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On the Deformation of Thurston's Circle Packings with Obtuse Intersection Angles

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Abstract
We study Thurston's circle packings with obtuse intersection angles on closed surfaces. By using combinatorial Ricci/Calabi flows and variational principle, we extend Thurston's existence theorem for circle packings with non-obtuse intersection angles to those with obtuse intersection angles. As consequences, we generalize the existence and convergence results related to Chow-Luo's combinatorial Ricci flows (J Differ Geom 63(1):97–129, 2018) and Ge's combinatorial Calabi flows (Combinatorial Methods and Geometric Equations, Thesis (Ph.D.), Peking University, Beijing, 2012, Trans Am math Soc 370(2):1377–1391, 2018, Adv Math 333:528–533, 2018).

Keywords Circle packings · Combinatorial Ricci/Calabi flow · Combinatorial Ricci potential

Mathematics Subject Classification 52C26 · 53C44

1 Introduction

1.1 Backgrounds

In the pioneering work of Thurston [32], he introduced circle packings (on triangulated surfaces with non-obtuse intersection angles) to construct hyperbolic metrics on a closed surface. The main idea is to take the triangulation as the nerve of a circle packing, from which the metric structure on the triangulated surface can be constructed via radii of circles and intersection angles between circles in the packing.

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We recall Thurston's construction first. Let M be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$, where V, E, F denote the sets of vertices, edges and faces, respectively. A circle packing is a positive function on the vertices which defined as $r : V \to (0, +\infty), v_i \mapsto r_i, i = 1, \ldots, N$, where $N = |V|$ is the number of vertices.. Fix a triangulated surface $(M, \mathcal{T})$. Each circle packing r endows a metric structure on $(M, \mathcal{T})$ as follows. Let $\mathbb{E}^2$ ($\mathbb{H}^2$ resp.) denote the Euclidean plane (the hyperbolic plane resp.) with constant Gaussian curvature 0 (-1 resp.). Using the cosine law in $\mathbb{E}^2$ ($\mathbb{H}^2$ resp.), one can equip each edge $\{ij\} \in E$ with the length

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_j \cos \Phi_{ij}} \tag{1.1}$$

$$(l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos \Phi_{ij}) \ \text{ resp.}). \tag{1.2}$$

Thurston proved a three-circle configuration theorem, which reads as for each face $\{ijk\} \in F$, the three edge lengths l_{ij}, l_{jk}, l_{ik} satisfy the triangle inequalities, see Lemma 13.7.2 in [32]. This makes each face in F isometric to a triangle in $\mathbb{E}^2$ ($\mathbb{H}^2$ resp.). Furthermore, a triangulated surface $(M, \mathcal{T})$ could be constructed by gluing these Euclidean (hyperbolic resp.) triangles coherently, i.e. along common edges. The resulting surface has a flat (hyperbolic resp.) cone metric with cone points in V. Obviously, this metric has no singularity on $M - V$ (one should notice that there is no singularity in the interior of each edges). The singularities of this metric are recorded at each vertex i in V by the so called combinatorial Gaussian curvature (also called discrete Gaussian curvature) K_i, which equals to 2π minus the cone angle at i. Denote θ_i^{jk} by the inner angle at the vertex i in the triangle $\{ijk\} \in F$, then we can express the combinatorial Gaussian curvature at i as

$$K_i = 2\pi - \sum_{\{ijk\} \in F} \theta_i^{jk}, \tag{1.3}$$

where the sum is taken over all triangles with i as one of its vertices. Similar to the smooth case, the following combinatorial version of Gauss-Bonnet formula holds true:

$$\sum_{i=1}^{N} K_i = 2\pi\chi(M) - \lambda \text{Area}(M), \tag{1.4}$$

where $\lambda = -1, 0$ correspond the two geometries, i.e., hyperbolic geometry $\mathbb{H}^2$ and Euclidean geometry $\mathbb{E}^2$. Consider the curvature map $K = K(r)$, where K varies as r varies. In Euclidean background geometry, we concerns a particular circle packing r_{av} that determines a constant combinatorial curvature $K(r_{av}) = (K_{av}, \ldots, K_{av})$ with

$$K_{av} = \frac{2\pi\chi(M)}{N}$$

for the reason that the corresponding Euclidean cone metric on $(M, \mathcal{T})$ curves the same way around each vertex.

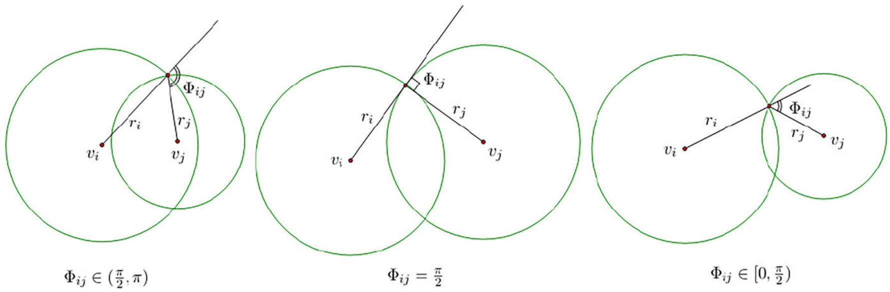

Fig. 1 The position relations of two circles

In hyperbolic background geometry, we concerns a particular circle packing r_{ze} that determines the zero curvature $K(r_{ze}) = 0$ for the reason that the corresponding hyperbolic cone metric has no singularities, and hence is a complete hyperbolic metric on $(M, \mathcal{T})$. However, Thurston find that there are combinatorial and topological obstacles (see [32], 13.6) for the existence of such circle packings r_{av} and r_{ze}. In fact, Thurston's existence theorem characterized perfectly the image of the curvature map $K = K(r)$ by the information of the combinatorics $\mathcal{T}$ and the topology of M. Thurston also suggested an algorithm to find particular circle patterns with polynomial convergence rate. Inspired by Hamilton's Ricci flow method, Chow-Luo [5] further introduced a combinatorial version of Ricci flow, which is the negative gradient flow of the Ricci potential. The flows can be used to deform the circle packing to a particular one with exponential convergence rate. Similarly, the thesis of Ge [8] (or see [9, 10]) introduced a combinatorial version of Calabi flow, which is the negative gradient flow of a more nature Calabi energy $\|K\|^2 = \sum_i K_i^2$. Since then, various discrete curvature flows were introduced and studied. We refer the the readers to Luo [30, 31], Guo [23], Glickenstein [21, 22], Ge-Xu [14, 16, 18], Ge-Jiang [11, 12], Lin-Zhang [28, 29].

