The Journal of Geometric Analysis (2024) 34:298
https://doi.org/10.1007/s12220-024-01722-6

Multiple Blowing-Up Solutions for Asymptotically Critical Lane-Emden Systems on Riemannian Manifolds

Wenjing Chen[1] · Zexi Wang[1]

Received: 9 January 2024 / Accepted: 14 June 2024 / Published online: 25 July 2024

Abstract

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 8$. We are concerned with the following elliptic system

$$\begin{cases} -\Delta_g u + h(x)u = v^{p-\alpha\varepsilon}, & \text{in } \mathcal{M}, \\ -\Delta_g v + h(x)v = u^{q-\beta\varepsilon}, & \text{in } \mathcal{M}, \\ u, v > 0, & \text{in } \mathcal{M}, \end{cases}$$

where $\Delta_g = div_g \nabla$ is the Laplace–Beltrami operator on $\mathcal{M}$, $h(x)$ is a C^1-function on $\mathcal{M}$, $\varepsilon > 0$ is a small parameter, $\alpha, \beta > 0$ are real numbers, $(p, q) \in (1, +\infty) \times (1, +\infty)$ satisfies $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Using the Lyapunov–Schmidt reduction method, we obtain the existence of multiple blowing-up solutions for the above problem.

Keywords Multiple blowing-up solutions · Asymptotically critical · Lane-Emden system · Riemannian manifolds

Mathematics Subject Classification 58J05 · 35J47 · 35B33

1 Introduction

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 8$, where g denotes the metric tensor. We consider the following elliptic system

$$\begin{cases} -\Delta_g u + h(x)u = v^{p-\alpha\varepsilon}, & \text{in } \mathcal{M}, \\ -\Delta_g v + h(x)v = u^{q-\beta\varepsilon}, & \text{in } \mathcal{M}, \\ u, v > 0, & \text{in } \mathcal{M}, \end{cases} \tag{1.1}$$

✉ Wenjing Chen
wjchen@swu.edu.cn

Zexi Wang
zxwangmath@163.com

[1] School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

where $\Delta_g = div_g \nabla$ is the Laplace–Beltrami operator on $\mathcal{M}$, $h(x)$ is a C^1-function on $\mathcal{M}$, $\varepsilon > 0$ is a small parameter, $\alpha, \beta > 0$ are real numbers, $(p, q) \in (1, +\infty) \times (1, +\infty)$ is a pair of numbers lying on the *critical hyperbola*

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.2}$$

Without loss of generality, we assume that $1 < p \le \frac{N+2}{N-2} \le q$. Moreover, by (1.2), we have $p > \frac{2}{N-2}$.

In the case $u = v$, $p = q$ and $\alpha = \beta = 1$, system (1.1) is reduced to the following equation

$$-\Delta_g u + h(x)u = u^{2^*-1-\varepsilon}, \quad u > 0, \quad \text{in } \mathcal{M}, \tag{1.3}$$

where $N \ge 3$, $2^* = \frac{2N}{N-2}$, $\varepsilon \in \mathbb{R}$ is a small parameter. If $h(x) = \frac{N-2}{4(N-1)} Scal_g$, where $Scal_g$ is the scalar curvature of the manifold, equation (1.3) is intensively studied as the well-known *Yamabe problem* whose positive solutions u are such the scalar curvature of the conformal metric $u^{2^*-2}g$ is constant, see e.g. [1, 31, 32, 34]. If $h(x) \neq \frac{N-2}{4(N-1)} Scal_g$, Micheletti et al. [26] first proved that (1.3) has a single blowing-up solution, provided the graph of $h(x)$ is distinct at some point from the graph of $\frac{N-2}{4(N-1)} Scal_g$. Here, we say that a family of solutions u_ε of (1.3) blows up at a point $\xi_0 \in \mathcal{M}$ if there exists a family of points $\xi_\varepsilon \in \mathcal{M}$ such that $\xi_\varepsilon \to \xi_0$ and $u_\varepsilon(\xi_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Soon after, Deng [9] considered the existence of multiple blowing-up solutions which are separate from each other for (1.3). Chen [4] discovered the existence of clustered solutions which concentrate at one point in $\mathcal{M}$ for (1.3). Moreover, Sign-changing bubble tower solutions of (1.3) have been established in [5, 27]. For more related results about (1.3), we refer the readers to [8, 10, 13, 30] and references therein.

Now, we return to the following elliptic system

$$\begin{cases} -\Delta u = |v|^{p-1}v, & \text{in } \Omega, \\ -\Delta v = |u|^{q-1}u, & \text{in } \Omega, \\ (u, v) \in \mathcal{X}_{p,q}(\Omega), \end{cases} \tag{1.4}$$

called the Lane-Emden system, where $N \ge 3$, (p, q) satisfies (1.2), Ω is either a smooth bounded domain or $\mathbb{R}^N$, and $\mathcal{X}_{p,q}(\Omega) = \dot{W}^{2,\frac{p+1}{p}}(\Omega) \times \dot{W}^{2,\frac{q+1}{q}}(\Omega)$. System (1.4) has received remarkable attention for decades. When $\Omega = \mathbb{R}^N$, by the Sobolev embedding theorem, there holds

$$\dot{W}^{2,\frac{p+1}{p}}(\mathbb{R}^N) \hookrightarrow \dot{W}^{1,p^*}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N),$$
$$\dot{W}^{2,\frac{q+1}{q}}(\mathbb{R}^N) \hookrightarrow \dot{W}^{1,q^*}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N),$$

with

$$\frac{1}{p^*}=\frac{p}{p+1}-\frac{1}{N}=\frac{1}{q+1}+\frac{1}{N},\quad \frac{1}{q^*}=\frac{q}{q+1}-\frac{1}{N}=\frac{1}{p+1}+\frac{1}{N}.$$

Thus the following energy functional is well defined in $\mathcal{X}_{p,q}(\mathbb{R}^N)$:

$$\mathcal{J}(u,v)=\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\mathrm{d}z-\frac{1}{p+1}\int_{\mathbb{R}^N}|v|^{p+1}\mathrm{d}z-\frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}\mathrm{d}z.$$

By applying the concentration compactness principle, Lions [25] found a positive least energy solution to (1.4) in $\mathcal{X}_{p,q}(\mathbb{R}^N)$, which is radially symmetric and radially decreasing. Moreover, Wang [33] and Hulshof and Van der Vorst [19] independently proved that the uniqueness of the positive least energy solution $(U_{1,0}(x), V_{1,0}(z))\in\mathcal{X}_{p,q}(\mathbb{R}^N)$, and all the positive least energy solutions $\big(U_{\delta,\xi}(z), V_{\delta,\xi}(x)\big)$ given by

$$\big(U_{\delta,\xi}(z), V_{\delta,\xi}(z)\big)=\big(\delta^{-\frac{N}{q+1}}U_{1,0}(\delta^{-1}(z-\xi)),\delta^{-\frac{N}{p+1}}V_{1,0}(\delta^{-1}(z-\xi))\big),\ \text{for any } \delta>0,\xi\in\mathbb{R}^N.$$

Frank et al. [12] established the non-degeneracy of (1.4) at each least energy solution, that is, the linearized system around a least energy solution has precisely the $(N+1)$-dimensional spaces of solutions in $\mathcal{X}_{p,q}(\mathbb{R}^N)$. Furthermore, by using the Lyapunov–Schmidt reduction method and the non-degeneracy result obtained in [12], Guo et al. [18] established the existence and non-degeneracy of multiple blowing-up solutions to (1.4) with two potentials. For more investigations of system (1.4) with $\Omega=\mathbb{R}^N$, we can see [7, 14].

If Ω is a smooth bounded domain, much attention has been paid to study (1.4). Kim and Pistoia [22] first built multiple blowing-up solutions for the Lane-Emden system

$$\begin{cases}-\Delta u=|v|^{p-1}v+\varepsilon(\alpha u+\beta_1 v), & \text{in } \Omega,\\ -\Delta v=|u|^{q-1}u+\varepsilon(\alpha v+\beta_2 u), & \text{in } \Omega,\\ u,v=0, & \text{on } \partial\Omega,\end{cases}\tag{1.5}$$

where $N\geq 8,\varepsilon>0,\alpha,\beta_1,\beta_2\in\mathbb{R}, 1<p<\frac{N-1}{N-2}$, and (p,q) satisfies (1.2). Furthermore, using the local Pohozaev identities for the system, Guo et al. [16] proved the non-degeneracy of the blowing-up solutions to (1.5) constructed in [22]. Jin and Kim [20] studied the Coron's problem for the critical Lane-Emden system, and established the existence, qualitative properties of positive solutions. More recently, inspired by [29], Guo and Peng [15] considered sign-changing solutions to the sightly supercritical Lane-Emden system with Neumann boundary conditions. For more classical results regarding Hamiltonian systems in bounded domains, the readers may refer to [3, 6, 17, 21, 28] for a good survey.

As far as we know, no existence result for the system (1.1)–(1.2) in the literature. Therefore, it is natural to ask that if the system possesses solutions on a smooth compact Riemannian manifold. Motivated by [22] and [26], in this paper, we give an affirmative answer for this question.

To state our main result, we first recall some definitions and results.

Definition 1.1 For $k \geq 2$ to be a positive integer, let $(u_\varepsilon, v_\varepsilon)$ be a family of solutions of (1.1)–(1.2), we say that $(u_\varepsilon, v_\varepsilon)$ blows up and concentrates at point $\bar{\xi}^0 = (\xi_1^0, \xi_2^0, \cdots, \xi_k^0) \in \mathcal{M}^k$ if there exist $\bar{\xi}^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon, \cdots, \xi_k^\varepsilon) \in \mathcal{M}^k$ and $(\delta_1^\varepsilon, \delta_2^\varepsilon, \cdots, \delta_k^\varepsilon) \in (\mathbb{R}^+)^k$ such that $\xi_j^\varepsilon \to \xi_j^0$ and $\delta_j^\varepsilon \to 0$ as $\varepsilon \to 0$ for $j = 1, 2, \cdots, k$, and

$$\left\| (u_\varepsilon, v_\varepsilon) - \Big(\sum_{j=1}^k W_{\delta_j^\varepsilon, \xi_j^\varepsilon}, \sum_{j=1}^k H_{\delta_j^\varepsilon, \xi_j^\varepsilon} \Big) \right\| \to 0, \quad \text{as } \varepsilon \to 0,$$

where $\|\cdot\|$ and $(W_{\delta,\xi}, H_{\delta,\xi})$ are defined in (2.1) and (2.5).

Definition 1.2 [23, Definition 0.1] Let $f \in C^1(\mathcal{M}, \mathbb{R})$, for any given integer $k \geq 2$, set $\bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_k)$, let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k \subset \mathcal{M}$ be k mutually disjoint closed subsets of critical points of f, we say that $(\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k) \subset \mathcal{M}^k$ is a C^1-stable critical set of function $F(\bar{\xi}) := \sum_{j=1}^k f(\xi_j)$, if for any $\varepsilon > 0$, there exists $\sigma > 0$ such that if $\Phi \in C^1(\mathcal{M}^k, \mathbb{R})$ with

$$\max_{d_g(\xi_j, \mathcal{C}_j) < \varepsilon, 1 \leq j \leq k} \big(|F(\bar{\xi}) - \Phi(\bar{\xi})| + |\nabla_g F(\bar{\xi}) - \nabla_g \Phi(\bar{\xi})| \big) < \delta,$$

then Φ has at least one critical point $\bar{\xi} \in \mathcal{M}^k$ with $d_g(\xi_j, \mathcal{C}_j) < \varepsilon$, where d_g is the geodesic distance on $\mathcal{M}$ with respect to the metric g.

Remark 1.3 [23, Remark 0.1] $(\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k) \subset \mathcal{M}^k$ is a C^1-stable critical set of function $F(\bar{\xi})$ if one of the following conditions holds:

(i) Every $\mathcal{C}_j$ is a strict local minimum (or local maximum) set of f, $j = 1, 2, \cdots, k$.
(ii) Every $\mathcal{C}_j = \{\xi_j^0\}$ is an isolated critical point of $f(\xi_j)$ with $\deg(\nabla_g f, B_g(\xi_j^0, \rho), 0) \neq 0$ for some $\rho > 0$, where deg is the Brouwer degree, and $B_g(\xi_j^0, \rho)$ is the ball in $\mathcal{M}$ centered at ξ_j^0 with radius ρ with respect to the distance induced by the metric g, $j = 1, 2, \cdots, k$.

Let $L_1, L_2, \cdots, L_7$ be positive numbers defined by

$$\begin{cases} L_1 = \int\limits_{\mathbb{R}^N} \nabla U_{1,0}(z) \nabla V_{1,0}(z) \mathrm{d}z, \\ L_2 = \int\limits_{\mathbb{R}^N} |z|^2 \nabla U_{1,0}(z) \nabla V_{1,0}(z) \mathrm{d}z, \\ L_3 = \int\limits_{\mathbb{R}^N} U_{1,0}(z) V_{1,0}(z) \mathrm{d}z, \end{cases} \quad \text{and} \quad \begin{cases} L_4 = \int\limits_{\mathbb{R}^N} |z|^2 V_{1,0}^{p+1}(z) \mathrm{d}z, \\ L_5 = \int\limits_{\mathbb{R}^N} |z|^2 U_{1,0}^{q+1}(z) \mathrm{d}z, \\ L_6 = \int\limits_{\mathbb{R}^N} V_{1,0}^{p+1}(z) \log V_{1,0}(z) \mathrm{d}z, \\ L_7 = \int\limits_{\mathbb{R}^N} U_{1,0}^{q+1}(z) \log U_{1,0}(z) \mathrm{d}z. \end{cases} \tag{1.6}$$

Our main result states as follows.

Theorem 1.4 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold, let $h(x)$ be a C^1-function on $\mathcal{M}$, (p, q) satisfies (1.2), for any given integer $k \geq 2$, set $\bar{\xi}^0 = (\xi_1^0, \xi_2^0, \cdots, \xi_k^0)$, let ξ_j^0 be an isolated critical point of*

$$\varphi(\xi_j) = h(\xi_j) - \Big(L_2 - \frac{L_4}{p+1} - \frac{L_5}{q+1}\Big) \frac{Scal_g(\xi_j)}{6NL_3} \tag{1.7}$$

with $\varphi(\xi_j^0) > 0$ and $\deg(\nabla_g \varphi, B_g(\xi_j^0, \rho), 0) \neq 0$ for some $\rho > 0$, $j = 1, 2, \cdots, k$, Assume that one of the following conditions holds:

(i) $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ *and* $N \geq 8$;
(ii) $p = \frac{N+2}{N-2}$ *and* $N \geq 10$;
(iii) $1 < p < \frac{N}{N-2}$ *and* $N \geq 8$.

Then for $\varepsilon > 0$ small enough, system (1.1) admits a family of solutions $(u_\varepsilon, v_\varepsilon)$, which blows up and concentrates at $\bar{\xi}^0$ as $\varepsilon \to 0$.

Remark 1.5 Under the assumptions on p, q and N of Theorem 1.4, we have that $L_i < +\infty$ for $i = 1, 2, \cdots, 7$.

Remark 1.6 From the proof of Theorem 1.4 (see Sect. 3), it's easy to find that if

$$\frac{\alpha}{(p+1)^2} + \frac{\beta}{(q+1)^2} > 0,$$

then Theorem 1.4 still holds true. However, in the proof of Proposition 3.1, we have to impose $\alpha, \beta > 0$ to guarantee the continuous embedding, see e.g. (4.9)–(4.10) and (5.18)–(5.19).

Remark 1.7 If $u = v$, $p = q = \frac{N+2}{N-2}$, $\alpha = \beta = 1$, then

$$\varphi(\xi_j) = h(\xi_j) - \frac{N-2}{4(N-1)} Scal_g(\xi_j),$$

and Theorem 1.4 is exactly the conclusion obtained in [9, Theorem 1.1].

The proof of our result relies on a well known finite dimensional Lyapunov–Schmidt reduction method, introduced in [2, 11]. The paper is organized as follows. In Sect. 2, we introduce the framework and present some preliminary results. The proof of Theorem 1.4 is given in Sect. 3. In Sect. 4, we perform the finite dimensional reduction, and Sect. 5 is devoted to the reduced problem. Throughout the paper, $C, C_i, i \in \mathbb{N}^+$ denote positive constants possibly different from line to line.

2 The Framework and Preliminary Results

Concerning the least energy solution $(U_{1,0}(z), V_{1,0}(z))$ of (1.4) with $\Omega = \mathbb{R}^N$, we have the following asymptotic behaviour and non-degeneracy result.

Lemma 2.1 [19, Theorem 2] *Assume that* $1 < p \leq \frac{N+2}{N-2}$. *If* $r \to +\infty$, *there hold*

$$V_{1,0}(r) = O(r^{2-N}),$$

and

$$U_{1,0}(r) = \begin{cases} O(r^{2-N}), & \text{if } p > \frac{N}{N-2}; \\ O(r^{2-N} \log r), & \text{if } p = \frac{N}{N-2}; \\ O(r^{2-(N-2)p}), & \text{if } p < \frac{N}{N-2}. \end{cases}$$

Lemma 2.2 [21, Lemma 2.2] *Assume that* $1 < p \leq \frac{N+2}{N-2}$. *If* $r \to +\infty$, *there hold*

$$V'_{1,0}(r) = O(r^{1-N}),$$

and

$$U'_{1,0}(r) = \begin{cases} O(r^{1-N}), & \text{if } p > \frac{N}{N-2}; \\ O(r^{1-N} \log r), & \text{if } p = \frac{N}{N-2}; \\ O(r^{1-(N-2)p}), & \text{if } p < \frac{N}{N-2}. \end{cases}$$

Lemma 2.3 [15, Remark 2.3] *Assume that* $1 < p \leq \frac{N+2}{N-2}$. *If* $r \to +\infty$, *there hold*

$$V''_{1,0}(r) = O(r^{-N}),$$

and

$$U''_{1,0}(r) = \begin{cases} O(r^{-N}), & \text{if } p > \frac{N}{N-2}; \\ O(r^{-N} \log r), & \text{if } p = \frac{N}{N-2}; \\ O(r^{-(N-2)p}), & \text{if } p < \frac{N}{N-2}. \end{cases}$$

Lemma 2.4 [12, Theorem 1] *Set*

$$(\Psi^1_{1,0},\Phi^1_{1,0})=\Big(z\cdot\nabla U_{1,0}+\frac{NU_{1,0}}{q+1},z\cdot\nabla V_{1,0}+\frac{NV_{1,0}}{p+1}\Big)$$

and

$$(\Psi^l_{1,0},\Phi^l_{1,0})=\big(\partial_l U_{1,0},\partial_l V_{1,0}\big),\quad \mathit{for} l=1,2,\cdots,N.$$

Then the space of solutions to the linear system

$$\begin{cases}-\Delta\Psi=pV_{1,0}^{p-1}\Phi, & in\ \mathbb{R}^N,\\ -\Delta\Phi=qU_{1,0}^{q-1}\Psi, & in\ \mathbb{R}^N,\\ (\Psi,\Phi)\in\dot W^{2,\frac{p+1}{p}}(\mathbb{R}^N)\times\dot W^{2,\frac{q+1}{q}}(\mathbb{R}^N)\end{cases}$$

is spanned by

$$\big\{(\Psi^0_{1,0},\Phi^0_{1,0}),(\Psi^1_{1,0},\Phi^1_{1,0}),\cdots,(\Psi^N_{1,0},\Phi^N_{1,0})\big\}.$$

Moreover, we have the following elementary inequality.

