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A Note on Almost Everywhere Convergence Along Tangential Curves to the Schrödinger Equation Initial Datum

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Abstract
In this short note, we give an easy proof of the following result: for $n \geq 2$, $\lim_{t\to 0} e^{it\Delta} f(x + \gamma(t)) = f(x)$ almost everywhere whenever γ is an α-Hölder curve with $\frac{1}{2} \leq \alpha \leq 1$ and $f \in H^s(\mathbb{R}^n)$, with $s > \frac{n}{2(n+1)}$. This is the optimal range of regularity up to the endpoint.

Keywords Schrödinger equation · Schrödinger maximal function · Almost everywhere convergence · Tangential convergence

Mathematics Subject Classification 35Q41 · 42B25 · 42B37

1 Introduction

Consider the linear Schrödinger equation on $\mathbb{R}^n \times \mathbb{R}$, $n \geq 1$, given by

$$\begin{cases} i\partial_t u(x,t) - \Delta_x u(x,t) = 0, \\ u(x,0) = f(x). \end{cases} \tag{1}$$

Its solution can be formally expressed as

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{2\pi i t|\xi|^2} \widehat{f}(\xi) d\xi. \tag{2}$$

It was first proposed by Carleson in 1980 [3] to find the values of $s > 0$ for which

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R}^n, \tag{3}$$

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holds true for all functions $f \in H^s(\mathbb{R}^n)$. Carleson [3] proved this convergence when $n = 1$ and $s \geq \frac{1}{4}$. Later, in 2006, Dahlberg and Kenig [6] showed that (3) was false whenever $s < \frac{1}{4}$.

Many researchers have worked in this problem throughout the years. Authors such as Carbery, Cowling, Vega, Sjölin, Moyua, Vargas, Tao, Lee and Bourgain to name a few. More recently, the problem has been solved in higher dimensions, except for the endpoint. In 2016, Bourgain [1] proved the necessity of $s \geq \frac{n}{2(n+1)}$ in order to have (3). In 2017, Du et al. [7] proved the sufficiency of the condition $s > \frac{1}{3}$ when $n = 2$. Later, in 2019, Du and Zhang [8] proved the sufficiency of $s > \frac{n}{2(n+1)}$ for general $n \geq 3$. A more detailed history of the problem can be found in [8] and the references therein.

Take a solution of (1). Consider a set of curves $\rho(x,t) = x + \gamma(t)$ that are bi-Lipschitz in $x \in \mathbb{R}^n$ and α-Hölder in $t \in \mathbb{R}$. Cho et al. [4] proved in 2012 that $u(\rho(x,t),t)$ converges to $f(x)$ almost everywhere as $t \to 0$ in $n = 1$ when $s > \max\left\{\frac{1}{2} - \alpha, \frac{1}{4}\right\}$. They also found this to be sharp up to the endpoint. Later, in 2021, Li and Wang [11] proved that convergence in dimension $n = 2$, for index $\frac{1}{2} \leq \alpha \leq 1$ and the range $s > \frac{3}{8}$. In 2023, Cao and Miao [2] gave a proof for general dimension n, index $\frac{1}{2} \leq \alpha \leq 1$, and $s > \frac{n}{2(n+1)}$. Their proof followed the argument presented in [8] and relied on techniques such as dyadic pigeonholing, broad-narrow analysis and induction on scales.

Our objective is to give an easy proof of the result in [2] without using the aforementioned techniques.

Fix $0 < \alpha \leq 1$ and $\tau \geq 1$. We consider the family of curves,

$$\Gamma_\tau^\alpha := \left\{\gamma : [0,1] \to \mathbb{R}^n : \text{for all } t, t' \in [0,1],\ |\gamma(t) - \gamma(t')| \leq \tau |t - t'|^\alpha \right\}. \quad (4)$$

The convergence result follows from the maximal bound below. Let $B_r^n(x_0)$ denote the ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$.

Theorem 1.1 *Let $n \geq 1$. Fix $\frac{1}{2} \leq \alpha \leq 1$ and $\tau \geq 1$. For any $\varepsilon > 0$, there exists a positive constant $C_{\varepsilon,\tau}$ such that, for every $\gamma \in \Gamma_\tau^\alpha$,*

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f(x + \gamma(t)) \right| \right\|_{L^2(B_1^n(0))} \leq C_{\varepsilon,\tau} \|f\|_{H^{\frac{n}{2(n+1)}+\epsilon}(\mathbb{R}^n)}, \quad (5)$$

holds for all $f \in H^{\frac{n}{2(n+1)}+\epsilon}(\mathbb{R}^n)$.

Remark 1.2 A change of variables shows that it is enough to consider the case $\tau = 1$. From now on we assume $\tau = 1$.

