_y \text{, i.e.; } g_y > 0 \text{ at } P^*$$

ii) $S_w f_r > (1 - L_w) g_r$ and $0 < L_w < 1$, so $g_r < 0$ at P^*

iii) $$\left(f_r g_y - g_r L_y\right) > \mu^{-1} S_w T_y$$

Proof. Recall that our system is locally asymptotically stable at P^* if $a_1 > 0$, $a_2 > 0$, and $a_1 a_2 - a_3 > 0$ where

$$a_1 = -TrJ^*,\ \ a_2 = B(J^*)\ \ \ a_3 = -DetJ^* \ \text{ and } \ J^* = J(P^*)$$

Since, from equation (3.8) follows that $f_y = L_y$, $B(J) > 0$ if and only if $\mu\alpha\left(f_r g_y - g_r f_y\right) > \alpha S_W T_y$. Thus, $a_2 = B(J^*) > 0$ if and only if iii) holds at point P^*. Moreover, since $T_y > 0$, (3.5) shows that $DetJ < 0$ if and only if $f_r g_m < g_r f_m$. Substituting for f_m and g_m using (3.8) and (3.10), we see that also $a_3 > 0$ if and only if $S_w f_r > (1 - L_w) g_r$ which is equivalent to the first inequality in ii) at $P = P^*$.

Since by (3.7) $S_w f_r < 0$, the (economic) assumption that $L_w < 1$ is then equivalent to having $g_r < 0$ at P^*.

Moreover, $a_2 > 0$ and $a_3 > 0$ imply, by the Archimedean Principle, that there exists a positive integer N defined by

(3.13) N = the first integer such that $Na_2 > a_3$, at P^*

Clearly, the desired conclusions follow if we show that assumption (3.12), at P^*, implies that $a_1 \geq N$.

But (3.13) is equivalent to $\alpha g_y + N \leq -\mu f_r$. That is, at P^*,

$$TrJ^* = \alpha g_y + \mu f_r \leq -N$$

So, $a_1 = -TrJ^* \geq N$ Q.E.D.

Corollary 1. Let the hypotheses of theorem 1 hold at a hyperbolic stationary point of the system from (3.1) to (3.3). Then the steady state for each $\alpha > 0$ and all $\mu \geq (-f_r)^{-1}[\alpha g_y(P^*) + N] \equiv \mu_\alpha(P^*) > 0$ is locally asymptotically stable.

Remark 2. By i) of Theorem 1, $g_y > 0$ is equivalent to $I_y > S_y\left[1 - T_y\right] + T_y > 0$ by (3.10). Thus, as expected, the marginal propensity to invest I_y is positive at P^*.

Moreover, as we saw earlier, hypothesis ii) implies the condition $g_r(P^*) < 0$. Since $g_r = I_r - S_w W_r$, it follows that

$$I_r < S_w W_r < 0 \text{ by (2.5) and (2.6).}$$

Therefore, on some neighborhood of P^*, the investment function I satisfies

(3.14) $$I_y > 0 \text{ and } I_r > 0$$

4. *Bifurcation*

Corollary 2. Assume Corollary 1 and let $J^* = J(P^*)$ as before. Then, there exists a value $\mu = \hat{\mu} > 0$ for which J^* has a pair of purely imaginary eigenvalues.

Proof. Recall that our assumptions imply, for all $\mu \geq \mu_\alpha(P^*)$, the conditions:

(4.1) $$a_1 = -TrJ^* > 0\,,\ a_2 = B(J^*) > 0\,, \qquad a_1\, a_2 > a_3$$

where $a_3 = -DetJ^* > 0$ also.

On the other hand, $TrJ^* = \mu\, f_r + \alpha\, g_y > 0$ with $g_y > 0$, by i) and $\mu\, f_r < 0$. Thus, $TrJ^* > 0$, if and only if $\alpha\, g_y > (-f_r)\,\mu$, that is,

$$0 < \mu < \frac{\alpha}{2}\left[g_y(P^*) \Big/ -f_r(P^*) \right]$$

Hence, for each $\alpha > 0$, we will have $a_1 < 0$ for all small enough μ.

Since (4.1) is also equivalent to the property that all eigenvalues of J^* have a negative real part and since $a_1 < 0$ for all (small) μ as above, it follows that some eigenvalue of J^* must have a positive real part. Hence, by continuity of the eigenvalue, there exists some adept quota $\hat{\mu} > 0$, such that the corresponding eigenvalue λ satisfies $\mathrm{Re}\{\lambda\} = 0$.

However, since the trace is real, either the real eigenvalues $\lambda = \lambda_1 = 0$ or else $\lambda = x \pm i\,y$ with $x = 0$. But $\lambda_1 = 0$ implies that $a_3 = 0$ contradicting the fact that $Det\,J^*$ cannot vanish as μ varies, because its sign is independent of μ. Therefore, at $\mu = \hat{\mu}$, $\lambda = \pm i y$ as desired. Q.E.D.

Our final result is as follows.

Theorem 2. Assume the hypotheses of Theorem 1 except that (3.12) is now replaced by

$$(4.2) \qquad 0 < \mu < \frac{\alpha}{2}\left[g_y(P^*) \Big/ -f_r(P^*) \right] \qquad \text{at } \mu = \hat{\mu}.$$

Then, there exists a continuous function $\mu(\delta)$ with $\mu(0) = \hat{\mu}$, and for all small enough $\delta \neq 0$ there exists a continuous family of non-constant positive periodic solutions

$$[r^*(t,\delta),\ y^*(t,\delta),\ m^*(t,\delta)]$$

for the dynamical system (3.1), (3.2), (3.3) which collapses to the stationary point P^* as $\delta \to 0$.

