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The Projectivity of Compact Kähler Manifolds with Mixed Curvature Condition

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Abstract
In a recent paper, Li–Ni–Zhu study the nefness and ampleness of the canonical line bundle of a compact Kähler manifold with $\mathrm{Ric}_k \leqslant 0$ and provide a direct alternate proof to a recent result of Chu–Lee–Tam. In this paper, we generalize the method of Li–Ni–Zhu to a more general setting which concerning the connection between the mixed curvature condition and the positivity of the canonical bundle. The key point is to do some a priori estimates to the solution of a Mong-Ampère type equation.

Keywords Canonical line bundle · Kähler manifolds · k-Ricci · Mixed curvature · A priori estimates

Mathematics Subject Classification 32Q10 · 32Q15 · 32W20 · 53C55

1 Introduction

There has been great interest in studying the connection between the curvature and properties of the manifold. In early 1970s, a conjecture of Yau asserts that the holomorphic sectional curvature determines the Ricci curvature in the following sense.

Conjecture 1.1 (Yau) *Let M be a projective manifold with a Kähler metric of negative holomorphic sectional curvature. Then its canonical line bundle K_M is ample.*

From the viewpoint of algebraic geometry, the abundance conjecture predicts that the canonical bundle is semiample if it is nef. From the viewpoint of hyperbolic geometry, there is also a conjceture due to Kobayashi [11].

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Conjecture 1.2 (Kobayashi) *If a compact complex manifold M is hyperbolic (in the sense of Kobayashi), then its canonical line bundle K_M is ample.*

Here a compact complex manifold M is hyperbolic (in the sense of Kobayashi) if and only if every holomorphic mapping $f : \mathbb{C} \to M$ is constant. Note that any compact Kähler manifold with holomorphic sectional curvature bounded from above by a negative constant is always Kobayashi hyperbolic.

In [25], Wong give an affirmative answer to Yau's conjecture (Conjecture 1.1) for Kähler surfaces. For higher dimension, Heier et al. [8] give an affirmative answer by assuming the validity of the abundance conjecture (which is known to hold for $n \leqslant 3$). For projective manifolds with Picard number 1, Wong et al. [26] generalize the result which only assume holomorphic sectional curvature to be non-positive on M and negative at some points. In this case, we say that the holomorphic sectional curvature is quasi-negative.

For moere general case, the conjecture was proved completely by Wu and Yau in [27]. Later on, Tosatti and Yang [23] extend Wu-Yau's result to all Kähler manifolds, not necessarily projective. Thus, the results of Wu-Yau and Tosatti-Yang can also be seen as a weak confirmation of the Kobayashi's conjecture (Conjecture 1.2) for Kähler manifolds. Diverio and Trapani [7] and Wu and Yau [28] further generalize the result to quasi-negative case. In the strictly negative case, Nomura [19] also give an alternative proof by means of the Kähler-Ricci flow. Recently, Zhang and Zhang-Zheng also consider more general cases and introduce a notion of almost non-positive (or quasi-negative) holomorphic sectional curvature. In [31], Zhang prove that a compact Kähler manifold of almost non-positive holomorphic sectional curvature has a nef canonical line bundle, contains no rational curves and satisfies some Miyaoka-Yau type inequalities. In [32], Zhang-Zheng extend Diverio-Trapani & Wu-Yau's result on quasi-negative case to compact Kähler manifolds of almost quasi-negative holomorphic sectional curvature.

On the other hand, it is also natural to ask what one can say in the positive case. When the holomorphic sectional curvature is positive there is also a conjecture by Yau in his Problem section.

Conjecture 1.3 (Yau, Problem 47, Problem section) *If M has a Kähler metric with positive holomorphic sectional curvature, then M is a projective and rationally connected manifold.*

A projective manifold X is called rationally connected if any two points of X can be connected by some rational curve. Heier and Wong [9] confirm Yau's conjecture 1.3 in the special case when X is projective. In [29], Yang give an affirmative answer to this conjecture for all Kähler manifolds, not necessarily projective.

There are also some other curvature notions related to the positivity of the canonical bundle. In 2018, Ni [16] introduced the notion of Ric_k in the study of the k-hyperbolicity of compact Kähler manifolds. A compact Kähler manifold M is defined to be k-hyperbolic if and only if any holomorphic map $f : \mathbb{C}^k \to M$ must be degenerate somewhere. The curvature notion Ric_k is defined as the Ricci curvature of the k-dimensional holomorphic subspaces of the holomorphic tangent bundle $T^{(1,0)}M$. One can regard Ric_k as an interpolation between holomorphic sectional curvature H

and the Ricci curvature Ric of M. We will discuss these curvature notions for more details in the next section. In [16], Ni establish the k-hyperbolicity on Kähler manifolds with $\mathrm{Ric}_k < 0$. Ni then ask if a Kähler manifold with negative Ric_k is projective? In the positive case, it was also proved by Ni [17] that a compact Kähler manifold with $\mathrm{Ric}_k > 0$ for some $1 \leqslant k \leqslant n$ must be projective and rationally connected. This generalizes the result of Yang (for k = 1) and the result of Campana [5] and Kollár et al. [12] (for k = n). Ni-Zheng also assert in [18] that any compact Kähler manifold with the second scalar curvature $S_2 > 0$ (the average of Ric_2) must be projective. Li [14] establish a vanishing theorem for uniformly RC k-positive Hermitian holomorphic vector bundles, and show that the holomorphic tangent bundle of a compact complex manifold equipped with a positive k-Ricci curvature Kähler metric (or more generally a positive k-Ricci curvature Kähler-like Hermitian metric) is uniformly RC k-positive.