1.2 Main Results

It is noticeable that the above work deal with circle packings with non-obtuse intersection angles, i.e. $\Phi \in [0, \frac{\pi}{2}]$. The purpose of this paper is to study combinatorial Ricci/Calabi flows under the condition $\Phi \in [0, \pi)$. See Fig. 1 for all possible arrangements of the circles.

The idea originated from Huang-Liu [26] and Zhou's [36] pioneering observation:

(HLZ): For each triangle $\{ijk\} \in F$, either $\Phi_{ij} + \Phi_{jk} + \Phi_{ik} \leq \pi$, or

$$\Phi_{ij} + \Phi_{jk} < \pi + \Phi_{ik}, \Phi_{ik} + \Phi_{jk} < \pi + \Phi_{ij}, \Phi_{ij} + \Phi_{ik} < \pi + \Phi_{jk}.$$

Under the **(HLZ)** condition, by a complicated calculation, Zhou proved that Thurston's three-circle configuration theorem (i.e. for each face $\{ijk\} \in F$, the three edge lengths l_{ij}, l_{jk}, l_{ik} satisfy the triangle inequalities, see Lemma 13.7.2 in [32]) is still valid. Geometrically, the three circles at the vertices have a power center O_{ijk}, see Fig. 2. Assuming **(HLZ)** and without further assumptions on Φ, the power center O_{ijk} may lie

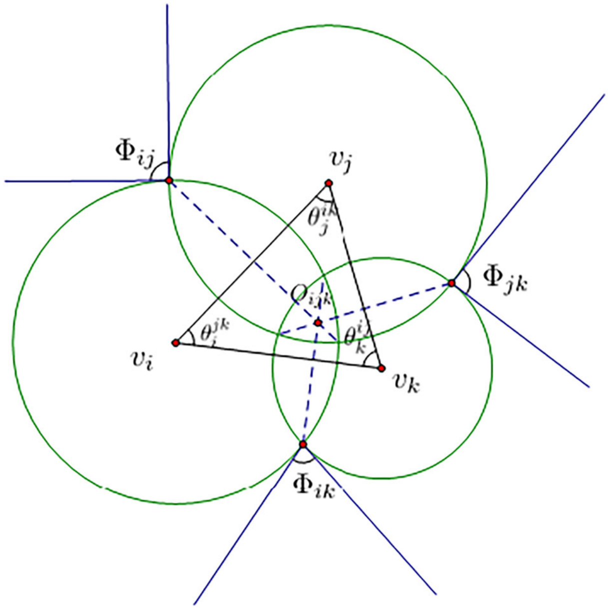

Fig. 2 A three-circle configuration

outside the triangle $\triangle v_i v_j v_k$. This is the main obscure to extend Andreev-Thurston's rigidity results and Chow-Luo and Ge's combinatorial Ricci/Calaib flow results. To overcome this obscure, Zhou first introduced the following condition

(Z) Set $I_{st} = \cos \Phi_{st}$ for $st = ij, jk, ik$. For each triangle $\{ijk\} \in F$, there holds

$$I_{ij} + I_{ik} I_{jk} \geq 0, \ \ I_{ik} + I_{ij} I_{jk} \geq 0, \ \ I_{jk} + I_{ij} I_{ik} \geq 0.$$

As pointed by Zhou, **(Z)** implies **(HLZ)**. Thus under **(Z)** condition, Thurston's three-circle configuration theorem is valid, and the power center O_{ijk} is inside the triangle $\triangle v_i v_j v_k$.

Our first result says that Thurston's existence theorem (i.e. the image of the curvature map $K = K(r)$ as a convex polytope whose boundary is characterized by the combinatorics of $\mathcal{T}$ and the topology of M) is the same as non-obtuse intersection angle case. Let F_A be the sub-complex constituted of those t-simplex ($t = 0, 1, 2$) that have at least one vertex in A, and $Lk(A)$ is the set of pairs (e, v) of an edge e and a vertex v satisfying (i) $v \in A$; (ii) both the two end points of e are not in A; (iii) v and the two end points of e forms a triangle in F. We have

Theorem 1.1 *Assume that* $\Phi \in [0, \pi)$ *satisfies* **(Z)**. *In Euclidean background geometry, the image of the curvature map* K *consists of vectors* $(K_1, K_2, \ldots, K_{|V|})$ *satisfying*

$$\sum_{i \in A} K_i > - \sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi \chi(F_A) \tag{1.5}$$

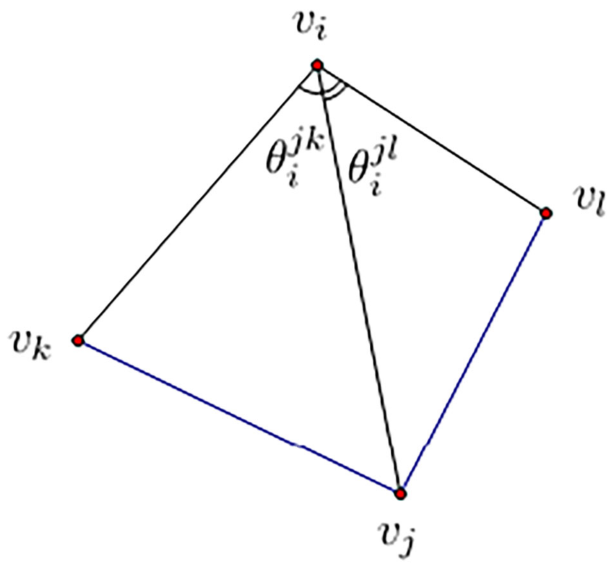

Fig. 3 Two adjacent triangles

for any non-empty subset A of V, where the equality holds if and only if $A = V$. While in hyperbolic background geometry, the image of the curvature map K consists of vectors $(K_1, K_2, \ldots, K_{|V|})$ satisfying

$$\sum_{i\in A} K_i > -\sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A) \tag{1.6}$$

for any non-empty subset A of V.

Remark 1 (1.5) was obtained by Ge-Jiang [13]. We restate here for completeness.