Then, we can reduce Theorem 1.1 as in [8]. We begin with a definition.

Definition 1.3 Fixed $0 < \alpha \leq 1$ and $R > 1$, we define

$$\Gamma^\alpha\left(R^{-1}\right) := \left\{\gamma : [0, R^{-1}] \to \mathbb{R}^n : \text{for all } t, t' \in [0, R^{-1}],\ |\gamma(t) - \gamma(t')| \leq |t - t'|^\alpha \right\}. \quad (6)$$

By Littlewood–Paley decomposition, the time localization lemma (e.g. Lemma 3.1 in Lee [9]) and parabolic rescaling, Theorem 1.1 can be reduced to the following Theorem 1.4.

Theorem 1.4 *Let $n \geq 1$ and $\frac{1}{2} \leq \alpha < 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that, for all $\gamma \in \Gamma^\alpha\left(R^{-1}\right)$,*

$$\left\| \sup_{0<t\leq R} \left| e^{it\Delta} f\left(x + R\gamma\left(\frac{t}{R^2}\right)\right)\right| \right\|_{L^2(B_R^n(0))} \leq C_\varepsilon R^{\frac{n}{2(n+1)}+\varepsilon} \|f\|_2. \tag{7}$$

holds for all $R \geq 1$ and all f with $\operatorname{supp} \widehat{f} \subset A(1) = \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$.

2 Intermediate Results

We consider the following result from [8].

Theorem 2.1 (Corollary 1.7 in [8]) *Let $n \geq 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds for all $R \geq 1$ and all f with* $\operatorname{supp} \widehat{f} \subset B_1^n(0)$. *Suppose that $X = \cup_k B_k$ is a union of lattice unit cubes in $B_R^{n+1}(0)$. Let $1 \leq \beta \leq n+1$ and*

$$\phi := \phi_{X,\beta} := \max_{\substack{B_r^{n+1}(x')\subset B_R^{n+1}(0)\\ x'\in\mathbb{R}^{n+1}, r\geq 1}} \frac{\#\left\{B_k : B_k \subset B_r^{n+1}\left(x'\right)\right\}}{r^\beta}. \tag{8}$$

Then

$$\left\| e^{it\Delta} f \right\|_{L^2(X,dxdt)} \leq C_\varepsilon \phi^{\frac{1}{n+1}} R^{\frac{\beta}{2(n+1)}+\varepsilon} \|f\|_2. \tag{9}$$

We generalize the above result to include α-Hölder curves.

Theorem 2.2 *Let $n \geq 1$ and $\frac{1}{2} \leq \alpha \leq 1$.*

For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds for any $R \geq 1$, every $\gamma \in \Gamma^\alpha\left(R^{-1}\right)$ and all f with $\operatorname{supp} \widehat{f} \subset B_1^n(0)$. *Suppose that $X = \cup_k B_k$ is a union of lattice unit cubes in $B_R^{n+1}(0)$. Let $1 \leq \beta \leq n+1$ and ϕ be given by (8). Then*

$$\left\| e^{it\Delta} f\left(x + R\gamma\left(\frac{t}{R^2}\right)\right)\right\|_{L^2(X,dxdt)} \leq C_\varepsilon \phi^{\frac{1}{n+1}} R^{\frac{\beta}{2(n+1)}+\varepsilon} \|f\|_2. \tag{10}$$

Proof of Theorem 2.2 Denote

$$\theta(t) := \theta_R(t) := R\gamma\left(\frac{t}{R^2}\right). \tag{11}$$

We begin with

$$\left\| e^{it\Delta} f\left(x + R\gamma\left(\frac{t}{R^2}\right)\right)\right\|_{L^2(X,dxdt)}^2 = \sum_k \int_{B_k} \left| e^{it\Delta} f\left(x + \theta(t)\right)\right|^2 dxdt. \tag{12}$$

Denote (x_k, t_k) to be the center of B_k. Then,

$$\leq \sum_k \int_{t_k-1}^{t_k+1} \int_{B_1^n(x_k)} \left| e^{it\Delta} f(x+\theta(t)) \right|^2 dxdt \tag{13}$$

$$= \sum_k \int_{t_k-1}^{t_k+1} \int_{B_1^n(x_k+\theta(t))} \left| e^{it\Delta} f(y) \right|^2 dydt. \tag{14}$$