Recently, Chu et al. [6] give an affirmative answer to Ni's question in [16] concerning the projectivity of a compact Kähler manifold M with $\mathrm{Ric}_k < 0$ for some integer k with $1 < k < n$. The result can be stated as follows.

Theorem 1.4 *Assume that* (M^n, ω) *is a compact Kähler manifold* ($n = \dim_{\mathbb{C}}(M)$) *with* $\mathrm{Ric}_k \leqslant -(k+1)\sigma$ *for some* $\sigma \geqslant 0$ *and some integer* $1 \leqslant k \leqslant n$*. Then* K_M *is nef and is ample if* $\sigma > 0$.

The above result generalizes the earlier work of Wu and Yau [27] and Tosatti and Yang [23]. The proof of Chu et al. in [6] is via the study of a twisted Kähler-Ricci flow. In [13], Li-Ni-Zhu provide an alternate proof by studying the a priori estimates of the Aubin-Yau solution [1, 30] to a complex Monge-Ampère type equation. In [21], Tang prove that a compact Kähler manifold of almost nonpositive k-Ricci curvature must have nef canonical line bundle. In [3], Broder-Tang introduced the concept of weighted orthogonal Ricci curvature and investigated its implications on the projectivity of Kähler manifolds, demonstrating several vanishing theorems in the process. In [15], Li also investigate the curvature operator of the second kind on Kähler manifolds. For the quasi-negativity case, Broder and Tang [4] consider the notion of (ε, δ)-quasi-negativity and obtain gap-type theorems for $\int_X c_1(K_X)^n > 0$ in terms of the real bisectional curvature and weighted orthogonal Ricci curvature.

As indicated in [13], their method may also prove effective in a broader context as discussed in [6] where the approach involves studying a twisted Kähler-Ricci flow. In this paper, we try to confirm this fact and give a complete proof in details. Borrowing the idea in [13] and modifying the techniques therein, we give a direct proof to the following result by using the a priori estimates to a Monge-Ampère type equation.

Theorem 1.5 *Let* (M, ω) *be a compact Kähler manifold. Suppose that*

$$\alpha |X|^2 \mathrm{Ric}(X, \overline{X}) + \beta R(X, \overline{X}, X, \overline{X}) \leqslant -\gamma |X|^4, \forall X \textit{ of } (1,0)\textit{-type}, \tag{1.1}$$

for some constants $\alpha, \beta > 0$ *and* $\gamma \geqslant 0$*. Then the canonical bundle* K_M *of* M *is nef. Moreover,* K_M *is ample if* $\gamma > 0$.

We shall call (1.1) the mixed curvature condition. Note that the assumption in Theorem 1.4 is stronger than the mixed curvature condition (1.1) in Theorem 1.5 (see [6]).

The rest of the paper is organized as follows: In Sect. 2, we will collect some basic notations and results in Kähler geometry to be used later. In Sect. 3, as an application of the a priori estimates for the solution of a Monge-Ampère type equation we will prove the nefness and ampleness of the canonical line bundle under the mixed curvature conditon.

2 Preliminaries

In this section, we collect some basic notations and results in Kähler geometry. One can find more details in the literature (e.g. [10, 22, 24]). Let (M, ω) be a compact n-dimensional Kähler manifold. Denote its Riemannian metric and Chern connection by g and ∇ respectively. Then we can define the curvature tensor R.

Definition 1 The curvature tensor R of ∇ is defined by

$$\begin{aligned} \mathrm{R}(X,Y)Z &:= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ \mathrm{R}(X,Y,Z,W) &:= g\left(\mathrm{R}(X,Y)Z, W\right), \end{aligned}$$

where $X, Y, Z, W \in TM$.

Let J be the induced almost complex structure on M. Then the curvature R satisfies a number of symmetries.

Proposition 2.1 *For any real vectors $X, Y, Z, W \in TM$, we have*

- $R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z)$.
- $R(X, Y)JZ = JR(X, Y)Z$.
- $R(X, Y, JZ, JW) = R(X, Y, Z, W) = R(X, Y, JZ, JW)$.

The holomorphic sectional curvature H is defined as

Definition 2

$$H_p(V) = \frac{R(V, \overline{V}, V, \overline{V})}{|V|_g^4},$$

for $V \in T_p^{(1,0)}(M) \setminus \{0\}$.