Thus the existence problem of a particular packing (such as r_{av} and r_{ze}) transfers to the problem whether K_{av} (zero resp.) is in the image of the curvature map $K(r)$. Recall $\mathcal{C}(r) = \|K - K_{av}\|^2$ ($\mathcal{C}(r) = \|K\|^2$ resp.) is the Euclidean (hyperbolic resp.) combinatorial Calabi energy, and a coordinate change $u_i = \ln r_i$ ($u_i = \ln\tanh\frac{r_i}{2}$ resp.) in Euclidean (hyperbolic resp.) background geometry. To find such a particular packing, we use Ge's combinatorial Calabi flow [8–10]

$$u_i'(t) = -\frac{1}{2}\partial_{u_i}\mathcal{C} = \Delta K_i, \tag{1.7}$$

for $i = 1, 2, \ldots, N$, where $\mathcal{C}$ is considered as a function of $u = (u_1, \ldots, u_N)$ in the expression $\partial_{u_i}\mathcal{C}$. Moreover, set

$$B_{ij} = \frac{\partial(\theta_i^{jk} + \theta_i^{jl})}{\partial u_j},$$

Δ is a discrete Laplacian and

$$\Delta K_i = \sum_{j\sim i} B_{ij}(K_j - K_i) \quad \text{in } \mathbb{E}^2;$$
$$\Delta K_i = \sum_{j\sim i} B_{ij}(K_j - K_i) - A_i K_i, \quad \text{in } \mathbb{H}^2$$

with

$$A_i = \frac{\partial}{\partial u_i}\Big(\sum_{\{ijk\}\in F} \text{Area}(\triangle v_i v_j v_k)\Big).$$

We have

Theorem 1.2 *Assume that $\Phi \in [0, \pi)$ satisfies **(Z)**. In Euclidean background geometry, the solution $r(t)$ to the combinatorial Calabi flow (1.7) exists for all the time $t \geq 0$, and the following properties (E_1)-(E_3) are equivalent:*

- (E_1) *$r(t)$ converges as $t \to +\infty$.*
- (E_2) *The vector $(K_{av}, \ldots, K_{av})$ belongs to the image of the curvature map.*
- (E_3) *If A is a proper non-empty subset of V, then*

$$2\pi\chi(M)\frac{|A|}{|V|} > -\sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A). \tag{1.8}$$

Moreover, if one of the above properties holds, then the combinatorial Calabi flow converges exponentially fast to a circle packing which produces an Euclidean cone metric on M with cone angles all equal to $2\pi - K_{av}$.

In hyperbolic background geometry, the results are more fruitful. Recall Chow-Luo's hyperbolic combinatorial Ricci flow [5]

$$\frac{dr_i(t)}{dt} = -K_i \sinh r_i \tag{1.9}$$

for $i = 1, 2, \ldots, N$. We have

Theorem 1.3 *Assume that $\Phi \in [0, \pi)$ satisfies **(Z)**. In hyperbolic background geometry, the solutions to the combinatorial Ricci flow (1.9) exists for all the time $t \geq 0$, and the following properties (H_1)-(H_6) are equivalent:*

- (H_1) *The solution $r(t)$ to the combinatorial Ricci flow (1.9) converges as $t \to +\infty$.*
- (H_2) *If A is a proper non-empty subset of V, then*

$$\sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) > 2\pi\chi(F_A). \tag{1.10}$$

(H_3) *The genus $g > 1$ and for any simple, null-homotopic closed path $e_1, e_2, \dots, e_s$, which is not the boundary of a triangle, there holds*

$$\sum_{i=1}^{s} \Phi(e_i) < (s-2)\pi. \tag{1.11}$$

(H_4) *The origin $(0, \dots, 0)$ belongs to the image of the curvature map.*
(H_5) *The solution $r(t)$ to the combinatorial Calabi flow (1.7) converges as $t \to +\infty$.*
(H_6) *The image of the curvature map contains a point with non-positive coordinates.*

Moreover, if one of the above properties holds, then the combinatorial Ricci/Calabi flow converges exponentially fast to a circle packing which produces a complete hyperbolic metric on M (with no cone points).

Remark 2 The results in the above theorem related to the combinatorial Ricci flows are essentially obtained by Ge-Hua-Zhou [19, 20]. In fact, they studied the combinatorial Ricci flow on surfaces of finite type. While our results are established on closed surfaces.

Our last result concerns the prescribed curvature problem. For any $\bar{K} = (\bar{K}_1, \dots, \bar{K}_N)$, the prescribed Calabi energy is $\bar{\mathcal{C}}(r) = \|K - \bar{K}\|^2$. Consider the prescribed Ricci flow

$$\frac{du_i}{dt} = (\bar{K}_i - K_i) \tag{1.12}$$

and the prescribed Calabi flow

$$u_i'(t) = -\frac{1}{2}\partial_{u_i}\bar{\mathcal{C}} = \Delta(K_i - \bar{K}_i) \tag{1.13}$$

in Euclidean background geometry, we have

Theorem 1.4 *Assume that $\Phi \in [0, \pi)$ satisfies* **(Z)**. *The solutions $r(t)$ to the prescribed Ricci flow (1.12) (Calabi flow (1.13) resp.) exists for all the time $t \geq 0$, and $r(t)$ converges if and only if $\bar{K}$ belongs to the image of the curvature map. Moreover, $r(t)$ converges exponentially fast to the unique (up to scaling in Euclidean background geometry) circle packing $\bar{r}$ with $K(\bar{r}) = \bar{K}$.*

This paper is organized as follows. In Sect. 2, we study the three-circle configurations and some useful lemmas. In Sect. 3, we give the proof of Theorem 1.2. In Sect. 4.1, we obtain an uniform estimate about the solutions to the combinatorial Calabi flow for hyperbolic background geometry. We give the proof of Theorem 1.3 in Sect. 4.2. In Appendix 1, we get the existence of the three-circle configurations for Euclidean background geometry; in Appendix 2, we obtain the image of $K(r)$ for hyperbolic background geometry; in Appendix 3, we give a direct detailed proof of the uniform bounded from above to the $\frac{\partial \theta_i^{jk}}{\partial u_j}$.

2 Preliminaries: Three-Circle Configurations

To endow a metric structure on $(M, \mathcal{T})$ with the help of circle packings, we first need a three-circle configuration.

Lemma 2.1 *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying **(HLZ)**. For any three positive numbers r_i, r_j, r_k, there exits a configuration of three mutually intersecting closed disks in both Euclidean and hyperbolic geometry, unique up to congruence, having radii r_i, r_j, r_k and meeting with exterior intersection angles Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$.*

Proof In hyperbolic geometry, this was proved by Zhou, see Lemma 2.4 in [36]. In Euclidean geometry, we postpone its proof to Appendix 1. □

Lemma 2.2 ([36], Proposition 5.1) *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying **(Z)**, then they satisfy **(HLZ)**. Consequently, the above three-circle configuration theorem is valid under the condition **(Z)**.*

Remark 3 Zhou's proof of Proposition in [36] can be used to both Euclidean and hyperbolic background geometry.

