The Journal of Geometric Analysis (2024) 34:296
https://doi.org/10.1007/s12220-024-01744-0

Normalized Solutions for Schrödinger–Poisson Systems Involving Critical Sobolev Exponents

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Received: 19 May 2024 / Accepted: 8 July 2024 / Published online: 20 July 2024

Abstract

In this paper, we are concerned with the existence and properties of ground states for the Schrödinger–Poisson system with combined power nonlinearities

$$\begin{cases} -\Delta u + \gamma\phi u = \lambda u + \mu|u|^{q-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

having prescribed mass

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2,$$

in the *Sobolev critical case*. Here $a > 0$, and $\gamma > 0$, $\mu > 0$ are parameters, $\lambda \in \mathbb{R}$ is an undetermined parameter. By using Jeanjean' theory, Pohozaev manifold method and Brezis and Nirenberg's technique to overcome the lack of compactness, we prove several existence results under the L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu|u|^{q-2}u$, under different assumptions imposed on the parameters γ, μ and the mass a, respectively. This study can be considered as a counterpart of the Brezis-Nirenberg problem in the context of normalized solutions of a Sobolev critical Schrödinger–Poisson problem perturbed with a subcritical term in the whole space $\mathbb{R}^3$.

Keywords Schrödinger–Poisson systems · Normalized solutions · Critical Sobolev exponent · Pohozaev manifold

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Mathematics Subject Classification 35J62 · 35J50 · 35B65

1 Introduction and Main Results

In this paper we study the nonlinear Schrödinger–Poisson system

$$\begin{cases} i\partial_t\Psi - \Delta\Psi + \gamma\phi(x)\Psi = af(|\Psi|^2)\Psi, & x \in \mathbb{R}^3, \\ -\Delta\phi = |\Psi|^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is the time-dependent wave function, $\gamma, a \in \mathbb{R}$ are parameters, the nonlinear term f simulates the interaction between many particles or external nonlinear perturbations. The nonlinear Schrödinger–Poisson system (1.1) attracted much attention in the last decade, starting from the fundamental contribution [13]. System (1.1) has many physical motivations, it derived from the approximation of the Hartree-Fock equation that describes a quantum mechanical of many particles, and is highly beneficial in the quantum description of the ground states of nonrelativistic atoms and molecules [34, 35, 39], and also arises in semiconductor theory [18].

When we are concerned with the standing wave solutions $\Psi(t,x) = e^{-i\lambda t}u(x)$, $\lambda \in \mathbb{R}$, then $u : \mathbb{R}^3 \to \mathbb{R}$ must verify

$$\begin{cases} -\Delta u + \lambda u + \gamma\phi u = af(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

At this time, there are two possible choices to deal with (1.2). One can fix $\lambda \in \mathbb{R}$ and to look for solutions as critical points of the associated energy functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - a\int_{\mathbb{R}^3} F(u)dx.$$

where $F(u) = \int_0^u f(s)ds$ is the primitive integral of f. Alternatively, one can search for solutions of Eq. (1.2) with prescribed L^2-norm. At this point, the parameter $\lambda \in \mathbb{R}$ cannot longer be fixed but instead appears as a Lagrange multiplier. Analogous to the first case, the solutions of (1.2) with $\|u\|_2^2 = m > 0$ can be obtained as critical points of the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - a\int_{\mathbb{R}^3} F(u)dx.$$

under the constraint L^2-sphere $S_m := \{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = m^2\}$. It is easy to check that J is a well-defined and C^1 functional on S_m. This approach is relevant from the physical point of view, in particular, since the L^2-norm is a preserved quantity of the evolution and since the variational characterization of such solutions is often a strong help to analyze their orbital stability, see for example, [6, 9–11, 29–31] and references therein.

As far as we know, the first work for normalized solution to Eq. (1.2) in the case $\gamma = 1$, and $f(u) = |u|^{p-2}u$ is due to Sánchez and Soler [41]. They showed that there exists a normalized solution of (1.2) provided that m is sufficiently small and $p = \frac{8}{3}$. Since then, there are some further studies for problem (1.2) in mass subcritical case. In this case, the corresponding functional is bounded from below on S_m, then a global minimizer can be obtained for some m. See for instance, [10, 11, 31, 32]. When the nonlinearity f in (1.2) is mass supercritical, the constrained functional $J|_{S_m}$ is no longer bounded from below and coercive. In this case, using a mountain-pass argument on S_m, Bellazzini, Jeanjean and Luo [12] proved the existence and the instability of standing waves for $m > 0$ sufficiently small. Bartsch and de Valeriola [6], Luo [38] studied the multiplicity of normalized solutions of (1.2).

At the same time, normalized solutions for Schrödinger–Poisson–Slater equation with general nonlinearity in case $\gamma = -1$, has also attracted much more attention. Xie, Chen and Shi [49] showed the existence and multiplicity results of solutions when f satisfies $\lim_{t\to 0} f(t)/t = 0$ and $\lim_{|t|\to\infty} F(t)/|t|^{10/3} = \infty$ under some mild conditions on f. Recently, Chen Tang and Yuan [17] investigated the existence of normalized solution by some new analytical techniques in case that f satisfies $\lim_{t\to 0} F(t)/t^2 = 0$ and $\lim_{|t|\to\infty} F(t)/|t|^{10/3} = \infty$.

Very recently, Wang and Qian [45] obtained the existence of normalized ground states and infinitely many radial solutions for (1.2) with *Sobolev subcritical term* f, by constructing a particular bounded Palais-Smale sequence when $\gamma < 0, a > 0$. Meanwhile, they obtained the nonexistence result in the case $\gamma < 0, a < 0$ and the existence result when$\gamma > 0, a < 0$ via variational methods. In [29], Jeanjean and Trung Le specialized in the existence of normalized solutions for problem (1.2) with L^2-supercritical growth:

$$\begin{cases} -\Delta u + \gamma(|x|^{-1} * |u|^2)u = \lambda u + a|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = c^2, \end{cases} \tag{1.3}$$

where $u \in H^1(\mathbb{R}^3), \gamma \in \mathbb{R}, a \in \mathbb{R}$ and $p \in (\frac{10}{3}, 6]$. The authors dealt with the following cases:

(a) If $\gamma < 0$ and $a > 0$, both in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$ and in the Sobolev critical case $p = 6$, they showed that there exists a $c_1 > 0$ such that, for any $c \in (0, c_1)$, (1.3) admits two solutions u_c^+ and u_c^-, which can be characterized as a local minimum and a mountain pass critical point of the associated energy functional, respectively.
(b) In the case $\gamma < 0$ and $a < 0$, they proved that, for any $p \in (\frac{10}{3}, 6]$ and any $c > 0$, (1.3) has a solution which is a global minimizer.
(c) Finally, in the case $\gamma > 0$, $a > 0$ and $p = 6$, they showed that (1.3) does not exist positive solutions.

When $\gamma = 1$, $p \in (\frac{10}{3}, 6)$, Bellazzini, Jeanjean and Luo [12] studied the existence of normalized solutions of (1.3) by a mountain-pass argument as $c > 0$ is sufficiently small and nonexistence as $c > 0$ is not small. In [31], Jeanjean and Luo considered the existence of minimizers with L^2-norm for (1.3) when $p \in [3, \frac{10}{3}]$, and they showed a threshold value of $c > 0$ separating existence and nonexistence of minimizers. For more results on normalized solutions of Schrödinger–Poisson systems, we refer to [1, 19, 27, 29, 31, 33, 37, 38, 50–53] and references therein.

After the above literature review, we find that, only the article [29] has considered the existence of normalized solutions of (1.3) in the case $p \in (\frac{10}{3}, 6]$, and $\gamma < 0$; and the no-existence of normalized solution of (1.3) with $p = 6$, $\gamma > 0$ and $a > 0$. Therefore, a natural and important question arising is how to obtain normalized solutions to system (1.3) in the case $\gamma > 0$, and in the presence of Sobolev critical exponent and mixed nonlinearities: $a|u|^{p-2}u + |u|^4u$? here $a|u|^{p-2}u$ is a subcritical perturbation term with $p \in (2, 6)$ and $a > 0$ a parameter. We notice that, this kind of critical nonlinearities has been used by Soave [42], Wei and Wu [47] to search for the normalized solutions for the Schrödinger equation

$$-\Delta u = \lambda u + a|u|^{p-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

with the prescribed L^2-norm $\int_{\mathbb{R}^3} |u|^2 dx = c^2$. But for the Schrödinger–Poisson system in presence of the Sobolev critical term $|u|^4u$, coupled with a subcritical perturbation term $a|u|^{p-2}u$, the existence of normalized solutions has not been studied in the existing literature, as far as we know. For more studies of existence of normalized solutions of the Schrödinger equation, see for example [28, 30, 42, 43, 54] and references therein.

Motivated by the works mentioned above, in this paper we focuss on studying the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \gamma \phi u = \lambda u + \mu |u|^{q-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

having prescribed L^2-norm

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2, \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is an undetermined parameter, $a > 0$ and $\mu, \gamma > 0$ are parameters, $\mu|u|^{q-2}u$ is a subcritical perturbation term with $q \in (2, 6)$. For this purpose, applying the reduction argument introduced in [40], system (1.4) is equivalent to the following single equation

$$-\Delta u + \gamma \phi_u u = \lambda u + \mu |u|^{q-2}u + |u|^4 u, \quad x \in \mathbb{R}^3, \tag{1.6}$$

where $\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy$. We shall look for solutions to (1.4)–(1.5), as a critical points of the action functional

$$I_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx,$$

under the L^2-norm constrained manifold

$$S(a) := \left\{ u \in H(\mathbb{R}^3) : \Psi(u) = \frac{1}{2}a^2 \right\},$$

where $\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^3} u^2 dx$. Physically, such type of solutions are the so-called normalized solutions to (1.4)–(1.5). In order to state our main results, we introduce some of the constants from the following Gagliardo-Nirenberg-Sobolev (GNS) inequality. That is, there exists a best constant $C(p)$ depending on p such that for any $u \in H^1(\mathbb{R}^3)$,

$$\|u\|_p^p \leq C(p)\|u\|_2^{(1-\gamma_p)p}\|\nabla u\|_2^{\gamma_p p}, \tag{1.7}$$

where $\gamma_p = \frac{3(p-2)}{2p}$. The constant $C(p)$ can be achieved by function Q_p, see [43]. For the problem obtained from (1.4)–(1.5) by removing the critical exponent term and the nonlocal term, we obtain one of its normalized solutions by rescaling Q_p. From $\gamma_p p = 2$, we get $p = \frac{10}{3}$ which is called the L^2-critical exponent for problem (1.4)–(1.5). Before presenting the existence result, we give the definition of ground states. If u^* is a solution to (1.4)–(1.5) having minimal energy among all the solutions which belongs to $S(a)$:

$$(I_\mu|_{S(a)})'(u^*) = 0 \text{ and } I_\mu(u^*) = \inf\{(I_\mu|_{S(a)})'(u) = 0, \quad \text{and } u \in S(a)\},$$

we say that u^* is a ground state of (1.4)–(1.5).

The following are the main results of this paper. In the L^2-subcritical case: $2 < q < \frac{10}{3}$, we have the following existence result of the normalized ground state solutions.

Theorem 1.1 *Let* $2 < q < \frac{10}{3}$, $\gamma > 0$, *and assume that* $0 < a < \min\{\alpha_1, \alpha_2\}$, *where*

$$\alpha_1 := \left\{ \frac{2q}{C(q)\mu(6-q\delta_q)} \left(\frac{(2-q\delta_q)S^3}{6-q\delta_q} \right)^{\frac{2-q\delta_q}{4}} \right\}^{\frac{1}{q(1-\delta_q)}},$$

and

$$\alpha_2 := \left\{ \frac{4}{C(q)\mu\delta_q(6-q\delta_q)} \left(\frac{q\delta_q S^{\frac{3}{2}}}{2-q\delta_q} \right)^{\frac{2-q\delta_q}{2}} \right\}^{\frac{1}{q(1-\delta_q)}},$$

where S *is defined in* (2.1). *Then there exists* $\tilde{\mu} > 0$ *such that* $\mu > \tilde{\mu}$, *problem* (1.4)–(1.5) *has a couple of solutions* $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$. *Moreover,*

$$I_\mu(u_a) = \inf_{u \in \mathcal{P}(a)} I_\mu(u) = \inf_{u \in \mathcal{P}(a)^+} I_\mu(u) = \inf_{u \in D_k} I_\mu(u),$$

for some suitable small constant $k > 0$, where $\mathcal{P}(a)$ is the Pohozaev manifold defined in Lemma 2.5, the set $\mathcal{P}(a)^+$ is defined in (3.1), and

$$D_k = \{u \in S(a) : \|\nabla u\|_{L^2(\mathbb{R}^3)} < k\}.$$

In the L^2-critical case: $q = \frac{10}{3}$, we have the following conclusion.

Theorem 1.2 *Let $q = \frac{10}{3}$, $\mu > 0$, and assume that $0 < a < \min\{\alpha_3, \alpha_4\}$, where*

$$\alpha_3 := \left(\frac{q}{2\mu C(q)} \right)^{\frac{1}{q(1-\delta_q)}},$$

and

$$\alpha_4 := \left(\frac{k^{\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}} \right)^{\frac{1}{3}},$$

where $\tilde{C}$ is defined as (2.3), and k is defined as

$$k = \min\left\{ \left(\frac{q(4\gamma\tilde{C}[C(12/5)]^{\frac{5}{3}})^{\frac{q(1-\delta_q)}{3}}}{32\mu C(q)} \right)^{\frac{6}{q(1-\delta_q)}}, \quad \left(\frac{3}{64}S^3\right)^{\frac{1}{2}} \right\}.$$

Then there exist $\tilde{\gamma_1}, \tilde{\gamma_2} > 0$ such that $0 < \gamma < \min\{\tilde{\gamma_1}, \tilde{\gamma_2}\}$, problem (1.4)–(1.5) has a couple of solutions $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$. Moreover,

$$I_\mu(u_a) = \inf_{u \in \mathcal{P}(a)} I_\mu(u) = \inf_{u \in \mathcal{P}(a)^-} I_\mu(u),$$

where $\mathcal{P}(a)^-$ is defined in (3.2).

In the L^2-supercritical case: $\frac{10}{3} < q < 6$, we have the following existence result.

Theorem 1.3 *Let $\frac{10}{3} < q < 6$, $\mu > 0$, and assume that $0 < a < \alpha_5$, where*

$$\alpha_5 := \left(\frac{{k^*}^{\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}} \right)^{\frac{1}{3}},$$

where k^* is defined in (5.1). Then there exist $\tilde{\gamma_1}, \tilde{\gamma_2} > 0$ such that $0 < \gamma < \min\{\tilde{\gamma_1}, \tilde{\gamma_2}\}$, problem (1.4)–(1.5) has a couple of solutions $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$. Moreover,

$$I_\mu(u_a) = \inf_{u \in \mathcal{P}(a)} I_\mu(u) = \inf_{u \in \mathcal{P}(a)^-} I_\mu(u).$$

Remark 1.1 In [29] Jeanjean and Le only studied the no-existence of normalized solutions of problem (1.4)–(1.5) with $\gamma > 0$ and $\mu = 0$. The existence of normalized solutions for (1.4)–(1.5) in the case $\gamma > 0$, $\mu > 0$ and $q \in (2, 6)$ has not been studied in the existing literature. Theorems 1.1–1.3 provide a complete description of the existence of normative solutions in the L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu|u|^{q-2}u$, respectively.

Remark 1.2 In Theorem 1.1, we assume that the parameter $\mu > 0$ is large enough, so as to ensure that the Lagrange multiplier sequence $\lambda_n \to \lambda < 0$ as $n \to \infty$, which plays a crucial role in our proof of the H^1-convergence of (PS)-sequence $\{u_n\} \subset S(a)$. In Theorem 1.2 and Theorem 1.3, it is necessary for the parameter $\gamma > 0$ to be appropriately small so that the Mountain Pass level is strictly less than $\frac{1}{3}S^{\frac{3}{2}}$. This characteristic is completely different from the critical Schrödinger–Poisson system without L^2-mass constrained, see for example [25, 26, 46, 55].

In order to prove Theorems 1.1–1.3, we apply the constrained variational methods. Note that the Sobolev critical terms $|u|^4u$ is L^2-supercritical, the functional I_μ is always unbounded from below on $S(a)$, and this causes difficulty to treat the existence of normalized solutions on the L^2- constraint. One of the main difficulties is to prove the convergence of constrained Palais-Smale sequences: Indeed, the Sobolev critical term $|u|^4u$ and nonlocal convolution term $\gamma\phi_u u$, make it more complex to estimate the critical value of mountain pass, and has to consider how the interaction between the nonlocal term and the mixed nonlinearities. In particularly, the energy balance between these competing terms needs to be controlled through moderate adjustments of parameter $\gamma > 0$. Another obstacle is that sequences of approximated Lagrange multipliers have to be controlled, since λ is not prescribed; and moreover, weak limits of Palais-Smale sequences could leave the constraint, since the embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ and also $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ are not compact.