Under the local complex coordinates $(z^1, \cdots, z^n)$, we write

$$g_{i\overline{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right), \quad \{g^{i\overline{j}}\} = \{g_{i\overline{j}}\}^{-1}.$$

Here $g^{i\overline{j}} g_{k\overline{j}} = \delta_{ik}$. Then the Kähler form ω is given by

$$\omega = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz^i \wedge d\overline{z}^j.$$

We define the Christoffel symbols Γ^i_{jk} by

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^k} = \Gamma^i_{jk} \frac{\partial}{\partial z^i}.$$

Then we have

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial \overline{z}^k} = \nabla_{\frac{\partial}{\partial \overline{z}^j}} \frac{\partial}{\partial z^k} = 0,$$
$$\nabla_{\frac{\partial}{\partial \overline{z}^j}} \frac{\partial}{\partial \overline{z}^k} = \Gamma^{\bar{i}}_{\overline{jk}} \frac{\partial}{\partial \overline{z}^i} = \overline{\Gamma^i_{jk}} \frac{\partial}{\partial \overline{z}^i},$$

and

$$\left\{ \Gamma^i_{jk} = g^{i\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial z^j} \right\}.$$

We use the following notion for the curvature tensor:

$$R_{i\bar{j}k\bar{l}} := R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \overline{z}^l} \right), \text{etc.}$$

Direct computations show that the curvature tensor R can be represented as

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -\, g_{m\bar{l}} \frac{\partial \Gamma^m_{ik}}{\partial \overline{z}^j} \\ &= -\, \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \overline{z}^j} + g^{s\bar{t}} \frac{\partial g_{s\bar{l}}}{\partial z^i} \frac{\partial g_{k\bar{t}}}{\partial \overline{z}^j}. \end{aligned}$$

We define the Ricci curvature

$$\mathrm{Ric}(\omega) := \sqrt{-1} R_{k\bar{l}} dz^k \wedge d\overline{z}^l$$

to be the trace of R, so we get

$$R_{k\bar{l}} = g^{i\bar{j}} R_{k\bar{l}i\bar{j}} = -\frac{\partial^2}{\partial z^k \partial \overline{z}^l} \left(\log \det g_{i\bar{j}} \right).$$

Therefore $\mathrm{Ric}(\omega)$ is a closed real (1, 1)-form. Let $\tilde{\omega}$ be another Kähler metric, then

$$\begin{aligned} \mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde{\omega}) &= \sqrt{-1} \partial \overline{\partial} \log \frac{\det \tilde{g}}{\det g} \\ &= \sqrt{-1} \partial \overline{\partial} \log \frac{\tilde{\omega}^n}{\omega^n}. \end{aligned}$$

Therefore we have that

Lemma 2.2 *The cohomology class* $[\mathrm{Ric}(\omega)] \in H^{1,1}_{\overline{\partial}}(M, \mathbb{R})$ *is independent of choice of* ω.

Then we can describe the first Chern class of a Kähler manifold M as follows.

Definition 3 We define the first Chern class of M to be

$$c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)].$$

If we denote by $K_M = \Lambda^n \left(T^{1,0}M\right)^*$ the canonical bundle of M, then the first Chern class of K_M satisfies

$$c_1(K_M) = -c_1(M).$$

We say K_M is ample is equivalent to the existence of one Kähler metric with negative Ricci curvature.

Next we define the Kähler cone of M.

Definition 4 The Kähler cone of M is defined to be

$$\mathcal{C}_M = \{[\vartheta] \mid \text{ there exists Kähler metric } \omega \text{ on } M \text{ with } [\omega] = [\vartheta]\}.$$

A class $[\zeta] \in \overline{\mathcal{C}_M}$ is called nef. A class $[\zeta] \in \overline{\mathcal{C}_M}$ is called nef and big if

$$\int_M \zeta^n > 0.$$

We have the following criteria about nef class.

Lemma 2.3 *Let* (M, ω) *be a compact Kähler manifold. Then a class* $[\zeta]$ *is nef if and only if for any given* $\varepsilon > 0$ *there exists* $\varphi_\varepsilon \in C^\infty(M, \mathbb{R})$ *such that*

$$\zeta + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon > -\varepsilon\omega.$$

For a function $u \in C^\infty(M)$, $\partial\overline{\partial}u$ is given in local coordinates by

$$\partial\overline{\partial}u = \frac{\partial^2 u}{\partial z^i \partial \overline{z}^j} dz^i \wedge d\overline{z}^j.$$

We define the canonical Laplacian of u respect to the Chern connection by

$$\Delta u := \frac{\frac{\sqrt{-1}}{2}\partial\overline{\partial}u \wedge \omega^{n-1}}{\omega^n} = g^{i\overline{j}} \frac{\partial^2 u}{\partial z^i \partial \overline{z}^j}.$$

The k-Ricci curvature Ric_k is defined as the Ricci curvature of the k-dimensional holomorphic subspaces of the holomorphic tangent bundle $T^{(1,0)}(M)$.

Definition 5 (*k-Ricci curvature*) For a point $p \in M$, let U be a k-dimensional subspace of $T_p^{(1,0)}(M)$ and $R|_U$ be the curvature tensor restricted to U. Then the k-Ricci curvature $\mathrm{Ric}_{k,U}$ on U is defined as

$$\mathrm{Ric}_{k,U}(X, \overline{Y}) = \mathrm{tr}_{g|_U} \mathrm{R}|_{\mathrm{U}}(X, \overline{Y}, \cdot, \cdot).$$

for $X, Y \in U$.

When $k = 1$, the k-Ricci curvature is the holomorphic sectional curvature (i.e. $\mathrm{Ric}_1 = \mathrm{H}$). When $k = n$, the k-Ricci curvature is the Ricci curvature (i.e. $\mathrm{Ric}_n = \mathrm{Ric}$). Therefore, one can regard Ric_k as an interpolation between H and Ric of M.