Lemma 2.3 ([33], Lemma 2.6) *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying **(Z)**. In Euclidean background geometry, the Jacobian matrix of functions θ_i^{jk}, θ_j^{ik}, θ_k^{ij} in terms of u_i, u_j, u_k is symmetric and semi-negative definite with rank 2 and kernel $\{t(1, 1, 1)|t \in \mathbb{R}\}$. Moreover, $\frac{\partial\theta_i^{jk}}{\partial u_i} < 0$ and $\frac{\partial\theta_i^{jk}}{\partial u_j} \geq 0$.*

Lemma 2.4 ([36], Lemma 5.5) *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying **(Z)**. In hyperbolic background geometry, the Jacobian matrix of functions θ_i^{jk}, θ_j^{ik}, θ_k^{ij} in terms of u_i, u_j, u_k is symmetric and negative definite. Moreover, $\frac{\partial\theta_i^{jk}}{\partial u_i} < 0$, $\frac{\partial\theta_i^{jk}}{\partial u_j} \geq 0$, $\frac{\partial(\theta_i^{jk}+\theta_j^{ik}+\theta_k^{ij})}{\partial u_i} < 0$.*

Lemma 2.5 ([33], Corollaries 2.7 and 3.8) *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying **(Z)**. In Euclidean background geometry, the Jacobian matrix $\Delta = -\frac{\partial(K_1,\ldots,K_N)}{\partial(u_1,\ldots,u_N)}$ is symmetric and semi-negative definite with rank $N-1$ and kernel $\{t\mathbf{1}|t \in \mathbb{R}\}$. In hyperbolic background geometry, the Jacobian matrix $\Delta = -\frac{\partial(K_1,\ldots,K_N)}{\partial(u_1,\ldots,u_N)}$ is symmetric and negative definite.*

3 Euclidean Geometry Background

Suppose $\triangle v_i v_j v_k$ is a topological triangle in F. We use l_{ij}, l_{jk}, l_{ik} (defined as (1.1)) to denote the lengths of the edge $v_i v_j$, $v_j v_k$, $v_i v_k$, respectively. From Lemma 2.4 in [36], we can see that there is no restriction on the radii such that the lengths l_{ij}, l_{jk}, l_{ik} for $\triangle v_i v_j v_k \in F$ satisfy the triangle inequalities.

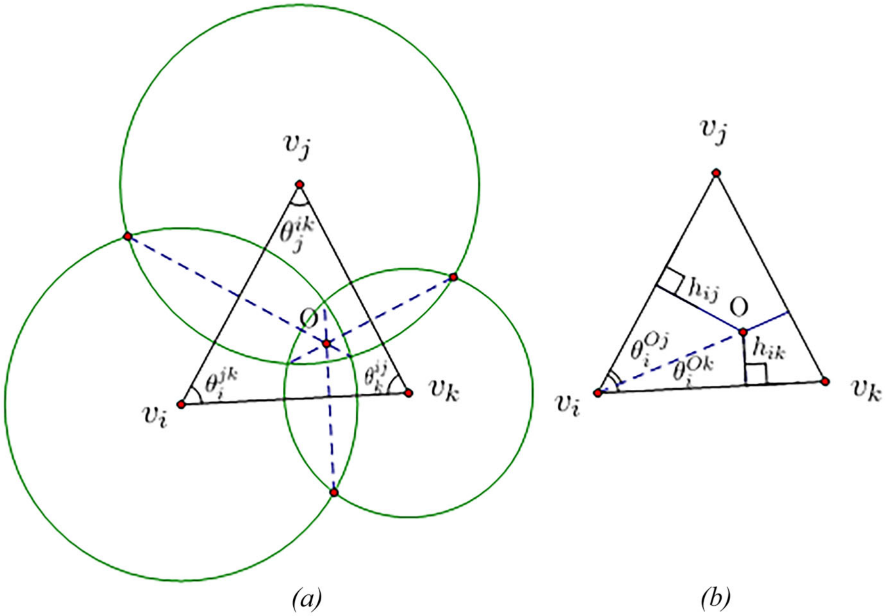

Fig. 4 The corresponding contact graph

Proposition 3.1 ([9], Proposition 4.1) *Along the Combinatorial Calabi flow, the discrete Gauss curvature evolves according to*

$$\frac{dK}{dt} = -L^2 K,$$

where $L = -\Delta$ *and* Δ *defined as in Lemma 2.5.*

Lemma 3.2 *For any topological triangle* $\triangle v_i v_j v_k \in F$ *with fixed weights* $\Phi_{ij}, \Phi_{jk}, \Phi_{ik} \in [0, \pi)$ *as intersection angles satisfying* **(Z)***, then there exists a constant* $C(\Phi)$ *which only depends on the given* Φ *such that*

$$0 < \frac{\partial \theta_i^{jk}}{\partial u_j} \leq C(\Phi).$$

Proof Here we give a geometric proof by following He [25]. Under the **(Z)** condition, let O be the power center which is inside the triangle $\triangle v_i v_j v_k$, then $\theta_i^{Oj} \in [0, \frac{\pi}{2})$. See Fig. 4, let h_{ij} be the length of the altitude from O onto side $v_i v_j$ and h_{ik} be the length of the altitude from O onto side $v_i v_k$. Thus $\frac{\partial \theta_i^{jk}}{\partial u_j} = \frac{h_{ij}}{l_{ij}}$.

If θ_i^{Oj} is not close to $\frac{\pi}{2}$, say, $\theta_i^{Oj} \in [0, 1)$, we have

$$\frac{\partial \theta_i^{ik}}{\partial u_j} = \frac{h_{ij}}{l_{ij}} < \tan \theta_i^{Oj} \leq C(\Phi),$$

where $C(\Phi)$ is a positive constant which only dependent on the given Φ.

If $\theta_i^{Oj} \in [1, \frac{\pi}{2})$, then $\theta_i^{jk} \geq \theta_i^{Oj} \geq 1$ is bounded from below. Since O lies in the convex hull of the union of circles C_i, C_j, we know $r_i < l_{ij}$ and $r_j < l_{ij}$. Thus $h_{ij} < 2l_{ij}$. It follows that

$$\frac{\partial \theta_i^{ik}}{\partial u_j} = \frac{h_{ij}}{l_{ij}} < 2.$$

Which completes the proof. □

By Lemma 2.5 in [33], we can see $B_{ij} = B_{ji}$. Directly from Lemma 3.2, we have

Lemma 3.3 *For any two adjacent topological triangles $\triangle v_i v_j v_k,\ \triangle v_i v_j v_l \in F$ with fixed weights $\Phi_{ij}, \Phi_{jk}, \Phi_{ik}, \Phi_{il}, \Phi_{jl} \in [0, \pi)$ as intersection angles satisfying* **(Z)**, *then there exists a constant $C(\Phi)$ which only depends on the given weight Φ such that*

$$0 < B_{ij} \leq C(\Phi).$$

3.1 Proof of Theorem 1.2

We first study the long time existence of the combinatorial Calabi flows (1.7) in Euclidean geometry background and get the following theorem.