Lemma 2.5 [24, Lemma 2.1] *For any $a>0$, b real, there holds*

$$\big||a+b|^\beta-b^\beta\big|\le\begin{cases}C(\beta)(a^{\beta-1}|b|+|b|^\beta), & \mathit{if}\ \beta\ge1,\\ C(\beta)\min\big\{a^{\beta-1}|b|,|b|^\beta\big\}, & \mathit{if}\ 0<\beta<1.\end{cases}$$

Now, we recall some definitions and results about the compact Riemannian manifold $(\mathcal{M},g)$.

Definition 2.6 Let $(\mathcal{M},g)$ be a smooth compact Riemannian manifold. On the tangent bundle of $\mathcal{M}$, define the exponential map $\exp:T\mathcal{M}\to\mathcal{M}$, which has the following properties:

(i) exp is of class C^∞;
(ii) there exists a constant $r_0>0$ such that $\exp_\xi|_{B(0,r_0)}\to B_g(\xi,r_0)$ is a diffeomorphism for all $\xi\in\mathcal{M}$.

Fix such r_0 in this paper with $r_0<i_g/2$, where i_g denotes the injectivity radius of $(\mathcal{M},g)$. For any $1<s<+\infty$ and $u\in L^s(\mathcal{M})$, we denote the L^s-norm of u by

$$\|u\|_s=\Big(\int_{\mathcal{M}}|u|^s dv_g\Big)^{1/s},$$

where $dv_g=\sqrt{\det g}dz$ is the volume element on $\mathcal{M}$ associated to the metric g. We introduce the Banach space

$$\mathcal{X}_{p,q}(\mathcal{M})=\dot W^{2,\frac{p+1}{p}}(\mathcal{M})\times\dot W^{2,\frac{q+1}{q}}(\mathcal{M})$$

equipped with the norm

$$\|(u,v)\| = \|\Delta_g u\|_{\frac{p+1}{p}} + \|\Delta_g v\|_{\frac{q+1}{q}}. \tag{2.1}$$

Denote by $\mathcal{I}^*$ the formal adjoint operator of the embedding $\mathcal{I} : \mathcal{X}_{q,p}(\mathcal{M}) \hookrightarrow L^{p+1}(\mathcal{M}) \times L^{q+1}(\mathcal{M})$. By the Calderón-Zygmund estimate, the operator $\mathcal{I}^*$ maps $L^{\frac{p+1}{p}}(\mathcal{M}) \times L^{\frac{q+1}{q}}(\mathcal{M})$ to $\mathcal{X}_{p,q}(\mathcal{M})$. Then we rewrite problem (1.1) as

$$(u,v) = \mathcal{I}^*\big(f_\varepsilon(v), g_\varepsilon(u)\big). \tag{2.2}$$

where $f_\varepsilon(u) := u_+^{p-\alpha\varepsilon}$, $g_\varepsilon(u) := u_+^{q-\beta\varepsilon}$ and $u_+ = \max\{u, 0\}$. Moreover, by the Sobolev embedding theorem, we have

$$\|\mathcal{I}^*\big(f_\varepsilon(v), g_\varepsilon(u)\big)\| \le C\|f_\varepsilon(v)\|_{\frac{p+1}{p}} + C\|g_\varepsilon(u)\|_{\frac{q+1}{q}}, \tag{2.3}$$

and

$$\mathcal{X}_{p,q}(\mathcal{M}) \hookrightarrow \dot{W}^{1,p^*}(\mathcal{M}) \times \dot{W}^{1,q^*}(\mathcal{M}), \quad \mathcal{X}_{p,q}(\mathcal{M}) \hookrightarrow L^2(\mathcal{M}) \times L^2(\mathcal{M}). \tag{2.4}$$

Let χ be a smooth cutoff function such that $0 \le \chi \le 1$ in $\mathbb{R}^+$, $\chi = 1$ in $[0, r_0/2]$, and $\chi = 0$ out of $[r_0, +\infty]$. For any $\xi \in \mathcal{M}$ and $\delta > 0$, we define the following functions on $\mathcal{M}$

$$\begin{aligned}(W_{\delta,\xi}(x), H_{\delta,\xi}(x)) := \big(&\chi(d_g(x,\xi))\delta^{-\frac{N}{q+1}} U_{1,0}(\delta^{-1}\exp_\xi^{-1}(x)),\\ &\chi(d_g(x,\xi))\delta^{-\frac{N}{p+1}} V_{1,0}(\delta^{-1}\exp_\xi^{-1}(x))\big)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(\Psi^i_{\delta,\xi}(x), \Phi^i_{\delta,\xi}(x)) := \big(&\chi(d_g(x,\xi))\delta^{-\frac{N}{q+1}} \Psi^i_{1,0}(\delta^{-1}\exp_\xi^{-1}(x)),\\ &\chi(d_g(x,\xi))\delta^{-\frac{N}{p+1}} \Phi^i_{1,0}(\delta^{-1}\exp_\xi^{-1}(x))\big),\end{aligned}$$

for $i = 0, 1, \cdots, N$, where $(\Psi^i_{1,0}, \Phi^i_{1,0})$ is given in Lemma 2.4.

For any $\varepsilon > 0$ and $\bar{t} = (t_1, t_2, \cdots, t_k) \in (\mathbb{R}^+)^k$, we set

$$\bar{\delta} = (\delta_1, \delta_2, \cdots, \delta_k) \in (\mathbb{R}^+)^k, \quad \delta_j = \sqrt{\varepsilon t_j}, \quad \varrho_1 < t_j < \frac{1}{\varrho_1}, \tag{2.6}$$

for fixed small $\varrho_1 > 0$. Moreover, for $\varrho_2 \in (0,1)$, we define the configuration space Λ by

$$\Lambda = \Big\{(\bar{\delta}, \bar{\xi}) : \bar{\delta} = (\delta_1, \delta_2, \cdots, \delta_k) \in (\mathbb{R}^+)^k,\ \bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_k) \in \mathcal{M}^k,$$

$$d_g(\xi_j, \xi_m) \geq \varrho_2 > 2r_0 \ \text{ for } j, m = 1, 2, \cdots, k \text{ and } j \neq m\Big\}.$$

Let $\mathcal{Y}_{\bar{\delta},\bar{\xi}}$ and $\mathcal{Z}_{\bar{\delta},\bar{\xi}}$ be two subspaces of $\mathcal{X}_{p,q}(\mathcal{M})$ given as

$$\mathcal{Y}_{\bar{\delta},\bar{\xi}} = span\Big\{(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}) : i = 0, 1, \cdots, N \text{ and } j = 1, 2, \cdots, k\Big\}$$

and

$$\mathcal{Z}_{\bar{\delta},\bar{\xi}} = \Big\{(\Psi, \Phi) \in \mathcal{X}_{p,q}(\mathcal{M}) : \langle(\Psi, \Phi), (\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j})\rangle_h = 0 \\ \text{for } i = 0, 1, \cdots, N \text{ and } j = 1, 2, \cdots, k\Big\},$$

where

$$\langle(u, v), (\varphi, \psi)\rangle_h = \int\limits_{\mathcal{M}} (\nabla_g u \cdot \nabla_g \psi + \nabla_g v \cdot \nabla_g \varphi) dv_g + \int\limits_{\mathcal{M}} (hu\psi + hv\varphi) dv_g$$

for any $(u, v), (\varphi, \psi) \in \mathcal{X}_{p,q}(\mathcal{M})$.

Lemma 2.7 *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{X}_{p,q}(\mathcal{M}) = \mathcal{Y}_{\bar{\delta},\bar{\xi}} \oplus \mathcal{Z}_{\bar{\delta},\bar{\xi}}$.*

Proof We shall prove that for any $(\Psi, \Phi) \in \mathcal{X}_{p,q}(\mathcal{M})$, there exists unique pair $(\Psi_0, \Phi_0) \in \mathcal{Z}_{\bar{\delta},\bar{\xi}}$ and coefficients $c_{10}, c_{11}, \cdots, c_{1N}, c_{20}, c_{21}, \cdots, c_{2N}, \cdots, c_{k0}, c_{k1}, \cdots, c_{kN}$ such that

$$(\Psi, \Phi) = (\Psi_0, \Phi_0) + \sum_{l=0}^{N} \sum_{m=1}^{k} c_{lm} (\Psi^l_{\delta_m,\xi_m}, \Phi^l_{\delta_m,\xi_m}). \tag{2.7}$$

The requirement that $(\Psi_0, \Phi_0) \in \mathcal{Z}_{\bar{\delta},\bar{\xi}}$ is equivalent to demanding

$$\begin{aligned} &\int\limits_{\mathcal{M}} \big(\nabla_g \Psi \cdot \nabla_g \Phi^i_{\delta_j,\xi_j} + \nabla_g \Phi \cdot \nabla_g \Psi^i_{\delta_j,\xi_j} + h\Psi \Phi^i_{\delta_j,\xi_j} + h\Phi \Psi^i_{\delta_j,\xi_j}\big) dv_g \\ &\quad = \sum_{l=0}^{N} \sum_{m=1}^{k} c_{lm} \int\limits_{\mathcal{M}} \big(\nabla_g \Psi^l_{\delta_m,\xi_m} \cdot \nabla_g \Phi^i_{\delta_j,\xi_j} + \nabla_g \Phi^l_{\delta_m,\xi_m} \cdot \nabla_g \Psi^i_{\delta_j,\xi_j} \\ &\qquad + h\Psi^l_{\delta_m,\xi_m} \Phi^i_{\delta_j,\xi_j} + h\Phi^l_{\delta_m,\xi_m} \Psi^i_{\delta_j,\xi_j}\big) dv_g \end{aligned} \tag{2.8}$$

for any $i = 0, 1, \cdots, N$ and $j = 1, 2, \cdots, k$.

We estimate the integral on the right-hand side of (2.8). By standard properties of the exponential map, there exists $C > 0$ such that for any $\xi \in \mathcal{M}$, $\delta > 0$, $z \in B(0, r_0/\delta)$, and $i, j, k \in \mathbb{N}^+$, there hold

$$|g^{ij}_{\delta,\xi}(z) - Eucl^{ij}| \leq C\delta^2 |z|^2, \quad \text{and} \quad |g^{ij}_{\delta,\xi}(z)(\Gamma_{\delta,\xi})^k_{ij}(z)| \leq C\delta^2 |z|, \tag{2.9}$$

where $g_{\delta,\xi}(z) = \exp_\xi^* g(\delta z)$ and $(\Gamma_{\delta,\xi})_{ij}^k$ stand for the Christoffel symbols of the metric $g_{\delta,\xi}$. Taking into account that there holds

$$\Delta_{g_{\delta,\xi}} = g_{\delta,\xi}^{ij}\Big(\frac{\partial^2}{\partial z_i \partial z_j} - (\Gamma_{\delta,\xi})_{ij}^k \frac{\partial}{\partial z_k}\Big), \tag{2.10}$$

by Lemma 2.4 and $dg(\xi_j, \xi_m) > 2r_0$ for $j \neq m$, we have

$$\begin{aligned}\int_{\mathcal{M}} \nabla_g \Psi_{\delta_m,\xi_m}^l \cdot \nabla_g \Phi_{\delta_j,\xi_j}^i dv_g &= \delta_{jm} \int_{\mathcal{M}} \nabla_g \Psi_{\delta_j,\xi_j}^l \cdot \nabla_g \Phi_{\delta_j,\xi_j}^i dv_g \\ &= \delta_{jm} \int_{B(0,r_0/\delta_j)} \nabla_{g_{\delta_j,\xi_j}} (\chi_{\delta_j} \Psi_{1,0}^l) \cdot \nabla_{g_{\delta_j,\xi_j}} (\chi_{\delta_j} \Phi_{1,0}^i) \mathrm{d}z \\ &= p\delta_{jm} \int_{B(0,r_0/\delta_j)} \chi_{\delta_j}^2 V_{1,0}^{p-1} \Phi_{1,0}^l \Phi_{1,0}^i \mathrm{d}z + O(\delta_j^2) \\ &= p\delta_{il}\delta_{jm} \int_{B(0,r_0/\delta_j)} \chi_{\delta_j}^2 V_{1,0}^{p-1} (\Phi_{1,0}^i)^2 \mathrm{d}z + O(\delta_j^2),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\int_{\mathcal{M}} h\Psi_{\delta_m,\xi_m}^l \Phi_{\delta_j,\xi_j}^i dv_g &= \delta_{jm} \int_{\mathcal{M}} h\Psi_{\delta_j,\xi_j}^l \Phi_{\delta_j,\xi_j}^i dv_g = \delta_{jm}\delta_j^2 \int_{B(0,r_0/\delta_j)} X_{\delta_j^2} h_{\delta_j,\xi_j} \Psi_{1,0}^l \Phi_{1,0}^i \mathrm{d}z \\ &= -\delta_{jm}\delta_j^2 \int_{B(0,r_0/\delta_j)} X_{\delta_j^2} h_{\delta_j,\xi_j} \frac{\Delta \Phi_{1,0}^l}{qU_{1,0}^{q-1}} \Phi_{1,0}^i \mathrm{d}z + o(\delta_j^2) = \delta_{il}\delta_{jm}\delta_j^2 \\ &\times \int_{B(0,r_0/\delta_j)} X_{\delta_j^2} h_{\delta_j,\xi_j} \frac{(\nabla \Phi_{1,0}^i)^2}{qU_{1,0}^{q-1}} \mathrm{d}z + o(\delta_j^2),\end{aligned} \tag{2.12}$$

where $\chi_{\delta_j}(x) = \chi(\delta_j |z|)$ and $h_{\delta_j,\xi_j}(z) = h(\exp_{\xi_j}(\delta_j z))$. Similarly, we have

$$\int_{\mathcal{M}} \nabla_g \Phi_{\delta_m,\xi_m}^l \cdot \nabla_g \Psi_{\delta_j,\xi_j}^i dv_g = q\delta_{il}\delta_{jm} \int_{B(0,r_0/\delta_j)} \chi_{\delta_j}^2 U_{1,0}^{q-1} (\Psi_{1,0}^i)^2 \mathrm{d}z + O(\delta_j^2), \tag{2.13}$$

and

$$\int_{\mathcal{M}} h\Phi_{\delta_m,\xi_m}^l \Psi_{\delta_j,\xi_j}^i dv_g = \delta_{il}\delta_{jm}\delta_j^2 \int_{B(0,r_0/\delta_j)} X_{\delta_j^2} h_{\delta_j,\xi_j} \frac{(\nabla \Psi_{1,0}^i)^2}{pV_{1,0}^{p-1}} \mathrm{d}z + o(\delta_j^2). \tag{2.14}$$

By plugging (2.11)–(2.14) into (2.8), we can see that the coefficients c_{lm} are uniquely determined for $l = 0, 1, \cdots, N$ and $m = 1, 2, \cdots, k$. By virtue of (2.7), so is (Ψ_0, Φ_0).

On the other hand, $\mathcal{Y}_{\bar{\delta},\bar{\xi}}$ and $\mathcal{Z}_{\bar{\delta},\bar{\xi}}$ are clearly closed subspaces of $\mathcal{X}_{p,q}(\mathcal{M})$, Therefore, they are topological complements of each other. □

3 Scheme of the Proof of Theorem 1.4

We look for solutions of system (1.1), or equivalently of (2.2), of the form

$$(u_\varepsilon, v_\varepsilon) = \big(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \quad \mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}\big), \mathcal{W}_{\bar{\delta},\bar{\xi}} = \sum_{j=0}^{k} W_{\delta_j,\xi_j},$$

$$\mathcal{H}_{\bar{\delta},\bar{\xi}} = \sum_{j=0}^{k} H_{\delta_j,\xi_j}, \quad (\bar{\delta}, \bar{\xi}) \in \Lambda, \tag{3.1}$$

where $\bar{\delta}$ is as in (2.6), $(W_{\delta_j,\xi_j}, H_{\delta_j,\xi_j})$ is as in (2.5), and $(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}}) \in \mathcal{Z}_{\bar{\delta},\bar{\xi}}$. By Lemma 2.7, $\mathcal{X}_{p,q}(\mathcal{M}) = \mathcal{Y}_{\bar{\delta},\bar{\xi}} \oplus \mathcal{Z}_{\bar{\delta},\bar{\xi}}$. Then we define the projections $\Pi_{\bar{\delta},\bar{\xi}}$ and $\Pi^{\perp}_{\bar{\delta},\bar{\xi}}$ of the Sobolev space $\mathcal{X}_{p,q}(\mathcal{M})$ onto $\mathcal{Y}_{\bar{\delta},\bar{\xi}}$ and $\mathcal{Z}_{\bar{\delta},\bar{\xi}}$ respectively. Therefore, we have to solve the couples of equations

$$\Pi_{\bar{\delta},\bar{\xi}}\Big[\big(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}\big) - \mathcal{I}^*\big(f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}), g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}})\big)\Big] = 0, \tag{3.2}$$

and

$$\Pi^{\perp}_{\bar{\delta},\bar{\xi}}\Big[\big(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}\big) - \mathcal{I}^*\big(f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}), g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}})\big)\Big] = 0. \tag{3.3}$$

The first step in the proof consists in solving equation (3.3). This requires Proposition 3.1 below, whose proof is postponed to Sect. 4.