Definition 6 We say that $\mathrm{Ric}_k \leqslant \tau$ if for any X and any k-dimensional subspace U containing X, we have

$$\mathrm{Ric}_{k,U}(X, \overline{X}) \leqslant \tau |X|_g^2.$$

Note that Ric_k is independent for different k. Thus, unlike its Riemannian analogue q-Ricci of Bishop and Wu [2], $\mathrm{Ric}_k \geqslant 0$ can not imply $\mathrm{Ric}_j \geqslant 0$ for $j \neq k$.

In [6], Chu-Lee-Tam derive the following curvature estimate by using a Royden's trick [20]. One can also provide another proof using the averaging technique (cf. Appendix of [16]) by modifying the techniques in [13].

Lemma 2.4 *Suppose (M, ω) is a compact Kähler manifold satisfying (1.1). If $\omega' = \omega_{g'}$ is another Kähler metric on M. Then the following estimate holds*

$$\begin{aligned} 2g'^{i\bar{j}}g'^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}} \leqslant \frac{-\lambda}{\beta}\left((\mathrm{tr}_{\omega'}\omega)^2 + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\right) \\ - \frac{\alpha}{\beta}\left(\mathrm{tr}_{\omega'}\omega \cdot (g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}) + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}\mathrm{Ric}_{i\bar{j}}\right). \end{aligned} \tag{2.1}$$

3 Proof of Theorem 1.5

In this section, we prove Theorem 1.5 using the a priori estimates for the solution of a Monge-Ampère type equation.

Proof We first prove the canonical line bundle K_M of (M, ω) is nef by contradiction. Suppose that K_M is not nef. Then we can find $\delta > 0$ such that $\delta[\omega] - c_1(M)$ is nef but not Kähler. Then by lemma 2.3, for any $\varepsilon > 0$ we have $(\varepsilon + \delta)[\omega] - c_1(M)$ is Kähler. Thus, $\forall \varepsilon > 0$ we can find a smooth function φ_ε such that

$$\omega_\varepsilon := (\delta + \varepsilon)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon > 0. \tag{3.1}$$

Then we can apply the Aubin and Yau theorem [1, 30]. For any $\varepsilon > 0$ we can solve the following complex Monge-Ampère equation for u_ε.

$$\begin{cases} \left(\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}u_\varepsilon\right)^n = \exp\left((1+\frac{\alpha}{2\beta})(\varphi_\varepsilon + u_\varepsilon)\right)\omega^n & (3.2) \\ \omega' := \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}u_\varepsilon > 0. & (3.3) \end{cases}$$

We need only prove

$$\sup_M(\mathrm{tr}_{\omega'}\,\omega) \leqslant C, \tag{3.4}$$

for some $C > 0$ independent of ε. In fact, this implies that

$$\omega' \geqslant C\omega \tag{3.5}$$

for some $C > 0$ independent of ε. Taking $\varepsilon \to 0$ we will see that this is a contradiction to that $\delta[\omega] - c_1(M)$ is not Kähler. This completes the proof of the nefness of the canonical line bundle K_M of (M, ω).

We apply the maximum principle to the following test function:

$$F := \log(\mathrm{tr}_{\omega'}\omega) - \frac{\alpha}{2\beta}(\varphi_\varepsilon + u_\varepsilon).$$

We denote by g and g' the Riemann metric respect to ω and ω'. Let p be the maximum point of F. We choose a local coordinate system $(V; z_1, z_2, \ldots, z_n)$ centered at p such that

$$g_{i\bar{j}}(p) = \delta_{ij}, \quad g'_{i\bar{j}}(p) = \theta_i\delta_{ij}, \quad \partial_k g'_{i\bar{j}}(p) = 0 \tag{3.6}$$

Since

$$\mathrm{tr}_{\omega'}\omega = g'^{i\bar{j}}g_{i\bar{j}},$$

at p we have

$$\begin{aligned} \Delta'(\mathrm{tr}_{\omega'}\,\omega) &= g'^{k\bar{l}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}\left(g'^{i\bar{j}}g_{i\bar{j}}\right) \\ &= g'^{k\bar{l}}g_{i\bar{j}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g'^{i\bar{j}} + g'^{k\bar{l}}g'^{i\bar{j}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g_{i\bar{j}} \\ &= -g'^{k\bar{l}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g'_{p\bar{q}} + g'^{k\bar{l}}g'^{i\bar{j}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g_{i\bar{j}} \\ &= -g'^{k\bar{l}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g'_{p\bar{q}} + g'^{k\bar{l}}g'^{i\bar{j}}g^{p\bar{q}}\partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}} \\ &\quad + g'^{k\bar{l}}g'^{i\bar{j}}\left(\frac{\partial^2}{\partial z_k\partial\bar{z}_l}g_{i\bar{j}} - \partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}}g^{p\bar{q}}\right) \end{aligned} \tag{3.7}$$