Theorem 3.4 *Given a triangulated surface $(M, \mathcal{T}, \Phi)$ in $\mathbb{E}^2$ with weight $\Phi \in [0, \pi)$ satisfying* **(Z)**. *For any initial circle packing metric $r(0) \in \mathbb{R}^N_{>0}$, the solution to the combinatorial Calabi flow (1.7) in $\mathbb{E}^2$ exists for all time $t \in [0, +\infty)$.*

Proof Let d_i denote the degree at vertex v_i, which is the number of edges adjacent to v_i. Set $d = \max\{d_1, \ldots, d_N\}$, then $(2-d)\pi < K_i < 2\pi$, and

$$|K_j - K_i| < d\pi, \text{ for all } i \in \{1, \ldots, N\}.$$

Hence, by Lemma 3.3, the combinatorial Calabi flow equation $\sum_{j\sim i} B_{ij}(K_j - K_i)$ are uniformly bounded by a positive constant $c_1 = 2\pi d C(\Phi)$, which depends only on the triangulation and the fixed weight Φ, where $C(\Phi)$ is a positive constant comes from Lemma 3.3. Then we have

$$c_0 e^{-c_1 t} \leq r_i(t) \leq c_0 e^{c_1 t},$$

where $c_0 = c(r(0))$, which implies that the combinatorial Calabi flow (1.7) has a solution for all time $t \in [0, +\infty)$ for any $r(0) \in \mathbb{R}^N_{>0}$. □

Now, we give the proof of Theorem 1.2.

Proof We first show "$(E_1) \Rightarrow (E_2)$". From Theorem 3.4, we know the solution of the combinatorial Calabi flow (1.7) in $\mathbb{E}^2$ exists for all the time. Then we can denote

$r(t),\ t \in [0, +\infty)$ as the solution of the combinatorial Calabi flow (1.7) in $\mathbb{E}^2$. We recall the definition of *combinatorial Calabi energy* in [9], that is

$$\mathcal{C}(r) = \|K - K(r_{av})\|^2 = \sum_{i=1}^{N} (K_i - K_{av})^2. \tag{3.1}$$

In fact, the combinatorial Calabi flow (1.7) is the negative gradient flow of combinatorial Calabi energy, and the Calabi energy (3.1) is descending along this flow.

If $\{r(t)|t \in [0, +\infty)\}$ converges, i.e.,

$$r(+\infty) = \lim_{t\to+\infty} r(t) \in \mathbb{R}^N_{>0}$$

exists, then both $K(+\infty) = \lim_{t\to+\infty} K(t) \in \mathbb{R}^N_{>0}$ and $L(+\infty) = \lim_{t\to+\infty} L(t) \in \mathbb{R}^N_{>0}$ exist. This leads to the existence of $\mathcal{C}(+\infty)$ and $\mathcal{C}'(+\infty)$. Combining with the fact that $\mathcal{C}(t)$ is uniformly bounded and using Lemma 2.5 and Proposition 3.1, we have

$$\mathcal{C}'(t) = 2\sum_{i=1}^{N} K_i' K_i = 2K^T K' = -2K^T L^2 K \leq 0,$$

and then

$$\mathcal{C}'(+\infty) = -2K^T(+\infty)L^2(+\infty)K(+\infty) = 0.$$

Hence

$$K(+\infty) \in Ker(L^2) = Ker(L).$$

By Lemma 2.3, we know $K(+\infty)$ is a constant and $r(+\infty)$ is a constant curvature metric.

Now we show "$(E_2) \Rightarrow (E_1)$". Assume there exists a constant curvature circle packing metric r_{av} which implies $K(r_{av}) \in K(\mathbb{R}^N_{>0})$. We claim $\{r(t)|t \in [0, +\infty)\} \subset\subset \mathbb{R}^N_{>0}$. Consider the combinatorial Ricci potential

$$F(u) = \int_{u_{av}}^{u} \sum_{i=1}^{N} (K_i - K_{av}) du_i,\ u \in \mathbb{R}^N, \tag{3.2}$$

where $u_{av} = \ln r_{av}$. This type of line integral was first introduced by Verdière in [6]. By Lemma 2.5, we know

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i},$$

the smooth differential 1-form $\sum_{i=1}^{N}(K_i - K_{av})du_i$ is closed, and hence then (3.2) is well defined and is independent on the choice of piecewise smooth paths in $\mathbb{R}^N$ from u_{av} to u. By Lemma B.2 in [9], we know that $F(u)$ is strictly convex, u_{av} is the unique critical point and

$$\lim_{\|u\|\to+\infty, u\in\mathbb{R}^N} F(u) = +\infty. \tag{3.3}$$

Set $\varphi(t) = F(u(t))$, then

$$\varphi'(t) = \sum_{i=1}^{N}(K_i - K_{av})\frac{du_i}{dt} = (K - K(r_{av}))^T(-LK) = -K^T LK \leq 0,$$

which means $\varphi(t)$ is descending as t increases. Combine with (3.3), we have $\{u(t)|t \in [0, +\infty)\} \subset\subset \mathbb{R}^N$, i.e., $\{r(t)|t \in [0, +\infty)\} \subset\subset \mathbb{R}^N_{>0}$. Hence $r(t)$ converges.