Proposition 3.1 *Under the assumptions of Theorem 1.4, if $(\bar{\delta}, \bar{\xi}) \in \Lambda$ and $\bar{\delta}$ is as in (2.6), then for any $\varepsilon > 0$ small enough, equation (3.3) admits a unique solution $(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}})$ in $\mathcal{Z}_{\bar{\delta},\bar{\xi}}$, which is continuously differentiable with respect to $\bar{t}$ and $\bar{\xi}$, such that*

$$\|(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}})\| \le C\varepsilon|\log\varepsilon|.$$

We now introduce the energy functional $\mathcal{J}_\varepsilon$ defined on $\mathcal{X}_{p,q}(\mathcal{M})$ by

$$\mathcal{J}_\varepsilon(u, v) = \int_{\mathcal{M}} \nabla_g u \cdot \nabla_g v dv_g + \int_{\mathcal{M}} huv dv_g$$

$$-\frac{1}{p+1-\alpha\varepsilon}\int_{\mathcal{M}} v^{p+1-\alpha\varepsilon}dv_g$$
$$-\frac{1}{q+1-\beta\varepsilon}\int_{\mathcal{M}} u^{q+1-\beta\varepsilon}dv_g.$$

It is clear that the critical points of $\mathcal{J}_\varepsilon$ are the solutions of system (1.1). Moreover,

$$\mathcal{J}_\varepsilon'(u,v)(\varphi,\psi) = \int_{\mathcal{M}}(\nabla_g u\cdot\nabla_g\psi + \nabla_g v\cdot\nabla_g\varphi)dv_g + \int_{\mathcal{M}}(hu\psi + hv\varphi)dv_g$$
$$-\int_{\mathcal{M}} u^{q-\beta\varepsilon}\varphi dv_g - \int_{\mathcal{M}} v^{p-\alpha\varepsilon}\psi dv_g,$$

for any $(u,v),(\varphi,\psi)\in\mathcal{X}_{p,q}(\mathcal{M})$. We also define the functional $\widetilde{\mathcal{J}}_\varepsilon : (\mathbb{R}^+)^k\times\mathcal{M}^k \to \mathbb{R}$

$$\widetilde{\mathcal{J}}_\varepsilon(\bar{t},\bar{\xi}) = \mathcal{J}_\varepsilon\big(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}\big), \tag{3.4}$$

where $(\mathcal{W}_{\bar{\delta},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}})$ is as (3.1), $(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}})$ is given in Proposition 3.1.

Definition 3.2 For a given C^1-function φ_ε, we say that the estimate $\varphi_\varepsilon = o(\varepsilon)$ is C^1-uniform if there hold $\varphi_\varepsilon = o(\varepsilon)$ and $\nabla\varphi_\varepsilon = o(\varepsilon)$ as $\varepsilon\to 0$.

We solve equation (3.2) in Proposition 3.3 below whose proof is postponed to Sect. 5.

Proposition 3.3 (i) *Under the assumptions of Theorem 1.4, if $\bar{\delta}$ is as in (2.6), for any $\varepsilon > 0$ small enough, if $(\bar{t},\bar{\xi})$ is a critical point of the functional $\widetilde{\mathcal{J}}_\varepsilon$, then $\big(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}\big)$ is a solution of system (1.1), or equivalently of (2.2).*

(ii) *Under the assumptions of Theorem 1.4, there holds*

$$\widetilde{\mathcal{J}}_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}}) = \frac{2k}{N}L_1 + c_1\varepsilon - c_2\varepsilon\log\varepsilon + \Psi_k(\bar{t},\bar{\xi})\varepsilon + o(\varepsilon),$$

as $\varepsilon\to 0$, C^1-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where

$$\Psi_k(\bar{t},\bar{\xi}) = \sum_{j=1}^{k}\Big\{L_3\varphi(\xi_j)t_j - \frac{NL_1}{2}\Big(\frac{\alpha}{(p+1)^2} + \frac{\beta}{(q+1)^2}\Big)\log(t_j)\Big\}, \tag{3.5}$$

and

$$c_1 = \Big[\Big(\frac{L_6\alpha}{p+1} + \frac{L_7\beta}{q+1}\Big) - \Big(\frac{\alpha}{(p+1)^2} + \frac{\beta}{(q+1)^2}\Big)L_1\Big]k,$$

$$c_2 = \frac{NL_1k}{2}\Big(\frac{\alpha}{(p+1)^2}+\frac{\beta}{(q+1)^2}\Big), \tag{3.6}$$

with L_1, L_3, L_6, L_7 are positive constants given in (1.6), $\varphi(\xi_j)$ is defined as (1.7), $j = 1, 2, \cdots, k$.

We now prove Theorem 1.4 by using Propositions 3.1 and 3.3.

Proof of Theorem 1.4 Define $\widetilde{\mathcal{J}} : (\mathbb{R}^+)^k \times \mathcal{M}^k \to \mathbb{R}$ by

$$\widetilde{\mathcal{J}}(\bar t,\bar\xi) = \sum_{j=1}^{k} f(t_j,\xi_j), \quad \text{with } f(t_j,\xi_j) = -\widetilde{C}\log t_j + L_3\varphi(\xi_j)t_j,$$

where $\widetilde{C} = \big(\frac{\alpha}{(p+1)^2}+\frac{\beta}{(q+1)^2}\big)\frac{NL_1}{2}$ and $L_1, L_3 > 0$ are given in (1.6). Since ξ_j^0 is an isolated critical point of the function $\varphi(\xi_j)$ with $\varphi(\xi_j^0) > 0$, and set $t_j^0 = \frac{\widetilde{C}}{L_3\varphi(\xi_j^0)}$, then $t_j^0 > 0$ and (t_j^0, ξ_j^0) is an isolated critical point of $f(t_j, \xi_j)$. Moreover, by $\deg(\nabla_g\varphi, B_g(\xi_j^0,\rho), 0) \neq 0$ for some $\rho > 0$, we obtain $\deg(\nabla_g f, B_g(\xi_j^0,\rho), 0) \neq 0$, $j = 1, 2, \cdots, k$. Hence, by Remark 1.3, $(\bar{t^0}, \bar{\xi^0})$ is a C^1-stable critical set of $\widetilde{\mathcal{J}}$, where $\bar{t^0} = (t_1^0, t_2^0, \cdots, t_k^0)$ and $\bar{\xi^0} = (\xi_1^0, \xi_2^0, \cdots, \xi_k^0)$. Using Proposition 3.3, we have

$$\Big|\partial_{\bar t}\big(\varepsilon^{-1}\widetilde{\mathcal{J}}_\varepsilon - \widetilde{\mathcal{J}}\big)\Big| + \Big|\partial_{\bar\xi}\big(\varepsilon^{-1}\widetilde{\mathcal{J}}_\varepsilon - \widetilde{\mathcal{J}}\big)\Big| \to 0,$$

as $\varepsilon \to 0$, uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$. By standard properties of the Brouwer degree, it follows that there exists a family of critical points $(\bar{t^\varepsilon}, \bar{\xi^\varepsilon})$ of $\widetilde{\mathcal{J}}_\varepsilon$ converging to $(\bar{t^0}, \bar{\xi^0})$ as $\varepsilon \to 0$. Using Proposition 3.3 again, we can see that the function $(u_\varepsilon, v_\varepsilon) = \big(\mathcal{W}_{\bar{\delta^\varepsilon},\bar{\xi^\varepsilon}} + \Psi_{\varepsilon,\bar{t^\varepsilon},\bar{\xi^\varepsilon}}, \mathcal{H}_{\bar{\delta^\varepsilon},\bar{\xi^\varepsilon}} + \Phi_{\varepsilon,\bar{t^\varepsilon},\bar{\xi^\varepsilon}}\big)$ is a pair of solutions of system (1.1) for any $\varepsilon > 0$ small enough, where $\bar{\delta^\varepsilon}$ is as in (2.6). Moreover, $(u_\varepsilon, v_\varepsilon)$ blows up and concentrates at $\bar{\xi^0}$ at $\varepsilon \to 0$. This ends the proof.

□

4 Proof of Proposition 3.1

This section is devoted to the proof of Proposition 3.1. For any $\varepsilon > 0$, $\bar t \in (\mathbb{R}^+)^k$, and $\bar\xi \in \mathcal{M}^k$, if $\bar\delta$ is as in (2.6), we introduce the map $\mathcal{L}_{\varepsilon,\bar t,\bar\xi} : \mathcal{Z}_{\bar\delta,\bar\xi} \to \mathcal{Z}_{\bar\delta,\bar\xi}$ defined by

$$\mathcal{L}_{\varepsilon,\bar t,\bar\xi}(\Psi,\Phi) = \Pi^{\perp}_{\bar\delta,\bar\xi}\Big[(\Psi,\Phi) - \mathcal{I}^*\big(f'_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})\Phi, g'_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\Psi\big)\Big]. \tag{4.1}$$

It's easy to check that $\mathcal{L}_{\varepsilon,\bar t,\bar\xi}$ is well defined in $\mathcal{Z}_{\bar\delta,\bar\xi}$. Next, we prove the invertibility of this map.

Lemma 4.1 *Under the assumptions on p, q and N of Theorem 1.4, if $(\bar\delta, \bar\xi) \in \Lambda$ and $\bar\delta$ is as in (2.6), then for any $\varepsilon > 0$ small enough, and $(\Psi, \Phi) \in \mathcal{Z}_{\bar\delta,\bar\xi}$, there holds*

$$\|\mathcal{L}_{\varepsilon,\bar t,\bar\xi}(\Psi,\Phi)\| \geq C\|(\Psi,\Phi)\|,$$

where $\mathcal{L}_{\varepsilon,\bar{t},\xi}(\Psi,\Phi)$ *is as in* (4.1).

Proof We assume by contradiction that there exist a sequence $\varepsilon_n \to 0$ as $n \to +\infty$, $(\bar{\delta_n}, \bar{\xi_n}) \in \Lambda$, $\bar{t_n} = (t_{1n}, t_{2n}, \cdots, t_{kn}) \in (\mathbb{R}^+)^k$, $\bar{\xi_n} = (\xi_{1n}, \xi_{2n}, \cdots, \xi_{kn}) \in \mathcal{M}^k$, and a sequence of functions $(\Psi_n, \Phi_n) \in \mathcal{Z}_{\bar{\delta_n},\bar{\xi_n}}$ such that

$$\|(\Psi_n, \Phi_n)\| = 1, \quad \|\mathcal{L}_{\varepsilon_n,\bar{t_n},\bar{\xi_n}}(\Psi_n, \Phi_n)\| \to 0, \quad \text{as } n \to +\infty.$$

Then $\|\Psi_n\|_{q+1} \le C$ and $\|\Phi_n\|_{p+1} \le C$.

Step 1: For any $n \in \mathbb{N}^+$ and $j = 1, 2, \cdots, k$, let

$$(\widetilde{\Psi}_n(x), \widetilde{\Phi}_n(x)) = \big(\chi(\delta_{jn}|x|)\delta_{jn}^{\frac{N}{q+1}} \Psi_n(\exp_{\xi_{jn}}(\delta_{jn}x)), \chi(\delta_{jn}|x|)\delta_{jn}^{\frac{N}{p+1}} \Phi_n(\exp_{\xi_{jn}}(\delta_{jn}x))\big),$$

where χ is a cutoff function as in (2.5). A direct computations shows

$$\begin{aligned}
\|\Delta\widetilde{\Psi}_n\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} &\le \int\limits_{B(0,r_0/\delta_{jn})} |\delta_{jn}^{\frac{N}{q+1}} \Delta\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}x))|^{\frac{p+1}{p}} \mathrm{d}x \\
&= \int\limits_{B(0,r_0)} \delta_{jn}^{-N} |\delta_{jn}^{2+\frac{N}{q+1}} \Delta\Psi_n(\exp_{\xi_{jn}}(y))|^{\frac{p+1}{p}} \mathrm{d}y \\
&= \int\limits_{B_g(\xi_{jn},r_0)} |\Delta_g\Psi_n|^{\frac{p+1}{p}} dv_g = \int\limits_{\mathcal{M}} |\Delta_g\Psi_n|^{\frac{p+1}{p}} dv_g \le C,
\end{aligned}$$

and

$$\begin{aligned}
\|\Delta\widetilde{\Phi}_n\|^{\frac{q+1}{q}}_{L^{\frac{q+1}{q}}(\mathbb{R}^N)} &\le \int\limits_{B(0,r_0/\delta_{jn})} |\delta_{jn}^{\frac{N}{p+1}} \Delta\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}x))|^{\frac{q+1}{q}} \mathrm{d}x \\
&= \int\limits_{B(0,r_0)} \delta_{jn}^{-N} |\delta_{jn}^{2+\frac{N}{p+1}} \Delta\Phi_n(\exp_{\xi_{jn}}(y))|^{\frac{q+1}{q}} \mathrm{d}y \\
&= \int\limits_{B_g(\xi_{jn},r_0)} |\Delta_g\Phi_n|^{\frac{q+1}{q}} dv_g = \int\limits_{\mathcal{M}} |\Delta_g\Phi_n|^{\frac{q+1}{q}} dv_g \le C.
\end{aligned}$$

Hence, $(\widetilde{\Psi}_n, \widetilde{\Phi}_n)$ is bounded in $\mathcal{X}_{p,q}(\mathbb{R}^N)$. Up to a subsequence, there exists $(\widetilde{\Psi}, \widetilde{\Phi}) \in \mathcal{X}_{p,q}(\mathbb{R}^N)$ such that $(\widetilde{\Psi}_n, \widetilde{\Phi}_n) \rightharpoonup (\widetilde{\Psi}, \widetilde{\Phi})$ in $\mathcal{X}_{p,q}(\mathbb{R}^N)$, $(\widetilde{\Psi}_n, \widetilde{\Phi}_n) \to (\widetilde{\Psi}, \widetilde{\Phi})$ in $L^s_{loc}(\mathbb{R}^N) \times L^t_{loc}(\mathbb{R}^N)$ for any $(s, t) \in [1, q+1] \times [1, p+1]$, and $(\widetilde{\Psi}_n, \widetilde{\Phi}_n) \to (\widetilde{\Psi}, \widetilde{\Phi})$ almost everywhere in $\mathbb{R}^N$. For convenience, we denote $(P_n, K_n) = \mathcal{L}_{\varepsilon_n,\bar{t_n},\bar{\xi_n}}(\Psi_n, \Phi_n)$. Furthermore, by $(P_n, K_n) \in \mathcal{Z}_{\bar{\delta_n},\bar{\xi_n}}$, there exist $c^0_{1n}, c^1_{1n}, \cdots, c^N_{1n}, c^0_{2n}, c^1_{2n}, \cdots, c^N_{2n}, \cdots, c^0_{kn}, c^1_{kn}, \cdots, c^N_{kn}$ such that

$$(\Psi_n, \Phi_n) - \mathcal{I}^*\big(f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta_n},\bar{\xi_n}})\Phi_n, g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta_n},\bar{\xi_n}})\Psi_n\big)$$

$$= (P_n, K_n) + \sum_{l=0}^{N}\sum_{m=1}^{k} c_{mn}^l (\Psi^l_{\delta_{mn},\xi_{mn}}, \Phi^l_{\delta_{mn},\xi_{mn}}), \tag{4.2}$$

which also reads

$$\begin{cases} -\Delta\Psi_n = f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n - \Delta P_n - \sum\limits_{l=0}^{N}\sum\limits_{m=1}^{k} c_{mn}^l \Delta\Psi^l_{\delta_{mn},\xi_{mn}}, & \text{in } \mathbb{R}^N, \\ -\Delta\Phi_n = g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n - \Delta K_n - \sum\limits_{l=0}^{N}\sum\limits_{m=1}^{k} c_{mn}^l \Delta\Phi^l_{\delta_{mn},\xi_{mn}}, & \text{in } \mathbb{R}^N. \end{cases} \tag{4.3}$$

Using $(\Psi_n, \Phi_n) \in \mathcal{Z}_{\bar{\delta}_n,\bar{\xi}_n}$ again, by an easy change of variable, for $i = 0, 1, \cdots, N$ and $j = 1, 2\cdots, k$, we have

$$\begin{aligned} 0 &= \int_{\mathcal{M}} (\nabla_g\Psi_n\cdot\nabla_g\Phi^i_{\delta_{jn},\xi_{jn}} + \nabla_g\Phi_n\cdot\nabla_g\Psi^i_{\delta_{jn},\xi_{jn}} + h\Psi_n\Phi^i_{\delta_{jn},\xi_{jn}} + h\Phi_n\Psi^i_{\delta_{jn},\xi_{jn}})dv_g \\ &= \int_{B(0,r_0/\delta_{jn})} \Big[\delta_{jn}^{N-2-\frac{N}{p+1}} \nabla_{g_n}\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\cdot\nabla_{g_n}(\chi_n\Phi^i_{1,0}) \\ &\quad + \delta_{jn}^{N-2-\frac{N}{q+1}} \nabla_{g_n}\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\cdot\nabla_{g_n}(\chi_n\Psi^i_{1,0}) \\ &\quad + \delta_{jn}^{N-\frac{N}{p+1}} h(\exp_{\xi_{jn}}(\delta_{jn}z))\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\chi_n\Phi^i_{1,0} \\ &\quad + \delta_{jn}^{N-\frac{N}{q+1}} h(\exp_{\xi_{jn}}(\delta_{jn}z))\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\chi_n\Psi^i_{1,0}\Big]\mathrm{d}z \\ &= \int_{B(0,r_0/\delta_{jn})} \Big[\nabla_{g_n}\widetilde{\Psi}_n\cdot\nabla_{g_n}(\chi_n\Phi^i_{1,0}) + \nabla_{g_n}\widetilde{\Phi}_n\cdot\nabla_{g_n}(\chi_n\Psi^i_{1,0}) \\ &\quad + \delta_{jn}^2 h_n\widetilde{\Psi}_n\Phi^i_{1,0} + \delta_{jn}^2 h_n\widetilde{\Phi}_n\Psi^i_{1,0}\Big]\mathrm{d}z, \end{aligned}$$

where $g_n(z) = \exp^*_{\xi_{jn}} g(\delta_{jn}z)$, $\chi_n(z) = \chi(\delta_{jn}|z|)$ and $h_n(z) = h(\exp_{\xi_{jn}}(\delta_{jn}z))$. By Lemma 2.4, passing to the limit for the above equality, we obtain

$$\int_{\mathbb{R}^N} \big(pV_{1,0}^{p-1}\Phi^i_{1,0}\widetilde{\Phi} + qU_{1,0}^{q-1}\Psi^i_{1,0}\widetilde{\Psi}\big)\mathrm{d}z = \int_{\mathbb{R}^N} \big(\nabla\widetilde{\Psi}\cdot\nabla\Phi^i_{1,0} + \nabla\widetilde{\Phi}\cdot\nabla\Psi^i_{1,0}\big)\mathrm{d}z = 0. \tag{4.4}$$