On the other hand, we notice that

$$\begin{aligned} R_{k\bar{l}i\bar{j}} &= -\partial_k\partial_{\bar{l}}g_{i\bar{j}} + \partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}}g^{p\bar{q}} \\ R'_{k\bar{l}i\bar{j}} &= -\partial_k\partial_{\bar{l}}g'_{i\bar{j}} + \partial_k g'_{i\bar{q}}\partial_{\bar{l}}g'_{p\bar{j}}g'^{p\bar{q}} = -\partial_k\partial_{\bar{l}}g'_{i\bar{j}} \end{aligned} \tag{3.8}$$

here $R_{k\bar{l}i\bar{j}} = g\left(\nabla_{\frac{\partial}{\partial z_k}}\nabla_{\frac{\partial}{\partial \bar{z}_l}}\frac{\partial}{\partial z_i} - \nabla_{\frac{\partial}{\partial \bar{z}_l}}\nabla_{\frac{\partial}{\partial z_k}}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$. Then

$$\begin{aligned} \Delta'(\mathrm{tr}_{\omega'}\omega) &= g'^{k\bar{l}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}R'_{k\bar{l}p\bar{q}} - g'^{k\bar{l}}g'^{i\bar{j}}R_{k\bar{l}i\bar{j}} + g'^{k\bar{l}}g'^{i\bar{j}}g^{p\bar{q}}\partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}} \\ &= \mathrm{Ric}'_{p\bar{q}}\, g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}} - g'^{k\bar{l}}g'^{i\bar{j}}R_{k\bar{l}i\bar{j}} + g'^{k\bar{l}}g'^{i\bar{j}}g^{p\bar{q}}\partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}} \end{aligned} \tag{3.9}$$

We deal with the first term involved 4-th order derivative in (3.9) by using the Monge-Ampère equation (3.2). Differentiate (3.2) twice we get

$$\begin{aligned} \mathrm{Ric}' &= \mathrm{Ric}(\omega) - \sqrt{-1}\left(1 + \frac{\alpha}{2\beta}\right)\partial\bar{\partial}(\varphi_\varepsilon + u_\varepsilon) \\ &= -\omega' + (\varepsilon+\delta)\omega - \sqrt{-1}\frac{\alpha}{2\beta}\partial\bar{\partial}(\varphi_\varepsilon + u_\varepsilon). \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} \Delta'(\mathrm{tr}_{\omega'}\omega) = &\left(-g'_{p\bar{q}} + (\varepsilon+\delta)g_{p\bar{q}} - \frac{\alpha}{2\beta}(\varphi_{\varepsilon_{p\bar{q}}} + u_{\varepsilon_{p\bar{q}}})\right)g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}} \\ &- g'^{k\bar{l}}g'^{i\bar{j}}R_{k\bar{l}i\bar{j}} + g'^{k\bar{l}}g'^{i\bar{j}}g^{p\bar{q}}\partial_k g_{i\bar{q}}\partial_{\bar{l}}g_{p\bar{j}} \end{aligned} \tag{3.11}$$

Next we deal with the second term involved the Riemannian curvature of ω in (3.11) by the mixed curvature condition. By Lemma 2.4 we have

$$\begin{aligned} g'^{i\bar{j}}g'^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}} \leqslant &\frac{-\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\omega)^2 + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\right) \\ &- \frac{\alpha}{2\beta}\left(\mathrm{tr}_{\omega'}\omega\cdot(g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}) + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}\mathrm{Ric}_{i\bar{j}}\right) \\ = &\frac{-\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\omega)^2 + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\right) - \frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\omega)(g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}) \\ &+ \frac{\alpha}{2\beta}\left(g'^{i\bar{i}}g'^{k\bar{k}}\mathrm{Ric}_{k\bar{k}} - \mathrm{Ric}_{k\bar{k}}g'^{k\bar{k}}g'^{k\bar{k}}\right) \\ = &\frac{-\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\omega)^2 + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\right) - \frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\omega)(g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}) \\ &+ \frac{\alpha}{2\beta}\left(\sum_k \mathrm{Ric}_{k\bar{k}}g'^{k\bar{k}}\left(\sum_i g'^{i\bar{i}} - g'^{k\bar{k}}\right)\right) \\ \leqslant &\frac{-\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\omega)^2 + g_{p\bar{q}}g'^{i\bar{q}}g'^{p\bar{j}}g_{i\bar{j}}\right) - \frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\omega)(g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{2\beta}\left(\sum_k\left((\varepsilon+\delta)+\left(\varphi_{\varepsilon_{k\bar k}}+u_{\varepsilon_{k\bar k}}\right)\right)g'^{k\bar k}\left(\sum_i g'^{i\bar i}-g'^{k\bar k}\right)\right)\\
=&\frac{-\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\,\omega)^2+g_{p\bar q}g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right)-\frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\,\omega)(g'^{i\bar j}\mathrm{Ric}_{i\bar j})\\
&+\frac{\alpha}{2\beta}\left((\varepsilon+\delta)(\mathrm{tr}_{\omega'}\,\omega)^2-(\varepsilon+\delta)g_{p\bar q}g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right)\\
&+\frac{\alpha}{2\beta}\left((\mathrm{tr}_{\omega'}\,\omega)\cdot\Delta'(\varphi_\varepsilon+u_\varepsilon)-(\varphi_{\varepsilon_{p\bar q}}+u_{\varepsilon_{p\bar q}})g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right).
\end{aligned}
\tag{3.12}
$$