Denote λ_1 as the minimum positive eigenvalue of $L = -\Delta$. Since the matrix L is semi-positive definite by Lemma 2.3, λ_1^2 is the minimum positive eigenvalue of L^2. Then

$$K^T L^2 K = (K - K(r_{av}))^T L^2 (K - K(r_{av})) \geq \lambda_1^2 \|K - K(r_{av})\|^2 = \lambda_1^2 \mathcal{C}.$$

Since $r(t)$ converges, $\lambda_1^2(t)$ has a uniform lower bound along the Combinatorial Calabi flow, i.e., $\lambda_1^2(t) \geq \lambda > 0$, where λ is a positive constant. Hence

$$\mathcal{C}'(t) = -2K^T L^2 K \leq -2\lambda_1^2(t)\mathcal{C}(t) \leq -2\lambda\mathcal{C}(t).$$

So $\mathcal{C}(t) \leq \mathcal{C}(0)e^{-2\lambda t}$ and using (3.1),

$$|K_i(t) - K_{av}| \leq |K(t) - K(r_{av})| \leq \sqrt{\mathcal{C}(t)} \leq \sqrt{\mathcal{C}(0)}e^{-\lambda t}.$$

Since $\{r(t)|t \in [0, +\infty)\} \subset\subset \mathbb{R}^N_{>0}$, r_i is bounded along the combinatorial Calabi flow. By Lemma 3.3, we have

$$\left|\frac{dr_i}{dt}\right| = |r_i \Delta K_i| = |r_i \sum_{j\sim i} B_{ij}(K_j - K_i)| \leq C(M, \Phi)e^{-\lambda t},$$

where $C(M, \Phi)$ is a positive constant and only dependent on M and Φ. This implies that the solution converges with exponential rate.

In view of $\sum_{i\in V} K_i = 2\pi\chi(M)$ and Theorem 1.1, "$(E_2) \Leftrightarrow (E_3)$" is obviously.

Moreover, from the step "$(E_1) \Rightarrow (E_2)$", we can see that the combinatorial Calabi flow converges exponentially fast to a circle packing which produces an Euclidean cone metric on M with cone angles all equal to $2\pi - K_{av}$. □

It is easy to see that we can also proof Theorem 1.4 by using the similar method of Theorems 3.4 and 1.2.

4 Hyperbolic Geometry Background

4.1 An Uniform Estimate

Suppose $\triangle v_i v_j v_k$ is a topological triangle in F. We use l_{ij}, l_{jk}, l_{ik} (defined as (1.2)) to denote the lengths of the edge $v_i v_j$, $v_j v_k$, $v_i v_k$, respectively. Zhou [36] obtained that there is no restriction on the radii such that the lengths l_{ij}, l_{jk}, l_{ik} for $\triangle v_i v_j v_k \in F$ satisfy the triangle inequalities.

If $\Phi_{ij}, \Phi_{ik}, \Phi_{jk} \in [0, \pi/2)$, the following result was first proved by Chow-Luo, Lemma 3.5 in [5], with a geometric argument. Moreover, Ge-Xu [17] Lemma 3.2, Ge-Jiang [13] Lemma 2.3 stated it by an analytic proof. Now we give a similar proof of Ge-Xu [17] Lemma 3.2 just for completeness.

Lemma 4.1 *Let $\triangle v_i v_j v_k$ be a hyperbolic triangle which is patterned by three circles with fixed weighted $\Phi_{ij}, \Phi_{ik}, \Phi_{jk} \in [0, \pi)$ as intersection angles which satisfies* **(Z)**. *Let θ_i^{jk} be the inner angle at v_i. Then for any $\epsilon > 0$, there exists a number l so that when $r_i > l$, the inner angle θ_i^{jk} is smaller than ϵ.*

Proof It is sufficient to prove that $\theta_i^{jk} \to 0$ uniformly as $r_i \to +\infty$. Set $a = \frac{\cosh(l_{ij}-l_{ik})}{\cosh(l_{ij}+l_{ik})}$ and $b = \frac{\cosh l_{jk}}{\cosh(l_{ij}+l_{ik})}$. By the hyperbolic cosine law, we have

$$\begin{aligned}\cos\theta_i^{jk} &= \frac{\cosh l_{ij}\cosh l_{ik} - \cosh l_{jk}}{\sinh l_{ij}\sinh l_{ik}}\\ &= \frac{\cosh(l_{ij}+l_{ik}) + \cosh(l_{ij}-l_{ik}) - 2\cosh l_{jk}}{\cosh(l_{ij}+l_{ik}) - \cosh(l_{ij}-l_{ik})}\\ &= \frac{1+a-2b}{1-a}.\end{aligned} \tag{4.1}$$

Noting that

$$0 < a < \frac{\cosh l_{ij}}{\cosh(l_{ij}+l_{ik})} < \frac{1}{\cosh l_{ik}} < \frac{1}{\cosh r_i},$$

we know $a \to 0$ uniformly as $r_i \to +\infty$. Now, we claim that $b \to 0$ uniformly as $r_i \to +\infty$.

Since $\cosh l_{jk} \le \cosh r_j \cosh r_k + \sinh r_j \sinh r_k = \cosh(r_j + r_k)$, we have

$$b \le \frac{\cosh(r_j + r_k)}{\cosh(l_{ij}+l_{ik})}. \tag{4.2}$$

Set $c_{ij} = \min\{\cos\Phi_{ij}, 0\}$, then $-1 < c_{ij} \le 0$. Since the triangulation $\mathcal{T}$ is a finite subdivision, we have

$$\begin{aligned}e^{l_{ij}} \ge \cosh l_{ij} &= \cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos\Phi_{ij}\\ &= (1+\cos\Phi_{ij})\cosh r_i \cosh r_j - \cosh(r_i - r_j)\cos\Phi_{ij}\end{aligned}$$

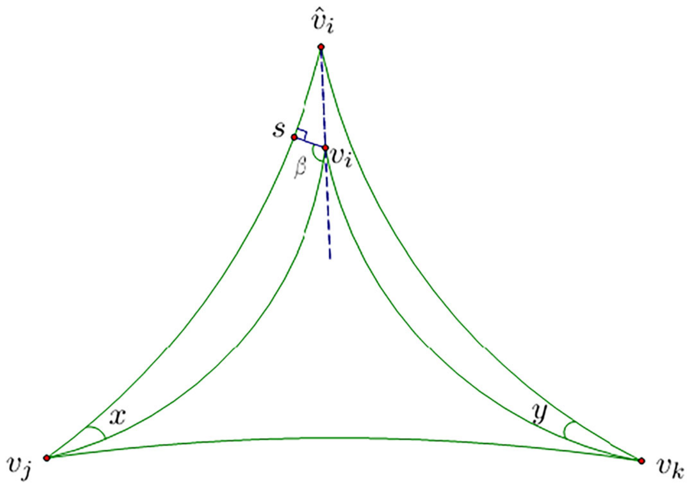

Fig. 5 Proof of Lemma 4.2

$$\geq (1 + c_{ij}) \cosh r_i \cosh r_j$$
$$\geq C_1 e^{r_i + r_j},$$

where $C_1 \in (0, 1/4]$ is a constant number and only depended on the triangulation $\mathcal{T}$. That is $l_{ij} \geq r_i + r_j + \ln C_1$. Similarly, we have $l_{ik} \geq r_i + r_k + \ln C_2$, where $C_1 \in (0, 1/4]$ is also a constant number and only depended on the triangulation $\mathcal{T}$. Hence, we get

$$l_{ij} + l_{ik} - (r_j + r_k) \geq 2r_i + \ln C_1 + \ln C_2 \to +\infty \tag{4.3}$$

uniformly as $r_i \to +\infty$. Combine (4.2) and (4.3), we know $b \to 0$ uniformly as $r_i \to +\infty$. Hence, by (4.1), we complete the proof. □