To overcome these difficulties, we shall employ Jeanjean's theory [28] by showing that the mountain pass geometry of $I_\mu|_{S(a)}$ allows to construct a Palais-Smale sequence of functions satisfying the Pohozaev identity, to obtain the boundedness, which is the first step to show strong H^1-convergence. To restore the loss of compactness caused by the critical growth, we shall utilize the concentration-compactness principle, mountain pass theorem and energy analysis to get the existence of normalized ground states of (1.4)–(1.5), by showing that, suitably combining some of the main ideas from [15, 42], compactness can be derived in the present setting.

This paper is organized as follows: In Sect. 2 we summarize some preliminary results which will often be used in the rest the paper. In Sect. 3, we investigate the existence of normalized ground state solutions for system (1.4)–(1.5) under the L^2-subcritical perturbation case: $q \in (2, \frac{10}{3})$ and complete the proof Theorem 1.1. In Sect. 4, we address the presence of the normalized ground state solutions for system (1.4)–(1.5) in L^2-critical perturbation case: $q = \frac{10}{3}$ and prove Theorem 1.2, by employing manifold and mountain road theorems. In Sect. 5, we tackle the existence of the normalized ground state solutions for problem (1.4)–(1.5) under L^2-supercritical perturbation case: $q \in (\frac{10}{3}, 6)$ and prove Theorem 1.3.

Notations. Throughout this paper, we denote $B_r(z)$ the open ball of radius r with center at z in $\mathbb{R}^3$, and $\|u\|_p$ is the usual norm of the space $L^p(\mathbb{R}^3)$ for $p \geq 1$. Moreover, we denote by $C, C_i > 0, i = 1, 2, \cdots$, different positive constants whose values may vary from line to line and are not essential to the problem.

2 Preliminary Stuff

In this section, we will give the functional space setting and introduce some notations and useful preliminary results, which are important to proving our Theorems. Let $H^1(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2 dx\right)^{\frac{1}{2}}.$$

And the homogeneous Sobolev space $D^{1,2}(\mathbb{R}^3)$ is defined by

$$D^{1,2}(\mathbb{R}^3) = \left\{u \in L^6(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty\right\},$$

endowed with the norm

$$\|u\|^2 := \|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \|\nabla u\|_2^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

The work space $H^1_{rad}(\mathbb{R}^3)$ is defined by

$$H^1_{rad}(\mathbb{R}^3) := \left\{u \in H^1(\mathbb{R}^3) : \ u \text{ is radially symmetric and decreasing}\right\}.$$

Let $\mathbb{H} = H \times \mathbb{R}$ with usual scalar product

$$\langle \cdot, \cdot \rangle_{\mathbb{H}} = \langle \cdot, \cdot \rangle_H + \langle \cdot, \cdot \rangle_{\mathbb{R}},$$

and the corresponding norm

$$\|(\cdot, \cdot)\|^2_{\mathbb{H}} = \|\cdot, \cdot\|^2_H + |\cdot, \cdot|^2_{\mathbb{R}}.$$

We denote the best Sobolev constant S by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}}. \tag{2.1}$$

It is well know that S is achieved by

$$U_\varepsilon(x) = \frac{C^*\varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \tag{2.2}$$

for any $\varepsilon > 0$ and C^* being normalized constant such that (see [15]):

$$\int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |U_\varepsilon|^6 dx = S^{\frac{3}{2}}.$$

In the following, we recall some useful inequalities, which play an important part in the proof of our main results.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality [34]) *Let $l, r > 1$ and $0 < \mu < N$ be such that $\frac{1}{r} + \frac{1}{l} + \frac{\mu}{N} = 2$, $f \in L^r(\mathbb{R}^N)$ and $h \in L^l(\mathbb{R}^N)$. Then there exists a constant $C(N, \mu, r, l) > 0$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)h(y)|x-y|^{-\mu} dxdy \right| \le C(N, \mu, r, l)\|f\|_r \|h\|_l.$$

From Proposition 2.1, with $l = r = \frac{6}{5}$, we have that:

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \le \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * u^2 \right) u^2 dx \le \tilde{C}\|u\|^4_{\frac{12}{5}}. \tag{2.3}$$

Next, we introduce the following Gagliardo-Nirenberg inequality.

Lemma 2.2 ([43]) *Let $p \in (2, 6)$. Then there exists a constant $C(p) > 0$ such that*

$$\|u\|_p^p \le C(p)\|\nabla u\|_2^{p\delta_p} \|u\|_2^{p(1-\delta_p)}, \quad \forall u \in H^1(\mathbb{R}^3), \tag{2.4}$$

where $\delta_p = \frac{3(p-2)}{2p}$.

Lemma 2.3 (Lemma 5.1 [23]) *If $u_n \rightharpoonup u$ in $H^1_{rad}(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u u^2 dx, \tag{2.5}$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u u \varphi dx, \;\; \forall \varphi \in H^1_{rad}(\mathbb{R}^3). \tag{2.6}$$

In the sequel, we define a useful fiber map (e.g. [42]) preserving the L^2-norm

$$(\iota \star u)(x) := e^{\frac{3\iota}{2}} u(e^\iota x), \quad x \in \mathbb{R}^3, \;\; \iota \in \mathbb{R}. \tag{2.7}$$

By simple calculation, we can infer that

$$\|(\iota\star u)\|_2^2 = \|u\|_2^2, \tag{2.8}$$
$$\|(\iota\star u)\|_q^q = e^{q\delta_q\iota}\|u\|_q^q, \tag{2.9}$$

and

$$\|\nabla(\iota\star u)(x)\|_2^2 = e^{2\iota}\|\nabla(\iota\star u)(x)\|_2^2. \tag{2.10}$$

Next, we define a auxiliary functional $E : \mathbb{H} \to \mathbb{R}$ by

$$\begin{aligned} E(u,\iota) :&= I_\mu((\iota\star u)) \\ &= \frac{1}{2}e^{2\iota}\|\nabla u\|_2^2 + \frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2 dx - \frac{\mu}{q}e^{q\delta_q\iota}\int_{\mathbb{R}^3}|u|^q dx - \frac{1}{6}e^{6\iota}\int_{\mathbb{R}^3}|u|^6 dx. \end{aligned} \tag{2.11}$$

Besides, we have the fact that

$$q\delta_q \begin{cases} < 2, \text{ as } 2 < q < \bar{q}; \\ = 2, \text{ as } q = \bar{q}; \\ > 2, \text{ as } \bar{q} < q < 6, \end{cases}$$

where $\bar{q} := \frac{10}{3}$ is the L^2-critical exponent.

The Pohozaev manifold plays an important role in the proof of our main results, so we introduce it below [22].

Proposition 2.4 *Let $u \in H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ be a weak solution of (1.4), then u satisfies the equality*

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{5\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3}|u|^2 dx + \frac{3\mu}{q}\int_{\mathbb{R}^3}|u|^q dx + \frac{1}{2}\int_{\mathbb{R}^3}|u|^6 dx. \tag{2.12}$$

Lemma 2.5 *Let $u \in H^1(\mathbb{R}^3)$ be a weak solution of (1.4)–(1.5), then we can construct the following Pohozaev manifold*

$$\mathcal{P}(a) = \{u \in S(a) : P_\mu(u) = 0\},$$

where

$$P_\mu(u) = \int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \mu\delta_q\int_{\mathbb{R}^3}|u|^q dx - \int_{\mathbb{R}^3}|u|^6 dx. \tag{2.13}$$

Proof Since u is the weak solution of (1.4)–(1.5), we have that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{5\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3}|u|^2 dx + \frac{3\mu}{q}\int_{\mathbb{R}^3}|u|^q dx + \frac{1}{2}\int_{\mathbb{R}^3}|u|^6 dx.$$

Moreover, since u is the weak solution of system (1.4)–(1.5), we have

$$\int_{\mathbb{R}^3}|\nabla u|^2dx+\gamma\int_{\mathbb{R}^3}\phi_u u^2dx=\lambda\int_{\mathbb{R}^3}|u|^2dx+\mu\int_{\mathbb{R}^3}|u|^qdx+\int_{\mathbb{R}^3}|u|^6dx.$$

Combining with (2.13) and the above equality, we obtain that

$$\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2dx=\mu\delta_q\int_{\mathbb{R}^3}|u|^qdx+\int_{\mathbb{R}^3}|u|^6dx.$$

The proof is completed. □

We define $\varphi_u(\iota):=E(u,\iota)$ for any $u\in S(a)$ and $\iota\in\mathbb{R}$, then

$$\begin{aligned}(\varphi_u)'(\iota)&=e^{2\iota}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2dx-\mu\delta_q e^{q\delta_q\iota}\int_{\mathbb{R}^3}|u|^qdx-e^{6\iota}\int_{\mathbb{R}^3}|u|^6dx\\&=\int_{\mathbb{R}^3}|\nabla(\iota\star u)|^2dx+\frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_{(\iota\star u)}|\iota\star u|^2dx-\mu\delta_q\int_{\mathbb{R}^3}|\iota\star u|^qdx-\int_{\mathbb{R}^3}|\iota\star u|^6dx\\&=P_\mu((\iota\star u)).\end{aligned}\tag{2.14}$$

Moreover, by direct calculation, we have

$$\begin{aligned}(\varphi_u)''(\iota)=2e^{2\iota}&\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2dx\\-\mu q\delta_q{}^2e^{q\delta_q\iota}&\int_{\mathbb{R}^3}|u|^qdx-6e^{6\iota}\int_{\mathbb{R}^3}|u|^6dx.\end{aligned}\tag{2.15}$$

Therefore, we have the following lemma:

Lemma 2.6 *For any $u\in S(a)$, $\iota\in\mathbb{R}$ is a critical point of $\varphi_u(\iota)$ if and only if $(\iota\star u)\in\mathcal{P}(a)$. Particularly, $u\in\mathcal{P}(a)$ if and only if 0 is a critical point for $\varphi_u(\iota)$.*

Finally, we state the following well-known embedding result.

Lemma 2.7 ([44]) *Let $N\geq 2$. The embedding $H^1_{rad}(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is compact for any $2<p<2^*$.*

Remark 2.8 ([11]) The map $(u,\iota)\in\mathbb{H}\to(\iota\star u)\in H$ is continuous.

3 L^2-Subcritical Perturbation Case

In this section, we shall address the L^2-subcritical perturbation case: $2<q<\frac{10}{3}$ and provide the proof of Theorem 1.1. First, we think about a decomposition of $\mathcal{P}(a)$ as in [42, 43]. By Lemma 2.6, we define the following sets:

$$\begin{aligned}\mathcal{P}(a)^+&:=\left\{u\in\mathcal{P}(a):2\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2dx>\mu q\delta_q{}^2\int_{\mathbb{R}^3}|u|^qdx+6\int_{\mathbb{R}^3}|u|^6dx\right\}\\&=\{u\in\mathcal{P}(a):(\varphi_u)''(0)>0\},\end{aligned}\tag{3.1}$$

$$\mathcal{P}(a)^0 := \left\{u \in \mathcal{P}(a) : 2\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx = \mu q {\delta_q}^2 \int_{\mathbb{R}^3} |u|^q dx + 6\int_{\mathbb{R}^3} |u|^6 dx\right\} \quad (3.2)$$
$$= \{u \in \mathcal{P}(a) : (\varphi_u)''(0) = 0\},$$
$$\mathcal{P}(a)^- := \left\{u \in \mathcal{P}(a) : 2\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx < \mu q {\delta_q}^2 \int_{\mathbb{R}^3} |u|^q dx + 6\int_{\mathbb{R}^3} |u|^6 dx\right\}$$
$$= \{u \in \mathcal{P}(a) : (\varphi_u)''(0) < 0\}. \quad (3.3)$$

We can easily get that

$$\mathcal{P}(a) = \mathcal{P}(a)^+ \cup \mathcal{P}(a)^0 \cup \mathcal{P}(a)^-.$$

Next, we will give some lemmas, which are useful for the proof of Theorem 1.1.

Lemma 3.1 *Let* $2 < q < \frac{10}{3}$, $\mu, \gamma > 0$, *and* $0 < a < \alpha_1$, *where*

$$\alpha_1 := \left\{\frac{2q}{C(q)\mu(6 - q\delta_q)}\left(\frac{(2 - q\delta_q)S^3}{6 - q\delta_q}\right)^{\frac{2-q\delta_q}{4}}\right\}^{\frac{1}{q(1-\delta_q)}}.$$

Then $\mathcal{P}(a)^0 = \varnothing$ *and* $\mathcal{P}(a)$ *is a smooth manifold of codimension* 2 *in* $H(\mathbb{R}^3)$.

Proof Suppose by contradiction that $\mathcal{P}(a)^0 \neq \varnothing$. Taking $u \in \mathcal{P}(a)^0$, one has

$$2\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx = \mu q {\delta_q}^2 \int_{\mathbb{R}^3} |u|^q dx + 6\int_{\mathbb{R}^3} |u|^6 dx,$$

and

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx = \mu \delta_q \int_{\mathbb{R}^3} |u|^q dx + \int_{\mathbb{R}^3} |u|^6 dx.$$

Since $\frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \geq 0$, so combining the above equalities with the GNS inequality (2.4) and (2.1), we can infer to

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq \frac{6 - q\delta_q}{2 - q\delta_q}\int_{\mathbb{R}^3} |u|^6 dx \leq \frac{6 - q\delta_q}{(2 - q\delta_q)S^3}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^3,$$

and

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq \frac{\mu\delta_q(6 - q\delta_q)}{4}\int_{\mathbb{R}^3} |u|^q dx$$
$$\leq \frac{\mu\delta_q(6 - q\delta_q)}{4} C(q) a^{q(1-\delta_q)}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{q\delta_q}{2}}.$$

By simple calculation and the fact $q\delta_q < 2$, we have

$$\begin{aligned} a^{q(1-\delta_q)} &\geq \frac{4}{\mu\delta_q C(q)(6-q\delta_q)} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{2-q\delta_q}{2}} \\ &\geq \frac{4}{\mu\delta_q C(q)(6-q\delta_q)} \left(\frac{(2-q\delta_q)S^3}{6-q\delta_q}\right)^{\frac{2-q\delta_q}{4}} \\ &\geq \frac{2q}{\mu C(q)(6-q\delta_q)} \left(\frac{(2-q\delta_q)S^3}{6-q\delta_q}\right)^{\frac{2-q\delta_q}{4}} \\ &:= \alpha_1^{q(1-\delta_q)}, \end{aligned}$$

which contradicts to $a < \alpha_1$.

Then, we verify that $\mathcal{P}(a)$ is a smooth manifold of codimension 2 in $H(\mathbb{R}^3)$. Let

$$\mathcal{P}(a) = \{u \in H : P_\mu(u) = 0, G(u) = 0\},$$

for $G(u) = \|u\|_2^2 - a^2$, with P_μ and G of class C^1 in H. Hence, we need to show that the differential $(dG(u), dP_\mu(u)) : H \to \mathbb{R}^2$ is surjective, for every $u \in \mathcal{P}(a)$. For this purpose, we will prove that for every $u \in \mathcal{P}(a)$, there exists $\varphi \in T_u S$, where

$$T_u S := \{v \in E : (u, v)_H = 0\},$$

which is the tangent space of $\mathcal{S}$ at a point $u \in \mathcal{S}$. Then, one has $dP_\mu(u)[\varphi] \neq 0$. Once the existence of φ is established, the system

$$\begin{cases} dG(u)[\alpha\varphi + \beta u] = x \\ dP_\mu(u)[\alpha\varphi + \beta u] = y \end{cases}$$

that is

$$\begin{cases} \beta a^2 = x \\ \alpha dP_\mu(u)[\varphi] + \beta dP_\mu(u)[u] = y \end{cases}$$

is solvable with respect to α, β for every $(x, y) \in \mathbb{R}^2$, so the surjective is proved. Next, suppose by contrary that for $u \in \mathcal{P}(a)$ such that a tangent vector φ does not exist, that is, $dP_\mu(u)[\varphi] = 0$ for every $\varphi \in T_u S$. Then u is a constrained critical point for the functional $P_\mu(u)$ on $S(a)$. Thus, by the Lagrange multipliers rule, there exists $\nu \in \mathbb{R}$ such that

$$-\Delta u + \frac{\gamma}{2}\phi_u u = \nu u + \frac{\mu q\delta_q}{2}|u|^{q-2}u + 3|u|^4 u, \quad \text{in } \mathbb{R}^3.$$

Then we can conclude the following Pohozaev type identity:

$$2\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2dx=\mu q{\delta_q}^2\int_{\mathbb{R}^3}|u|^q dx+6\int_{\mathbb{R}^3}|u|^6dx,$$

which is contradiction to the fact that $u\in\mathcal{P}(a)$. □

In virtue of the GNS inequality (2.4) and (2.1), for every $u\in H(\mathbb{R}^3)\bigcap S(a)$, we have

$$\begin{aligned}I_\mu(u)&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\gamma}{4}\int_{\mathbb{R}^3}\phi_u u^2dx-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^q dx-\frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx\\&\geq\frac{1}{2}\|\nabla u\|_2^2-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q}-\frac{1}{6S^3}\|\nabla u\|_2^6\\&:=g(\|\nabla u\|_2),\end{aligned}\tag{3.4}$$

where

$$g(t)=\frac{1}{2}t^2-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}t^{q\delta_q}-\frac{1}{6S^3}t^6.$$

By the fact $q\delta_q<2$, we can derive that $g(0^+)=0^-$ and $g(+\infty)=-\infty$.