Recall that $\gamma \in \Gamma^\alpha\left(R^{-1}\right)$, and $\alpha \geq \frac{1}{2}$. Thus, if $t \in (t_k-1, t_k+1)$, then $|\theta(t) - \theta(t_k)| \leq 1$. Hence,

$$\leq \sum_k \int_{t_k-1}^{t_k+1} \int_{B_3^n(x_k+\theta(t_k))} \left| e^{it\Delta} f(y) \right|^2 dydt \tag{15}$$

$$\leq \int_{\mathbb{R}^n} \sum_k \chi_{B_4^{n+1}(x_k+\theta(t_k),t_k)}(y) \left| e^{it\Delta} f(y) \right|^2 dydt \tag{16}$$

$$\leq C \int_{\bigcup B_4^{n+1}(x_k+\theta(t_k),t_k)} \left| e^{it\Delta} f(y) \right|^2 dydt, \tag{17}$$

for some $C > 0$. Note that, if $B_4(x_k + \theta(t_k), t_k) \cap B_4(x_i + \theta(t_i), t_i) \neq \emptyset$, then $|t_k - t_i| \leq 8$. Since $\alpha \geq \frac{1}{2}$, we have $|\theta(t_k) - \theta(t_i)| \leq 8$. Hence, $|x_k - x_i| \leq 16$. Therefore, the balls $\{B_4(x_k + \theta(t_k), t_k)\}_k$ have finite overlap $C = C_n$.

Define $Y = \bigcup_l Q_l$ to be the minimal union of lattice unit cubes satisfying that $\bigcup_k B_4^{n+1}(x_k + \theta(t_k), t_k) \subset Y$. We have proven that

$$\left\| e^{it\Delta} f(x+\theta(t)) \right\|_{L^2(X,dxdt)} \leq C_n \left\| e^{it\Delta} f(x) \right\|_{L^2(Y,dxdt)}. \tag{18}$$

Hence by Theorem 2.1,

$$\leq C_{\epsilon,n} \phi_{Y,\beta}^{\frac{1}{n+1}} R^{\frac{\beta}{2(n+1)}+\epsilon} \|f\|_2. \tag{19}$$

We **claim** that

$$\phi_{Y,\beta} \leq c_n \phi_{X,\beta}, \tag{20}$$

for some $C > 0$. This would conclude the proof.

To prove (20), note that, if $Q_l \subset B_r^{n+1}(y_0, s_0)$, $r \geq 1$, and $Q_l \cap B_4^{n+1}(x_k + \theta(t_k), t_k) \neq \emptyset$, then $B_k = B_1^{n+1}(x_k, t_k) \subset B_{r+5}^{n+1}(y_0 - \theta(s_0), s_0)$.

Therefore,

$$\frac{\#\{Q_l : Q_l \subset B_r^{n+1}(y_0, s_0)\}}{r^\beta} \leq c_n \frac{\#\left\{B_k : B_k \subset B_{r+5}^{n+1}(y_0 - \theta(s_0), s_0)\right\}}{(r+5)^\beta} \cdot \frac{(r+5)^\beta}{r^\beta}$$
$$\leq c_n \phi_{X,\beta}. \tag{21}$$

□

3 Proof of Theorem 1.4

Before the proof, let us introduce a stability property of the Schrödinger operator. More general versions of the following appeared in an article of Tao [12] from 1999 and an article of Christ [5] from 1988.

Suppose that $\widehat{f}$ is supported inside a ball of radius 1.

If $|x'-y'| \leq 4$ and $|t'-s'| \leq 4$, then,

$$\left|e^{it'\Delta} f(x')\right| \leq \sum_{\mathfrak{l}\in\mathbb{Z}^n} \frac{1}{(1+|\mathfrak{l}|)^{n+1}} \left|e^{is'\Delta} f_{\mathfrak{l}}(y')\right|, \tag{22}$$

where $\widehat{f_{\mathfrak{l}}}(\xi) = e^{2\pi i \mathfrak{l}\xi}\widehat{f}(\xi)$.

Now, fix $\alpha \geq 1/2$ and $\gamma \in \Gamma^\alpha\left(R^{-1}\right)$. Define θ as in (11). Whenever $|x-y| \leq 2$ and $|t-s| \leq 2$, we have that $|x+\theta(t)-(y+\theta(s))| \leq 4$. Thus,

$$\left|e^{it\Delta} f(x+\theta(t))\right| \leq \sum_{\mathfrak{l}\in\mathbb{Z}^n} \frac{1}{(1+|\mathfrak{l}|)^{n+1}} \left|e^{is\Delta} f_{\mathfrak{l}}(y+\theta(s))\right|. \tag{23}$$

Therefore, if $|x-x_0| \leq 1$ and $|t-t_0| \leq 1$, then,

$$\left|e^{it\Delta} f(x+\theta(t))\right| \leq \sum_{\mathfrak{l}\in\mathbb{Z}^n} \frac{1}{(1+|\mathfrak{l}|)^{n+1}} \int_{t_0}^{t_0+1}\int_{B_1(x_0)} \left|e^{is\Delta} f_{\mathfrak{l}}(y+\theta(s))\right| dyds. \tag{24}$$