Step 2: For any $l = 0, 1, \cdots, N$ and $m = 1, 2, \cdots, k$, $c_{mn}^l \to 0$ as $n \to \infty$. For any $n \in \mathbb{N}^+$, since (Ψ_n, Φ_n) and (P_n, K_n) belong to $\mathcal{Z}_{\bar{\delta}_n,\bar{\xi}_n}$, multiplying (4.2) by

$(\Psi^i_{\delta_{jn},\xi_{jn}}, \Phi^i_{\delta_{jn},\xi_{jn}})$, $0 \le i \le N$, $1 \le j \le k$, using (2.11)–(2.14), we have

$$
\begin{aligned}
&-\int\limits_{\mathcal{M}} \big(f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n\Phi^i_{\delta_{jn},\xi_{jn}} + g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\Psi^i_{\delta_{jn},\xi_{jn}}\big)dv_g \\
=&\sum_{l=0}^{N}\sum_{m=1}^{k} c^l_{mn}\delta_{il}\delta_{jm}\int\limits_{B(0,r_0/\delta_{jn})}\big(p\chi_n^2 V_{1,0}^{p-1}(\Phi^i_{1,0})^2 + q\chi_n^2 U_{1,0}^{q-1}(\Psi^i_{1,0})^2\big)\mathrm{d}z + O(\varepsilon_n).
\end{aligned}
\tag{4.5}
$$

Moreover, by (4.4), we have

$$
\begin{aligned}
&\int\limits_{\mathcal{M}} \big(f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n\Phi^i_{\delta_{jn},\xi_{jn}} + g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\Psi^i_{\delta_{jn},\xi_{jn}}\big)dv_g \\
=&\int\limits_{\mathcal{M}} \big((p-\alpha\varepsilon_n)\mathcal{H}^{p-1-\alpha\varepsilon_n}_{\bar{\delta}_n,\bar{\xi}_n}\Phi_n\Phi^i_{\delta_{jn},\xi_{jn}} + (q-\beta\varepsilon_n)\mathcal{W}^{q-1-\beta\varepsilon_n}_{\bar{\delta}_n,\bar{\xi}_n}\Psi_n\Psi^i_{\delta_{jn},\xi_{jn}}\big)dv_g \\
=&\sum_{m=1}^{k}\int\limits_{\mathcal{M}} \big((p-\alpha\varepsilon_n)H^{p-1-\alpha\varepsilon_n}_{\delta_{mn},\xi_{mn}}\Phi_n\Phi^i_{\delta_{jn},\xi_{jn}} + (q-\beta\varepsilon_n)W^{q-1-\beta\varepsilon_n}_{\delta_{mn},\xi_{mn}}\Psi_n\Psi^i_{\delta_{jn},\xi_{jn}}\big)dv_g \\
=&\delta_{jm}\int\limits_{B(0,r_0/\delta_{jn})}\Big[(p-\alpha\varepsilon_n)\delta_{jn}^{N-\frac{N(p-\alpha\varepsilon_n)}{p+1}-\frac{N}{p+1}}(\chi_n V_{1,0})^{p-1-\alpha\varepsilon_n}\chi_n\delta_{jn}^{\frac{N}{p+1}}\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\Phi^i_{1,0} \\
&+(q-\beta\varepsilon_n)\delta_{jn}^{N-\frac{N(q-\beta\varepsilon_n)}{q+1}-\frac{N}{q+1}}(\chi_n U_{1,0})^{q-1-\beta\varepsilon_n}\chi_n\delta_{jn}^{\frac{N}{q+1}}\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\Psi^i_{1,0}\Big]\mathrm{d}z \\
=&\delta_{jm}\int\limits_{B(0,r_0/\delta_{jn})}\Big[(p-\alpha\varepsilon_n)\delta_{jn}^{N-\frac{N(p-\alpha\varepsilon_n)}{p+1}-\frac{N}{p+1}}(\chi_n V_{1,0})^{p-1-\alpha\varepsilon_n}\widetilde{\Phi}_n(z)\Phi^i_{1,0} \\
&+(q-\beta\varepsilon_n)\delta_{jn}^{N-\frac{N(q-\beta\varepsilon_n)}{q+1}-\frac{N}{q+1}}(\chi_n U_{1,0})^{q-1-\beta\varepsilon_n}\widetilde{\Psi}_n(z)\Psi^i_{1,0}\Big]\mathrm{d}z \\
\to&\delta_{jm}\int\limits_{\mathbb{R}^N}(pV_{1,0}^{p-1}\Phi^i_{1,0}\widetilde{\Phi} + qU_{1,0}^{q-1}\Psi^i_{1,0}\widetilde{\Psi})\mathrm{d}z = 0, \quad \text{as } n \to +\infty.
\end{aligned}
\tag{4.6}
$$

It follows from (4.5) and (4.6) that for any $l = 0, 1, \cdots, N$ and $m = 1, 2, \cdots, k$, $c^l_{mn} \to 0$ as $n \to \infty$.

Step 3: $(\widetilde{\Psi}, \widetilde{\Phi}) = (0, 0)$. For any $j = 1, 2, \cdots, k$, there hold

$$
\begin{aligned}
\Delta\widetilde{\Psi}_n = \delta_{jn}^{\frac{Np}{p+1}}\big[&\chi_n\Delta\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z)) + \nabla\chi_n\cdot\nabla\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}x)) \\
&+ \Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\Delta\chi_n\big],
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\widetilde{\Phi}_n = \delta_{jn}^{\frac{Nq}{q+1}}\big[&\chi_n\Delta\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z)) + \nabla\chi_n\cdot\nabla\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}x)) \\
&+ \Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\Delta\chi_n\big].
\end{aligned}
$$

Thus we obtain a system of equations satisfied by $(\widetilde{\Psi}_n, \widetilde{\Phi}_n)$. For any $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ and $j = 1, 2, \cdots, k$, by the dominated convergence theorem, we obtain

$$
\begin{aligned}
&\lim_{n\to+\infty}(p-\alpha\varepsilon)\delta_{jn}^{\frac{Np}{p+1}} \int\limits_{\{z\in\mathbb{R}^N:\varphi(z)\neq 0\}} \left(\chi_n\delta_{jn}^{-\frac{N}{p+1}}V_{1,0}\right)^{p-1-\alpha\varepsilon}\chi_n\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\varphi \mathrm{d}z \\
&= p\int\limits_{\{x\in\mathbb{R}^N:\varphi(z)\neq 0\}} V_{1,0}^{p-1}\widetilde{\Phi}\varphi \mathrm{d}z,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to+\infty}(q-\beta\varepsilon)\delta_{jn}^{\frac{Nq}{q+1}} \int\limits_{\{z\in\mathbb{R}^N:\psi(z)\neq 0\}} \left(\chi_n\delta_{jn}^{-\frac{N}{q+1}}U_{1,0}\right)^{q-1-\beta\varepsilon}\chi_n\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\psi \mathrm{d}z \\
&= q\int\limits_{\{z\in\mathbb{R}^N:\psi(z)\neq 0\}} U_{1,0}^{q-1}\widetilde{\Psi}\psi \mathrm{d}z.
\end{aligned}
$$

Using (4.3), $\|(P_n, K_n)\| \to 0$, $c_{mn}^l \to 0$ as $n \to \infty$ for any $l = 0, 1, \cdots, N$ and $m = 1, 2, \cdots, k$, we deduce that $(\widetilde{\Psi}, \widetilde{\Phi})$ satisfies

$$
\begin{cases}
-\Delta\widetilde{\Psi} = pV_{1,0}^{p-1}\widetilde{\Phi}, & \text{in } \mathbb{R}^N, \\
-\Delta\widetilde{\Phi} = qU_{1,0}^{q-1}\widetilde{\Psi}, & \text{in } \mathbb{R}^N.
\end{cases}
$$

This together with (4.4) and Lemma 2.4 yields that $(\widetilde{\Psi}, \widetilde{\Phi}) = (0, 0)$.

Step 4: $\|\mathcal{I}^*\big(f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n, g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\big)\| \to 0$ as $n \to \infty$. By (2.3), we know

$$
\|\mathcal{I}^*\big(f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n, g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\big)\| \le C\left\|f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n\right\|_{\frac{p+1}{p}} + C\left\|g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\right\|_{\frac{q+1}{q}}.
$$

For any fixed $R > 0$ and $j = 1, 2, \cdots, k$, by the Hölder inequality, $\widetilde{\Phi}_n \to 0$ in $L_{loc}^{\frac{p+1}{1+\alpha\varepsilon_n}}(\mathbb{R}^N)$ and $\widetilde{\Psi}_n \to 0$ in $L_{loc}^{\frac{q+1}{1+\beta\varepsilon_n}}(\mathbb{R}^N)$, we have

$$
\begin{aligned}
&\left\|f'_{\varepsilon_n}(\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n})\Phi_n\right\|_{\frac{p+1}{p}}^{\frac{p+1}{p}} \\
&= \int\limits_{\mathcal{M}} \left|(p-\alpha\varepsilon_n)\mathcal{H}_{\bar{\delta}_n,\bar{\xi}_n}^{p-1-\alpha\varepsilon_n}\Phi_n\right|^{\frac{p+1}{p}} dv_g \\
&= \sum_{j=1}^k \delta_{jn}^{\frac{N\alpha\varepsilon_n}{p}} \int\limits_{B(0,r_0/\delta_{jn})} \left|(p-\alpha\varepsilon_n)\chi_n^{p-2-\alpha\varepsilon}V_{1,0}^{p-1-\alpha\varepsilon_n}\chi_n\delta_{jn}^{\frac{N}{p+1}}\Phi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\right|^{\frac{p+1}{p}} \mathrm{d}z \\
&= \sum_{j=1}^k \delta_{jn}^{\frac{N\alpha\varepsilon_n}{p}} \int\limits_{B(0,r_0/\delta_{jn})} \left|(p-\alpha\varepsilon_n)\chi_n^{p-2-\alpha\varepsilon}V_{1,0}^{p-1-\alpha\varepsilon_n}\widetilde{\Phi}_n(z)\right|^{\frac{p+1}{p}} \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&\le C\left(\int\limits_{B(0,r_0/\delta_{jn})} V_{1,0}^{p+1}\mathrm{d}z\right)^{\frac{p-1-\alpha\varepsilon_n}{p}}\left(\int\limits_{B(0,r_0/\delta_{jn})}|\widetilde{\Phi}_n(z)|^{\frac{p+1}{1+\alpha\varepsilon_n}}\mathrm{d}z\right)^{\frac{1+\alpha\varepsilon_n}{p}}\\
&\le C\left(\int\limits_{B(0,R)}|\widetilde{\Phi}_n(z)|^{\frac{p+1}{1+\alpha\varepsilon_n}}\right)^{\frac{1+\alpha\varepsilon_n}{p}} + C\varepsilon_n^{\frac{[(N-2)p-2](p-1-\alpha\varepsilon_n)}{2p}} \to 0, \quad \text{as } n\to+\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|g'_{\varepsilon_n}(\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n})\Psi_n\right\|_{\frac{q+1}{q}}^{\frac{q+1}{q}}\\
&=\int\limits_{\mathcal{M}}\left|(q-\beta\varepsilon_n)\mathcal{W}_{\bar{\delta}_n,\bar{\xi}_n}^{q-1-\beta\varepsilon_n}\Psi_n\right|^{\frac{q+1}{q}}dv_g\\
&=\sum_{j=1}^{k}\delta_{jn}^{\frac{N\beta\varepsilon_n}{q}}\int\limits_{B(0,r_0/\delta_{jn})}\left|(q-\beta\varepsilon_n)\chi_n^{q-2-\beta\varepsilon}U_{1,0}^{q-1-\beta\varepsilon_n}\chi_n\delta_{jn}^{\frac{N}{q+1}}\Psi_n(\exp_{\xi_{jn}}(\delta_{jn}z))\right|^{\frac{q+1}{q}}\mathrm{d}z\\
&=\sum_{j=1}^{k}\delta_{jn}^{\frac{N\beta\varepsilon_n}{q}}\int\limits_{B(0,r_0/\delta_{jn})}\left|(q-\beta\varepsilon_n)\chi_n^{q-2-\beta\varepsilon}U_{1,0}^{q-1-\beta\varepsilon_n}\widetilde{\Psi}_n(z)\right|^{\frac{q+1}{q}}\mathrm{d}z\\
&\le C\left(\int\limits_{B(0,r_0/\delta_{jn})}U_{1,0}^{q+1}\mathrm{d}z\right)^{\frac{q-1-\beta\varepsilon_n}{q}}\left(\int\limits_{B(0,r_0/\delta_{jn})}|\widetilde{\Psi}_n(z)|^{\frac{q+1}{1+\beta\varepsilon_n}}\mathrm{d}z\right)^{\frac{1+\beta\varepsilon_n}{q}}\\
&\le\begin{cases}
C\left(\int\limits_{B(0,R)}|\widetilde{\Psi}_n(z)|^{\frac{q+1}{1+\beta\varepsilon_n}}\mathrm{d}z\right)^{\frac{1+\beta\varepsilon_n}{q}} + C\varepsilon_n^{\frac{[(N-2)q-2](q-1-\beta\varepsilon_n)}{2q}}, & \text{if } p>\dfrac{N}{N-2},\\
C\left(\int\limits_{B(0,R)}|\widetilde{\Psi}_n(z)|^{\frac{q+1}{1+\beta\varepsilon_n}}\mathrm{d}z\right)^{\frac{1+\beta\varepsilon_n}{q}} + C\varepsilon_n^{\frac{[(N-3)q-3](q-1-\beta\varepsilon_n)}{2q}}, & \text{if } p=\dfrac{N}{N-2},\\
C\left(\int\limits_{B(0,R)}|\widetilde{\Psi}_n(z)|^{\frac{q+1}{1+\beta\varepsilon_n}}\mathrm{d}z\right)^{\frac{1+\beta\varepsilon_n}{q}} + C\varepsilon_n^{\frac{Np(q-1-\beta\varepsilon_n)}{2q}}, & \text{if } p<\dfrac{N}{N-2}.
\end{cases}\\
&\to 0 \quad \text{as } n\to+\infty.
\end{aligned}
$$

From the above arguments, we get $\|(\Psi_n, \Phi_n)\| \to 0$ as $n \to +\infty$, which is an absurd. Thus, we complete the proof. □

For any $\varepsilon > 0$ small enough, $\bar{t} \in (\mathbb{R}^+)^k$, and $\bar{\xi} \in \mathcal{M}^k$, if $\bar{\delta}$ is as in (2.6), then equation (3.3) is equivalent to

$$
\mathcal{L}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi) = \mathcal{N}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi) + \mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}},
$$

where

$$\mathcal{N}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi) = \Pi^{\perp}_{\bar{\delta},\bar{\xi}}\mathcal{I}^*\big[f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}+\Phi) - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}) - f'_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\Phi, g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}+\Psi) \\ -g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}) - g'_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\Psi\big], \tag{4.7}$$

and

$$\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}} = \Pi^{\perp}_{\bar{\delta},\bar{\xi}}\Big[\mathcal{I}^*\big(f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}), g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\big) - -(\mathcal{W}_{\bar{\delta},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}})\Big]. \tag{4.8}$$

In the following lemma, we estimate the reminder term $\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}$.

Lemma 4.2 *Under the assumptions on p, q and N of Theorem 1.4, if $(\bar{\delta},\bar{\xi}) \in \Lambda$ and $\bar{\delta}$ is as in (2.6), then for any $\varepsilon > 0$ small enough, there holds*

$$\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\| \le C\varepsilon|\log\varepsilon|,$$

where $\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}$ is as in (4.8).