In the following, we show the properties of the function g and give some technical lemmas.

Lemma 3.2 *Let $2<q<\frac{10}{3}$, $\mu,\gamma>0$, and $0<a<\alpha_1$. Then the function g has a local strict minimum at negative level and a global strict maximum at positive level, and there exist two positive constants R_1, R_2 both depending on a, with $R_1<R_2$, such that $g(R_1)=g(R_2)=0$ and $g(t)>0$ for $t\in(R_1,R_2)$.*

Proof Note that

$$\begin{aligned}g(t)&=\frac{1}{2}t^2-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}t^{q\delta_q}-\frac{1}{6S^3}t^6\\&=t^{q\delta_q}\left(\frac{1}{2}t^{2-q\delta_q}-\frac{1}{6S^3}t^{6-q\delta_q}-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}\right)\\&=t^{q\delta_q}m(t),\end{aligned}$$

where

$$m(t)=\frac{1}{2}t^{2-q\delta_q}-\frac{1}{6S^3}t^{6-q\delta_q}-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}.$$

It is easy to see that $g(t)>0$ if and only if $m(t)>0$ for all $t>0$. So, by direct calculation, we have

$$m'(t)=\frac{2-q\delta_q}{2}t^{1-q\delta_q}-\frac{6-q\delta_q}{6S^3}t^{5-q\delta_q}.$$

Let $m'(t) = 0$, it follows that

$$t_1 = \left(\frac{3S^3(2 - q\delta_q)}{6 - q\delta_q}\right)^{\frac{1}{4}},$$

and we know that m is strictly increasing on $(0, t_1)$ and decreasing on (t_1, ∞). Moreover, the maximum value of m on $(0, +\infty)$ is

$$\begin{aligned}
m(t_1) &= \frac{1}{2}\left(\frac{3S^3(2 - q\delta_q)}{6 - q\delta_q}\right)^{\frac{2-q\delta_q}{4}} - \frac{1}{6S^3}\left(\frac{3S^3(2 - q\delta_q)}{6 - q\delta_q}\right)^{\frac{6-q\delta_q}{4}} - \frac{\mu}{q}C(q)a^{q(1-\delta_q)} \\
&= \frac{2}{6 - q\delta_q}\left(\frac{3S^3(2 - q\delta_q)}{6 - q\delta_q}\right)^{\frac{2-q\delta_q}{4}} - \frac{\mu}{q}C(q)a^{q(1-\delta_q)} \\
&> \frac{2}{6 - q\delta_q}\left(\frac{S^3(2 - q\delta_q)}{6 - q\delta_q}\right)^{\frac{2-q\delta_q}{4}} - \frac{\mu}{q}C(q)a^{q(1-\delta_q)} \\
&= \frac{\mu}{q}C(q)\alpha_1^{q(1-\delta_q)} - \frac{\mu}{q}C(q)a^{q(1-\delta_q)}.
\end{aligned}$$

By virtue of $a < \alpha_1$, we deduce that there exist two constants R_1 and R_2 such that

$$g(t)\begin{cases} < 0, & \text{if } t \in (0, R_1) \text{ or } (R_2, \infty); \\ = 0, & \text{if } t = R_1 \text{ or } R_2; \\ > 0, & \text{if } t \in (R_1, R_2). \end{cases}$$

Based on above analysis and the fact $g(0^+) = 0^-$, we infer that $g(t)$ has a global maximum at positive level in (R_1, R_2) and a local minimum at negative level in $(0, R_1)$. It is easy to see that $R_1 < t_1 < R_2$. Besides, by a simple calculation, we have

$$\begin{aligned}
g'(t) =& q\delta_q t^{q\delta_q - 1}\left(\frac{1}{2}t^{2-q\delta_q} - \frac{1}{6S^3}t^{6-q\delta_q} - \frac{\mu}{q}C(q)a^{q(1-\delta_q)}\right) \\
&+ t^{q\delta_q}\left(\frac{2 - q\delta_q}{2}t^{1-q\delta_q} - \frac{6 - q\delta_q}{6S^3}t^{5-q\delta_q}\right) \\
=& t^{q\delta_q - 1}\left(t^{2-q\delta_q} - \frac{1}{S^3}t^{6-q\delta_q} - \mu\delta_q C(q)a^{q(1-\delta_q)}\right) \\
:=& t^{q\delta_q - 1}h(t),
\end{aligned}$$

where

$$h(t) = t^{2-q\delta_q} - \frac{1}{S^3}t^{6-q\delta_q} - \mu\delta_q C(q)a^{q(1-\delta_q)}.$$

It is easy to see that $g'(t) = 0$ if and only if $h(t) = 0$ for $t > 0$. So, by direct calculation, we get

$$h'(t) = (2 - q\delta_q)t^{1-q\delta_q} - \frac{6 - q\delta_q}{S^3} t^{5-q\delta_q}.$$

From $h'(t) = 0$, there exists a unique solution $t_2 > 0$ with the expression:

$$t_2 = \left(\frac{S^3(2 - q\delta_q)}{6 - q\delta_q} \right)^{\frac{1}{4}},$$

and we know that h is a strictly increasing on $(0, t_2)$ and decreasing on $(t_2, +\infty)$. Hence, h has at most two zeros on $(0, +\infty)$, which are necessarily the previously found local minimum and the global maximum of g. □

Lemma 3.3 *Let* $2 < q < \frac{10}{3}$, $\mu, \gamma > 0$, *and* $0 < a < \alpha_1$. *Then for every* $u \in S(a)$, $\varphi_u(\iota)$ *has two critical points* $s_u < t_u \in \mathbb{R}$ *and two zeros* $c_u < d_u$ *with* $s_u < c_u < t_u < d_u$. *Besides,*

(i) $s_u \star u \in \mathcal{P}(a)^+$, $t_u \star u \in \mathcal{P}(a)^-$, *and if* $\iota \star u \in \mathcal{P}(a)$, *then either* $\iota = s_u$ *or* $\iota = t_u$;
(ii) $\|\nabla u\|_2 \leq R_1$ *for every* $\iota < c_u$ *and*

$$I_\mu(s_u \star u) = \min\{I_\mu(\iota \star u) : \iota \in \mathbb{R} \text{ and } \|\nabla u\|_2 \leq R_1\} < 0; \tag{3.5}$$

(iii) *we have*

$$I_\mu(t_u \star u) = \max\{I_\mu(\iota \star u) : \iota \in \mathbb{R}\} > 0, \tag{3.6}$$

and $\varphi_u(\iota)$ *is strictly decreasing and concave on* $(t_u, +\infty)$;
(iv) *the maps* $u \in \mathcal{P}(a) \mapsto s_u \times \mathbb{R}$ *and* $u \in \mathcal{P}(a) \mapsto t_u \times \mathbb{R}$ *are of class* C^1.

Proof We claim that $\varphi_u(\iota)$ has two critical points. In view of (3.4), one has

$$\begin{aligned}
\varphi_u(\iota) &= I_\mu((\iota \star u)) \\
&\geq \frac{1}{2}\|\nabla(\iota \star u)\|_2^2 - \frac{\mu}{q} C(q) a^{q(1-\delta_q)} \|\nabla(\iota \star u)\|_2^{q\delta_q} - \frac{1}{6S^3}\|\nabla(\iota \star u)\|_2^6 \\
&= g(\|\nabla(\iota \star u)\|_2) = g(e^\iota \|\nabla u\|_2),
\end{aligned}$$

we can obtain $\varphi_u(\iota) > 0$ on $(\xi(R_1), \xi(R_2))$ from Lemma 3.2, where

$$\xi(R) = \log R - \log \|\nabla u\|_2.$$

Since $\varphi_u(\iota)$ is a C^2 function, and by the fact that $\varphi_u(-\infty) = 0^-$, $\varphi_u(+\infty) = -\infty$, it follows that $\varphi_u(\iota)$ has at least critical points s_u, t_u with $s_u < t_u$. Moreover, we know that s_u is a local minimum point on $(-\infty, \xi(R_1))$ at negative level and t_u is a global maximum point at positive level. Hence, we derive to

$$I_\mu(t_u \star u) = \max\{I_\mu(\iota \star u) : \iota \in \mathbb{R}\} > 0,$$

$$\|\nabla(s_u \star u)\|_2 = e^{s_u}\|\nabla u\|_2 \leq e^{\xi(R_1)}\|\nabla u\|_2 = R_1, \tag{3.7}$$

and

$$I_\mu(s_u \star u) = \min\{I_\mu(\iota \star u) : \iota \in \mathbb{R} \text{ and } \|\nabla u\|_2 \leq R_1\} < 0.$$

Arguing as in the proof of Lemma 3.2, we can deduce that $\varphi_u(\iota)$ has no other critical points. In view of $(\varphi_u)''(s_u) \geq 0$, $(\varphi_u)''(t_u) \leq 0$ and the fact that $\mathcal{P}(a)^0 = \emptyset$, we have $s_u \star u \in \mathcal{P}(a)^+$ and $t_u \star u \in \mathcal{P}(a)^-$.

Next, we claim that $\varphi_u(\iota)$ has two zeros $c_u < d_u$. Since $\varphi_u(s_u) < 0$, $\varphi_u(t_u) > 0$ and $\varphi_u(+\infty) = -\infty$, it is easy to get that $\varphi_u(\iota)$ has two zeros $c_u < d_u$ with $s_u < c_u < t_u < d_u$. Furthermore, $\varphi_u(\iota)$ has no other zeros. Indeed, if $\varphi_u(\iota)$ has other zeros, then it will have other critical point, which leads to a contradiction.

Recalling that

$$\begin{aligned}(\varphi_u)''(\iota) = 2e^{2\iota}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2dx\\ -\mu q{\delta_q}^2 e^{q\delta_q\iota}\int_{\mathbb{R}^3}|u|^q dx - 6e^{6\iota}\int_{\mathbb{R}^3}|u|^6dx,\end{aligned}$$

we have $(\varphi_u)''(-\infty) = 0^-$. Since $(\varphi_u)''(s_u) > 0$ and $(\varphi_u)''(t_u) < 0$, we get $(\varphi_u)''(\iota)$ has two zeros, which means that $\varphi_u(\iota)$ has two inflection points. Arguing as before, $(\varphi_u)''(\iota)$ has exactly two inflection points. Hence, $\varphi_u(\iota)$ is is strictly decreasing and concave on $(t_u, +\infty)$. The items (i)–(iii) are proved.

Finally, we will prove that maps $u \in \mathcal{P}(a) \mapsto s_u \times \mathbb{R}$ and $u \in \mathcal{P}(a) \mapsto t_u \times \mathbb{R}$ are of class C^1. Applying the implicit function theorem, let $\Phi(\iota, u) := (\varphi_u)'(\iota) > 0$, since $\Phi(s_u, u) = 0$ and $\partial_\iota\Phi(s_u, u) > 0$, we know that $u \in \mathcal{P}(a) \mapsto s_u \times \mathbb{R}$ is of class C^1. Similarly, we have $u \in \mathcal{P}(a) \mapsto t_u \times \mathbb{R}$ is of class C^1. □

Thus, we can easily deduce the following conclusion.

Corollary 3.4 $\sup_{u\in\mathcal{P}(a)^+} I_\mu(u) \leq 0 \leq \inf_{u\in\mathcal{P}(a)^-} I_\mu(u)$ *and* $\mathcal{P}(a)^+ \subset D_{R_1}$, *where*

$$D_{R_1} := \{u \in S(a) : \|\nabla u\|_2 < R_1\}, \quad \textit{for} \quad R_1 > 0.$$

Lemma 3.5 *There holds that* $-\infty < m_\mu(a) = \inf_{u\in\mathcal{P}(a)} I_\mu(u) = \inf_{u\in\mathcal{P}(a)^+} I_\mu(u) < 0$, *and*

$$m_\mu(a) < \inf_{u\in\overline{D_{R_1}}\setminus D_{R_1-\rho}} I_\mu(u),$$

for $\rho > 0$ small enough, where

$$m_\mu(a) := \inf_{u \in D_{R_1}} I_\mu(u).$$

Proof For $u \in D_{R_1}$, in view of (3.4), we have

$$I_\mu(u) \geq g(\|\nabla u\|_2) \geq \min_{t \in [0, R_1]} g(t) > -\infty.$$

Besides, for any $u \in S(a)$, we get $\|\nabla u\|_2 < R_1$ and $I_\mu(s_u \star u) < 0$. Hence, we can infer to

$$m_\mu(a) < 0.$$

On one hand, since $\mathcal{P}(a)^+ \subset D_{R_1}$, we get that $m_\mu(a) \leq \inf_{\mathcal{P}(a)^+} I_\mu$. On the other hand, if $u \in D_{R_1}$, then $s_u \star u \in \mathcal{P}(a)^+ \subset D_{R_1}$, and

$$I_\mu(s_u \star u) = \min\{I_\mu(\iota \star u) : \iota \in \mathbb{R} \text{ and } \|\nabla u\|_2 \leq R_1\} \leq I_\mu(u),$$

which implies that $\inf_{u \in \mathcal{P}(a)^+} I_\mu(u) \leq m_\mu(a)$. Combining with the fact $0 \leq \inf_{u \in \mathcal{P}(a)^-} I_\mu(u)$, we obtain

$$\inf_{u \in \mathcal{P}(a)} I_\mu(u) = \inf_{u \in \mathcal{P}(a)^+} I_\mu(u).$$

Finally, due to the continuity of g and $g(R_1) = 0$, there exists $\rho > 0$ such that

$$g(t) \geq \frac{m_\mu(a)}{2}, \quad t \in [R_1 - \rho, R_1].$$

Therefore, by (3.4), we have

$$I_\mu(u) \geq g(\|\nabla u\|_2) \geq \frac{m_\mu(a)}{2} \geq m_\mu(a),$$

for any $u \in \overline{D}_{R_1} \setminus D_{R_1 - \rho}$. The proof is completed. □

Proof of Theorem 1.1 First, we take a minimizing sequence $\{v_n\} \subset H \cap S(a)$ for $I_\mu|_{D_{R_1}}$ and assume that $\{v_n\} \subset H_r$ are radially decreasing for every n. Otherwise, we can let $v_n := |v_n|^*$, which is the Schwarz rearrangement of $|v_n|$. In view of Lemmas 3.3 and 3.5, we know that there exists a sequence $\{s_{v_n}\}$ such that $s_{v_n} \star v_n \in \mathcal{P}(a)^+$ and $I_\mu(s_{v_n} \star v_n) \leq I_\mu(v_n)$ for every n. Furthermore, we have $s_{v_n} \star v_n \notin \overline{D}_{R_1} \setminus D_{R_1 - \rho}$. Based on above analysis, we get a new minimizing sequence $\{\overline{v}_n := s_{v_n} \star v_n\}$ for $I_\mu|_{D_{R_1}}$, satisfying

$$\overline{v}_n \in H_r \cap \mathcal{P}(a)^+ \quad \text{and} \quad \|\nabla \overline{v}_n\|_2 \leq R_1 - \rho.$$

By Ekeland's variational principle, there exists a new minimizing sequence $\{u_n\}$, with $\|u_n - \overline{v}_n\| \to 0$ as $n \to \infty$, which is also a PS sequence for I_μ on $S(a)$. Since $\{u_n\} \subset D_{R_1}$, we see that $\{u_n\}$ is bounded in H. So from $\|u_n - \overline{v}_n\| \to 0$ and the boundedness of $\{u_n\}$, we can obtain

$$P_\mu(u_n) = P_\mu(\overline{v}_n) + o_n(1) \to 0 \qquad \text{as } n \to \infty.$$

In fact,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u_n|^2 dx &= \int_{\mathbb{R}^3} |\nabla \overline{v}_n|^2 dx + \int_{\mathbb{R}^3} |\nabla (u_n - \overline{v}_n)|^2 dx + \int_{\mathbb{R}^3} \nabla u_n \nabla \overline{v}_n dx \\
&= \int_{\mathbb{R}^3} |\nabla \overline{v}_n|^2 dx + o_n(1), \\
\frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx &= \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{\overline{v}_n} \overline{v}_n^2 dx + \gamma \int_{\mathbb{R}^3} \phi_{(\overline{v}_n + \theta_n^1 (u_n - \overline{v}_n))} |\overline{v}_n + \theta_n^1 (u_n - \overline{v}_n)| (u_n - \overline{v}_n) dx \\
&= \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{\overline{v}_n} \overline{v}_n^2 dx + o_n(1),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_n|^p dx &= \int_{\mathbb{R}^3} |\overline{v}_n|^p dx + \int_{\mathbb{R}^3} p |\overline{v}_n + \theta_n^2 (u_n - \overline{v}_n)|^{p-1} (u_n - \overline{v}_n) dx \\
&= \int_{\mathbb{R}^3} |\overline{v}_n|^p dx + o_n(1),
\end{aligned}$$

for every $p \in [2, 6]$, where $\theta_n^1,\ \theta_n^2 \in [0, 1]$. Moreover, $\{u_n\}$ satisfies

$$\begin{cases} I_\mu(u_n) \to m_\mu(a) & \text{as } n \to \infty \\ I'_\mu|_{S(a)}(u_n) \to 0 & \text{as } n \to \infty \end{cases} \tag{3.8}$$

Then, using the Lagrange multipliers rule, there exists a sequence $\lambda_n \in \mathbb{R}$ such that

$$I'_\mu(u_n) - \lambda_n \Psi'(u_n) \to 0 \quad \text{in } H^{-1}. \tag{3.9}$$

Since $\{u_n\} \subset D_{R_1}$, we have $\{u_n\}$ is bounded in H. So there exists $u_a \in H$, such that, for some subsequence, $u_n \rightharpoonup u_a$ in H. In the following, we will proceed with our argument in three steps.