Proof. The proof follows directly from the Hopf Bifurcation Theorem; see Guckenheimer and Holmes, (1983). By definition, our system is sufficiently smooth, and we assumed the existence of a stationary point P^*, independent of μ. Moreover, by Corollary 2, the Jacobian J^* has a pair of purely imaginary eigenvalues $\lambda = \pm i y$ at $\mu = \hat{\mu}$, for some $\hat{\mu} > 0$. It follows from $DetJ^*$ different from zero for $0 < \mu < \frac{\alpha}{2}\left[g_y(P^*) \big/ -f_r(P^*) \right]$ that the real eigenvalues are always different from zero.

In order to apply Hopf's Theorem, it remains to verify that, at $\mu = \hat{\mu}$, $\mathrm{Re}\{\lambda\} = 0$ is not stationary with respect to μ; that is,

$$\frac{d\,\mathrm{Re}\{\lambda\}}{d\mu} \neq 0 \ \text{ at } \ \mu = \hat{\mu} \ \text{ and } \ P = P^*$$

Following Benhabib and Miyao (1981), this last condition is equivalent to

$\dfrac{d(a_1 a_2 - a_3)}{d\mu} \neq 0$ at $\mu = \hat{\mu}$ and $P = P^*$ where $a_1 a_2 - a_3 = 0$ and the a_i are defined as in (4.1).

Claim. Our assumptions imply that

$$\frac{d(a_1 a_2 - a_3)}{d\mu} < 0 \text{ at } \mu = \hat{\mu} \text{ and } P = P^*$$

By definition of a_1 and (3.6), $\dfrac{da_1}{d\mu} = -f_r$. Also $a_2 = B(J^*)$ is given by (3.11) at $P = P^*$. Therefore,

$$\left[\frac{da_1}{d\mu}\right] a_2 = \alpha f_r \left\{S_w T_y - \mu\left[f_r g_y - g_r L_y)\right]\right\}.$$

Similarly, we have

$$a_1\left[\frac{da_2}{d\mu}\right] = -\alpha\left\{[\mu f_r + \alpha g_y][f_r g_y - g_r L_y]\right\}$$

and

$$-\left[\frac{da_3}{d\mu}\right] = \left[\frac{dDet(J^*)}{d\mu}\right] = -\alpha S_w T_y f_r - \alpha T_y g_r (L_w - 1)$$

where we used also (3.11) and (3.10).

So, adding up and canceling the term $\alpha S_w T_y f_r$, yields

$$\frac{d(a_1 a_2 - a_3)}{d\mu} = -\left[2I_1 + I_2 + I_3\right]$$

where

$$I_1 = \alpha\mu f_r [f_r g_y - g_r L_y]$$

$$I_2 = \alpha^2 g_y [g_y f_r - g_r L_y]$$

and

$$I_3 = \alpha T_y g_r (L_w - 1)$$

By (3.9) and assumption ii) of Theorem 1, it is clear that $I_3 > 0$ at P^*. Therefore, to prove our Claim, it suffices to show that $2I_1 + I_2 > 0$ also. By assumption iii) of Theorem 1, the common factor $[g_y f_r - g_r L_y] > 0$ at P^*. Thus dividing by this factor, $2I_1 + I_2 > 0$ if and only if $2\mu\alpha f_r + \alpha^2 g_y > 0$.

Finally, since $f_r < 0$, this last condition is equivalent to (4.2).This verifies the Claim and the Proof of Theorem 2 is complete. Q.E.D.

5. *Conclusions*

We studied the occurrence of a cycle in a "general" three-dimensional dynamic version of the *IS-LM* model, with standard markets of goods and money including also a government budget constraint. We have modeled investment behavior as a general non-linear function without introducing any Kaldor type assumption (see Torre, 1977; Schinasi, 1982; Cai, 2005) and avoiding the Schinasi "strong" assumption that the money market adjusts very quickly to equilibrium relative to the product market (see Torre, 1977; Schinasi, 1982).

In order to understand how the control parameters of the model might be varied by governments and market mechanisms to predict the occurrence of a cycle (see Cai, 2005; De Cesare and Sportelli, 2005), as mathematical tool (Johnson, 1991) we have used the Hopf bifurcation theorem. We have proved the existence of a limit cycle bifurcation for some value of the adjustment parameter in the money market.

It follows from the elaborated analysis that the orbit emanating from the bifurcation will be locally attractive (locally stable) if it is super-critical and will be locally repellent (locally unstable) if it is sub-critical.

Whether the super-critical or sub-critical case holds depends on the higher order non-linear terms in the Taylor expansion of the dynamical system at the stationary point (see Mattana and Venturi, 1999).

From an economic point of view, sub-critical or supercritical orbits are both reasonable. Since the third real root of the Jacobian matrix is negative, the existence of a super-critical Hopf bifurcations becomes very interesting in the analysis of macroeconomic fluctuations for an *IS-LM* model. A stable economy, by the increase or decrease of its control parameters, could be destabilized into a stable cycle in the dynamic of R^3 (see also Benhabib and Nishimura, 1979; Benhabib, 1992; Johnson, 1991; Mattana and Venturi, 1999; De Cesare and Sportelli, 2005; Cai, 2005).

The sub-critical Hopf bifurcations may correspond to the *Keynesian corridor* (Leijonhufvud, 1973): the economy has stability inside the corridor while it will loose stability outside the corridor.

In such a case the dynamics are either converging to an equilibrium point or the trajectories go somewhere else, and it is also possible that another attracting set exists, but often the alternative is diverging trajectories.

We have seen that fluctuations derive from the mechanisms through which money markets reflect and respond to the developments in the real economy.

Our analysis provides an example of the classical thesis concerning endogenous explanations of the existence of fluctuations in some real world economic variables.

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