Lemma 4.2 *Let $\triangle v_i v_j v_k$ be a hyperbolic triangle which is patterned by three circles with fixed weights $\Phi_{ij}, \Phi_{jk}, \Phi_{ik} \in [0, \pi)$ as intersection angles and satisfying* **(Z)***. There exists a constant $C > 0$ which is only depending on the triangulation $\mathcal{T}$, such that if $r_i \geq C$, then*

$$\frac{\partial}{\partial r_i}(2Area(\triangle v_i v_j v_k) + \theta_i^{jk}) \geq 0. \tag{4.4}$$

Proof The proof is similar to the proof of Lemma 3.2 in [10]. Assume the triangle $\triangle v_i v_j v_k$ is embedded in $\mathbb{H}^2$, with v_j, v_k and the corresponding radii r_j, r_k fixed. Let $\hat{v}_i$ be the new vertex with a larger radius $\hat{r}_i > r_i$. Since $\hat{r}_i > r_i$, we have $l_{\hat{v}_i v_j} > l_{v_i v_j}$ and $l_{\hat{v}_i v_k} > l_{v_i v_k}$. We draw two triangles, $\triangle v_i v_j v_k$ and $\triangle \hat{v}_i v_j v_k$; with common edge $v_j v_k$ in the same half hyperbolic plane separated by the (extended) geodesic $v_j v_k$. By Lemma 2.4, for fixed r_j and r_k the angles θ_j^{ik} and θ_i^{jk} are increasing in r_i which implies that the vertex v_i lies in the interior of the triangle $\triangle v_i v_j v_k$, see Fig. 5. Denote

$\hat{\theta}_i^{jk}$, $\hat{\theta}_j^{ik}$ and $\hat{\theta}_k^{ij}$ as three inner angles of the new triangle $\triangle \hat{v}_i v_j v_k$ respectively. To prove (4.4), it suffices to show that for any $\hat{r}_i > r_i$, sufficiently close to r_i,

$$2\text{Area}(\triangle \hat{v}_i v_j v_k) - 2\text{Area}(\triangle v_i v_j v_k) + \hat{\theta}_i^{jk} - \theta_i^{jk} \geq 0. \tag{4.5}$$

Setting $x = \hat{\theta}_j^{ik} - \theta_j^{ik}$ and $y = \hat{\theta}_k^{ij} - \theta_k^{ij}$, we get

$$\begin{aligned} & 2\text{Area}(\triangle \hat{v}_i v_j v_k) - 2\text{Area}(\triangle v_i v_j v_k) + \hat{\theta}_i^{jk} - \theta_i^{jk} \\ =& \text{Area}(\triangle \hat{v}_i v_i v_j) + \text{Area}(\triangle \hat{v}_i v_i v_k) - x - y \\ =& [\text{Area}(\triangle \hat{v}_i v_i v_j) - x] + [\text{Area}(\triangle \hat{v}_i v_i v_k) - y]. \end{aligned} \tag{4.6}$$

Then it suffices to prove that

$$\text{Area}(\triangle \hat{v}_i v_i v_j) \geq x, \text{ and } \text{Area}(\triangle \hat{v}_i v_i v_k) \geq y.$$

By the symmetry, without loss of generality, we show that

$$\text{Area}(\triangle \hat{v}_i v_i v_j) \geq x. \tag{4.7}$$

Let s be the point on the geodesic $\hat{v}_i v_j$ which attains the minimum distance from the vertex v_i to a point on the geodesic $\hat{v}_i v_j$. Since $l_{\hat{v}_i v_j} > l_{v_i v_j}$, s is in the interior of the geodesic $\hat{v}_i v_j$, see Fig. 5. We assume that $\hat{r}_i$ is sufficiently close to r_i such that $l_{\hat{v}_i v_i} \leq 1$.

By the hyperbolic cosine law,

$$\cos x = \frac{\cosh l_{v_i v_j} \cosh l_{\hat{v}_i v_j} - \cosh l_{\hat{v}_i v_i}}{\sinh l_{v_i v_j} \sinh l_{\hat{v}_i v_j}} \to 1,$$

uniformly as $r_i \to \infty$. Hence there is a universal constant C_1 such that if $r_i \geq C_1$, then $x \leq \frac{\pi}{8}$. Set $\beta = \angle s v_i v_j$.

If $\beta < \frac{\pi}{4}$, then $\beta + x + x < \frac{\pi}{2}$. By the Gauss-Bonnet theorem in the hyperbolic case,

$$\beta + x + \frac{\pi}{2} = \pi - \text{Area}(\triangle s v_i v_j),$$

we get $\text{Area}(\triangle \hat{v}_i v_i v_j) \geq \text{Area}(\triangle s v_i v_j) > x$, which yields (4.7).

If $\beta \geq \frac{\pi}{4}$, then

$$\frac{\sinh l_{sj}}{\sinh l_{ij}} = \sin \beta \geq \frac{\sqrt{2}}{2}.$$

Using the cosine law in the hyperbolic right triangle $\triangle s v_i v_j$, we have

$$\cos \beta = \sin x \cosh l_{sj}, \ \cos x = \tanh l_{sj} / \tanh l_{ij}, \ \sin \beta = \sinh l_{sj} / \sinh l_{ij}.$$

This leads to

$$\begin{aligned}\sinh(\text{Area}(\triangle s v_i v_j)) &= \cos(x+\beta) = \cos x \cos\beta - \sin x \sin\beta \\ &= \frac{\tanh l_{sj}}{\tanh l_{ij}} \sin x \cosh l_{sj} - \sin x \frac{\sinh l_{sj}}{\sinh l_{ij}} \\ &= \sin x (\cosh l_{ij} - 1) \frac{\sinh l_{sj}}{\sinh l_{ij}}. \end{aligned} \tag{4.8}$$

Set $c_{ij} = \min\{\cos\Phi_{ij}, 0\}$, then $-1 < c_{ij} \le 0$. So, we have

$$\begin{aligned}\cosh l_{ij} &= \cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos\Phi_{ij} \\ &= (1+\cos\Phi_{ij}) \cosh r_i \cosh r_j - \cosh(r_i - r_j) \cos\Phi_{ij} \\ &\ge (1 + c_{ij}) \cosh r_i \cosh r_j. \end{aligned}$$

Hence, combine with (4.8), there exists a large enough universal constant C_2 which depending only on the triangulation $\mathcal{T}$, such that if $r_i \ge C_2$, then

$$\sin(\text{Area}(\triangle s v_i v_j)) \ge \sin x.$$

Noting that both x and $\text{Area}(\triangle s v_i v_j)$ are in $(0, \frac{\pi}{2})$, we obtain (4.7).