Step 1 We show that, up to subsequence, $\lim_{n \to +\infty} \lambda_n = \lambda_a < 0$. By (3.9) and the fact $\{u_n\}$ is bounded in H, we get

$$I'_\mu(u_n) u_n - \lambda_n \Psi'(u_n) u_n = o_n(1). \tag{3.10}$$

Then, we infer to

$$\lambda_n \|u_n\|_2^2 = \|\nabla u_n\|_2^2 + \gamma \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \mu \|u_n\|_q^q - \|u_n\|_6^6 + o_n(1). \tag{3.11}$$

Again by $\{u_n\}$ is bounded in H, we see that $\{\lambda_n\}$ is bounded. Thus, up to subsequence, there exists $\lambda_a \in \mathbb{R}$ such that $\lambda_n \to \lambda_a \in \mathbb{R}$. Next, we prove $\lambda_a < 0$. Before this, we show

$$\int_{\mathbb{R}^3} |u_n|^q dx \to \int_{\mathbb{R}^3} |u_a|^q dx \neq 0, \text{ i.e. } u_a \neq 0.$$

Assume by contradiction that, $\int_{\mathbb{R}^3} |u_n|^q dx \to 0$. In view of the proof of Lemma 3.2, we have $\|\nabla u_n\|_2 \leq R_1 < t_1$, and $t_1 = \left(\frac{3S^3(2-q\delta_q)}{6-q\delta_q}\right)^{\frac{1}{4}} < S^{\frac{3}{4}}$. Then we deduce

$$\|\nabla u_n\|_2 < S^{\frac{3}{4}}.$$

From the definition of I_μ and above inequality, we infer that

$$\begin{aligned} 0 > m_\mu(a) &= \lim_{n\to\infty} I_\mu(u_n) \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n|^q dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx\right] \\ &\geq \lim_{n\to\infty}\left[\frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{6}S^{-3}\|\nabla u_n\|_2^6 - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n|^q dx\right] \\ &\geq -\frac{\mu}{q}\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n|^q dx = 0, \end{aligned}$$

which is absurd.

We claim that there exists $\tilde{\mu} > 0$ independently on $n \in \mathbb{N}$ such that, if $\mu > \tilde{\mu}$, the lagrange multiplier $\lambda_a < 0$. In fact, since $\{u_n\} \subset D_{R_1}$, by (2.3) and the GNS inequality (2.4), there exists $T_1 > 0$ independently on $n \in \mathbb{N}$ such that

$$\begin{aligned} T_1 \leq \int_{\mathbb{R}^3} |u_n|^q dx &\leq C(q)\|\nabla u_n\|_2^{q\delta_q}\|u_n\|_2^{q(1-\delta_q)} \\ &\leq C(q) R_1^{q\delta_q} a^{q(1-\delta_q)}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq \tilde{C}\|u_n\|_{\frac{12}{5}}^4 &\leq \tilde{C}[C(12/5)]^{\frac{5}{3}}\|\nabla u_n\|_2\|u_n\|_2^3 \\ &\leq \tilde{C}[C(12/5)]^{\frac{5}{3}} R_1 a^3 := T_2, \end{aligned} \tag{3.13}$$

where $T_2 = T_2(R_1, a) > 0$. We define the constant

$$\tilde{\mu} := \frac{3\gamma T_2}{4(1-\delta_q)T_1}. \tag{3.14}$$

By (3.12)–(3.14) we have

$$\tilde{\mu} \geq \lim_{n\to+\infty} \left\{ \frac{3\gamma \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx}{4(1-\delta_q)\int_{\mathbb{R}^3} |u_n|^q dx} \right\} = \frac{3\gamma \int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx}{4(1-\delta_q)\int_{\mathbb{R}^3} |u_a|^q dx} > 0. \tag{3.15}$$

By the fact $P_\mu(u_n) \to 0$, (3.11), Lemma 2.7 and $\delta_q < 1$, if $\mu > \tilde{\mu}$, then

$$\begin{aligned} \lambda_a a^2 &= \lim_{n\to+\infty} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \mu \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^6 dx \right) \\ &= \lim_{n\to+\infty} \left(\frac{3}{4}\gamma \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \mu(1-\delta_q) \int_{\mathbb{R}^3} |u_n|^q dx \right) \\ &= \frac{3}{4}\gamma \int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx - \mu(1-\delta_q) \int_{\mathbb{R}^3} |u_a|^q dx \\ &< \frac{3}{4}\gamma \int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx - \tilde{\mu}(1-\delta_q) \int_{\mathbb{R}^3} |u_a|^q dx \leq 0. \end{aligned}$$

Thus, if $\mu > \tilde{\mu}$, we have $\lim_{n\to+\infty} \lambda_n = \lambda_a < 0$.

Step 2 Since $\lambda_a < 0$, we define an equivalent norm of H as:

$$\|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^3} |u|^2 dx.$$

In view of the fact $u_n \rightharpoonup u_a$ in H and (3.9), then u_a satisfies

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla u_a \nabla v dx + \gamma \int_{\mathbb{R}^3} \phi_{u_a} u_a v dx - \lambda_a \int_{\mathbb{R}^3} u_a v dx \\ &-\mu \int_{\mathbb{R}^3} |u_a|^{q-2} u_a v dx - \int_{\mathbb{R}^3} |u_a|^4 u_a v dx = 0, \end{aligned} \tag{3.16}$$

for $\forall v \in H$. It follows from the Pohozaev identity that $P_\mu(u_a) = 0$. Let $v_n = u_n - u_a \rightharpoonup 0$, by Brezis–Lieb Lemma [48], we conclude

$$\begin{cases} \|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2 - \|\nabla u_a\|_2^2 + o_n(1), \\ \|v_n\|_6^6 = \|u_n\|_6^6 - \|u_a\|_6^6 + o_n(1). \end{cases} \tag{3.17}$$

By the fact (2.5), Lemma 2.7 and $P_\mu(v_n) = P_\mu(u_n) - P_\mu(u_a) \to 0$, we obtain

$$\|\nabla v_n\|_2^2 = \|v_n\|_6^6 + o_n(1).$$

Thus, for some subsequence, we suppose that

$$\|\nabla v_n\|_2^2 = \|v_n\|_6^6 \to \tau.$$

By using (2.1), we derive to

$$\tau^{\frac{1}{3}} \leq \frac{\tau}{S},$$

then, one has

$$\tau \geq S^{\frac{3}{2}} \quad \text{or} \quad \tau = 0.$$

If $\tau \geq S^{\frac{3}{2}}$, by (3.17), we have

$$\begin{aligned} m_\mu(a) &= \lim_{n\to+\infty} I_\mu(u_n) \\ &= \lim_{n\to+\infty} \left(I_\mu(u_a) + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{6}\|v_n\|_6^6 \right) \\ &= I_\mu(u_a) + \frac{1}{3}\tau \\ &\geq I_\mu(u_a) + \frac{1}{3}S^{\frac{3}{2}}. \end{aligned} \tag{3.18}$$

In what follows, we verify that $\tau \geq S^{\frac{3}{2}}$, which will lead to a contradiction. In fact, by the GNS inequality (2.5) and $P_\mu(u_a) = 0$, we get

$$\begin{aligned} I_\mu(u_a) &= I_\mu(u_a) - \frac{1}{6}P_\mu(u_a) \\ &= \frac{1}{3}\|\nabla u_a\|_2^2 + \frac{5}{24}\gamma \int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx - \mu\left(\frac{1}{q} - \frac{\delta_q}{6}\right)\|u_a\|_q^q \\ &\geq \frac{1}{3}\|\nabla u_a\|_2^2 - \frac{\mu(6 - q\delta_q)}{6q}\|u_a\|_q^q \\ &\geq \frac{1}{3}\|\nabla u_a\|_2^2 - \frac{\mu(6 - q\delta_q)}{6q} C(q) a^{q(1-\delta_q)} \|\nabla u_a\|_2^{q\delta_q} \\ &:= f(\|\nabla u\|_2), \end{aligned}$$

where

$$f(t) = \frac{1}{3}t^2 - \frac{\mu(6 - q\delta_q)}{6q} C(q) a^{q(1-\delta_q)} t^{q\delta_q}.$$

By $f'(t) = 0$, there exists a unique $t_3 > 0$ such that

$$f'(t_3) = \frac{2}{3}t_3 - \frac{\mu(6 - q\delta_q)}{6q} C(q) a^{q(1-\delta_q)} q\delta_q t_3^{q\delta_q - 1} = 0,$$

with

$$t_3 = \left(\frac{3}{2} \frac{\mu(6 - q\delta_q)}{6q} C(q) a^{q(1-\delta_q)} q\delta_q \right)^{\frac{1}{2 - q\delta_q}}.$$

Then, we see that $f(t)$ is strictly decreasing on $(0, t_3)$ and increasing on $(t_3, +\infty)$. Moreover, $f(t)$ gets the minimum on $(0, +\infty)$, that is

$$f(t_3) = -\frac{2 - q\delta_q}{2}\left(\frac{\mu(6 - q\delta_q)}{6q}C(q)a^{q(1-\delta_q)}\right)^{\frac{2}{2-q\delta_q}}\left(\frac{3q\delta_q}{2}\right)^{\frac{q\delta_q}{2-q\delta_q}}.$$

Define

$$\alpha_2 := \left\{\frac{4}{C(q)\mu\delta_q(6 - q\delta_q)}\left(\frac{q\delta_q S^{\frac{3}{2}}}{2 - q\delta_q}\right)^{\frac{2-q\delta_q}{2}}\right\}^{\frac{1}{q(1-\delta_q)}}.$$

Since $a < \alpha_2$, we get

$$f(t) > -\frac{1}{3}S^{\frac{3}{2}} \quad \text{on} \quad (0, +\infty). \tag{3.19}$$

Combining (3.18) and (3.19), we infer that $m_\mu(a) > 0$, which is a contradiction.

Step 3 From the above analysis, we know that $\tau = 0$. In other words, we have

$$\|u_n\|_6^6 \to \|u_a\|_6^6.$$

Then, by (3.16), we have

$$I'_\mu(u_a)u_a - \lambda_a\Psi'(u_a)u_a = 0. \tag{3.20}$$

Combining (3.10) and (3.20), one has

$$\|u_n\|^2 \to \|u_a\|^2.$$

Since $u_n \rightharpoonup u_a$ in H, we have $u_n \to u_a$ in H. Moreover, by the fact that $I_\mu(u_a) = \inf_{u\in\mathcal{P}(a)} I_\mu(u)$, we know that u_a is a ground state.

Finally, in view of Lemma 3.5, one has

$$I_\mu(u_a) = \inf_{u\in\mathcal{P}(a)} I_\mu(u) = \inf_{u\in D_{R_1}} I_\mu(u).$$

The proof is completed. □

4 L^2-Critical Perturbation Case

In this section, we shall address the L^2-critical perturbation case: $q = \frac{10}{3}$ and provide the proof of Theorem 1.2. To begin with, we give some useful lemmas, and show that $E(u, \iota)$ has the mountain pass geometry on $S_r(a) \times \mathbb{R}$, where $S_r(a) = H^1_{rad}(\mathbb{R}^3) \cap S(a)$.

Lemma 4.1 *Let* $q = \frac{10}{3}$, $\mu, \gamma > 0$ *and* $u \in S(a)$, *then*

(i) $\|\nabla(\iota\star u)\|_2 \to 0^+$ *and* $I_\mu((\iota\star u)) \to 0^+$ *if* $\iota \to -\infty$;
(ii) $\|\nabla(\iota\star u)\|_2 \to +\infty$ *and* $I_\mu((\iota\star u)) \to -\infty$ *if* $\iota \to +\infty$.

Proof By (2.10), we have

$$\int_{\mathbb{R}^3} |\nabla(\iota\star u)|^2 dx = e^{2\iota} \int_{\mathbb{R}^3} |\nabla u|^2 dx,$$

Then, it is easy to obtain

$$\|\nabla(\iota\star u)\|_2 \to 0^+ \quad \text{if } \iota \to -\infty,$$

and

$$\|\nabla(\iota\star u)\|_2 \to +\infty \quad \text{if } \iota \to +\infty.$$

From (2.11), we have

$$I_\mu((\iota\star u)) = \frac{1}{2}e^{2\iota}\|u\|^2 + \frac{\gamma}{4}e^{\iota} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q}e^{q\delta_q \iota} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6}e^{6\iota} \int_{\mathbb{R}^3} |u|^6 dx.$$

From the fact $q\delta_q = 2$, it follows that

$$I_\mu((\iota\star u)) \to 0^+ \quad \text{if } \iota \to -\infty,$$

and

$$I_\mu((\iota\star u)) \to -\infty \quad \text{if } \iota \to +\infty.$$

The proof is completed. □

Lemma 4.2 *Let* $q = \frac{10}{3}$, $\mu, \gamma > 0$, *and assume that* $0 < a < \min\{\alpha_3, \alpha_4\}$, *where*

$$\alpha_3 := \left(\frac{q}{2\mu C(q)}\right)^{\frac{1}{q(1-\delta_q)}},$$

and

$$\alpha_4 := \left(\frac{k^{\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}}\right)^{\frac{1}{3}}.$$

There exist $0 < k_1 < k_2 < k$ *such that*

$$P_\mu(u), I_\mu(u) > 0 \text{ for all } u \in A_{k_1} \text{ and } 0 < \sup_{u\in A_{k_1}} I_\mu(u) < \inf_{u\in B_{k_2}} I_\mu(u),$$

where

$$A_k := \{u \in S_r(a) : \|\nabla u\|_2^2 \le k\} \quad \textit{and} \quad B_k := \{u \in S_r(a) : \|\nabla u\|_2^2 = 2k\}.$$

Proof Take $k > 0$, which will be determined later. Assume that $u, v \in S_r(a)$ such that $\|\nabla u\|_2^2 \le k$ and $\|\nabla v\|_2^2 = 2k$. By (2.1), the GNS inequality (2.4) and $q\delta_q = 2$, we derive to

$$\begin{aligned} P_\mu(u) &= \|\nabla u\|_2^2 + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \mu\delta_q \|u\|_q^q - \|u\|_6^6 \\ &\ge \|\nabla u\|_2^2 - C(q)\mu\delta_q a^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q} - S^{-3}\|\nabla u\|_2^6 \\ &= \left(1 - \frac{2C(q)\mu}{q}a^{q(1-\delta_q)}\right)\|\nabla u\|_2^2 - S^{-3}\|\nabla u\|_2^6, \end{aligned}$$

and

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q}\|u\|_q^q - \frac{1}{6}\|u\|_6^6 \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - C(q)\frac{\mu}{q}a^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q} - \frac{1}{6}S^{-3}\|\nabla u\|_2^6 \\ &= \left(\frac{1}{2} - \frac{C(q)\mu}{q}a^{q(1-\delta_q)}\right)\|\nabla u\|_2^2 - \frac{1}{6}S^{-3}\|\nabla u\|_2^6. \end{aligned}$$