Here we used Lemma 2.4 in the first inequality and (3.3) in the second inequality. Plugging this into (3.11) we get

$$
\begin{aligned}
\Delta'(\mathrm{tr}_{\omega'}\,\omega)\geqslant&\left(-g'_{p\bar q}+(\varepsilon+\delta)g_{p\bar q}-\frac{\alpha}{2\beta}(\varphi_{\varepsilon_{p\bar q}}+u_{\varepsilon_{p\bar q}})\right)g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\\
&+\frac{\lambda}{2\beta}\left((\mathrm{tr}_{\omega'}\,\omega)^2+g_{p\bar q}g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right)+\frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\,\omega)(g'^{i\bar j}\mathrm{Ric}_{i\bar j})\\
&-\frac{\alpha}{2\beta}\left((\varepsilon+\delta)(\mathrm{tr}_{\omega'}\,\omega)^2-(\varepsilon+\delta)g_{p\bar q}g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right)\\
&-\frac{\alpha}{2\beta}\left((\mathrm{tr}_{\omega'}\,\omega)\cdot\Delta'(\varphi_\varepsilon+u_\varepsilon)-(\varphi_{\varepsilon_{p\bar q}}+u_{\varepsilon_{p\bar q}})g'^{i\bar q}g'^{p\bar j}g_{i\bar j}\right)\\
&+g'^{k\bar l}g'^{i\bar j}g^{p\bar q}\partial_k g_{i\bar q}\partial_{\bar l}g_{p\bar j}\\
=&\left(-g'_{p\bar q}+(\varepsilon+\delta)g_{p\bar q}\right)g'^{i\bar q}g'^{p\bar j}g_{i\bar j}+\frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\,\omega)(g'^{i\bar j}\mathrm{Ric}_{i\bar j})\\
&+\frac{\lambda-\alpha(\varepsilon+\delta)}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)^2+\frac{\lambda+\alpha(\varepsilon+\delta)}{2\beta}(g_{p\bar q}g'^{i\bar q}g'^{p\bar j}g_{i\bar j})\\
&-\frac{\alpha}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)\cdot\Delta'(\varphi_\varepsilon+u_\varepsilon)+g'^{k\bar l}g'^{i\bar j}g^{p\bar q}\partial_k g_{i\bar q}\partial_{\bar l}g_{p\bar j}\\
=&-\mathrm{tr}_{\omega'}\,\omega+(\varepsilon+\delta)\left(\sum_i\frac{1}{\theta_i^2}\right)+\frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\,\omega)(g'^{i\bar j}\mathrm{Ric}_{i\bar j})\\
&+\frac{\lambda-\alpha(\varepsilon+\delta)}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)^2+\frac{\lambda+\alpha(\varepsilon+\delta)}{2\beta}\left(\sum_i\frac{1}{\theta_i^2}\right)\\
&-\frac{\alpha}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)\cdot\Delta'(\varphi_\varepsilon+u_\varepsilon)+g'^{k\bar l}g'^{i\bar j}g^{p\bar q}\partial_k g_{i\bar q}\partial_{\bar l}g_{p\bar j}\\
\geqslant&-\mathrm{tr}_{\omega'}\,\omega+\frac{\alpha}{\beta}(\mathrm{tr}_{\omega'}\,\omega)(g'^{i\bar j}\mathrm{Ric}_{i\bar j})+\frac{\lambda-\alpha(\varepsilon+\delta)}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)^2\\
&-\frac{\alpha}{2\beta}(\mathrm{tr}_{\omega'}\,\omega)\cdot\Delta'(\varphi_\varepsilon+u_\varepsilon)+g'^{k\bar l}g'^{i\bar j}g^{p\bar q}\partial_k g_{i\bar q}\partial_{\bar l}g_{p\bar j}
\end{aligned}
\tag{3.13}
$$

Now we deal with the third order term in (3.13). Using the condition (3.6), we have $\mathrm{tr}_{\omega'}\,\omega(p)=\sum_{i=1}^n\frac{1}{\theta_i}$ and

$$
\begin{aligned}
&\Delta' \log(\mathrm{tr}_{\omega'} \omega) \\
&= \frac{\Delta'(\mathrm{tr}_{\omega'} \omega)}{\mathrm{tr}_{\omega'} \omega} - \frac{|\nabla'(\mathrm{tr}_{\omega'} \omega)|^2}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&\geqslant \frac{\Delta'(\mathrm{tr}_{\omega'} \omega)}{\mathrm{tr}_{\omega'} \omega} - \frac{g'^{i\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{l}} g_{p\bar{q}} g'^{p\bar{q}} g'^{k\bar{l}}}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&= \frac{\Delta'(\mathrm{tr}_{\omega'} \omega)}{\mathrm{tr}_{\omega'} \omega} - \frac{\frac{1}{\theta_j}\frac{1}{\theta_k}\frac{1}{\theta_p} \partial_k g_{j\bar{j}} \partial_{\bar{k}} g_{p\bar{p}}}{(\mathrm{tr}_{\omega'} \omega)^2}
\end{aligned}
\tag{3.14}
$$