Proof By (2.3), we know there exists $C > 0$ such that for $\varepsilon > 0$ small enough, $\bar{t} \in (\mathbb{R}^+)^k$, and $\bar{\xi} \in \mathcal{M}^k$, there holds

$$\begin{aligned}
\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\| &\le C\big\|f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}) + \Delta_g\mathcal{W}_{\bar{\delta},\bar{\xi}} - h\mathcal{W}_{\bar{\delta},\bar{\xi}}\big\|_{\frac{p+1}{p}} + C\big\|g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}) + \Delta_g\mathcal{H}_{\bar{\delta},\bar{\xi}} - h\mathcal{H}_{\bar{\delta},\bar{\xi}}\big\|_{\frac{q+1}{q}} \\
&= C\sum_{j=1}^{k}\big\|f_\varepsilon(H_{\delta_j,\xi_j}) + \Delta_g W_{\delta_j,\xi_j} - hW_{\delta_j,\xi_j}\big\|_{\frac{p+1}{p}} \\
&\quad + C\sum_{j=1}^{k}\big\|g_\varepsilon(W_{\delta_j,\xi_j}) + \Delta_g H_{\delta_j,\xi_j} - hH_{\delta_j,\xi_j}\big\|_{\frac{q+1}{q}} \\
&=: C\sum_{j=1}^{k}(I_j + II_j).
\end{aligned}$$

By an easy change of variable, and using Lemma 2.4, for any $j = 1, 2, \cdots, k$, we have

$$\begin{aligned}
I_j^{\frac{p+1}{p}} &\le C\int_{B(0,r_0/\delta_j)} \big|\delta_j^{\frac{N\alpha\varepsilon}{p+1}}\chi_{\delta_j}^{p-\alpha\varepsilon}V_{1,0}^{p-\alpha\varepsilon}\big|^{\frac{p+1}{p}}dz + C\int_{B(0,r_0/\delta_j)} \big|\chi_{\delta_j}\Delta_{g_{\delta_j,\xi_j}}U_{1,0}\big|^{\frac{p+1}{p}}dz \\
&\quad + C\int_{B(0,r_0/\delta_j)} \big|\delta_j^2 U_{1,0}\Delta_{g_{\delta_j,\xi_j}}\chi_{\delta_j}\big|^{\frac{p+1}{p}}dz + C\int_{B(0,r_0/\delta_j)} \big|\delta_j\nabla_{g_{\delta_j,\xi_j}}\chi_{\delta_j}\cdot\nabla_{g_{\delta_j,\xi_j}}U_{1,0}\big|^{\frac{p+1}{p}}dz \\
&\quad + C\int_{B(0,r_0/\delta_j)} \big|\delta_j^2 h_{\delta_j}\chi_{\delta_j}U_{1,0}\big|^{\frac{p+1}{p}}\mathrm{d}z \\
&\le C\Big[\int_{B(0,r_0/\delta_j)} \big|\delta_j^{\frac{N\alpha\varepsilon}{p+1}}\chi_{\delta_j}^{p-\alpha\varepsilon}(V_{1,0}^{p-\alpha\varepsilon} - V_{1,0}^{p})\big|^{\frac{p+1}{p}}dz
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{B(0,r_0/\delta_j)} \left|(\delta_j^{\frac{N\alpha\varepsilon}{p+1}} \chi_{\delta_j}^{p-\alpha\varepsilon} - \chi_{\delta_j}) V_{1,0}^p\right|^{\frac{p+1}{p}} dz \\
&+ \int\limits_{B(0,r_0/\delta_j)} \left|\chi_{\delta_j}(\Delta_{g_{\delta_j,\xi_j}} U_{1,0} - \Delta_{Eucl} U_{1,0})\right|^{\frac{p+1}{p}} \mathrm{d}x \\
&+ \int\limits_{B(0,r_0/\delta_j)} \left|\delta_j^2 U_{1,0} \Delta_{g_{\delta_j,\xi_j}} \chi_{\delta_j}\right|^{\frac{p+1}{p}} dz \\
&+ \int\limits_{B(0,r_0/\delta_j)} \left|\delta_j \nabla_{g_{\delta_j,\xi_j}} \chi_{\delta_j} \cdot \nabla_{g_{\delta_j,\xi_j}} U_{1,0}\right|^{\frac{p+1}{p}} dz + \\
&\int\limits_{B(0,r_0/\delta_j)} \left|\delta_j^2 h_{\delta_j,\xi_j} \chi_{\delta_j} U_{1,0}\right|^{\frac{p+1}{p}} \mathrm{d}z \Big] \\
&=: C(A_1 + A_2 + A_3 + A_4 + A_5 + A_6),
\end{aligned}
$$

where $g_{\delta_j,\xi_j}(z) = \exp_{\xi_j}^* g(\delta_j z)$, $\chi_{\delta_j}(z) = \chi(\delta_j |z|)$ and $h_{\delta_j,\xi_j}(z) = h(\exp_{\xi_j}(\delta_j z))$. We are led to estimate each A_i, $i = 1, 2, \cdots, 6$. First, for any fixed $R > 0$ large enough and $j = 1, 2, \cdots, k$, by Lemma 2.1 and Taylor formula, we have

$$
\begin{aligned}
A_1 &\le C \int\limits_{B(0,r_0/\delta_j)} \left|(V_{1,0}^{p-\alpha\varepsilon} - V_{1,0}^p)\right|^{\frac{p+1}{p}} dz = O\left(\varepsilon^{\frac{p+1}{p}} \int\limits_{B(0,r_0/\delta_j)} \left|V_{1,0}^{p+1} \log V_{1,0}^{\frac{p+1}{p}}\right| dz\right) \\
&= O(\varepsilon^{\frac{p+1}{p}}) + O\left(\varepsilon^{\frac{p+1}{p}} \int\limits_{B(0,r_0/\delta_j)\setminus B(0,R)} \left|V_{1,0}^{p+1} \log V_{1,0}^{\frac{p+1}{p}}\right| dz\right) \\
&= O(\varepsilon^{\frac{p+1}{p}}) + O\left(\varepsilon^{\frac{p+1}{p}} \int_R^{r_0/\delta_j} r^{N-1-\frac{(N-2)(p+1)^2}{p}} dr\right) = O(\varepsilon^{\frac{p+1}{p}}),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly with respect to $\xi_j \in \mathcal{M}$ and $t_j \in [a, b]$, $0 < a < b < +\infty$, where we have used the fact that $N < \frac{(N-2)(p+1)^2}{p}$, since $p > \frac{2}{N-2}$. Using Lemma 2.1 and Taylor formula again, for $j = 1, 2, \cdots, k$, we obtain

$$
\begin{aligned}
A_2 = O\big(|\varepsilon \log \varepsilon|^{\frac{p+1}{p}}\big) &+ O\left(|\varepsilon \log \varepsilon|^{\frac{p+1}{p}} \int\limits_{B(0,r_0/2\delta_j)\setminus B(0,R)} V_{1,0}^{p+1} \mathrm{d}z\right) \\
&+ O\left(\int\limits_{B(0,r_0/\delta_j)\setminus B(0,r_0/2\delta_j)} V_{1,0}^{p+1} \mathrm{d}z\right)
\end{aligned}
$$

$$
\begin{aligned}
&= O\big(|\varepsilon\log\varepsilon|^{\frac{p+1}{p}}\big) + O\Big(|\varepsilon\log\varepsilon|^{\frac{p+1}{p}}\int_R^{r_0/2\delta_j} r^{N-1-(N-2)(p+1)}dr\Big)\\
&\quad + O\left(\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(N-2)(p+1)}dr\right)\\
&= O\big(|\varepsilon\log\varepsilon|^{\frac{p+1}{p}}\big) + O\big(\varepsilon^{\frac{(N-2)p-2}{2}}\big),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly with respect to $\xi_j \in \mathcal{M}$ and $t_j \in [a,b]$. Since $N \geq 8$, then $A_2 \leq |\varepsilon\log\varepsilon|^{\frac{p+1}{p}}$. For any fixed $R > 0$ large enough and $j = 1, 2, \cdots, k$, it follows from (2.9) and (2.10) that

$$
A_3 = \begin{cases}
O(\varepsilon^{\frac{p+1}{p}}) + O\Big(\varepsilon^{\frac{p+1}{p}}\int_R^{r_0/\delta_j} r^{N-1-\frac{(N-2)(p+1)}{p}}dr\Big) = O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p > \frac{N}{N-2};\\
O(\varepsilon^{\frac{p+1}{p}}) + O\Big(\varepsilon^{\frac{p+1}{p}}\int_R^{r_0/\delta_j} r^{N-1-\frac{(N-3)(p+1)}{p}}dr\Big) = O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p = \frac{N}{N-2};\\
O(\varepsilon^{\frac{p+1}{p}}) + O\Big(\varepsilon^{\frac{p+1}{p}}\int_R^{r_0/\delta_j} r^{N-1-(N-2)(p+1)+\frac{2p+2}{p}}dr\Big) = O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p < \frac{N}{N-2},
\end{cases}
$$

as $\varepsilon \to 0$, uniformly with respect to $\xi_j \in \mathcal{M}$ and $t_j \in [a,b]$, where we have used the fact that $N \geq 8$ and $p > 1$. Since there hold $|\chi'_{\delta_j}| \leq C\delta_j$ and $|\chi''_{\delta_j}| \leq C\delta_j^2$ for any $j = 1, 2, \cdots, k$, we have

$$
A_4 = \begin{cases}
O\Big(\varepsilon^{\frac{2(p+1)}{p}}\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-2)(p+1)}{p}}dr\Big) = O(\varepsilon^{\frac{2(p+1)}{p}}), & \text{if } p > \frac{N}{N-2};\\
O\Big(\varepsilon^{\frac{2(p+1)}{p}}\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-3)(p+1)}{p}}dr\Big) = O(\varepsilon^{\frac{2(p+1)}{p}}), & \text{if } p = \frac{N}{N-2};\\
O\Big(\varepsilon^{\frac{2(p+1)}{p}}\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(N-2)(p+1)+\frac{2p+2}{p}}dr\Big) = O(\varepsilon^{\frac{2(p+1)}{p}}), & \text{if } p < \frac{N}{N-2},
\end{cases}
$$

and

$$A_5=\begin{cases} O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-1)(p+1)}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p>\frac{N}{N-2};\\ O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-2)(p+1)}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p=\frac{N}{N-2};\\ O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(N-2)(p+1)+\frac{p+1}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p<\frac{N}{N-2}, \end{cases}$$

as $\varepsilon\to 0$, uniformly with respect to $\xi_j\in\mathcal{M}$ and $t_j\in[a,b]$. Moreover, for any fixed $R>0$ large enough and $j=1,2,\cdots,k$, it's easy to obtain

$$A_6=\begin{cases} O(\varepsilon^{\frac{p+1}{p}})+O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{R}^{r_0/\delta_j} r^{N-1-\frac{(N-2)(p+1)}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p>\frac{N}{N-2};\\ O(\varepsilon^{\frac{p+1}{p}})+O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{R}^{r_0/\delta_j} r^{N-1-\frac{(N-3)(p+1)}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p=\frac{N}{N-2};\\ O(\varepsilon^{\frac{p+1}{p}})+O\Big(\varepsilon^{\frac{p+1}{p}}\int\limits_{R}^{r_0/\delta_j} r^{N-1-(N-2)(p+1)+\frac{2p+2}{p}}dr\Big)=O(\varepsilon^{\frac{p+1}{p}}), & \text{if } p<\frac{N}{N-2}, \end{cases}$$

as $\varepsilon\to 0$, uniformly with respect to $\xi_j\in\mathcal{M}$ and $t_j\in[a,b]$. From the above arguments, we obtain $I_j=O(\varepsilon|\log\varepsilon|)$ for any $j=1,2,\cdots,k$.

Similarly, we can prove that

$$\begin{aligned} II_j^{\frac{q+1}{q}} \le C\Big[&\int\limits_{B(0,r_0/\delta_j)}\big|\delta_j^{\frac{N\beta\varepsilon}{q+1}}\chi_{\delta_j}^{q-\beta\varepsilon}(U_{1,0}^{q-\beta\varepsilon}-U_{1,0}^{q})\big|^{\frac{q+1}{q}}dz\\ &+\int\limits_{B(0,r_0/\delta_j)}\big|(\delta_j^{\frac{N\beta\varepsilon}{q+1}}\chi_{\delta_j}^{q-\beta\varepsilon}-\chi_{\delta_j})U_{1,0}^{q}\big|^{\frac{q+1}{q}}dz\\ &+\int\limits_{B(0,r_0/\delta_j)}\big|\chi_{\delta_j}(\Delta_{g_{\delta_j,\xi_j}}V_{1,0}-\Delta_{Eucl}V_{1,0})\big|^{\frac{q+1}{q}}dz+\int\limits_{B(0,r_0/\delta_j)}\big|\delta_j^2V_{1,0}\Delta_{g_{\delta_j,\xi_j}}\chi_{\delta_j}\big|^{\frac{q+1}{q}}dz\\ &+\int\limits_{B(0,r_0/\delta_j)}\big|\delta_j\nabla_{g_{\delta_j,\xi_j}}\chi_{\delta_j}\cdot\nabla_{g_{\delta_j,\xi_j}}V_{1,0}\big|^{\frac{q+1}{q}}dz+\int\limits_{B(0,r_0/\delta_j)}\big|\delta_j^2h_{\delta_j,\xi_j}\chi_{\delta_j}V_{1,0}\big|^{\frac{q+1}{q}}\mathrm{d}z\Big]\\ =:\,& C(B_1+B_2+B_3+B_4+B_5+B_6). \end{aligned}$$

For any fixed $R > 0$ large enough and $j = 1, 2, \cdots, k$, by $N \geq 8$ and $q > 1$, we have

$$B_1 = \begin{cases} O(\varepsilon^{\frac{q+1}{q}}) + O\Big(\varepsilon^{\frac{q+1}{q}} \int_R^{r_0/\delta_j} r^{N-1-\frac{(N-2)(q+1)^2}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}), & \text{if } p > \frac{N}{N-2}; \\ O(\varepsilon^{\frac{q+1}{q}}) + O\Big(\varepsilon^{\frac{q+1}{q}} \int_R^{r_0/\delta_j} r^{N-1-\frac{(N-3)(q+1)^2}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}), & \text{if } p = \frac{N}{N-2}; \\ O(\varepsilon^{\frac{q+1}{q}}) + O\Big(\varepsilon^{\frac{q+1}{q}} \int_R^{r_0/\delta_j} r^{N-1-\frac{[(N-2)p-2](q+1)^2}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}), & \text{if } p < \frac{N}{N-2}, \end{cases}$$

and

$$B_2 = \begin{cases} O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big) + O\Big(\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(N-2)(q+1)} dr\Big) = O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big), & \text{if } p > \frac{N}{N-2}; \\ O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big) + O\Big(\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(N-3)(q+1)} dr\Big) = O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big), & \text{if } p = \frac{N}{N-2}, N \geq 10; \\ O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big) + O\Big(\int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-(Np+N)} dr\Big) = O\big(|\varepsilon \log \varepsilon|^{\frac{q+1}{q}}\big), & \text{if } p < \frac{N}{N-2}, N \geq 12, \end{cases}$$

as $\varepsilon \to 0$, uniformly with respect to $\xi_j \in \mathcal{M}$ and $t_j \in [a, b]$. Similar arguments as above, we have

$$B_3 = O(\varepsilon^{\frac{q+1}{q}}) + O\Big(\varepsilon^{\frac{q+1}{q}} \int_R^{r_0/\delta_j} r^{N-1-\frac{(N-2)(q+1)}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}),$$

$$B_4 = O\Big(\varepsilon^{\frac{2(q+1)}{q}} \int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-2)(q+1)}{q}} dr\Big) = O(\varepsilon^{\frac{2(q+1)}{q}}),$$

$$B_5 = O\Big(\varepsilon^{\frac{q+1}{q}} \int_{r_0/2\delta_j}^{r_0/\delta_j} r^{N-1-\frac{(N-1)(q+1)}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}),$$

and

$$B_6 = O(\varepsilon^{\frac{q+1}{q}}) + O\Big(\varepsilon^{\frac{q+1}{q}} \int_R^{r_0/\delta_j} r^{N-1-\frac{(N-2)(q+1)}{q}} dr\Big) = O(\varepsilon^{\frac{q+1}{q}}),$$

as $\varepsilon \to 0$, uniformly with respect to $\xi_j \in \mathcal{M}$ and $t_j \in [a,b]$. Hence $II_j = O(\varepsilon|\log\varepsilon|)$ for any $j = 1, 2, \cdots, k$. This ends the proof. □

We now prove Proposition 3.1 by using Lemmas 4.1 and 4.2.

Proof of Proposition 3.1 For any $\varepsilon > 0$ small enough, $\bar{t} \in (\mathbb{R}^+)^k$, and $\bar{\xi} \in \mathcal{M}^k$, if $\bar{\delta}$ is as in (2.6), we define the map $\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}} : \mathcal{Z}_{\bar{\delta},\bar{\xi}} \to \mathcal{Z}_{\bar{\delta},\bar{\xi}}$ by

$$\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi) = \mathcal{L}^{-1}_{\varepsilon,\bar{t},\bar{\xi}}(\mathcal{N}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi) + \mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}),$$

where $\mathcal{L}_{\varepsilon,\bar{t},\bar{\xi}}$, $\mathcal{N}_{\varepsilon,\bar{t},\bar{\xi}}$ and $\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}$ are as in (4.1), (4.7) and (4.8), respectively. We also set

$$\mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma) = \left\{(\Psi,\Phi) \in \mathcal{Z}_{\bar{\delta},\bar{\xi}} : \|(\Psi,\Phi)\| \le \gamma\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\|\right\},$$

where $\gamma > 0$ is a fixed constant large enough. We prove that the map $\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}$ admits a fixed point $(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}})$. Therefore, we shall prove that, for any $\varepsilon > 0$ small, there hold:

(i) $\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}(\mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma)) \subset \mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma)$;

(ii) $\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}$ is a contraction map on $\mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma)$.

For (i), by (2.3) and Lemma 4.1, for any $\varepsilon > 0$ small enough, and $(\Psi,\Phi) \in \mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma)$, we have

$$\begin{aligned}
&\|\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}(\mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma))\| \le C\|\mathcal{N}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi,\Phi)\| + C\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\| \\
&\le C\Big[\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\| + \left\|f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi) - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}) - f'_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\Phi\right\|_{\frac{p+1}{p}} \\
&\quad + \left\|g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi) - g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}) - g'_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\Psi\right\|_{\frac{q+1}{q}}\Big] =: C(\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\| + I + II).
\end{aligned}$$

By the mean value formula, Lemmas 2.5, 4.2, and the Sobolev embedding theorem, we obtain

$$I \le C\|\Phi\|^{p-\alpha\varepsilon}_{\frac{(p+1)(p-\alpha\varepsilon)}{p}} \le C\|\Phi\|^{p-\alpha\varepsilon}_{p+1} \le C\gamma^{p-\alpha\varepsilon}\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\|^{p-\alpha\varepsilon} \le \gamma\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\|,$$

and

$$II \le \begin{cases} C\|\Psi\|^{q-\beta\varepsilon}_{\frac{(q+1)(q-\beta\varepsilon)}{q}} + C\|\Psi\|^2_{\frac{2(q+1)}{2+\beta\varepsilon}} \sum\limits_{j=1}^{k} \|W_{\delta_j,\xi_j}\|^{q-2-\beta\varepsilon}_{q+1} \le \gamma\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\|, & \text{if } q > 2, \\ C\|\Psi\|^{q-\beta\varepsilon}_{\frac{(q+1)(q-\beta\varepsilon)}{q}} \le \gamma\|\mathcal{R}_{\varepsilon,\bar{t},\bar{\xi}}\|, & \text{if } q \le 2, \end{cases}$$

where we have used the fact that $\|W_{\delta_j,\xi_j}\|_{q+1} < +\infty$ for any $1 < p \le \frac{N+2}{N-2} \le q$ and $j = 1, 2, \cdots, k$. So we have (i).