If $a < \alpha_3$, we can deduce that

$$P_\mu(u) > 0 \ \text{ and } \ I_\mu(u) > 0,$$

for $k > 0$ small enough. Next, if $a < \alpha_4$, we have

$$\begin{aligned} I_\mu(v) - I_\mu(u) \ge& \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q}\|v\|_q^q - \frac{1}{6}\|v\|_6^6 \\ \ge& \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{4}\tilde{C}[C(12/5)]^{\frac{5}{3}}a^3\|\nabla u\|_2 \\ &- \frac{\mu}{q}C(q)a^{q(1-\delta_q)}\|\nabla v\|_2^{q\delta_q} - \frac{1}{6}S^{-3}\|\nabla v\|_2^6 \\ \ge& k - \frac{1}{2}k - \frac{\gamma}{4}\tilde{C}[C(12/5)]^{\frac{5}{3}}\left(\frac{k^{\frac{1}{2}}}{4\gamma\tilde{C}[C(12/5)]^{\frac{5}{3}}}\right)k^{\frac{1}{2}} \\ &- \frac{C(q)\mu}{q}\left(\frac{k^{\frac{1}{2}}}{4\gamma\tilde{C}[C(12/5)]^{\frac{5}{3}}}\right)^{\frac{q(1-\delta_q)}{3}} 2k - \frac{1}{6}S^{-3}(2k)^3 \\ =& \frac{1}{2}k - \frac{1}{16}k - \left(\frac{2C(q)\mu}{q\left(4\gamma\tilde{C}[C(12/5)]^{\frac{5}{3}}\right)^{\frac{q(1-\delta_q)}{3}}}k^{\frac{q(1-\delta_q)}{6}}\right)k - \left(\frac{4}{3}S^{-3}k^2\right)k \\ \ge& \frac{5}{16}k > 0. \end{aligned}$$

If we take

$$k = \min\left\{ \left(\frac{q(4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}})^{\frac{q(1-\delta_q)}{3}}}{32\mu C(q)} \right)^{\frac{6}{q(1-\delta_q)}}, \ \left(\frac{3}{64}S^3\right)^{\frac{1}{2}} \right\}, \tag{4.1}$$

then, for $0 < k_1 < k_2 < k$ small enough and $0 < a < \min\{\alpha_3, \alpha_4\}$, we infer to

$$P_\mu(u),\ I_\mu(u) \text{ for all } u \in A_{k_1} \text{ and } 0 < \sup_{u \in A_{k_1}} I_\mu(u) < \inf_{u \in B_{k_2}} I_\mu(u).$$

The proof is completed. □

In the following, we study the characteristics of the mountain pass levels for $E(u, \iota)$ and $I_\mu(u)$. Here, we define a closed set $I_\mu^d := \{u \in S_r(a) : I_\mu(u) \leq d\}$.

Proposition 4.3 *Let* $q = \frac{10}{3}$, $\mu, \gamma > 0$, *and assume that* $0 < a < \min\{\alpha_3, \alpha_4\}$. *Take*

$$\widetilde{\sigma}_\mu(a) := \inf_{\widetilde{\zeta} \in \widetilde{\Gamma}_a} \max_{t \in [0,1]} E(\widetilde{\zeta}(t)),$$

where

$$\widetilde{\Gamma}_a = \{\widetilde{\zeta} \in C([0,1], S_r(a) \times \mathbb{R}) : \ \widetilde{\zeta}(0) \in (A_{k_1}, 0),\ \widetilde{\zeta}(1) \in (I_\mu^0, 0)\},$$

and

$$\sigma_\mu(a) := \inf_{\zeta \in \Gamma_a} \max_{t \in [0,1]} I_\mu(\zeta(t)),$$

where

$$\Gamma_a = \{\zeta \in C([0,1], S_r(a)) : \ \zeta(0) \in A_{k_1},\ \zeta(1) \in I_\mu^0\}.$$

Then we have

$$\widetilde{\sigma}_\mu(a) = \sigma_\mu(a).$$

Proof Since $\Gamma_a \times \{0\} \subset \widetilde{\Gamma}_a$, it is easy to know that $\widetilde{\sigma}_\mu(a) \leq \sigma_\mu(a)$. Then we only need to verify $\widetilde{\sigma}_\mu(a) \geq \sigma_\mu(a)$. For $\widetilde{\zeta}(t) = (\widetilde{\zeta}_1(t), \widetilde{\zeta}_2(t)) \in \widetilde{\Gamma}_a$, one has,

$$\widetilde{\zeta}(0) = (\widetilde{\zeta}_1(0), \widetilde{\zeta}_2(0)) \in (A_{k_1}, 0) \text{ and } \widetilde{\zeta}(1) = (\widetilde{\zeta}_1(1), \widetilde{\zeta}_2(1)) \in (I_\mu^0, 0).$$

So, set $\zeta(t) = (\widetilde{\zeta}_2(t) \star \widetilde{\zeta}_1(t))$, we have $\zeta(t) \in \Gamma_a$, and so,

$$\max_{t \in [0,1]} E(\widetilde{\zeta}(t)) = \max_{t \in [0,1]} I_\mu(\widetilde{\zeta}_2(t) \star \widetilde{\zeta}_1(t)) = \max_{t \in [0,1]} I_\mu(\zeta(t)),$$

which implies that $\widetilde{\sigma}_\mu(a) \geq \sigma_\mu(a)$. The proof is completed. □

Next, we will verify the existence of the $(PS)_{\widetilde{\sigma}_\mu(a)}$ sequence for $E(u,\iota)$ on $S_r(a)\times\mathbb{R}$, which is demonstrated by a standard argument by using Ekeland's variational principle and constructing pseudo-gradient flow (Proposition 2.2 [28]).

Proposition 4.4 *Let* $\{\xi_n\}\subset\widetilde{\Gamma}_a$ *be such that*

$$\max_{t\in[0,1]} E(\xi_n(t))\le\widetilde{\sigma}_\mu(a)+\frac{1}{n},$$

then there exists a sequence $\{(u_n,\iota_n)\}\subset S_r(a)\times\mathbb{R}$ *satisfying*

(i) $E(u_n,\iota_n)\in[\widetilde{\sigma}_\mu(a)-\frac{1}{n},\widetilde{\sigma}_\mu(a)+\frac{1}{n}]$;
(ii) $\min_{t\in[0,1]}\|(u_n,\iota_n)-\xi_n(t)\|_{\mathbb{H}}\le\frac{1}{\sqrt{n}}$;
(iii) $\|E'|_{S_r(a)\times\mathbb{R}}(u_n,\iota_n)\|\le\frac{2}{\sqrt{n}}$, *i.e.,*

$$|\langle E'(u_n,\iota_n),z\rangle_{\mathbb{H}^{-1}\times\mathbb{H}}|\le\frac{2}{\sqrt{n}}\|z\|_{\mathbb{H}},$$

for all

$$z\in\widetilde{T}_{(u_n,\iota_n)}:=\{(z_1,z_2)\in\mathbb{H}:\langle u_n,z_1\rangle_{L^2}=0\}.$$

With the help of Proposition 4.4, we can obtain a $(PS)_{\sigma_\mu(a)}$ sequence for $I_\mu(u)$ on $S_r(a)$ in the following.

Proposition 4.5 *Let* $q=\frac{10}{3}$, $\mu,\gamma>0$, *and assume that* $0<a<\min\{\alpha_3,\alpha_4\}$. *There exists a sequence* $\{w_n\}\subset S_r(a)$ *such that*

(i) $I_\mu(w_n)\to\sigma_\mu(a)$ *as* $n\to\infty$;
(ii) $P_\mu(w_n)\to 0$ *as* $n\to\infty$;
(iii) $I'_\mu|_{S_r(a)}(w_n)\to 0$ *as* $n\to\infty$, *i.e.,*

$$|\langle I'_\mu(w_n),z\rangle_{H^{-1}\times H}|\to 0,$$

uniformly for all $h\in T_{w_n}$ *and* $\|h\|\le 1$, *where* $T_{w_n}:=\{h\in H:\langle w_n,h\rangle_{L^2}=0\}$.

Proof By Proposition 4.3, we have $\widetilde{\sigma}_\mu(a)=\sigma_\mu(a)$. Now, we take $\{\xi_n=((\xi_n)_1,0)\}\in\widetilde{\Gamma}_a$ such that

$$\max_{t\in[0,1]} E(\xi_n(t))\le\widetilde{\sigma}_\mu(a)+\frac{1}{n}.$$

From Proposition 4.4, we know that there exists a sequence $\{(u_n,\iota_n)\}\subset S_r(a)\times\mathbb{R}$ such that as $n\to\infty$, we have

$$E(u_n,\iota_n)\to\sigma_\mu(a), \tag{4.2}$$

$$\iota_n\to 0, \tag{4.3}$$

$$\partial_\iota E(u_n, \iota_n) \to 0. \tag{4.4}$$

Let $w_n = \iota_n \star u_n$, then $I_\mu(w_n) = E(u_n, \iota_n)$, so item (i) follows.

Next, we show item (ii). Since

$$\begin{aligned}
\partial_\iota E(u_n, \iota_n) &= e^{2\iota_n} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\gamma}{4} e^{\iota_n} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \mu\delta_q e^{q\delta_q \iota_n} \int_{\mathbb{R}^3} |u_n|^q dx \\
&\quad - e^{6\iota_n} \int_{\mathbb{R}^3} |u_n|^6 dx \\
&= \int_{\mathbb{R}^3} |\nabla(\iota_n \star u_n)|^2 dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{(\iota_n \star u_n)} (\iota_n \star u_n)^2 dx - \mu\delta_q \int_{\mathbb{R}^3} |\iota_n \star u_n|^q dx \\
&\quad - \int_{\mathbb{R}^3} |\iota_n \star u_n|^6 dx \\
&= P_\mu(w_n),
\end{aligned}$$

it follows that item (ii) holds.

To prove item (iii), we set $h_n \in T_{w_n}$, then

$$\begin{aligned}
&\langle I'_\mu(w_n), h_n \rangle_{H^{-1} \times H} \\
&= \int_{\mathbb{R}^3} \nabla w_n \nabla h_n dx + \gamma \int_{\mathbb{R}^3} \phi_{w_n} w_n h_n dx - \mu \int_{\mathbb{R}^3} |w_n|^{q-2} w_n h_n dx - \int_{\mathbb{R}^3} |w_n|^4 w_n h_n dx \\
&= e^{-\frac{\iota_n}{2}} \int_{\mathbb{R}^3} \nabla u_n(x) \nabla h_n(e^{-\iota_n} x) dx + \gamma e^{-\frac{\iota_n}{2}} \int_{\mathbb{R}^3} \phi_{u_n(x)} u_n(x) h_n(e^{-\iota_n} x) dx \\
&\quad - \mu e^{\frac{3(q-3)}{2}\iota_n} \int_{\mathbb{R}^3} |u_n(x)|^{q-2} u_n(x) h_n(e^{-\iota_n} x) dx - e^{\frac{9}{2}\iota_n} \int_{\mathbb{R}^3} |u_n(x)|^4 u_n(x) h_n(e^{-\iota_n} x) dx \\
&= e_n^\iota \int_{\mathbb{R}^3} \nabla u_n(x) \nabla \left(e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x) \right) dx + \gamma e_n^\iota \int_{\mathbb{R}^3} \phi_{u_n(x)} u_n(x) e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x) dx \\
&\quad - \mu e^{q\delta_q \iota_n} \int_{\mathbb{R}^3} |u_n(x)|^{q-2} u_n(x) e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x) dx \\
&- e^{6\iota_n} \int_{\mathbb{R}^3} |u_n(x)|^4 u_n(x) e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x) dx.
\end{aligned}$$

Let $\widetilde{h}_n(x) = e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x)$, we obtain

$$\langle I'_\mu(w_n), h_n \rangle_{H^{-1} \times H} = \langle E'(u_n, \iota_n), (\widetilde{h}_n, 0) \rangle_{\mathbb{H}^{-1} \times \mathbb{H}}.$$

Moreover, we get

$$\begin{aligned}
\langle u_n, \widetilde{h}_n \rangle_{L^2} &= \int_{\mathbb{R}^3} u_n(x) e^{-\frac{3}{2}\iota_n} h_n(e^{-\iota_n} x) dx \\
&= \int_{\mathbb{R}^3} e^{\frac{3}{2}\iota_n} u_n(e^{\iota_n} x) h_n(x) dx \\
&= \int_{\mathbb{R}^3} w_n(x) h_n(x) dx = 0.
\end{aligned}$$

Thus, we obtain $(\widetilde{h}_n, 0) \in \widetilde{T}_{(u_n,\iota_n)}$. On the other hand,

$$
\begin{aligned}
\|(\widetilde{h}_n, 0)\|_{\mathbb{H}}^2 &= \|\widetilde{h}_n(x)\|_H^2 \\
&= \|h_n(x)\|_2^2 + e^{-2\iota_n}\|h_n(x)\|^2 \\
&\leq C\|h_n(x)\|^2,
\end{aligned}
$$

where the last inequality can be established by (4.3). So the item (iii) is proved. □

Now, we construct the relationship between $\sigma_\mu(a)$ and $m_{\mu,r}(a)$, where

$$
m_{\mu,r}(a) = \inf_{u\in\mathcal{P}_r(a)} I_\mu(u),
$$

and

$$
\mathcal{P}_r(a) = \mathcal{P}(a) \cap S_r(a).
$$

Lemma 4.6 *Let* $q = \frac{10}{3}$, $\mu, \gamma > 0$, *and assume that* $0 < a < \min\{\alpha_3, \alpha_4\}$. *Then we have*

$$
m_{\mu,r}(a) = \inf_{u\in\mathcal{P}_r(a)^-} I_\mu(u) = \sigma_\mu(a) > 0,
$$

where

$$
\mathcal{P}_r(a)^- = \mathcal{P}(a)^- \cap S_r(a).
$$

Proof In the following, we split the proof into four steps.

Step 1 We verify that for each $u \in S_r(a)$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{P}_r(a)$, with t_u is the strict maximum point for the function $\varphi_u(\iota)$ on $(0, +\infty)$ at positive level. Moreover, $\mathcal{P}_r(a) = \mathcal{P}_r(a)^-$.

In fact, by Lemma 4.1, we have

$$
\varphi_u(-\infty) = 0^+ \quad \text{and} \quad \varphi_u(+\infty) = -\infty. \tag{4.5}
$$

Since $\varphi_u(\iota)$ is a C^2 function, we can deduce that $\varphi_u(\iota)$ has at least one critical point t_u, with t_u is a global maximum point at positive level. In view of Lemma 2.6, we have $t_u \star u \in \mathcal{P}_r(a)$. Next, we prove that $\varphi_u(\iota)$ has no other critical points. Indeed, recall $(\varphi_u)'(\iota)$ and $(\varphi_u)''(\iota)$ as follow:

$$
(\varphi_u)'(\iota) = e^{2\iota}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2dx - \mu\delta_q e^{q\delta_q\iota}\int_{\mathbb{R}^3}|u|^q dx - e^{6\iota}\int_{\mathbb{R}^3}|u|^6dx,
$$

and

$$
(\varphi_u)''(\iota) = 2e^{2\iota}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_u u^2dx
$$

$$-\mu q {\delta_q}^2 e^{q\delta_q \iota}\int_{\mathbb{R}^3}|u|^q dx - 6e^{6\iota}\int_{\mathbb{R}^3}|u|^6 dx.$$

Assume by contradiction, there exists other critical point $e_u \in \mathbb{R}$ with $t_u < e_u$ and e_u is also a global maximum point of $\varphi_u(\iota)$. Then, we see that there exists a critical point f_u, such that $t_u < f_u < e_u$ and f_u is a minimum point of $\varphi_u(\iota)$. Consequently, we have

$$(\varphi_u)'(e_u) = 0, \quad (\varphi_u)'(f_u) = 0,$$
$$(\varphi_u)''(e_u) = -\frac{\gamma}{4}e^{e_u}\int_{\mathbb{R}^3}\phi_u u^2 dx - 4e^{6e_u}\int_{\mathbb{R}^3}|u|^6 dx < 0,$$

and

$$(\varphi_u)''(f_u) = -\frac{\gamma}{4}e^{f_u}\int_{\mathbb{R}^3}\phi_u u^2 dx - 4e^{6f_u}\int_{\mathbb{R}^3}|u|^6 dx < 0,$$

which is a contradiction.

Step 2 We show that $I_\mu(u) \le 0$ implies $P_\mu(u) < 0$. In fact, since $\varphi_u(0) = I_\mu(0\star u) = I_\mu(u) \le 0$, by the properties of the function $\varphi_u(\iota)$ presented in Step 1 and by (4.5), we infer that $t_u < 0$. Besides, since

$$P_\mu(t_u\star u) = (\varphi_u)'(t_u) = 0 \ \text{ and } \ P_\mu(u) = P_\mu(0\star u) = (\varphi_u)'(0),$$

we obtain that $P_\mu(u) < 0$.