Plugging (3.13) into (3.14) we have

$$
\begin{aligned}
&\Delta' \log(\mathrm{tr}_{\omega'} \omega) \\
&\geqslant -1 + \frac{\alpha}{\beta}(g'^{i\bar{j}} \mathrm{Ric}_{i\bar{j}}) + \frac{\lambda - \alpha(\varepsilon + \delta)}{2\beta}(\mathrm{tr}_{\omega'} \omega) - \frac{\alpha}{2\beta}\Delta'(\varphi_\varepsilon + u_\varepsilon) \\
&\quad + \frac{\frac{1}{\theta_k}\frac{1}{\theta_j} \partial_k g_{j\bar{p}} \partial_{\bar{k}} g_{p\bar{j}}}{\mathrm{tr}_{\omega'} \omega} - \frac{\frac{1}{\theta_j}\frac{1}{\theta_k}\frac{1}{\theta_p} \partial_k g_{j\bar{j}} \partial_{\bar{k}} g_{p\bar{p}}}{(\mathrm{tr}_{\omega'} \omega)^2}
\end{aligned}
\tag{3.15}
$$

We notice that

$$
\begin{aligned}
&\frac{\frac{1}{\theta_k}\frac{1}{\theta_j} \partial_k g_{j\bar{p}} \partial_{\bar{k}} g_{p\bar{j}}}{\mathrm{tr}_{\omega'} \omega} - \frac{\frac{1}{\theta_j}\frac{1}{\theta_k}\frac{1}{\theta_p} \partial_k g_{j\bar{j}} \partial_{\bar{k}} g_{p\bar{p}}}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&\geqslant \frac{\frac{1}{\theta_k}\frac{1}{\theta_j} \partial_k g_{j\bar{j}} \partial_{\bar{k}} g_{j\bar{j}}}{\mathrm{tr}_{\omega'} \omega} - \frac{\frac{1}{\theta_j}\frac{1}{\lambda_k}\frac{1}{\theta_p} \partial_k g_{j\bar{j}} \partial_{\bar{k}} g_{p\bar{p}}}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&\geqslant \frac{\frac{1}{\theta_k}\frac{1}{\theta_j} (\partial_k g_{j\bar{j}})^2}{\mathrm{tr}_{\omega'} \omega} - \frac{\frac{1}{\theta_k}\left(\sum_{j=1}^n \frac{\partial_k g_{j\bar{j}}}{\theta_j}\right)^2}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&= \frac{\frac{1}{\theta_k}\left\{\left(\sum_{j=1}^n \frac{1}{\theta_j}\right)\left(\sum_{j=1}^n \frac{1}{\theta_j}(\partial_k g_{j\bar{j}})^2\right) - \left(\sum_{j=1}^n \frac{\partial_k g_{j\bar{j}}}{\theta_j}\right)^2\right\}}{(\mathrm{tr}_{\omega'} \omega)^2} \\
&\geqslant 0.
\end{aligned}
\tag{3.16}
$$

Here we use Cauchy-Schwartz inequality in the last inequality. Finally, we derive that

$$
\begin{aligned}
&\Delta' \log(\mathrm{tr}_{\omega'} \omega) \\
&\geqslant -1 + \frac{\alpha}{\beta}(g'^{i\bar{j}} \mathrm{Ric}_{i\bar{j}}) + \frac{\lambda - \alpha(\varepsilon + \delta)}{2\beta}(\mathrm{tr}_{\omega'} \omega) - \frac{\alpha}{2\beta}\Delta'(\varphi_\varepsilon + u_\varepsilon)
\end{aligned}
\tag{3.17}
$$

For the zero order term in the test function F we observe that

$$
\begin{aligned}
&\Delta'(\varphi_\varepsilon + u_\varepsilon) \\
&= g'^{i\bar{j}}(\varphi_{\varepsilon_{i\bar{j}}} + u_{\varepsilon_{i\bar{j}}})
\end{aligned}
$$

$$
\begin{aligned}
&= g'^{i\bar{j}}\left(g'_{i\bar{j}} + \mathrm{Ric}_{i\bar{j}} - (\varepsilon+\delta)g_{i\bar{j}}\right)\\
&= n + g'^{i\bar{j}}\mathrm{Ric}_{i\bar{j}} - (\varepsilon+\delta)\,\mathrm{tr}_{\omega'}\,\omega. \qquad (3.18)
\end{aligned}
$$

Combing (3.17) and (3.18), we see that

$$
\begin{aligned}
\Delta' F &= \Delta'\left(\log(\mathrm{tr}_{\omega'}\,\omega) - \frac{\alpha}{2\beta}(\varphi_\varepsilon + u_\varepsilon)\right)\\
&\geqslant \frac{\lambda + (\varepsilon+\delta)\alpha}{2\beta}\,\mathrm{tr}_{\omega'}\,\omega - C(\alpha,\beta,n) \qquad (3.19)\\
&\geqslant \frac{\lambda + \delta\alpha}{2\beta}\,\mathrm{tr}_{\omega'}\,\omega - C(\alpha,\beta,n).
\end{aligned}
$$