Similarly, by (2.3) and Lemma 4.1, for any $\varepsilon > 0$ small enough, and $(\Psi_1,\Phi_1), (\Psi_2,\Phi_2) \in \mathcal{B}_{\varepsilon,\bar{t},\bar{\xi}}(\gamma)$, we have

$$\|\mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi_1,\Phi_1) - \mathcal{T}_{\varepsilon,\bar{t},\bar{\xi}}(\Psi_2,\Phi_2)\|$$

$$
\begin{aligned}
&\le C\|\mathcal{N}_{\varepsilon,\bar t,\bar\xi}(\Psi_1,\Phi_1)-\mathcal{N}_{\varepsilon,\bar t,\bar\xi}(\Psi_2,\Phi_2)\|\\
&\le C\Big[\big\|f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_1)-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_2)-f'_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})(\Phi_1-\Phi_2)\big\|_{\frac{p+1}{p}}\\
&\quad+C\big\|g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_1)-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_2)-g'_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})(\Psi_1-\Psi_2)\big\|_{\frac{q+1}{q}}\Big]\\
&=: C(III+IV).
\end{aligned}
$$

By the mean value formula, Lemma 2.5, and the Sobolev embedding theorem, we obtain

$$
\begin{aligned}
III &\le C\Big(\|\Phi_1\|^{p-1-\alpha\varepsilon}_{\frac{(p-1-\alpha\varepsilon)(p+1)}{p-1}}+\|\Phi_2\|^{p-1-\alpha\varepsilon}_{\frac{(p-1-\alpha\varepsilon)(p+1)}{p-1}}\Big)\|\Phi_1-\Phi_2\|_{p+1}\\
&\le C\gamma^{p-1-\alpha\varepsilon}\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|^{p-1-\alpha\varepsilon}\|\Phi_1-\Phi_2\|,
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
IV &\le \begin{cases} C\Big(\|\Psi_1\|^{q-1-\beta\varepsilon}_{\frac{(q-1-\beta\varepsilon)(q+1)}{q-1}}+\|\Psi_2\|^{q-1-\beta\varepsilon}_{\frac{(q-1-\beta\varepsilon)(q+1)}{q-1}}\Big)\|\Psi_1-\Psi_2\|_{q+1} & \\ \quad +C(\|\Psi_1\|_{\frac{q+1}{1+\beta\varepsilon}}+\|\Psi_2\|_{\frac{q+1}{1+\beta\varepsilon}})\|\Psi_1-\Psi_2\|_{q+1}\sum_{j=1}^{k}\|W_{\delta_j,\xi_j}\|^{q-2-\beta\varepsilon}_{q+1} & \text{if } q>2,\\ C\Big(\|\Psi_1\|^{q-1-\beta\varepsilon}_{\frac{(q-1-\beta\varepsilon)(q+1)}{q-1}}+\|\Psi_2\|^{q-1-\beta\varepsilon}_{\frac{(q-1-\beta\varepsilon)(q+1)}{q-1}}\Big)\|\Psi_1-\Psi_2\|_{q+1}, & \text{if } q\le 2,\end{cases}\\
&\le \begin{cases} C\gamma^{q-1-\beta\varepsilon}\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|^{q-1-\beta\varepsilon}\|\Psi_1-\Psi_2\|+C\gamma\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|\|\Psi_1-\Psi_2\|, & \text{if } q>2,\\ C\gamma^{q-1-\beta\varepsilon}\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|^{q-1-\beta\varepsilon}\|\Psi_1-\Psi_2\|, & \text{if } q\le 2.\end{cases}
\end{aligned}
\tag{4.10}
$$

By Lemma 4.2, we know $C\gamma\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|, C\gamma^{p-1-\alpha\varepsilon}\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|^{p-1-\alpha\varepsilon}, C\gamma^{q-1-\beta\varepsilon}\|\mathcal{R}_{\varepsilon,\bar t,\bar\xi}\|^{q-1-\beta\varepsilon}\in(0,1)$. This proves (ii). Finally, by using the implicit function theorem, we can prove the regularity of $(\Psi_{\varepsilon,\bar t,\bar\xi},\Phi_{\varepsilon,\bar t,\bar\xi})$ with respect to $\bar t$ and $\bar\xi$. Thus we complete the proof. □

5 Proof of Proposition 3.3

This section is devoted to the proof of Proposition 3.3. As a first step, we have

Lemma 5.1 *Under the assumptions on p, q and N of Theorem 1.4, if $\bar\delta$ is as in (2.6), then for any $\varepsilon>0$ small enough, if $(\bar t,\bar\xi)$ is a critical point of the functional $\widetilde{\mathcal{J}}_\varepsilon$, then $\big(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi}\big)$ is a solution of system (1.1), or equivalently of (2.2).*

Proof Let $(\bar t,\bar\xi)$ is a critical point of $\widetilde{\mathcal{J}}_\varepsilon$, where $\bar t=(t_1,t_2,\cdots,t_k)\in(\mathbb{R}^+)^k$ and $\bar\xi=(\xi_1,\xi_2,\cdots,\xi_k)\in\mathcal{M}^k$. Let $\bar\xi(y)=\big(\exp_{\xi_1}(y^1),\exp_{\xi_2}(y^2),\cdots,\exp_{\xi_k}(y^k)\big)$, $y=(y^1,y^2,\cdots,y^k)\in B(0,r)^k$, and $\xi_j(y^j)=\exp_{\xi_j}(y^j)$ for any $j=1,2,\cdots,k$, then $\bar\xi(0)=\bar\xi$. Since $(\bar t,\bar\xi)$ is a critical point of $\widetilde{\mathcal{J}}_\varepsilon$, for any $m=1,2,\cdots,k$ and $l=1,2,\cdots,N$, there hold

$$
\mathcal{J}'_\varepsilon\big(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi}\big)\big(\partial_{t_m}\mathcal{W}_{\bar\delta,\bar\xi}+\partial_{t_m}\Psi_{\varepsilon,\bar t,\bar\xi},\partial_{t_m}\mathcal{H}_{\bar\delta,\bar\xi}+\partial_{t_m}\Phi_{\varepsilon,\bar t,\bar\xi}\big)=0,
$$

and

$$\mathcal{J}_\varepsilon'\big(\mathcal{W}_{\bar{\delta},\bar{\xi}}+\Psi_{\varepsilon,\bar{t},\bar{\xi}},\mathcal{H}_{\bar{\delta},\bar{\xi}}+\Phi_{\varepsilon,\bar{t},\bar{\xi}}\big)\big(\partial_{y_l^m}\mathcal{W}_{\bar{\delta},\bar{\xi}}+\partial_{y_l^m}\Psi_{\varepsilon,\bar{t},\bar{\xi}},\partial_{y_l^m}\mathcal{H}_{\bar{\delta},\bar{\xi}}+\partial_{y_l^m}\Phi_{\varepsilon,\bar{t},\bar{\xi}}\big)=0.$$

For any $(\varphi,\psi)\in\mathcal{X}_{p,q}(\mathcal{M})$, by Proposition 3.1, there exist some constants $c_{10},c_{11},\cdots,c_{1N},c_{20},c_{21},\cdots,c_{2N},\cdots,c_{k0},c_{k1},\cdots,c_{kN}$ such that

$$\mathcal{J}_\varepsilon'(\mathcal{W}_{\bar{\delta},\bar{\xi}}+\Psi_{\varepsilon,\bar{t},\bar{\xi}},\mathcal{H}_{\delta,\xi}+\Phi_{\varepsilon,\bar{t},\bar{\xi}})(\varphi,\psi)=\sum_{l=0}^N\sum_{m=1}^k c_{lm}\big\langle(\Psi^l_{\delta_m,\xi_m},\Phi^l_{\delta_m,\xi_m}),(\varphi,\psi)\big\rangle_h.$$

Let ∂_s denote ∂_{t_m} or $\partial_{y_l^m}$ for any $m=1,2,\cdots,k$ and $l=1,2,\cdots,N$. Then

$$\begin{aligned}
&\partial_s\widetilde{\mathcal{J}}_\varepsilon(\bar{t},\bar{\xi}(y))\\
&=\mathcal{J}_\varepsilon'\Big(\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)},\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}+\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}\Big)\Big(\partial_s\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\partial_s\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)},\partial_s\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}\\
&\quad+\partial_s\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}\Big)\\
&=\Big\langle\Big(\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)},\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}+\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}\Big)-\mathcal{I}^*\Big(f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}\\
&\quad+\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}),g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)})\Big),\\
&\quad\times\Big(\partial_s\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\partial_s\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)},\partial_s\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}+\partial_s\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}\Big)\Big\rangle\\
&=\sum_{i=0}^N\sum_{j=1}^k c_{ij}\Big\langle\Big(\Psi^i_{\delta_j,\xi_j(y^j)},\Phi^i_{\delta_j,\xi_j(y^j)}\Big),\Big(\partial_s\mathcal{W}_{\bar{\delta},\bar{\xi}(y)}+\partial_s\Psi_{\varepsilon,\bar{t},\bar{\xi}(y)},\partial_s\mathcal{H}_{\bar{\delta},\bar{\xi}(y)}\\
&\quad+\partial_s\Phi_{\varepsilon,\bar{t},\bar{\xi}(y)}\Big)\Big\rangle_h.
\end{aligned}\tag{5.1}$$

We prove that if we compute (5.1) at $y=0$, then for any $\varepsilon>0$ small enough, there holds

$$c_{ij}=0,\quad \text{for any } i=0,1,\cdots,N \text{ and } j=1,2,\cdots,k.$$

Since $(\bar{t},\bar{\xi})$ is a critical point of $\widetilde{\mathcal{J}}_\varepsilon$, then

$$\partial_s\widetilde{\mathcal{J}}_\varepsilon(\bar{t},\bar{\xi}(y))|_{y=0}=0.\tag{5.2}$$

For any $m=1,2,\cdots,k$ and $l=1,2,\cdots,N$, we can easily check that there hold

$$\big(\partial_{t_m}\mathcal{W}_{\bar{\delta},\bar{\xi}},\partial_{t_m}\mathcal{H}_{\bar{\delta},\bar{\xi}}\big)=-\frac{1}{2t_m}\big(\Psi^0_{\delta_m,\xi_m},\Phi^0_{\delta_m,\xi_m}\big),\tag{5.3}$$

and

$$\big(\partial_{y_l^m}(\mathcal{W}_{\bar{\delta},\bar{\xi}(y)})\big|_{y=0},\partial_{y_l^m}(\mathcal{H}_{\bar{\delta},\bar{\xi}(y)})\big|_{y=0}\big)=\frac{1}{\delta_m}\big(\Psi^l_{\delta_m,\xi_m}+R_1,\Phi^l_{\delta_m,\xi_m}+R_2\big),\tag{5.4}$$

where $\|(R_1, R_2)\| = o(\varepsilon^{\frac{\vartheta}{2}})$ as $\varepsilon \to 0$ for all $\vartheta \in (0,1)$. Using (2.11)–(2.14), we have

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\partial_{t_m}\mathcal{W}_{\bar\delta,\bar\xi}, \partial_{t_m}\mathcal{H}_{\bar\delta,\bar\xi}\big)\big\rangle_h \\
&= -\frac{1}{2t_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\Psi^0_{\delta_m,\xi_m}, \Phi^0_{\delta_m,\xi_m}\big)\big\rangle_h \\
&= -\frac{1}{2t_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\delta_{i0}\delta_{jm}\int\limits_{B(0,r_0/\delta_m)} \big(p\chi^2_{\delta_m}V^{p-1}_{1,0}(\Phi^0_{1,0})^2 + q\chi^2_{\delta_m}U^{q-1}_{1,0}(\Psi^0_{1,0})^2\big)\mathrm{d}x + O(\varepsilon),
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\partial_{y^m_l}\mathcal{W}_{\bar\delta,\bar\xi(y)}\big|_{y=0}, \partial_{y^m_l}\mathcal{H}_{\bar\delta,\bar\xi(y)}\big|_{y=0}\big)\big\rangle_h \\
&=\frac{1}{\delta_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\Psi^l_{\delta_m,\xi_m}+R_1, \Phi^l_{\delta_m,\xi_m}+R_2\big)\big\rangle_h \\
&=\frac{1}{\delta_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\delta_{il}\delta_{jm}\int\limits_{B(0,r_0/\delta_m)} \big(p\chi^2_{\delta_m}V^{p-1}_{1,0}(\Phi^l_{1,0})^2 + q\chi^2_{\delta_m}U^{q-1}_{1,0}(\Psi^l_{1,0})^2\big)\mathrm{d}x + O(\varepsilon),
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\partial_s\Psi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}, \partial_s\Phi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}\big)\big\rangle_h \\
&= -\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\partial_s\Psi^i_{\delta_j,\xi_j(y^j)}\big|_{y=0}, \partial_s\Phi^i_{\delta_j,\xi_j(y^j)}\big|_{y=0}\big), \big(\Psi_{\varepsilon,\bar t,\bar\xi}, \Phi_{\varepsilon,\bar t,\bar\xi}\big)\big\rangle_h,
\end{aligned}
\tag{5.7}
$$

where $\chi_{\delta_m}(x) = \chi(\delta_m|x|)$. For any $\vartheta \in (0,1)$, with the aid of Proposition 3.1, it's easy to check

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\partial_{t_m}\Psi^i_{\delta_j,\xi_j}, \partial_{t_m}\Phi^i_{\delta_j,\xi_j}\big), \big(\Psi_{\varepsilon,\bar t,\bar\xi}, \Phi_{\varepsilon,\bar t,\bar\xi}\big)\big\rangle_h \\
&\le\frac{1}{2t_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\delta_{jm}\Big(\big\|\partial_\delta\big(\delta^{-\frac{N}{q+1}}\Psi^i_{1,0}(\delta^{-1}y)\big)\big|_{\delta=1}\big\|_{\dot W^{1,p^*}(\mathbb{R}^N)}\|\nabla_g\Phi_{\varepsilon,\bar t,\bar\xi}\|_{q^*} \\
&\qquad + \big\|\partial_\delta\big(\delta^{-\frac{N}{p+1}}\Phi^i_{1,0}(\delta^{-1}y)\big)\big|_{\delta=1}\big\|_{\dot W^{1,q^*}(\mathbb{R}^N)}\|\nabla_g\Psi_{\varepsilon,\bar t,\bar\xi}\|_{p^*}\Big) + O(\varepsilon^2 log\varepsilon) \\
&=o\big(\varepsilon^{\vartheta}\big),
\end{aligned}
\tag{5.8}
$$

and

$$\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\partial_{y_l^m}\Psi^i_{\delta_j,\xi_j(y^j)}\big|_{y=0},\partial_{y_l^m}\Phi^i_{\delta_j,\xi_j(y^j)}\big|_{y=0}\big),\big(\Psi_{\varepsilon,\bar t,\bar\xi},\Phi_{\varepsilon,\bar t,\bar\xi}\big)\big\rangle_h$$
$$\leq\frac{1}{\delta_m}\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\delta_{jm}\Big(\big\|\partial_{y_l}\Psi^i_{1,0}\big\|_{\dot W^{1,p^*}(\mathbb{R}^N)}\|\nabla_g\Phi_{\varepsilon,\bar t,\bar\xi}\|_{q^*}+\big\|\partial_{y_l}\Phi^i_{1,0}\big\|_{\dot W^{1,q^*}(\mathbb{R}^N)}\|\nabla_g\Psi_{\varepsilon,\bar t,\bar\xi}\|_{p^*}\Big)$$
$$+O(\varepsilon^{3/2}log\varepsilon)=o\big(\varepsilon^{\vartheta}\big). \tag{5.9}$$

Therefore, by (5.2) and (5.5)–(5.9), we deduce that the linear system in (5.1) has only a trivial solution when $y=0$ provided that $\varepsilon>0$ small enough. This ends the proof. □

In the next lemma, we give the asymptotic expansion of $\mathcal{J}_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi})$ as $\varepsilon\to 0$ for $(\bar\delta,\bar\xi)\in\Lambda$, where $\bar\delta$ is as in (2.6).

Lemma 5.2 *Under the assumptions on p,q and N of Theorem 1.4, if $(\bar\delta,\bar\xi)\in\Lambda$ and $\bar\delta$ is as in (2.6), then there holds*

$$\mathcal{J}_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi})=\frac{2k}{N}L_1+c_1\varepsilon-c_2\varepsilon\log\varepsilon+\Psi_k(\bar t,\bar\xi)\varepsilon+o(\varepsilon),$$

as $\varepsilon\to 0$, C^1-uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$, where the function $\Psi_k(\bar t,\bar\xi)$ is defined as (3.5), c_1 and c_2 are given in (3.6).