Step 3 We claim that $m_{\mu,r}(a) = \sigma_\mu(a)$. Indeed, let $u \in S_r(a)$, we take $\iota^- \ll 0$ and $\iota^+ \gg 0$ such that $\iota^-\star u \in A_{k_1}$ and $I_\mu(\iota^+\star u) < 0$, respectively. Then we can define a path

$$\zeta_u : t \in [0,1] \mapsto ((1-t)\iota^- + t\iota^+)\star u \in \Gamma_a. \tag{4.6}$$

Hence, we get

$$\max_{t\in[0,1]} I_\mu(\zeta_u(t)) \ge \sigma_\mu(a),$$

and so, we have $m_{\mu,r}(a) \ge \sigma_\mu(a)$. Moreover, for any $\widetilde{\zeta}(t) = (\widetilde{\zeta}_1(t), \widetilde{\zeta}_2(t)) \in \widetilde{\Gamma}_a$, one has,

$$\widetilde{\zeta}(0) = (\widetilde{\zeta}_1(0), \widetilde{\zeta}_2(0)) \in (A_{k_1},\ 0) \ \text{ and } \ \widetilde{\zeta}(1) = (\widetilde{\zeta}_1(1), \widetilde{\zeta}_2(1)) \in (I_\mu^0,\ 0).$$

Now, we define the function

$$\widetilde{P}_\mu(t) = P_\mu(\widetilde{\zeta}_2(t)\star\widetilde{\zeta}_1(t)).$$

Since $(\widetilde{\zeta}_2(0)\star\widetilde{\zeta}_1(0)) = \widetilde{\zeta}_1(0) \in A_{k_1}$ and $(\widetilde{\zeta}_2(1)\star\widetilde{\zeta}_1(1)) = \widetilde{\zeta}_1(1) \in I_\mu^0$, in view of Lemma 4.2 and Step 2, we have

$$\widetilde{P}_\mu(0) = \widetilde{P}_\mu(\widetilde{\zeta}_1(0)) > 0,$$

and

$$\widetilde{P}_\mu(1) = \widetilde{P}_\mu(\widetilde{\zeta}_1(1)) < 0.$$

Since $\widetilde{P}_\mu(t)$ is continuous and by Remark 2.8, we infer that there exists $t^* \in (0, 1)$ so as to $\widetilde{P}_\mu(t^*) = 0$, which implies that $(\widetilde{\zeta}_2(t^*)\star\widetilde{\zeta}_1(t^*)) \in \mathcal{P}_r(a)$. Consequently, one has

$$\max_{t\in[0,1]} E(\widetilde{\zeta}(t)) = \max_{t\in[0,1]} I_\mu(\widetilde{\zeta}_2(t)\star\widetilde{\zeta}_1(t)) \geq \inf_{u\in\mathcal{P}_r(a)} I_\mu(u).$$

Hence, we have $\sigma_\mu(a) \geq m_{\mu,r}(a)$. In conclusion, one has $\sigma_\mu(a) = m_{\mu,r}(a)$.

Step 4 We claim that $m_{\mu,r}(a) > 0$. Let $u \in \mathcal{P}_r(a)$, we have $P_\mu(u) = 0$. By the GNS inequality (2.4) and (2.1), we infer to

$$\begin{aligned}\|\nabla u\|_2^2 &= -\frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx + \mu\delta_q \|u\|_q^q + \|u\|_6^6 \\ &\leq \mu\delta_q C(q) a^{q(1-\delta_q)} \|\nabla u\|_2^{q\delta_q} + S^{-3}\|\nabla u\|_2^6.\end{aligned}$$

And by $q\delta_q = 2$, one has

$$\left(1 - \mu\delta_q C(q) a^{q(1-\delta_q)}\right) \|\nabla u\|_2^2 \leq S^{-3}\|\nabla u\|_2^6.$$

By $a < \alpha_3$, there exists $\rho > 0$ such that

$$\inf_{u\in\mathcal{P}_r(a)} \|\nabla u\|_2^2 \geq \rho.$$

So, for any $u \in \mathcal{P}_r(a)$, it follows that

$$\begin{aligned}I_\mu(u) &= I_\mu(u) - \frac{1}{6}P_\mu(u) \\ &= \frac{1}{3}\|\nabla u\|_2^2 + \frac{5\gamma}{24}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{2\mu}{3q}\int_{\mathbb{R}^3} |u|^q dx \\ &\geq \frac{1}{3}\|\nabla u\|_2^2 - \frac{2\mu}{3q} C(q) a^{q(1-\delta_q)} \|\nabla u\|_2^{q\delta_q} \\ &= \frac{1}{3}\left(1 - \frac{2\mu}{q} C(q) a^{q(1-\delta_q)}\right) \|\nabla u\|_2^2 \\ &\geq \frac{1}{3}\left(1 - \frac{2\mu}{q} C(q) a^{q(1-\delta_q)}\right) \rho > 0.\end{aligned}$$

Consequently, we obtain $\sigma_\mu(a) > 0$, which completes the proof. □

Next, we give an upper bounded estimate for the mountain pass level $\sigma_\mu(a)$ in the following Lemma, which plays an important role in the proof of Theorem 1.2.

Lemma 4.7 *Let* $q = \frac{10}{3}$*,* $\mu > 0$*, and assume that* $0 < a < \min\{\alpha_3, \alpha_4\}$*. Then there exists* $\widetilde{\gamma_1} > 0$*, such that* $\sigma_\mu(a) < \frac{1}{3}S^{\frac{3}{2}}$ *for* $\gamma \in (0, \widetilde{\gamma_1})$ *small enough.*

Proof Recall (2.1) and (2.2), we have the best constant S is attained by

$$U_\varepsilon(x) = \frac{C^*\varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}$$

for any $\varepsilon > 0$ and C^* being normalized constant such that

$$\int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |U_\varepsilon|^6 dx = S^{\frac{3}{2}}.$$

We take

$$u_\varepsilon = \varphi U_\varepsilon,$$

where $\varphi(x) \in C_0^\infty(B_2(0))$ is a radial cutoff function such that $0 \le \varphi(x) \le 1$ and $\varphi(x) \equiv 1$ on $B_1(0)$. Let

$$v_\varepsilon = a\frac{u_\varepsilon}{\|u_\varepsilon\|_2} \in S(a) \cap H^1_{rad}(\mathbb{R}^3).$$

As showed in [15], we have

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx = S^{\frac{3}{2}} + O(\varepsilon), \tag{4.7}$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon|^6 dx = S^{\frac{3}{2}} + O(\varepsilon^3). \tag{4.8}$$

From Lemma 7.1 [30], we have the following estimations:

$$\int_{\mathbb{R}^3} |u_\varepsilon|^q dx = \begin{cases} O\left(\varepsilon^{\frac{q}{2}}\right), & \text{if } 2 < q < 3; \\ O\left(\varepsilon^{\frac{3}{2}}|\log\varepsilon|\right), & \text{if } q = 3; \\ O\left(\varepsilon^{\frac{6-q}{2}}\right), & \text{if } 3 < q < 6. \end{cases} \tag{4.9}$$

and when $q = 2$, one has

$$\int_{\mathbb{R}^3} |u_\varepsilon|^2 dx = C\varepsilon. \tag{4.10}$$

Recall the function

$$\varphi_{v_\varepsilon}(\iota) := I_\mu((\iota\star v_\varepsilon)) = \frac{1}{2}e^{2\iota}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx + \frac{\gamma}{4}e^{\iota}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx$$
$$-\frac{\mu}{q}e^{\frac{3(q-2)}{2}\iota}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx - \frac{1}{6}e^{6\iota}\int_{\mathbb{R}^3}|v_\varepsilon|^6dx, \tag{4.11}$$

Similar to the first step in proving Lemma 4.6, we can conclude that φ_{v_ε} can obtain its global positive maximum at some ι_ε. And so, by (2.14), we have

$$(\varphi)'_{v_\varepsilon}(\iota_\varepsilon) = P_\mu(\iota_\varepsilon\star v_\varepsilon) = 0. \tag{4.12}$$

By (4.12) and $q\delta_q = 2$, we deduce

$$\begin{aligned} e^{4\iota_\varepsilon} &= \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} + \frac{\gamma}{4}e^{-\iota_\varepsilon}\frac{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx}{\|v_\varepsilon\|_6^6} - \mu\delta_q e^{(q\delta_q-2)\iota_\varepsilon}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\ &\geq \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\ &= \frac{\|u_\varepsilon\|_2^4\|\nabla u_\varepsilon\|_2^2}{a^4\|u_\varepsilon\|_6^6} - \mu\delta_q\frac{\|u_\varepsilon\|_2^{6-q}\|u_\varepsilon\|_q^q}{a^{6-q}\|u_\varepsilon\|_6^6} \\ &= \frac{\|u_\varepsilon\|_2^4}{a^4\|u_\varepsilon\|_6^6}\left(\|\nabla u_\varepsilon\|_2^2 - \mu\delta_q\frac{\|u_\varepsilon\|_2^{2-q}\|u_\varepsilon\|_q^q}{a^{2-q}}\right) \\ &= \frac{\|u_\varepsilon\|_2^4}{a^4\|u_\varepsilon\|_6^6}\left(\|\nabla u_\varepsilon\|_2^2 - \frac{\mu\delta_q}{a^{2-q}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q-2}}\right). \end{aligned} \tag{4.13}$$

In view of (4.7) and (4.8), we have that there exist positive constants C_1, C_2 and C_3 depending on s and q, such that

$$C_1 \leq (\|\nabla u_\varepsilon\|_2^2)^{\frac{6-q\delta_q}{4}} \leq \frac{1}{C_1}, \tag{4.14}$$

$$C_2 \leq (\|u_\varepsilon\|_6^6)^{\frac{q\delta_q-2}{4}} \leq \frac{1}{C_2}, \tag{4.15}$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}} = C_3\varepsilon^{\frac{6-q}{4}}. \tag{4.16}$$

Hence, by (4.13)–(4.16) and $q\delta_q = 2$, we obtain

$$e^{4\iota_\varepsilon} \geq C\frac{\|u_\varepsilon\|_2^4}{a^4}\left(C_1 - \frac{\mu\delta_q}{a^{2-q}}C_3\varepsilon^{\frac{2}{3}}\right) \geq C\frac{\|u_\varepsilon\|_2^4}{a^4}. \tag{4.17}$$

In the following, we shall make an upper estimation of $\max_{\iota\in\mathbb{R}} \varphi_{v_\varepsilon}(\iota)$. Firstly, we define the function $\varphi^0_{v_\varepsilon}(\iota)$ as follow:

$$\varphi^0_{v_\varepsilon}(\iota) := \frac{e^{2\iota}}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{e^{6\iota}}{6}\int_{\mathbb{R}^3} |v_\varepsilon|^6 dx. \tag{4.18}$$

By simple calculation, we derive that the function $\varphi^0_{v_\varepsilon}(\iota)$ has a unique critical point ι^0_ε, which is a strict maximum point given by

$$e^{\iota^0_\varepsilon} = \left(\frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6}\right)^{\frac{1}{4}}. \tag{4.19}$$

Applying the fact that

$$\sup_{\theta\geq 0}\left(\frac{\theta^2}{2}a - \frac{\theta^6}{6}b\right) = \frac{1}{3}\left(\frac{a}{b^{\frac{1}{3}}}\right)^{\frac{3}{2}},$$

for any fixed $a, b > 0$. In view of (4.7) and (4.8), we infer that

$$\begin{aligned}\varphi^0_{v_\varepsilon}(\iota^0_\varepsilon) &= \frac{1}{3}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{(\|v_\varepsilon\|_6^6)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \frac{1}{3}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{(\|u_\varepsilon\|_6^6)^{\frac{1}{3}}}\right)^{\frac{3}{2}} \\ &= \frac{1}{3}\left(\frac{S^{\frac{3}{2}} + O(\varepsilon)}{(S^{\frac{3}{2}} + O(\varepsilon^2))^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon).\end{aligned} \tag{4.20}$$

Secondly, we make an estimation for $\varphi_{v_\varepsilon}(\iota)$. By (2.3), (4.12) and Hölder inequality, we derive to

$$\begin{aligned} e^{4\iota_\varepsilon} &= \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} + \frac{\gamma}{4}e^{-\iota_\varepsilon}\frac{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2 dx}{\|v_\varepsilon\|_6^6} - \mu\delta_q e^{(q\delta_q-2)\iota_\varepsilon}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\ &\leq \frac{1}{\|v_\varepsilon\|_6^6}\left(\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{-\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2 dx\right) \\ &\leq \frac{1}{\|v_\varepsilon\|_6^6}\left(\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{-\iota_\varepsilon}\widetilde{C}\|v_\varepsilon\|_{\frac{12}{5}}^4\right) \\ &\leq \frac{1}{\|v_\varepsilon\|_6^6}\left(\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{-\iota_\varepsilon}\widetilde{C}\|v_\varepsilon\|_2^3\|v_\varepsilon\|_6\right) \\ &= \frac{1}{a^4\|u_\varepsilon\|_6^6}\left(\|\nabla u_\varepsilon\|_2^2\|u_\varepsilon\|_2^4 + \frac{\gamma}{4}e^{-\iota_\varepsilon}\widetilde{C}a^2\|u_\varepsilon\|_2^5\|u_\varepsilon\|_6\right).\end{aligned} \tag{4.21}$$

By virtue of (4.7)–(4.8), (4.10) and (4.21), we can see that ι_ε can not go to $+\infty$, namely, there exists some $\iota^* \in \mathbb{R}$ such that

$$\iota_\varepsilon \le \iota^*, \quad \text{for all} \;\; \varepsilon > 0. \tag{4.22}$$

Based on above analysis, by (2.3)–(2.4), (4.7)–(4.8), (4.17), (4.20), (4.22) and the fact $q\delta_q = 2$, we derive to

$$\begin{aligned}
\sup_{\iota\in\mathbb{R}} \varphi_{v_\varepsilon}(\iota) &= \varphi_{v_\varepsilon}(\iota_\varepsilon) = \varphi^0_{v_\varepsilon}(\iota_\varepsilon) + \frac{\gamma}{4} e^{\iota_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx - \frac{\mu}{q} e^{q\delta_q \iota_\varepsilon} \|v_\varepsilon\|_q^q \\
&\le \sup_{\iota\in\mathbb{R}} \Psi^0_{v_\varepsilon}(\iota) + \frac{\gamma}{4} e^{\iota_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx - \frac{\mu}{q} e^{2\iota_\varepsilon} \|v_\varepsilon\|_q^q \\
&\le \Psi^0_{v_\varepsilon}(\iota_{v^0_\varepsilon}) + \frac{\gamma}{4} e^{\iota_\varepsilon} \widetilde{C} \|v_\varepsilon\|^4_{\frac{12}{5}} - \frac{C\mu}{q} \frac{\|u_\varepsilon\|_2^2}{a^2} \|v_\varepsilon\|_q^q \\
&\le \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon) + C\gamma \frac{a^4}{\|u_\varepsilon\|_2^4} \|u_\varepsilon\|^4_{\frac{12}{5}} - \frac{C\mu}{q} \frac{a^{q-2}}{\|u_\varepsilon\|_2^{q-2}} \|u_\varepsilon\|_q^q \\
&\le \frac{1}{3} S^{\frac{3}{2}} + C^1 \varepsilon + C^2 \gamma \frac{\varepsilon^{\frac{6}{5}\times\frac{5}{3}}}{\varepsilon^2} - C^3 \varepsilon^{\frac{6-q}{4}} \\
&= \frac{1}{3} S^{\frac{3}{2}} + C^1 \varepsilon + C^2 \gamma - C^3 \varepsilon^{\frac{2}{3}} < \frac{1}{3} S^{\frac{3}{2}},
\end{aligned} \tag{4.23}$$

if we choose $\gamma = \varepsilon^\alpha$ for some constant $\alpha \ge 1$, and use the fact $0 < \frac{6-q}{4} < 1$.

Finally, by Lemma 4.1, we take $\iota_1 < 0$ and $\iota_2 > 0$ such that $\iota_1 \star v_\varepsilon \in A_k$ and $I_\mu(\iota_2 \star v_\varepsilon) < 0$, respectively. Define a path

$$\eta_{v_\varepsilon} : t \in [0, 1] \mapsto ((1 - t)\iota_1 + t\iota_2) \star v_\varepsilon \in \Gamma_a.$$

Consequently, by (4.23), we have that there exists some $\widetilde{\gamma_1} > 0$, such that

$$m_{\mu,r}(a) = \sigma_\mu(a) \le \max_{t\in[0,1]} I_\mu(\eta_{v_\varepsilon}(t)) \le \sup_{\iota\in\mathbb{R}} \varphi_{v_\varepsilon}(\iota) < \frac{1}{3} S^{\frac{3}{2}},$$

for $\gamma \in (0, \widetilde{\gamma_1})$ small enough. □

Now, based on the above preparation, we are ready to accomplish the proof of Theorem 1.2.