Proof of Theorem 1.4. For the sake of briefness, given $(x,t)\in\mathbb{R}^{n+1}$ let us denote

$$E'f(x,t) := E'_{\gamma,R}f(x,t) := e^{it\Delta} f\left(x+R\gamma\left(\frac{t}{R^2}\right)\right). \tag{25}$$

Now, we can write

$$\left\|\sup_{0<t\leq R}\left|e^{it\Delta} f\left(x+R\gamma\left(\frac{t}{R^2}\right)\right)\right|\right\|^2_{L^2(B^n_R(0))} = \left\|\sup_{0<t\leq R}|E'f(x,t)|\right\|^2_{L^2(B^n_R(0))} \tag{26}$$

$$= \int_{B^n_R(0)} \left(\sup_{0<t\leq R}|E'f(x,t)|^2\right) dx \tag{27}$$

$$= \sum_{\substack{x_0\in\mathbb{Z}^n\\ |x_0|<R}} \int_{B^n_1(x_0)} \left(\sup_{0<t\leq R}|E'f(x,t)|^2\right) dx \tag{28}$$

$$\leq C_n \sum_{\substack{x_0\in\mathbb{Z}^n\\ |x_0|<R}} \left[\sup_{|x-x_0|\leq 1}\left(\sup_{0<t\leq R}|E'f(x,t)|^2\right)\right]. \tag{29}$$

For each x_0, there exists $\widetilde{t_0} = \widetilde{t_0}(x_0) \in \mathbb{Z} \cap [0, R]$ such that the supremum on each term of the above sum is almost attained inside $B_1^{n+1}(x_0, \widetilde{t_0})$. Therefore,

$$\left\| \sup_{0<t\leq R} |E' f(x,t)| \right\|^2_{L^2(B_R^n(0))} \lesssim \sum_{\substack{x_0\in\mathbb{Z}^n \\ |x_0|<R}} \sup_{(x,t)\in B_1^{n+1}(x_0,\widetilde{t_0})} \left|E' f(x,t)\right|^2, \tag{30}$$

which is, by (24),

$$\lesssim \sum_{\substack{x_0\in\mathbb{Z}^n \\ |x_0|<R}} \left| \sum_{\mathfrak{l}\in\mathbb{Z}^n} \frac{1}{(1+|\mathfrak{l}|)^N} \int_{B_1^n(x_0)} \int_{\widetilde{t_0}}^{\widetilde{t_0}+1} \left|E' f_{\mathfrak{l}}(y,s)\right| ds dy \right|^2. \tag{31}$$

Therefore, denoting $C_{\mathfrak{l}} = \frac{1}{(1+|\mathfrak{l}|)^N}$ and using Cauchy–Schwarz,

$$\lesssim \sum_{\substack{x_0\in\mathbb{Z}^n \\ |x_0|<R}} \sum_{\mathfrak{l}\in\mathbb{Z}^n} C_{\mathfrak{l}} \left\| E' f_{\mathfrak{l}}(y,t) \right\|^2_{L^2(B_2^{n+1}(x_0,\widetilde{t_0}))}. \tag{32}$$

Let us choose $X = \bigcup_{\substack{x_0\in\mathbb{Z}^n \\ |x_0|<R}} B_2^{n+1}\left(x_0, \widetilde{t_0}\right)$. The above lets us deduce that

$$\left\| \sup_{0<t\leq R} |E' f(x,t)| \right\|^2_{L^2(B_R^n(0),dx)} \lesssim \sum_{\mathfrak{l}\in\mathbb{Z}^n} C_{\mathfrak{l}} \left\| E' f_{\mathfrak{l}}(y,t) \right\|^2_{L^2(X)}. \tag{33}$$

By Theorem 2.2, this is,

$$\lesssim C_\epsilon \sum_{\mathfrak{l}\in\mathbb{Z}^n} C_{\mathfrak{l}} (\phi_{X,n})^{\frac{2}{n+1}} \| f_{\mathfrak{l}} \|^2_{L^2(\mathbb{R}^n)} R^{\frac{2n}{2(n+1)}+\epsilon}. \tag{34}$$

Recall that, given $x_0 \in \mathbb{Z}^n \cap B_R^n(0)$, we have chosen exactly one $\widetilde{t_0} \in \mathbb{Z} \cap [0, R]$. Consequently, $\phi_{X,n} \leq 1$. Therefore, the above inequalities yield

$$\left\| \sup_{0<t\leq R} |E' f(x,t)| \right\|^2_{L^2(B_R^n(0),dx)} \lesssim C_\epsilon R^{\frac{2n}{2(n+1)}+\epsilon} \| f \|_2^2. \tag{35}$$

□

Data Availability Data availability is not applicable to this manuscript.

Conflict of Interest I declare that I have no conflict of interest in relation to this manuscript.

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