Proof For any $\xi\in\mathcal{M}$, there holds

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B(\xi,r)}d\sigma_g=1-\frac{1}{6N}Scal_g(\xi)r^2+O(r^4)$$

as $r\to 0$, where ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$. Furthermore, by standard properties of the exponential map, the reminder $O(r^4)$ can be made C^1-uniform with respect to ξ. Under the assumptions on p,q and N of Theorem 1.4, we can compute

$$\int_{\mathcal{M}}\nabla_g\Big(\sum_{j=1}^{k}W_{\delta_j,\xi_j}\Big)\cdot\nabla_g\Big(\sum_{j=1}^{k}H_{\delta_j,\xi_j}\Big)dv_g$$
$$=\sum_{j=1}^{k}\int_{\mathcal{M}}\nabla_g W_{\delta_j,\xi_j}\cdot\nabla_g H_{\delta_j,\xi_j}dv_g$$
$$=\sum_{j=1}^{k}\Big[\int_{B(0,r_0/2\delta_j)}\nabla_{g_{\delta_j,\xi_j}}U_{1,0}\cdot\nabla_{g_{\delta_j,\xi_j}}V_{1,0}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2+O(\delta_j^4|z|^4)\Big)\mathrm{d}z$$

$$
\begin{aligned}
&+\int\limits_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)} \nabla_{g_{\delta_j,\xi_j}}(\chi_{\delta_j}U_{1,0})\cdot\nabla_{g_{\delta_j,\xi_j}}(\chi_{\delta_j}V_{1,0})\\
&\times\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2+O(\delta_j^4|z|^4)\Big)\mathrm{d}z\Big]\\
=&\Big(\sum_{j=1}\Big[\int\limits_{\mathbb{R}^N}\nabla_{g_{\delta_j,\xi_j}}U_{1,0}\cdot\nabla_{g_{\delta_j,\xi_j}}V_{1,0}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2+O(\delta_j^4|z|^4)\Big)\mathrm{d}z\\
&-\int\limits_{B^c(0,r_0/2\delta_j)}\nabla_{g_{\delta_j,\xi_j}}U_{1,0}\cdot\nabla_{g_{\delta_j,\xi_j}}V_{1,0}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2+O(\delta_j^4|z|^4)\Big)\mathrm{d}z\\
&+\int\limits_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)} \nabla_{g_{\delta_j,\xi_j}}(\chi_{\delta_j}U_{1,0})\cdot\nabla_{g_{\delta_j,\xi_j}}(\chi_{\delta_j}V_{1,0})\\
&\times\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2+O(\delta_j^4|z|^4)\Big)\mathrm{d}z\Big]\\
=&\,kL_1-\sum_{j=1}^{k}\Big\{\frac{L_2Scal_g(\xi_j)}{6N}\delta_j^2+o(\delta_j^2)\Big\},
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\int\limits_{\mathcal{M}}\nabla_g\Big(\sum_{j=1}^{k}W_{\delta_j,\xi_j}\Big)\cdot\nabla_g\Big(\sum_{j=1}^{k}H_{\delta_j,\xi_j}\Big)dv_g\Big\}\\
&\quad=-\sum_{j=1}^{k}\Big\{\frac{L_2Scal_g(\xi_j)}{3N}\delta_j\delta_j'+o(\delta_j\delta_j')\Big\},
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
&\int\limits_{\mathcal{M}}h\Big(\sum_{j=1}^{k}W_{\delta_j,\xi_j}\Big)\Big(\sum_{j=1}^{k}H_{\delta_j,\xi_j}\Big)dv_g=\sum_{j=1}^{k}\int\limits_{\mathcal{M}}hW_{\delta_j,\xi_j}H_{\delta_j,\xi_j}dv_g\\
=&\sum_{j=1}^{k}\Big\{\delta_j^2\int\limits_{\mathbb{R}^N}h_{\delta_j,\xi_j}U_{1,0}V_{1,0}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z-\delta_j^2\int\limits_{B^c(0,r_0/2\delta_j)}h_{\delta_j,\xi_j}U_{1,0}V_{1,0}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\\
&+\delta_j^2\int\limits_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)}h_{\delta_j,\xi_j}\chi_{\delta_j}^2U_{1,0}V_{1,0}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\Big\}\\
=&\sum_{j=1}^{k}\big\{L_3h(\xi_j)\delta_j^2+o(\delta_j^2)\big\},
\end{aligned}
\tag{5.12}
$$

$$
\frac{d}{dt}\Big\{\int\limits_{\mathcal{M}}h\Big(\sum_{j=1}^{k}W_{\delta_j,\xi_j}\Big)\Big(\sum_{j=1}^{k}H_{\delta_j,\xi_j}\Big)dv_g\Big\}=\sum_{j=1}^{k}\big\{2L_3h(\xi_j)\delta_j\delta_j'+o(\delta_j\delta_j')\big\},
\tag{5.13}
$$

as $\varepsilon \to 0$, C^1-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where $g_{\delta_j,\xi_j}(z) = \exp_{\xi_j}^* g(\delta_j z)$, $\chi_{\delta_j}(z) = \chi(\delta_j|z|)$, and $h_{\delta_j,\xi_j}(z) = h(\exp_{\xi_j}(\delta_j z))$. Using the Taylor formula, we have

$$
\begin{aligned}
&\frac{1}{p+1-\alpha\varepsilon}\int_{\mathcal{M}}\Big(\sum_{j=1}^{k} H_{\delta_j,\xi_j}\Big)^{p+1-\alpha\varepsilon} dv_g = \sum_{j=1}^{k}\frac{1}{p+1-\alpha\varepsilon}\int_{\mathcal{M}} H_{\delta_j,\xi_j}^{p+1-\alpha\varepsilon} dv_g\\
&= \sum_{j=1}^{k}\Big\{\frac{1}{p+1}\int_{\mathcal{M}} H_{\delta_j,\xi_j}^{p+1} dv_g + \alpha\varepsilon\int_{\mathcal{M}}\Big[\frac{H_{\delta_j,\xi_j}^{p+1}}{(p+1)^2} - \frac{H_{\delta_j,\xi_j}^{p+1}\log H_{\delta_j,\xi_j}}{p+1}\Big]dv_g + o(\delta_j^2)\Big\}\\
&= \sum_{j=1}^{k}\Big\{\frac{1}{p+1}\int_{\mathbb{R}^N} V_{1,0}^{p+1}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2 + O(\delta_j^4|z|^4)\Big)\mathrm{d}z\\
&- \frac{1}{p+1}\int_{B^c(0,r_0/2\delta_j)} V_{1,0}^{p+1}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2 + O(\delta_j^4|z|^4)\Big)\mathrm{d}z\\
&+ \frac{1}{p+1}\int_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)} \chi_{\delta_j}^{p+1} V_{1,0}^{p+1}\Big(1-\frac{1}{6N}Scal_g(\xi_j)\delta_j^2|z|^2 + O(\delta_j^4|z|^4)\Big)\mathrm{d}z\\
&+ \frac{\alpha\varepsilon}{(p+1)^2}\int_{\mathbb{R}^N} V_{1,0}^{p+1}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z - \frac{\alpha\varepsilon}{(p+1)^2}\int_{B^c(0,r_0/2\delta_j)} V_{1,0}^{p+1}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\\
&+ \frac{\alpha\varepsilon}{(p+1)^2}\int_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)} \chi_{\delta_j}^{p+1} V_{1,0}^{p+1}\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\\
&- \frac{\alpha\varepsilon}{p+1}\int_{\mathbb{R}^N} V_{1,0}^{p+1}\log\big(\delta_j^{-\frac{N}{p+1}} V_{1,0}\big)\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\\
&+ \frac{\alpha\varepsilon}{p+1}\int_{B^c(0,r_0/2\delta_j)} V_{1,0}^{p+1}\log\big(\delta_j^{-\frac{N}{p+1}} V_{1,0}\big)\big(1+\delta_j^2|z|^2\big)\mathrm{d}z\\
&- \frac{\alpha\varepsilon}{p+1}\int_{B(r_0/\delta_j)\setminus B(r_0/2\delta_j)} \chi_{\delta_j}^{p+1} V_{1,0}^{p+1}\log\big(\chi_{\delta_j}\delta_j^{-\frac{N}{p+1}} V_{1,0}\big)\big(1+\delta_j^2|z|^2\big)\mathrm{d}z + o(\delta_j^2)\Big\}\\
&= \frac{kL_1}{p+1} + \frac{kL_1\alpha}{(p+1)^2}\varepsilon - \frac{kL_6\alpha}{p+1}\varepsilon + \sum_{j=1}^{k}\Big\{-\frac{L_4 Scal_g(\xi_j)}{6N(p+1)}\delta_j^2 + \frac{NL_1\alpha}{(p+1)^2}\varepsilon\log\delta_j + o(\delta_j^2)\Big\},
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{p+1-\alpha\varepsilon}\int_{\mathcal{M}}\Big(\sum_{j=1}^{k} H_{\delta_j,\xi_j}\Big)^{p+1-\alpha\varepsilon} dv_g\Big\}\\
&= \sum_{j=1}^{k}\Big\{-\frac{L_4 Scal_g(\xi_j)}{3N(p+1)}\delta_j\delta_j' + \frac{NL_1\alpha\delta_j'\varepsilon}{(p+1)^2\delta_j} + o(\delta_j\delta_j')\Big\},
\end{aligned}
\tag{5.15}
$$

as $\varepsilon \to 0$, C^1-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$. Similarly, we can prove that

$$\frac{1}{q+1-\beta\varepsilon}\int_{\mathcal{M}}\Big(\sum_{j=1}^{k} W_{\delta_j,\xi_j}\Big)^{q+1-\beta\varepsilon} dv_g$$
$$= \frac{kL_1}{q+1} + \frac{kL_1\beta}{(q+1)^2}\varepsilon - \frac{kL_7\beta}{q+1}\varepsilon + \sum_{j=1}^{k}\Big\{ - \frac{L_5 Scal_g(\xi_j)}{6N(q+1)}\delta_j^2 + \frac{NL_1\beta}{(q+1)^2}\varepsilon\log\delta_j + o(\delta_j^2)\Big\}, \tag{5.16}$$

and

$$\frac{d}{dt}\Big(\frac{1}{q+1-\beta\varepsilon}\int_{\mathcal{M}}\Big(\sum_{j=1}^{k} W_{\delta_j,\xi_j}\Big)^{q+1-\beta\varepsilon} dv_g\Big)$$
$$= \sum_{j=1}^{k}\Big\{ - \frac{L_5 Scal_g(\xi_j)}{3N(q+1)}\delta_j\delta_j' + \frac{NL_1\beta\delta_j'\varepsilon}{(q+1)^2\delta_j} + o(\delta_j\delta_j')\Big\}, \tag{5.17}$$

as $\varepsilon \to 0$, C^1-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where we have used the fact that $N \geq 10$ if $p = \frac{N}{N-2}$ and $N \geq 12$ if $p < \frac{N}{N-2}$. From (5.10)–(5.17), we conclude the result. □

We now give the asymptotic expansion of the function $\widetilde{\mathcal{J}}_\varepsilon$ defined in (3.4) as $\varepsilon \to 0$.

Lemma 5.3 *Under the assumptions on p, q and N of Theorem 1.4, if $(\bar{\delta}, \bar{\xi}) \in \Lambda$ and $\bar{\delta}$ is as in (2.6), then there holds*

$$\widetilde{\mathcal{J}}_\varepsilon(\bar{t}, \bar{\xi}) = \mathcal{J}_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}}) + o(\varepsilon),$$

as $\varepsilon \to 0$, C^0-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof It's easy to verify

$$\widetilde{\mathcal{J}}_\varepsilon(\bar{t}, \bar{\xi}) - \mathcal{J}_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}, \mathcal{H}_{\bar{\delta},\bar{\xi}})$$
$$= \int_{\mathcal{M}}\big(- \Delta_g \mathcal{W}_{\bar{\delta},\bar{\xi}} + h\mathcal{W}_{\bar{\delta},\bar{\xi}} - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\big)\Phi_{\varepsilon,\bar{t},\bar{\xi}} dv_g$$
$$+ \int_{\mathcal{M}}\big(- \Delta_g \mathcal{H}_{\bar{\delta},\bar{\xi}} + h\mathcal{H}_{\bar{\delta},\bar{\xi}} - g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\big)\Psi_{\varepsilon,\bar{t},\bar{\xi}} dv_g$$
$$+ \int_{\mathcal{M}}\big(\nabla_g \Psi_{\varepsilon,\bar{t},\bar{\xi}}\cdot\nabla_g \Phi_{\varepsilon,\bar{t},\xi} + h\Psi_{\varepsilon,\bar{t},\bar{\xi}}\Phi_{\varepsilon,\bar{t},\bar{\xi}}\big)dv_g$$
$$- \int_{\mathcal{M}}\big(F_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\xi}) - F_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}}) - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\Phi_{\varepsilon,\bar{t},\bar{\xi}}\big)dv_g$$

$$-\int_{\mathcal{M}}\big(G_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi})-G_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\Psi_{\varepsilon,\bar t,\bar\xi}\big)dv_g,$$

where $F_\varepsilon(u)=\int_0^u f_\varepsilon(s)ds$, $G_\varepsilon(u)=\int_0^u g_\varepsilon(s)ds$. By the Hölder inequality, Proposition 3.1, Lemma 4.2, and (2.4), for any $\vartheta\in(0,1)$, we get

$$\begin{aligned}&\int_{\mathcal{M}}\big(-\Delta_g\mathcal{W}_{\bar\delta,\bar\xi}+h\mathcal{W}_{\bar\delta,\bar\xi}-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})\big)\Phi_{\varepsilon,\bar t,\bar\xi}dv_g\\&\quad\le\big\|-\Delta_g\mathcal{W}_{\bar\delta,\bar\xi}+h\mathcal{W}_{\bar\delta,\bar\xi}-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})\big\|_{\frac{p+1}{p}}\|\Phi_{\varepsilon,\bar t,\bar\xi}\|_{p+1}=o(\varepsilon^{2\vartheta}),\\&\int_{\mathcal{M}}\big(-\Delta_g\mathcal{H}_{\bar\delta,\bar\xi}+h\mathcal{H}_{\bar\delta,\bar\xi}-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\big)\Psi_{\varepsilon,\bar t,\xi}dv_g\\&\quad\le\big\|-\Delta_g\mathcal{H}_{\bar\delta,\bar\xi}+h\mathcal{H}_{\bar\delta,\bar\xi}-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\big\|_{\frac{q+1}{q}}\|\Psi_{\varepsilon,\bar t,\bar\xi}\|_{q+1}=o(\varepsilon^{2\vartheta}),\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathcal{M}}\big(\nabla_g\Psi_{\varepsilon,\bar t,\bar\xi}\cdot\nabla_g\Phi_{\varepsilon,\bar t,\bar\xi}+h\Psi_{\varepsilon,\bar t,\bar\xi}\Phi_{\varepsilon,\bar t,\bar\xi}\big)dv_g\\&\quad\le\|\nabla_g\Psi_{\varepsilon,\bar t,\bar\xi}\|_{p^*}\|\nabla_g\Phi_{\varepsilon,\bar t,\bar\xi}\|_{q^*}+C\|\Psi_{\varepsilon,\bar t,\bar\xi}\|_2\|\Phi_{\varepsilon,\bar t,\bar\xi}\|_2=o(\varepsilon^{2\vartheta}),\end{aligned}$$

as $\varepsilon\to0$, uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$. Moreover, by the mean value formula, Lemma 2.5, (5.14), (5.16) and the Sobolev embedding theorem, for any $\vartheta\in(0,1)$, we obtain

$$\begin{aligned}&\int_{\mathcal{M}}\big(F_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi})-F_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})\Phi_{\varepsilon,\bar t,\bar\xi}\big)dv_g\\&\quad\le C\int_{\mathcal{M}}\mathcal{H}_{\bar\delta,\bar\xi}^{p-1-\alpha\varepsilon}\Phi_{\varepsilon,\bar t,\bar\xi}^2dv_g+C\int_{\mathcal{M}}\Phi_{\varepsilon,\bar t,\bar\xi}^{p+1-\alpha\varepsilon}dv_g\\&\quad\le C\|\Phi_{\varepsilon,\bar t,\bar\xi}\|_{p+1-\alpha\varepsilon}^2\sum_{j=1}^k\|H_{\delta_j,\xi_j}\|_{p+1-\alpha\varepsilon}^{p-1-\alpha\varepsilon}+C\|\Phi_{\varepsilon,\bar t,\bar\xi}\|_{p+1-\alpha\varepsilon}^{p+1-\alpha\varepsilon}=o(\varepsilon^{2\vartheta}),\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}&\int_{\mathcal{M}}\big(G_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi})-G_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\Psi_{\varepsilon,\bar t,\bar\xi}\big)dv_g\\&\quad\le C\int_{\mathcal{M}}\mathcal{W}_{\bar\delta,\bar\xi}^{q-1-\beta\varepsilon}\Psi_{\varepsilon,\bar t,\bar\xi}^2dv_g+C\int_{\mathcal{M}}\Psi_{\varepsilon,\bar t,\bar\xi}^{q+1-\beta\varepsilon}dv_g\end{aligned}$$

$$\leq C\|\Psi_{\varepsilon,\bar{t},\bar{\xi}}\|^2_{q+1-\beta\varepsilon}\sum_{j=1}^{k}\|W_{\delta_j,\xi_j}\|^{q-1-\beta\varepsilon}_{q+1-\beta\varepsilon}+C\|\Psi_{\varepsilon,\bar{t},\bar{\xi}}\|^{q+1-\beta\varepsilon}_{q+1-\beta\varepsilon}=o(\varepsilon^{2\vartheta}), \quad (5.19)$$

as $\varepsilon \to 0$, uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$. This ends the proof. □

Next, we estimate the gradient of the reduced energy.

Lemma 5.4 *Under the assumptions on p, q and N of Theorem 1.4, if $(\bar{\delta}, \bar{\xi}) \in \Lambda$ and $\bar{\delta}$ is as in (2.6), then for any $m = 1, 2, \cdots, k$, there holds*

$$\partial_{t_m}\widetilde{\mathcal{J}}_\varepsilon(\bar{t},\bar{\xi})=\partial_{t_m}\Psi_k(\bar{t},\bar{\xi})+o(\varepsilon),$$

and set $\bar{\xi}(y) = \big(\exp_{\xi_1}(y^1), \exp_{\xi_2}(y^2), \cdots, \exp_{\xi_k}(y^k)\big)$, $y = (y^1, y^2, \cdots, y^k) \in B(0,r)^k$, for any $l = 1, 2, \cdots, N$, it holds that

$$\partial_{y_l^m}\widetilde{\mathcal{J}}_\varepsilon(\bar{t},\bar{\xi}(y))\big|_{y=0}=\partial_{y_l^m}\Psi_k(\bar{t},\bar{\xi}(y))\big|_{y=0}+o(\varepsilon),$$

as $\varepsilon \to 0$, C^0-uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where the function $\Psi_k(\bar{t},\bar{\xi})$ is defined as (3.5).