Proof of Theorem 1.2 Take a PS sequence $\{u_n\}$ as in Proposition 4.5, we have

$$I'_\mu|_{S_r(a)}(u_n) \to 0 \;\; \text{as} \;\; n \to \infty,$$

Using the Lagrange multipliers rule, we have that there exists a sequence $\{\lambda_n\} \in \mathbb{R}$ such that

$$I'_\mu(u_n) - \lambda_n \Psi'(u_n) \to 0 \;\; \text{in} \;\; H^{-1}. \tag{4.24}$$

Again by Proposition 4.5, we get

$$I_\mu(u_n) \to \sigma_\mu(a) \ \text{ as } \ n \to \infty.$$

which means $I_\mu(u_n)$ is bounded. So we deduce to

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n|^q dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx \le C. \tag{4.25}$$

From $P_\mu(u_n) \to 0$ and $q\delta_q = 2$, we infer that

$$|I_\mu(u_n) + P_\mu(u_n)| \le C,$$

that is,

$$\frac{3}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\gamma}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{3\mu}{q}\int_{\mathbb{R}^3} |u_n|^q dx - \frac{7}{6}\int_{\mathbb{R}^3} |u_n|^6 dx \ge -C. \tag{4.26}$$

Combining (4.25)–(4.26), one has

$$\frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx + \frac{2}{3}\int_{\mathbb{R}^3} |u_n|^6 dx \le 4C. \tag{4.27}$$

Thus, we know that $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx$ and $\int_{\mathbb{R}^3} |u_n|^6 dx$ are bounded. Then, by (4.25) and the GNS inequality (2.4), we infer to

$$\begin{aligned} C &\ge \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n|^q dx \\ &\ge \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\mu}{q} C(q) a^{q(1-\delta_q)} \|\nabla u_n\|_2^{q\delta_q} \\ &= \left(\frac{1}{2} - \frac{\mu}{q} C(q) a^{q(1-\delta_q)}\right) \|\nabla u_n\|_2^2. \end{aligned}$$

Since $a < \alpha_3$, it is easy to get $\|\nabla u_n\|_2 \le R^*$ for some $R^* > 0$ independently on $n \in \mathbb{N}$. Consequently, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, and so, up to subsequence, there exists u_a such that

$$\begin{cases} u_n \rightharpoonup u_a \ \text{ in } \ H^1(\mathbb{R}^3), \\ u_n \to u_a \ \text{ in } \ L^p(\mathbb{R}^3), \ \forall p \in (2, 6). \end{cases}$$

From (4.24), we have

$$\lambda_n \|u_n\|_2^2 = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx$$

$$-\mu \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^6 dx + o_n(1), \tag{4.28}$$

that is

$$\lambda_n = \frac{1}{a^2}\left[\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx - \mu \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^6 dx\right] + o_n(1).$$

From the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^3)$, we have that $\{\lambda_n\}$ is bounded. Now, we verify

$$\int_{\mathbb{R}^3} |u_n|^q dx \to \int_{\mathbb{R}^3} |u_a|^q dx \neq 0, \quad \text{i.e. } u_a \neq 0.$$

Assume by contradiction that, $\int_{\mathbb{R}^3} |u_n|^q dx \to 0$. By (2.3) and the interpolation inequality, we obtain $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \to 0$. Combining with

$$P_\mu(u_n) = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx - \mu\delta_q \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^6 dx = o_n(1),$$

we get

$$\lambda_n = \frac{1}{a^2}\left[\frac{3\gamma}{4} \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx + \mu(\delta_q - 1) \int_{\mathbb{R}^3} |u_n|^q dx\right] + o_n(1).$$

So, $\lambda_n \to 0$ as $n \to \infty$. Then (4.28) becomes

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} |u_n|^6 dx = o_n(1).$$

Set

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^6 dx = l,$$

then,

$$\begin{aligned}
\lim_{n\to\infty} I_\mu(u_n) &= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx\right] \\
&= \frac{1}{2} l - \frac{1}{6} l \\
&= \frac{1}{3} l.
\end{aligned} \tag{4.29}$$

Since $I_\mu(u_n) \to \sigma_\mu(a)$ as $n \to \infty$, and $\sigma_\mu(a) < \frac{1}{3}S^{\frac{3}{2}}$ from Lemma 4.7 and (4.29), we obtain

$$l < S^{\frac{3}{2}}.$$

On the other hand, by virtue of the Sobolev inequality (2.1), we have

$$S \le \frac{\lim_{n\to\infty} \|\nabla u_n\|_2^2}{(\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^6 dx)^{\frac{1}{3}}} = \frac{l}{l^{\frac{1}{3}}} = l^{\frac{2}{3}},$$

which leads to a contradiction. Hence, $\int_{\mathbb{R}^3} |u_n|^q dx \to \int_{\mathbb{R}^3} |u_a|^q dx \neq 0$. Then, by the boundedness of $\{\lambda_n\}$, up to subsequence, there exists λ_a such that $\lambda_n \to \lambda_a$. Consequently, by $\|\nabla u_n\|_2 \le R^*$, (2.3), the GNS inequality (2.4) and $q\delta_q = 2$, we have

$$\begin{aligned} T_3 \le \int_{\mathbb{R}^3} |u_n|^q dx &\le C(q)\|\nabla u_n\|_2^2 \|u_n\|_2^{q(1-\delta_q)} \\ &\le C(q) {R^*}^2 a^{q(1-\delta_q)}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \le \widetilde{C}\|u_n\|_{\frac{12}{5}}^4 &\le \widetilde{C}[C(12/5)]^{\frac{5}{3}} \|\nabla u_n\|_2 \|u_n\|_2^3 \\ &\le \widetilde{C}[C(12/5)]^{\frac{5}{3}} R^* a^3 \\ &:= T_4, \end{aligned}$$

where $T_3 > 0$ and $T_4 = T_4(R^*, a)$. We define the positive constant

$$\widetilde{\gamma_2} := \frac{4\mu(1-\delta_q)T_3}{3T_4}.$$

So, we get

$$\widetilde{\gamma_2} \le \lim_{n\to\infty} \frac{4\mu(1-\delta_q)\int_{\mathbb{R}^3} |u_n|^q dx}{3\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx} = \frac{4\mu(1-\delta_q)\int_{\mathbb{R}^3} |u_a|^q dx}{3\int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx}.$$

In view of (4.24) and $\{u_n\}$ is bounded in $H(\mathbb{R}^3)$, we have

$$\lambda_n \|u_n\|_2^2 = \|\nabla u_n\|_2^2 + \gamma \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx - \mu\|u_n\|_q^q - \|u_n\|_6^6 + o_n(1).$$

Combining with $P_\mu(u_n) \to 0$ and Lemma 2.7, if $\gamma \in (0, \widetilde{\gamma_2})$, one has

$$\lambda_a a^2 = \lim_{n\to\infty} \left(\|\nabla u_n\|_2^2 + \gamma \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 dx - \mu\|u_n\|_q^q - \|u_n\|_6^6 \right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(\frac{3}{4}\gamma\int_{\mathbb{R}^3}\phi_{u_n}{u_n}^2dx + \mu(\delta_q - 1)\|u_n\|_q^q\right)\\
&= \frac{3}{4}\gamma\int_{\mathbb{R}^3}\phi_{u_a}{u_a}^2dx + \mu(\delta_q - 1)\|u_a\|_q^q\\
&< \frac{3}{4}\widetilde{\gamma}_2\int_{\mathbb{R}^3}\phi_{u_a}{u_a}^2dx + \mu(\delta_q - 1)\|u_a\|_q^q \le 0,
\end{aligned}
$$

which proves that $\lim_{n\to\infty}\lambda_n = \lambda_a < 0$. Let $v_n = u_n - u_a \rightharpoonup 0$, by (3.17), (2.5), Lemma 2.7 and $P_\mu(v_n) = P_\mu(u_n) - P_\mu(u_a) \to 0$, we infer to

$$\|\nabla v_n\|_2^2 = \|v_n\|_6^6 + o_n(1).$$

Up to subsequence, we assume that

$$\|\nabla v_n\|_2^2 = \|v_n\|_6^6 \to \tau.$$

So, by (2.1), we have

$$\tau^{\frac{1}{3}}S \le \tau,$$

that is,

$$\tau \ge S^{\frac{3}{2}} \quad \text{or} \quad \tau = 0.$$

If $\tau \ge S^{\frac{3}{2}}$, in view of (3.17), we derive as

$$
\begin{aligned}
\sigma_\mu(a) &= \lim_{n\to\infty} I_\mu(u_n)\\
&= \lim_{n\to\infty}\left(I_\mu(u_a) + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{6}\|v_n\|_6^6\right)\\
&= I_\mu(u_a) + \frac{1}{3}\tau\\
&\ge I_\mu(u_a) + \frac{1}{3}S^{\frac{3}{2}}.
\end{aligned}
$$

Besides,

$$I_\mu(u_a) = I_\mu(u_a) + \frac{1}{2}P_\mu(u_a) = \frac{\gamma}{8}\int_{\mathbb{R}^3}\phi_{u_a}{u_a}^2dx + \frac{1}{3}\|u_a\|_6^6 > 0,$$

which is contradicted to Lemma 4.7. Thus, we have $\tau = 0$. By a similar argument as in the end of the proof of Theorem 1.1, we infer to

$$u_n \to u_a \ \text{ in } \ H^1(\mathbb{R}^3).$$

Next, we claim that $m_\mu(a) = m_{\mu,r}(a)$. Since $\mathcal{P}_r(a) \subset \mathcal{P}(a)$, it is easy to see that $m_\mu(a) \le m_{\mu,r}(a)$. Then, we only need to verify $m_\mu(a) \ge m_{\mu,r}(a)$. Suppose by contradiction, there exists $w \in \mathcal{P}(a)\backslash S_r(a)$ such that

$$I_\mu(w) < \inf_{\mathcal{P}_r(a)} I_\mu(u). \tag{4.30}$$

Then, let $v := |w|^*$, by virtue of the schwarz rearrangement, it follows that

$$I_\mu(v) \le I_\mu(w) \quad \text{and} \quad P_\mu(v) \le P_\mu(w) = 0.$$

If $P_\mu(v) = 0$, we know $v \in \mathcal{P}(a)$, $v := |w|^* \in \mathcal{P}_r(a)$ and

$$I_\mu(v) \ge \inf_{\mathcal{P}_r(a)} I_\mu(u) > I_\mu(w) \ge I_\mu(v),$$

which is a contradiction. If $P_\mu(v) < 0$, we see that $(\varphi_v)'(0) = P_\mu(v) < 0$, by the claim of Step 1 of the proof of Lemma 4.6, we have that $t_v < 0$. Since $t_v \star v \in \mathcal{P}_r(a)$, by (4.29), we deduce to

$$\begin{aligned} I_\mu(w) < I_\mu(t_v \star v) &= I_\mu(t_v \star v) - \frac{1}{2} P_\mu(t_v \star v) \\ &= \frac{\gamma}{8} e^{t_v} \int_{\mathbb{R}^3} \phi_v v^2 dx + \frac{1}{3} e^{6t_v} \int_{\mathbb{R}^3} |v|^6 dx \\ &= \frac{\gamma}{8} e^{t_v} \int_{\mathbb{R}^3} \phi_w w^2 dx + \frac{1}{3} e^{6t_v} \int_{\mathbb{R}^3} |w|^6 dx \\ &\le e^{t_v} \left(I_\mu(w) - \frac{1}{2} P_\mu(w) \right) \\ &= e^{t_v} I_\mu(w) < I_\mu(w), \end{aligned}$$

which leads to a contradiction. Again by the the claim of step 1 of the proof of Lemma 4.6, we have $\mathcal{P}(a) = \mathcal{P}(a)^-$. Consequently, we get

$$I_\mu(u_a) = \sigma_\mu(a) = m_{\mu,r}(a) = m_\mu(a) = \inf_{\mathcal{P}(a)} I_\mu(u) = \inf_{\mathcal{P}(a)^-} I_\mu(u),$$

and u_a is a ground state. □

5 L^2-Supcritical Perturbation Case

In this section, we consider the L^2-supercritical case: $\frac{10}{3} < q < 6$ and prove Theorem 1.3. For the sake of convenience, we still utilize the notations and definitions in Section 4.

In Lemma 4.1, the conclusion remains valid when $\frac{10}{3} < q < 6$. In the following, we show that $E(u, \iota)$ has the mountain pass geometry on $S_r(a) \times \mathbb{R}$.

Lemma 5.1 *Let* $\frac{10}{3} < q < 6$, $\mu, \gamma > 0$, *and assume that* $0 < a < \alpha_5$, *where*

$$\alpha_5 := \left(\frac{k^{*\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}} \right)^{\frac{1}{3}}.$$

There exist $0 < k_1^* < k_2^* < k^*$ *such that*

$$P_\mu(u), I_\mu(u) \ \textit{for all} \ u \in A_{k_1^*} \ \textit{and} \ 0 < \sup_{u \in A_{k_1^*}} I_\mu(u) < \inf_{u \in B_{k_2^*}} I_\mu(u),$$

where

$$A_{k^*} := \{u \in S_r(a) : \|\nabla u\|_2^2 \leq k^*\} \quad \textit{and} \ \ B_{k^*} := \{u \in S_r(a) : \|\nabla u\|_2^2 = 2k^*\}$$

Proof Take $k^* > 0$, which will be determined later. Suppose that $u, v \in S_r(a)$ such that $\|\nabla u\|_2^2 \leq k^*$ and $\|\nabla v\|_2^2 = 2k^*$. The proof here is similar to Lemma 4.2, which we briefly outline.

$$\begin{aligned} P_\mu(u) &= \|\nabla u\|_2^2 + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \mu \delta_q \|u\|_q^q - \|u\|_6^6 \\ &\geq \|\nabla u\|_2^2 - C(q) \mu \delta_q a^{q(1-\delta_q)} \|\nabla u\|_2^{q\delta_q} - S^{-3} \|\nabla u\|_2^6, \end{aligned}$$

and

$$\begin{aligned} I_\mu(u) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q} \|u\|_q^q - \frac{1}{6} \|u\|_6^6 \\ &\geq \frac{1}{2} \|\nabla u\|_2^2 - C(q) \frac{\mu}{q} a^{q(1-\delta_q)} \|\nabla u\|_2^{q\delta_q} - \frac{1}{6} S^{-3} \|\nabla u\|_2^6. \end{aligned}$$

Moreover,

$$\begin{aligned} I_\mu(v) - I_\mu(u) &\geq \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{2} \|\nabla u\|_2^2 - \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{q} \|v\|_q^q - \frac{1}{6} \|v\|_6^6 \\ &\geq \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{2} \|\nabla u\|_2^2 - \frac{\gamma}{4} \tilde{C}[C(12/5)]^{\frac{5}{3}} a^3 \|\nabla u\|_2 \\ &\quad - \frac{\mu}{q} C(q) a^{q(1-\delta_q)} \|\nabla u\|_2^{q\delta_q} - \frac{1}{6} S^{-3} \|\nabla u\|_2^6 \\ &\geq k^* - \frac{1}{2} k^* - \frac{\gamma}{4} \tilde{C}[C(12/5)]^{\frac{5}{3}} \left(\frac{k^{*\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}} \right) k^{*\frac{1}{2}} \\ &\quad - \frac{C(q)\mu}{q} \left(\frac{k^{*\frac{1}{2}}}{4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}} \right)^{\frac{q(1-\delta_q)}{3}} (2k^*)^{\frac{q\delta_q}{2}} - \frac{1}{6} S^{-3} (2k^*)^3 \end{aligned}$$

$$= \frac{7}{16}k^* - \left(\frac{2^{\frac{q\delta_q}{2}} C(q)\mu}{q\left(4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}}\right)^{\frac{q(1-\delta_q)}{3}}} k^{*\frac{2q-6}{3}} \right) k^* - \left(\frac{4}{3} S^{-3} k^{*2} \right) k^*$$
$$\geq \frac{5}{16}k^* > 0,$$

for $a < \alpha_5$, and we take

$$k^* = \min \left\{ \left(\frac{q(4\gamma \tilde{C}[C(12/5)]^{\frac{5}{3}})^{\frac{q(1-\delta_q)}{3}}}{16\mu C(q) 2^{\frac{q\delta_q}{2}}} \right)^{\frac{3}{2q-6}}, \ \left(\frac{3}{64} S^3 \right)^{\frac{1}{2}} \right\}, \tag{5.1}$$

then, for $0 < k_1^* < k_2^* < k^*$ small enough and $0 < a < \alpha_5$, we have

$$P_\mu(u), \ I_\mu(u) \ \text{ for all } \ u \in A_{k_1^*} \ \text{ and } \ 0 < \sup_{u \in A_{k_1^*}} I_\mu(u) < \inf_{u \in B_{k_2^*}} I_\mu(u).$$

The proof is completed. □

Lemma 5.2 *Let* $\frac{10}{3} < q < 6$, $\mu, \gamma > 0$, *and assume that* $0 < a < \alpha_5$. *Then we have*

(i) *There exists a sequence* $\{w_n\} \in S_r(a)$ *such that*

$$I_\mu(w_n) \to \sigma_\mu(a) \quad as \quad n \to \infty, \tag{5.2}$$
$$P_\mu(w_n) \to 0 \quad as \quad n \to \infty, \tag{5.3}$$
$$I'_\mu|_{S_r(a)}(w_n) \to 0 \quad as \quad n \to \infty. \tag{5.4}$$

(ii) $\sigma_\mu(a) = m_{\mu,r}(a) > 0$, *where* $\sigma_\mu(a)$ *and* $m_{\mu,r}(a)$ *is defined in Section* 4.