Applying the maximum principle to F we get, at p,

$$
(\mathrm{tr}_{\omega'}\,\omega)(p) \leqslant C, \qquad (3.20)
$$

for C independent of ε. Then we can estimate, using the Monge-Ampère equation (3.2),

$$
\begin{aligned}
\sup_M F &= F(p)\\
&= \log(\mathrm{tr}_{\omega'}\omega) - \frac{\alpha}{2\beta}(\varphi_\varepsilon + u_\varepsilon)\\
&= \log(\mathrm{tr}_{\omega'}\omega) - \frac{\alpha}{2\beta+\alpha}\log\frac{\omega'^n}{\omega^n} \qquad (3.21)\\
&= \log(\mathrm{tr}_{\omega'}\omega) + \frac{\alpha}{2\beta+\alpha}\log\frac{1}{\prod_i \theta_i}.
\end{aligned}
$$

By the inequality of arithmetic and geometric means, we have

$$
\sqrt[n]{\prod_i \frac{1}{\theta_i}} \leqslant \frac{\sum_i \frac{1}{\theta_i}}{n} = \frac{\mathrm{tr}_{\omega'}\,\omega}{n}.
$$

Plugging this into (3.21), then we get

$$
\begin{aligned}
\sup_M F &\leqslant \log(\mathrm{tr}_{\omega'}\omega) + \frac{\alpha n}{2\beta+\alpha}\log\frac{\mathrm{tr}_{\omega'}\,\omega}{n} \qquad (3.22)\\
&\leqslant C,
\end{aligned}
$$

for C independent of ε.

Now we need to estimate the upper bound for $(\varphi_\varepsilon + u_\varepsilon)$. Let q be the maximum point of $(\varphi_\varepsilon + u_\varepsilon)$. Using the maximum principle, at q, we obtain

$$
\sqrt{-1}\partial\bar{\partial}(\varphi_\varepsilon + u_\varepsilon) \leqslant 0.
$$

This implies

$$
\begin{aligned}
&((\delta+\varepsilon)\omega-\mathrm{Ric}(\omega))(q)\\
&\quad\geqslant(\delta+\varepsilon)\omega-\mathrm{Ric}(\omega)+\sqrt{-1}\partial\bar{\partial}(\varphi_\varepsilon+u_\varepsilon)\\
&\quad=\omega'\geqslant 0.
\end{aligned}
\tag{3.23}
$$

Then by the Monge-Ampère equation (3.2), we have

$$
\begin{aligned}
&\left(1+\frac{\alpha}{2\beta}\right)\sup_M(\varphi_\varepsilon+u_\varepsilon)\\
&\quad=\left(1+\frac{\alpha}{2\beta}\right)(\varphi_\varepsilon+u_\varepsilon)(q)\\
&\quad=\log\frac{\omega'^n}{\omega^n}\\
&\quad=\log\frac{((\delta+\varepsilon)\omega-\mathrm{Ric}(\omega)+\sqrt{-1}\partial\bar{\partial}(\varphi_\varepsilon+u_\varepsilon))^n}{\omega^n}\\
&\quad\leqslant\log\frac{((\varepsilon+\delta)\omega-\mathrm{Ric}(\omega))^n}{\omega^n}\\
&\quad\leqslant C,
\end{aligned}
\tag{3.24}
$$

for some C independent of ε. Thus we obtain the upper bound for $(\varphi_\varepsilon+u_\varepsilon)$.

Combingning (3.22) and (3.24), we obtain

$$
\sup_M\log(\mathrm{tr}_{\omega'}\,\omega)=\sup_M\left(F+\frac{\alpha}{2\beta}(\varphi_\varepsilon+u_\varepsilon)\right)\leqslant C,
\tag{3.25}
$$

for some C independent of ε. Hence

$$
\sup_M(\mathrm{tr}_{\omega'}\,\omega)\leqslant C,
\tag{3.26}
$$

for some C independent of ε. This completes the proof of nefness of K_M.

Next we prove the canonical line bundle K_M is ample if $\gamma>0$. Using the fact that K_M is nef, we can take $\delta=0$ and repeat a similar arguments as above. We solve the similar Mong-Ampère equation and do the similar estimates. Note that we still can have the uniform estimates for $\mathrm{tr}_{\omega'}\,\omega$ since $\gamma>0$ guarantees that the coefficient of $\mathrm{tr}_{\omega'}\,\omega$ in (3.19) is still strictly positive when $\varepsilon\to 0$.

Then we have

$$
\mathrm{tr}_\omega\,\omega'\leqslant\frac{1}{(n-1)!}(\mathrm{tr}_{\omega'}\,\omega)^{n-1}\frac{\omega'^n}{\omega^n}\leqslant C.
$$

Hence

$$
C^{-1}\omega\leqslant\omega'\leqslant C\omega,
\tag{3.27}
$$

for some $C>0$ independent of ε. Higher order estimates follow from Evans-Krylov theory and Schauder estimate.

Then one can apply the Arzela-Ascoli Theorem to get a convergent subsequence out of $(\varphi_\varepsilon + u_\varepsilon)$ as $\varepsilon \to 0$. Taking limit on both side of the Monge-Ampère equation (3.2), we can find a function u_0 and a Kähler form ω_0 satisfies

$$\mathrm{Ric}(\omega_0) = -\omega_0 - \frac{\alpha}{2\beta}\sqrt{-1}\partial\bar{\partial}u_0.$$

This shows that the canonical line bundle K_M of (M, ω) is ample under the assumption that $\gamma > 0$ in (1.1). $\square$

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