Proof For any $(\varphi,\psi) \in \mathcal{X}_{p,q}(\mathcal{M})$, by Proposition 3.1, there exist $c_{10}, c_{11}, \cdots, c_{1N}, c_{20}, c_{21}, \cdots, c_{2N}, \cdots, c_{k0}, c_{k1}, \cdots, c_{kN}$ such that

$$\mathcal{J}'_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}}+\Psi_{\varepsilon,\bar{t},\bar{\xi}},\mathcal{H}_{\delta,\xi}+\Phi_{\varepsilon,\bar{t},\bar{\xi}})(\varphi,\psi)=\sum_{l=0}^{N}\sum_{m=1}^{k}c_{lm}\big\langle\big(\Psi^l_{\delta_m,\xi_m},\Phi^l_{\delta_m,\xi_m}\big),(\varphi,\psi)\big\rangle_h. \quad (5.20)$$

We claim that: for any $\vartheta \in (0,1)$, there holds

$$\sum_{l=0}^{N}\sum_{m=1}^{k}|c_{lm}|=O(\varepsilon^{\vartheta}). \quad (5.21)$$

Taking $(\varphi,\psi) = (\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j})$, $0 \leq i \leq N$, $1 \leq j \leq k$, by (2.11)–(2.14), we have

$$\begin{aligned}&\sum_{l=0}^{N}\sum_{m=1}^{k}c_{lm}\big\langle\big(\Psi^l_{\delta_m,\xi_m},\Phi^l_{\delta_m,\xi_m}\big),\big(\Psi^i_{\delta_j,\xi_j},\Phi^i_{\delta_j,\xi_j}\big)\big\rangle_h\\&=\sum_{l=0}^{N}\sum_{m=1}^{k}c_{lm}\delta_{il}\delta_{jm}\int\limits_{B(0,r_0/\delta_j)}\big(p\chi^2_{\delta_j}V_{1,0}^{p-1}(\Phi^i_{1,0})^2+q\chi^2_{\delta_j}U_{1,0}^{q-1}(\Psi^i_{1,0})^2\big)\mathrm{d}x+O(\varepsilon),\end{aligned} \quad (5.22)$$

as $\varepsilon \to 0$, uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where $\chi_{\delta_j}(x) = \chi(\delta_j|x|)$. On the other hand, it follows from $(\Psi_{\varepsilon,\bar{t},\bar{\xi}}, \Phi_{\varepsilon,\bar{t},\bar{\xi}}) \in \mathcal{Z}_{\bar{\delta},\bar{\xi}}$ that

$$\begin{aligned}
&\mathcal{J}'_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}, \mathcal{H}_{\delta,\xi} + \Phi_{\varepsilon,\bar{t},\bar{\xi}})(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}) \\
&\quad = \int\limits_{\mathcal{M}} \big(- \Delta_g \mathcal{W}_{\bar{\delta},\bar{\xi}} + h\mathcal{W}_{\bar{\delta},\bar{\xi}} - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\big)\Phi^i_{\delta_j,\xi_j} dv_g \\
&\qquad + \int\limits_{\mathcal{M}} \big(- \Delta_g \mathcal{H}_{\bar{\delta},\bar{\xi}} + h\mathcal{H}_{\bar{\delta},\bar{\xi}} - g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\big)\Psi^i_{\delta_j,\xi_j} dv_g \\
&\qquad - \int\limits_{\mathcal{M}} \big(f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}} + \Phi_{\varepsilon,\bar{t},\bar{\xi}}) - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\big)\Phi^i_{\delta_j,\xi_j} dv_g \\
&\qquad - \int\limits_{\mathcal{M}} \big(g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}} + \Psi_{\varepsilon,\bar{t},\bar{\xi}}) - g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\big)\Psi^i_{\delta_j,\xi_j} dv_g \\
&\quad \le \big\| - \Delta_g \mathcal{W}_{\bar{\delta},\bar{\xi}} + h\mathcal{W}_{\bar{\delta},\bar{\xi}} - f_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\big\|_{\frac{p+1}{p}} \|\Phi^i_{\delta_j,\xi_j}\|_{p+1} \\
&\qquad + \big\| - \Delta_g \mathcal{H}_{\bar{\delta},\bar{\xi}} + h\mathcal{H}_{\bar{\delta},\bar{\xi}} - g_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\big\|_{\frac{q+1}{q}} \|\Psi^i_{\delta_j,\xi_j}\|_{q+1} \\
&\qquad + C\|\Phi^i_{\delta_j,\xi_j}\|_{p+1}\|\Phi_{\varepsilon,\bar{t},\bar{\xi}}\|_{\frac{p+1}{1+\alpha\varepsilon}} \sum_{j=1}^{k} \|H_{\delta_j,\xi_j}\|_{p+1}^{p-1-\alpha\varepsilon} \\
&\qquad + C\|\Phi^i_{\delta_j,\xi_j}\|_{p+1}\|\Phi_{\varepsilon,\bar{t},\bar{\xi}}\|^{p-\alpha\varepsilon}_{\frac{(p-\alpha\varepsilon)(p+1)}{p}} \\
&\qquad + C\|\Psi^i_{\delta_j,\xi_j}\|_{q+1}\|\Psi_{\varepsilon,\bar{t},\bar{\xi}}\|_{\frac{q+1}{1+\beta\varepsilon}} \sum_{j=1}^{k} \|W_{\delta_j,\xi_j}\|_{q+1}^{q-1-\beta\varepsilon} \\
&\qquad + C\|\Psi^i_{\delta_j,\xi_j}\|_{q+1}\|\Psi_{\varepsilon,\bar{t},\bar{\xi}}\|^{q-\beta\varepsilon}_{\frac{(q-\beta\varepsilon)(q+1)}{q}} = o(\varepsilon^{\vartheta}),
\end{aligned} \tag{5.23}$$

as $\varepsilon \to 0$, uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{t}$ in compact subsets of $(\mathbb{R}^+)^k$, where we have used the fact that $\|\Psi^i_{\delta_j,\xi_j}\|_{q+1} < +\infty$ and $\|\Phi^i_{\delta_j,\xi_j}\|_{p+1} < +\infty$ for any $1 < p \le \frac{N+2}{N-2} \le q$, $i = 0, 1, \cdots, N$ and $j = 1, 2, \cdots, k$. From (5.22) and (5.23), we prove the claim.

By (5.3) and (5.4), we can compute

$$\begin{aligned}
&\partial_{t_m} \widetilde{\mathcal{J}}_\varepsilon(\bar{t}, \bar{\xi}) - \partial_{t_m} \Psi_k(\bar{t}, \bar{\xi}) \\
&\quad = -\frac{1}{2t_m}\Big(\int\limits_{\mathcal{M}} \big(- \Delta_g \Psi^0_{\delta_m,\xi_m} + h\Psi^0_{\delta_m,\xi_m} - f'_\varepsilon(\mathcal{H}_{\bar{\delta},\bar{\xi}})\Phi^0_{\delta_m,\xi_m}\big)\Phi_{\varepsilon,\bar{t},\bar{\xi}} dv_g \\
&\qquad + \int\limits_{\mathcal{M}} \big(- \Delta_g \Phi^0_{\delta_m,\xi_m} + h\Phi^0_{\delta_m,\xi} - g'_\varepsilon(\mathcal{W}_{\bar{\delta},\bar{\xi}})\Psi^0_{\delta_m,\xi_m}\big)\Psi_{\varepsilon,\bar{t},\bar{\xi}} dv_g
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathcal{M}}\big(f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi})-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})-f_\varepsilon'(\mathcal{H}_{\bar\delta,\bar\xi})\Phi_{\varepsilon,\bar t,\bar\xi}\big)\Phi^0_{\delta_m,\xi_m}dv_g\\
&-\int_{\mathcal{M}}\big(g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi})-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})-g_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi})\Psi_{\varepsilon,\bar t,\bar\xi}\big)\Psi^0_{\delta_m,\xi_m}dv_g\Big)\\
&+\mathcal{J}_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi})\big(\partial_{t_m}\Psi_{\varepsilon,\bar t,\bar\xi},\partial_{t_m}\Phi_{\varepsilon,\bar t,\bar\xi}\big),
\end{aligned}
\tag{5.24}
$$

and

$$
\begin{aligned}
&\partial_{y_l^m}\widetilde{\mathcal{J}}_\varepsilon(\bar t,\bar\xi(y))\big|_{y=0}-\partial_{y_l^m}\Psi_k(\bar t,\bar\xi(y))\big|_{y=0}\\
=&\frac{1}{\delta_m}\Big(\int_{\mathcal{M}}\big(-\Delta_g\Psi^l_{\delta_m,\xi_m}+h\Psi^l_{\delta_m,\xi_m}-f_\varepsilon'(\mathcal{H}_{\bar\delta,\bar\xi})\Phi^l_{\delta_m,\xi_m}\big)\Phi_{\varepsilon,\bar t,\bar\xi}dv_g\\
&+\int_{\mathcal{M}}\big(-\Delta_g\Phi^l_{\delta_m,\xi_m}+h\Phi^l_{\delta_m,\xi_m}-g_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi})\Psi^l_{\delta_m,\xi_m}\big)\Psi_{\varepsilon,\bar t,\bar\xi}dv_g\\
&-\int_{\mathcal{M}}\big(f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi})-f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})-f_\varepsilon'(\mathcal{H}_{\bar\delta,\bar\xi})\Phi_{\varepsilon,\bar t,\bar\xi}\big)\Phi^l_{\delta_m,\xi_m}dv_g\\
&-\int_{\mathcal{M}}\big(g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi})-g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})-g_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi})\Psi_{\varepsilon,\bar t,\bar\xi}\big)\Psi^l_{\delta_m,\xi_m}dv_g\Big)\\
&+\mathcal{J}_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi}+\Psi_{\varepsilon,\bar t,\bar\xi},\mathcal{H}_{\bar\delta,\bar\xi}+\Phi_{\varepsilon,\bar t,\bar\xi})\big(\partial_{y_l^m}\Psi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0},\partial_{y_l^m}\Phi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}\big)+o(\varepsilon^{\frac{3\vartheta}{2}}),
\end{aligned}
\tag{5.25}
$$

as $\varepsilon\to 0$. Next, we estimate (5.24) and (5.25). By the Hölder inequality, Proposition 3.1, and the Sobolev embedding theorem, arguing as Lemma 4.2, for any $l=0,1,\cdots,N$, we have

$$
\begin{aligned}
&\int_{\mathcal{M}}\big(-\Delta_g\Psi^l_{\delta_m,\xi_m}+h\Psi^l_{\delta_m,\xi_m}-f_\varepsilon'(\mathcal{H}_{\bar\delta,\bar\xi})\Phi^l_{\delta_m,\xi_m}\big)\Phi_{\varepsilon,\bar t,\bar\xi}dv_g\\
&\quad\le\big\|-\Delta_g\Psi^l_{\delta_m,\xi_m}+h\Psi^l_{\delta_m,\xi_m}-f_\varepsilon'(\mathcal{H}_{\bar\delta,\bar\xi})\Phi^l_{\delta_m,\xi_m}\big\|_{\frac{p+1}{p}}\|\Phi_{\varepsilon,\bar t,\bar\xi}\|_{p+1}=o(\varepsilon^{2\vartheta}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathcal{M}}\big(-\Delta_g\Phi^l_{\delta_m,\xi_m}+h\Phi^l_{\delta_m,\xi_m}-g_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi})\Psi^l_{\delta_m,\xi_m}\big)\Psi_{\varepsilon,\bar t,\bar\xi}dv_g\\
&\quad\le\big\|-\Delta_g\Phi^l_{\delta_m,\xi_m}+h\Phi^l_{\delta_m,\xi_m}-g_\varepsilon'(\mathcal{W}_{\bar\delta,\bar\xi})\Psi^l_{\delta_m,\xi_m}\big\|_{\frac{q+1}{q}}\|\Psi_{\varepsilon,\bar t,\bar\xi}\|_{q+1}=o(\varepsilon^{2\vartheta}),
\end{aligned}
$$

as $\varepsilon\to 0$, uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$. Moreover, by the mean value formula, Lemma 2.5, (5.14), (5.16) and the Sobolev embedding theorem, for any $l = 0, 1, \cdots, N$, we obtain

$$\begin{aligned}
&\int_{\mathcal{M}} \big(f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi} + \Phi_{\varepsilon,\bar t,\bar\xi}) - f_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi}) - f'_\varepsilon(\mathcal{H}_{\bar\delta,\bar\xi})\Phi_{\varepsilon,\bar t,\bar\xi}\big)\Phi^l_{\delta_m,\xi_m} dv_g \\
&\quad \le C \int_{\mathcal{M}} \mathcal{H}^{p-2-\alpha\varepsilon}_{\bar\delta,\bar\xi} \Phi^2_{\varepsilon,\bar t,\bar\xi} \Phi^l_{\delta_m,\xi_m} dv_g \le C\|\Phi^l_{\delta_m,\xi_m}\|_{p+1}\|\Phi_{\varepsilon,\bar t,\bar\xi}\|^2_{p+1} \\
&\quad\quad \times \sum_{j=1}^{k} \|H_{\delta_j,\xi_j}\|^{p-2-\alpha\varepsilon}_{\frac{(p-2-\alpha\varepsilon)(p+1)}{p-2}} = o(\varepsilon^{2\vartheta}),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{M}} \big(g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi} + \Psi_{\varepsilon,\bar t,\bar\xi}) - g_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi}) - g'_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi})\Psi_{\varepsilon,\bar t,\bar\xi}\big)\Psi^l_{\delta_m,\xi_m} dv_g \\
&\le \begin{cases} C\displaystyle\int_{\mathcal{M}} \mathcal{W}^{q-2-\beta\varepsilon}_{\bar\delta,\bar\xi}\Psi^2_{\varepsilon,\bar t,\bar\xi}\Psi^l_{\delta_m,\xi_m} dv_g + C\int_{\mathcal{M}} \Psi^{q-\beta\varepsilon}_{\varepsilon,\bar t,\bar\xi}\Psi^l_{\delta_m,\xi_m} dv_g, & \text{if } q > 2, \\ C\displaystyle\int_{\mathcal{M}} \mathcal{W}^{q-2-\beta\varepsilon}_{\bar\delta,\bar\xi}\Psi^2_{\varepsilon,\bar t,\bar\xi}\Psi^l_{\delta_m,\xi_m} dv_g, & \text{if } q \le 2, \end{cases} \\
&\le \begin{cases} C\|\Psi^l_{\delta_m,\xi_m}\|_{q+1}\|\Psi_{\varepsilon,\bar t,\bar\xi}\|^2_{q+1}\displaystyle\sum_{j=1}^{k}\|W_{\delta_j,\xi_j}\|^{q-2-\beta\varepsilon}_{\frac{(q-2-\beta\varepsilon)(q+1)}{q-2}} + \|\Psi^l_{\delta_m,\xi_m}\|_{q+1}\|\Psi_{\varepsilon,\bar t,\bar\xi}\|^{q-\beta\varepsilon}_{\frac{(q-\beta\varepsilon)(q+1)}{q}}, & \text{if } q > 2, \\ C\|\Psi^l_{\delta_m,\xi_m}\|_{q+1}\|\Psi_{\varepsilon,\bar t,\bar\xi}\|^2_{q+1}\sum_{j=1}^{k}\|W_{\delta_j,\xi_j}\|^{q-2-\beta\varepsilon}_{\frac{(q-2-\beta\varepsilon)(q+1)}{q-2}}, & \text{if } q \le 2, \end{cases} \\
&= o(\varepsilon^{2\vartheta}),
\end{aligned}$$

as $\varepsilon \to 0$, uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$. Using (2.4), (5.8), (5.9), (5.20) and (5.21), for any $l = 1, 2, \cdots, N$, we get

$$\begin{aligned}
&\mathcal{J}'_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi} + \Psi_{\varepsilon,\bar t,\bar\xi}, \mathcal{H}_{\bar\delta,\bar\xi} + \Phi_{\varepsilon,\bar t,\bar\xi})\big(\partial_{t_m}\Psi_{\varepsilon,\bar t,\bar\xi}, \partial_{t_m}\Phi_{\varepsilon,\bar t,\bar\xi}\big) \\
&\quad = \sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\Psi^i_{\delta_j,\xi_j}, \Phi^i_{\delta_j,\xi_j}\big), \big(\partial_{t_m}\Psi_{\varepsilon,\bar t,\bar\xi}, \partial_{t_m}\Phi_{\varepsilon,\bar t,\bar\xi}\big)\big\rangle_h \\
&\quad = -\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\big\langle\big(\partial_{t_m}\Psi^i_{\delta_j,\xi_j}, \partial_{t_m}\Phi^i_{\delta_j,\xi_j}\big), \big(\Psi_{\varepsilon,\bar t,\bar\xi}, \Phi_{\varepsilon,\bar t,\bar\xi}\big)\big\rangle_h \\
&\quad = o\Big(\varepsilon^{\vartheta}\sum_{i=0}^{N}\sum_{i=1}^{k}|c_{ij}|\Big) = o(\varepsilon^{2\vartheta}),
\end{aligned}$$

and

$$\mathcal{J}'_\varepsilon(\mathcal{W}_{\bar\delta,\bar\xi} + \Psi_{\varepsilon,\bar t,\bar\xi}, \mathcal{H}_{\bar\delta,\bar\xi} + \Phi_{\varepsilon,\bar t,\bar\xi})\big(\partial_{y^m_l}\Psi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}, \partial_{y^m_l}\Phi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}\big)$$

$$
\begin{aligned}
&=\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\Big\langle\big(\Psi^i_{\delta_j,\xi_j},\,\Phi^i_{\delta_j,\xi_j}\big),\,\big(\partial_{y_l^m}\Psi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0},\,\partial_{y_l^m}\Phi_{\varepsilon,\bar t,\bar\xi(y)}\big|_{y=0}\big)\Big\rangle_h\\
&=-\sum_{i=0}^{N}\sum_{j=1}^{k} c_{ij}\Big\langle\big(\partial_{y_l^m}\Psi^i_{\delta_j,\xi_j}\big|_{y=0},\,\partial_{y_l^m}\Phi^i_{\delta_j,\xi_j}\big|_{y=0}\big),\,\big(\Psi_{\varepsilon,\bar t,\bar\xi(y)},\,\Phi_{\varepsilon,\bar t,\bar\xi(y)}\big)\Big\rangle_h\\
&=o\Big(\varepsilon^{\frac{2\vartheta-1}{2}}\sum_{i=0}^{N}\sum_{j=1}^{k}|c_{ij}|\Big)=o\big(\varepsilon^{\frac{4\vartheta-1}{2}}\big),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar t$ in compact subsets of $(\mathbb{R}^+)^k$. Taking $\frac{3}{4} < \vartheta < 1$, we complete the proof. □

Acknowledgements The research has been supported by Natural Science Foundation of Chongqing, China CSTB2024NSCQ-LZX0038.

Data Availability Data sharing is not applicable to this article as no new data were created analyzed in this study.

Conflict of Interest The authors declare that they have no conflict of interest.

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