The proof of this lemma is similar to that of Propositions 4.3–4.5 and Lemmas 4.2–4.6 utilizing $q\delta_q > 2$, and thus it is omitted here.

Now, we make an upper bounded estimation for the mountain pass level $\sigma_\mu(a)$ in the following.

Lemma 5.3 *Let* $\frac{10}{3} < q < 6$, $\mu, \gamma > 0$, *and assume that* $0 < a < \alpha_5$. *Then we have* $\sigma_\mu(a) < \frac{1}{3} S^{\frac{3}{2}}$ *for* $\gamma \in (0, \widetilde{\gamma}_1)$ *small enough, where* $\widetilde{\gamma}_1$ *is defined in Lemma* 4.7.

Proof As in the proof of Lemma 4.7, we conclude that $\varphi_{v_\varepsilon}(\iota)$ achieves its global positive maximum at some ι_ε, and the critical point ι_ε is unique. In view of $(\varphi_{v_\varepsilon})'(\iota_\varepsilon) =$

$P_\mu(\iota_\varepsilon \star v_\varepsilon) = 0$, one has

$$\begin{aligned} e^{6\iota_\varepsilon}\|v_\varepsilon\|_6^6 &= e^{2\iota_\varepsilon}\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx - \mu\delta_q e^{q\delta_q}\|v_\varepsilon\|_q^q \\ &\le e^{2\iota_\varepsilon}\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx \\ &= e^{2\iota_\varepsilon}\left(\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma}{4}e^{-\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx\right) \\ &\le e^{2\iota_\varepsilon}2\max\left\{\|\nabla v_\varepsilon\|_2^2, \frac{\gamma}{4}e^{-\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx\right\}. \end{aligned} \tag{5.5}$$

Then, we consider the following possible cases.

Case 1 If $\|\nabla v_\varepsilon\|_2^2 > \frac{\gamma}{4}e^{-\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx$, we have

$$e^{6\iota_\varepsilon}\|v_\varepsilon\|_6^6 \le e^{2\iota_\varepsilon}2\|\nabla v_\varepsilon\|_2^2,$$

that is,

$$e^{4\iota_\varepsilon} \le \frac{2\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6}. \tag{5.6}$$

By $(\varphi_{v_\varepsilon})'(\iota_\varepsilon) = 0$, we infer that

$$\begin{aligned} e^{4\iota_\varepsilon} &= \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} + \frac{\gamma}{4}\frac{e^{-\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx}{\|v_\varepsilon\|_6^6} - \mu\delta_q e^{(q\delta_q-2)\iota_\varepsilon}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\ &\ge \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q\left(\frac{2\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6}\right)^{\frac{q\delta_q-2}{4}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\ &= \frac{\|u_\varepsilon\|_2^4\|\nabla u_\varepsilon\|_2^2}{a^4\|u_\varepsilon\|_6^6} - \mu\delta_q\left(\frac{2\|u_\varepsilon\|_2^4\|\nabla u_\varepsilon\|_2^2}{a^4\|u_\varepsilon\|_6^6}\right)^{\frac{q\delta_q-2}{4}}\frac{\|u_\varepsilon\|_2^{6-q}\|u_\varepsilon\|_q^q}{a^{6-q}\|u_\varepsilon\|_6^6} \\ &= \frac{\|u_\varepsilon\|_2^4(\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q-2}{4}}}{a^4\|u_\varepsilon\|_6^6}\left[(\|\nabla u_\varepsilon\|_2^2)^{\frac{6-q\delta_q}{4}} - \frac{\mu\delta_q 2^{\frac{q\delta_q-2}{4}}a^{q(1-\delta_q)}\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}(\|u_\varepsilon\|_6^6)^{\frac{q\delta_q-2}{4}}}\right], \\ &= \frac{\|u_\varepsilon\|_2^4(\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q-2}{4}}}{a^4\|u_\varepsilon\|_6^6}\left[(\|\nabla u_\varepsilon\|_2^2)^{\frac{6-q\delta_q}{4}} - \frac{\mu\delta_q 2^{\frac{q\delta_q-2}{4}}a^{q(1-\delta_q)}}{(\|u_\varepsilon\|_6^6)^{\frac{q\delta_q-2}{4}}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}}\right]. \end{aligned} \tag{5.7}$$

By virtue of (4.14)–(4.16) and (5.7), we get

$$e^{4\iota_\varepsilon} \ge C\frac{\|u_\varepsilon\|_2^4}{a^4}\left[C_1 - \mu\delta_q a^{q(1-\delta_q)}2^{\frac{q\delta_q-2}{4}}\frac{C_3}{C_2}\varepsilon^{\frac{6-q}{4}}\right] \ge C\frac{\|u_\varepsilon\|_2^4}{a^4}. \tag{5.8}$$

Case 2 If $\|\nabla v_\varepsilon\|_2^2 \le \frac{\gamma}{4} e^{-\iota_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx$, we have

$$e^{6\iota_\varepsilon} \|v_\varepsilon\|_6^6 \le 2e^{2\iota_\varepsilon} \frac{\gamma}{4} e^{-\iota_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx,$$

that is,

$$e^{5\iota_\varepsilon} \le \frac{\gamma}{2} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx}{\|v_\varepsilon\|_6^6}. \tag{5.9}$$

Again by $(\varphi_{v_\varepsilon})'(\iota_\varepsilon) = 0$, (2.3) and Hölder inequality, we infer to

$$\begin{aligned}
e^{4\iota_\varepsilon} &= \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} + \frac{\gamma}{4} e^{-\iota_\varepsilon} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx}{\|v_\varepsilon\|_6^6} - \mu\delta_q e^{(q\delta_q - 2)\iota_\varepsilon} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\
&\ge \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q \left(\frac{\gamma}{2} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 dx}{\|v_\varepsilon\|_6^6} \right)^{\frac{q\delta_q - 2}{5}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\
&\ge \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q \left(\frac{\gamma}{2} \frac{\widetilde{C} \|v_\varepsilon\|_{\frac{12}{5}}^4}{\|v_\varepsilon\|_6^6} \right)^{\frac{q\delta_q - 2}{5}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\
&\ge \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q \left(\frac{\gamma}{2} \frac{\widetilde{C} \|v_\varepsilon\|_2^3 \|v_\varepsilon\|_6}{\|v_\varepsilon\|_6^6} \right)^{\frac{q\delta_q - 2}{5}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\
&\ge \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{2} \right)^{\frac{q\delta_q - 2}{5}} a^{\frac{3(q\delta_q - 2)}{5}} \left(\frac{1}{\|v_\varepsilon\|_6^5} \right)^{\frac{q\delta_q - 2}{5}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_6^6} \\
&\ge \frac{\|u_\varepsilon\|_2^4 \|\nabla u_\varepsilon\|_2^2}{a^4 \|u_\varepsilon\|_6^6} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{2} \right)^{\frac{q\delta_q - 2}{5}} a^{\frac{5(q-6) - 2(q\delta_q - 2)}{5}} \frac{\|u_\varepsilon\|_q^q \|u_\varepsilon\|_2^{4 - q(1-\delta_q)}}{\|u_\varepsilon\|_6^{q\delta_q + 4}} \\
&= \frac{\|u_\varepsilon\|_2^4 (\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q - 2}{4}}}{a^4 \|u_\varepsilon\|_6^6} \\
&\quad \left[(\|\nabla u_\varepsilon\|_2^2)^{\frac{6 - q\delta_q}{4}} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{2} \right)^{\frac{q\delta_q - 2}{5}} a^{\frac{5q - 2q\delta_q - 26}{5}} \frac{\|u_\varepsilon\|_q^q \|u_\varepsilon\|_2^{q(\delta_q - 1)}}{\|u_\varepsilon\|_6^{q\delta_q - 2} (\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q - 2}{4}}} \right] \\
&= \frac{\|u_\varepsilon\|_2^4 (\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q - 2}{4}}}{a^4 \|u_\varepsilon\|_6^6} \\
&\quad \left[(\|\nabla u_\varepsilon\|_2^2)^{\frac{6 - q\delta_q}{4}} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{2} \right)^{\frac{q\delta_q - 2}{5}} \frac{a^{\frac{5q - 2q\delta_q - 26}{5}}}{\|u_\varepsilon\|_6^{q\delta_q - 2} (\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q - 2}{4}}} \frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}} \right],
\end{aligned} \tag{5.10}$$

By (4.7)–(4.8), we deduce that there exists constant $C_4 > 0$ such that

$$C_4 \leq (\|\nabla u_\varepsilon\|_2^2)^{\frac{q\delta_q-2}{4}} \|u_\varepsilon\|_6^{q\delta_q-2} \leq \frac{1}{C_4}. \tag{5.11}$$

So, in view of (4.14)–(4.16) and (5.11), we get

$$\begin{aligned} e^{4\iota_\varepsilon} &\geq C\frac{\|u_\varepsilon\|_2^4}{a^4}\left[C_1 - \mu\delta_q\left(\frac{\gamma\widetilde{C}}{2}\right)^{\frac{q\delta_q-2}{5}} a^{\frac{5q-2q\delta_q-26}{5}} \frac{C_3}{C_4}\varepsilon^{\frac{6-q}{4}}\right] \\ &\geq \frac{C\|u_\varepsilon\|_2^4}{a^4}. \end{aligned} \tag{5.12}$$

Based on above analysis, we will make an upper estimation for $\varphi_{v_\varepsilon}(\iota)$. Firstly, as in Lemma 4.7, we can define the function $\varphi^0_{v_\varepsilon}(\iota)$ and make an estimation for $\varphi^0_{v_\varepsilon}(\iota)$, that is

$$\varphi^0_{v_\varepsilon}(\iota^0_\varepsilon) = \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon). \tag{5.13}$$

where ι^0_ε is a unique strict maximum point of $\varphi^0_{v_\varepsilon}(\iota)$. Secondly, we make an estimation for $\varphi_{v_\varepsilon}(\iota)$. By virtue of (4.22), we know that there exists some $\iota^* \in \mathbb{R}$ such that

$$\iota_\varepsilon \leq \iota^*, \quad \text{for all } \varepsilon > 0.$$

So, by (2.3)–(2.4), (5.11)–(5.13), (4.9)–(4.10), (4.16) and above inequality, we obtain

$$\begin{aligned} \sup_{\iota\in\mathbb{R}} \varphi_{v_\varepsilon}(\iota) &= \varphi_{v_\varepsilon}(\iota_\varepsilon) \\ &= \varphi^0_{v_\varepsilon}(\iota_\varepsilon) + \frac{\gamma}{4}e^{\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx - \frac{\mu}{q}e^{q\delta_q\iota_\varepsilon}\|v_\varepsilon\|_q^q \\ &\leq \sup_{\iota\in\mathbb{R}}\Psi^0_{v_\varepsilon}(\theta) + \frac{\gamma}{4}e^{\iota_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx - \frac{\mu}{q}e^{q\delta_q\iota_\varepsilon}\|v_\varepsilon\|_q^q \\ &\leq \Psi^0_{v_\varepsilon}(\iota_{\varepsilon,0}) + \frac{\gamma}{4}\widetilde{C}\|v_\varepsilon\|^4_{\frac{12}{5}} - \frac{\mu}{q}e^{q\delta_q\iota_\varepsilon}\|v_\varepsilon\|_q^q \\ &\leq \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon) + C\gamma\frac{a^4}{\|u_\varepsilon\|_2^4}\|u_\varepsilon\|^4_{\frac{12}{5}} - \frac{C\mu a^{q(1-\delta_q)}}{q}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}} \\ &\leq \frac{1}{3}S^{\frac{3}{2}} + C^1\varepsilon + C^2\gamma\frac{\|u_\varepsilon\|^4_{\frac{12}{5}}}{\|u_\varepsilon\|_2^4} - C^3\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_q)}} \\ &= \frac{1}{3}S^{\frac{3}{2}} + C^1\varepsilon + C^2\gamma\frac{\varepsilon^{\frac{6}{5}\times\frac{5}{3}}}{\varepsilon^2} - C^3\varepsilon^{\frac{6-q}{4}} \\ &= \frac{1}{3}S^{\frac{3}{2}} + C^1\varepsilon + C^2\gamma - C^3\varepsilon^{\frac{6-q}{4}} \end{aligned}$$

$$< \frac{1}{3} S^{\frac{3}{2}}, \tag{5.14}$$

if we choose $\gamma = \varepsilon^\alpha$ for some constant $\alpha \geq 1$, and using the fact $0 < \frac{6-q}{4} < 1$.

Since $v_\varepsilon \in S_r(a)$, from Lemma 5.2 we take $\iota_3 < 0$ and $\iota_4 > 0$ such that $\iota_3 \star v_\varepsilon \in A_k$ and $I_\mu(\iota_4 \star v_\varepsilon) < 0$, respectively. We define a path

$$\eta^*_{v_\varepsilon} : t \in [0,1] \mapsto ((1-t)\iota_3 + t\iota_4) \star v_\varepsilon \in \Gamma_a.$$

Consequently, by (5.14), we obtain that there exists some $\widetilde{\gamma}_1 > 0$, such that

$$\sigma_\mu(a) \leq \max_{t \in [0,1]} I_\mu(\eta^*_{v_\varepsilon}(t)) \leq \sup_{\iota \in \mathbb{R}} \varphi_{v_\varepsilon}(\iota) < \frac{1}{3} S^{\frac{3}{2}},$$

for $\gamma \in (0, \widetilde{\gamma}_1)$ small enough, which completes the proof. □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 By virtue of (5.4), we have that there exists a sequence $\{\lambda_n\} \in \mathbb{R}$ such that

$$I'_\mu(u_n) - \lambda_n \Psi'(u_n) \to 0 \ \text{ in } \ H^{-1}. \tag{5.15}$$

Then we claim that $\{u_n\}$ is bounded in H. Indeed, by (5.2)and (5.3), we have

$$|2I_\mu(u_n) + P_\mu(u_n)| \leq C, \tag{5.16}$$

that is,

$$\begin{aligned} &2\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{4}\gamma \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{\mu(3q-2)}{2q} \int_{\mathbb{R}^3} |u_n|^q dx \\ &- \frac{4}{3} \int_{\mathbb{R}^3} |u_n|^6 dx \geq -C. \end{aligned} \tag{5.17}$$

By (5.17) and the bounded of $I_\mu(u_n)$, we infer to

$$-C \leq -\frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{\mu(3q-10)}{2q} \int_{\mathbb{R}^3} |u_n|^q dx - \frac{2}{3} \int_{\mathbb{R}^3} |u_n|^6 dx + 4C,$$

it follows that

$$\frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx + \frac{\mu(3q-10)}{2q} \int_{\mathbb{R}^3} |u_n|^q dx + \frac{2}{3} \int_{\mathbb{R}^3} |u_n|^6 dx \leq 5C,$$

which implies that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx, \quad \int_{\mathbb{R}^3} |u_n|^q dx \ \text{ and } \ \int_{\mathbb{R}^3} |u_n|^6 dx$$

are all bounded. Thus, we deduce that $\int_{\mathbb{R}^3} |\nabla u_n|^2 dx$ is also bounded. For convenience, we still take $\|\nabla u_n\|_2 \leq R^*$. We can proceed exactly as in the proof of Theorem 1.2 utilizing Lemmas 5.2–5.3, so complete the proof of Theorem 1.3. □

Acknowledgements We would like to thank the anonymous reviewers for careful reading the manuscript and their valuable comments. This work is supported by NSFC (12171497, 11771468, 11971027).

Data Availibility We declare that the manuscript has no associated data.

Declarations

Conflict of interest The authors have no conflict of interest to declare for this article.

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