By setting $\max\{C_1, C_2\}$, combining all cases above, we complete the proof. □

Lemma 4.3 *There exists a universal number $C > 0$, such that if $r_i > C$, then*

$$A_i \ge \sum_{j \sim i} B_{ij}.$$

Proof By (4.4), we have

$$\begin{aligned} A_i - \sum_{j\sim i} B_{ij} &= \sum_{\{ijk\}\in F} \frac{\partial \text{Area}(\triangle v_i v_j v_k)}{\partial r_i} \sinh r_i - \sum_{j\sim i} \left(\frac{\partial \theta_i^{jk}}{\partial r_j} \sinh r_j + \frac{\partial \theta_i^{jl}}{\partial r_j} \sinh r_j \right) \\ &= \sum_{\{ijk\}\in F} \frac{\partial \text{Area}(\triangle v_i v_j v_k)}{\partial r_i} \sinh r_i - \sum_{\{ijk\}\in F} \left(\frac{\partial \theta_i^{jk}}{\partial r_j} \sinh r_j + \frac{\partial \theta_i^{kj}}{\partial r_k} \sinh r_k \right) \\ &= \sum_{\{ijk\}\in F} \frac{\partial \left(\text{Area}(\triangle v_i v_j v_k) - \theta_j^{ik} - \theta_k^{ij} \right)}{\partial r_i} \sinh r_i \\ &= \sum_{\{ijk\}\in F} \frac{\partial \left(2\text{Area}(\triangle v_i v_j v_k) + \theta_i^{jk} \right)}{\partial r_i} \sinh r_i \ge 0. \end{aligned}$$

□

Theorem 4.4 *Let $(M, \mathcal{T})$ be a triangulated compact hyperbolic surface with an edge weight $\Phi : E \to [0, \pi)$ which satisfies* **(Z)**. *Let $r(t)$ be the unique solution to the combinatorial Calabi flow on a maximal time interval $[0, T)$. Then all $r_i(t)$ are uniformly bounded above on $[0, T)$.*

Proof By contradiction. Suppose it is not true, then there exists at least one vertex $i \in V$, such that

$$\limsup_{t \to T} r_i(t) = +\infty. \tag{4.9}$$

For this vertex i, using Lemma 4.1, we can choose a large enough positive number l such that $r_i > l$, the inner angle θ_i is smaller than $\frac{\pi}{d_i}$, where d_i is the degree of the vertex i. Then we have $K_i > \pi$.

Set $L = \max\{l, c, r_i(0) + 1\}$, where c is given in Lemma 4.3. Now, we claim that for any $t \in (0, T)$ and if $r_i(t) > l$, then

$$\frac{dr_i}{dt} < 0. \tag{4.10}$$

Since

$$\begin{aligned}
\frac{1}{\sinh r_i} \frac{dr_i}{dt} &= \Delta K_i = \sum_{j \sim i} B_{ij}(K_j - K_i) - A_i K_i < \sum_{j \sim i} B_{ij}(2\pi - K_i) - A_i K_i \\
&= 2\pi \sum_{j \sim i} B_{ij} - \left(\sum_{j \sim i} B_{ij} + A_i \right) K_i \leq 2\pi \sum_{j \sim i} B_{ij} - \pi \left(\sum_{j \sim i} B_{ij} + A_i \right) \\
&= \pi \left(\sum_{j \sim i} B_{ij} - A_i \right) \leq 0.
\end{aligned}$$

Hence we proved the claim.

By (4.10), we may choose $t_0 \in (0, T)$ such that $r_i(t_0) > c$. Let $t_1 \in [0, t_0]$ attain the maximum of $r_i(t)$ in $[0, t_0]$. By the definition of L, $t_1 > 0$. Hence

$$\frac{dr_i}{dt}(t_1) \geq 0,$$

which contradicts to (4.10). This proves the theorem. □

4.2 Proof of Theorem 1.3

Proof "$(H_1) \Rightarrow (H_2)$". Suppose the solution $r(t)$ to the combinatorial Ricci flow (1.9) converges as $t \to +\infty$. Let u^* be the corresponding u-coordinate of r^*, then $u(t)$ converges to u^*.

$$u_i(n+1) - u_i(n) = u_i'(\xi_n) = -K_i(\xi_n) \to 0, \quad \text{as } n \to +\infty.$$

As $u(t) \to u^*$, we have $K(t) \to K(u^*)$. Thus $K_i(u^*) = 0$ for each vertex $i \in V$ and then u^* has zero curvature. This implies that there exists a particular circle pattern with all its curvatures $K_i \leq 0$. By Theorem 1.1, we know the image of the curvature map K consists of vectors $(K_1, K_2, \ldots, K_{|V|})$ satisfying

$$\sum_{i \in A} K_i > -\sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A)$$

for any non-empty subset A of V, which implies (1.10).

"$(H_2) \Leftrightarrow (H_3)$". First we prove $(H_2) \Rightarrow (H_3)$. We follow the way in [20] and [19]. Assume

$$-\sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A) < 0$$

for each $A \subset V$, we need to prove $\sum_{i=1}^{s} \Phi(e_i) < (s-2)\pi$, whenever $e_1, e_2, \ldots, e_s$ form a simple, null-homotopic closed path which is not the boundary of a triangle. Given such a path, we take $A \subset V$ as the interior vertices that are bounded by the simple, null-homotopic closed path $e_1, e_2, \ldots, e_s$. Because the path is not a boundary of a triangle, A is nonempty. In addition, $\chi(F_A) = 1$ since F_A is contractible. Moreover, it is easy to see

$$\sum_{l=1}^{s} \big(\pi - \Phi(e_l)\big) = \sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) > 2\pi,$$

which implies what we need to prove.

Next we prove $(H_3) \Rightarrow (H_2)$. Assume $\sum_{i=1}^{s} \Phi(e_i) < (s-2)\pi$, or equivalently,

$$\sum_{i=1}^{s} \big(\pi - \Phi(e_i)\big) > 2\pi$$

whenever $e_1, e_2, \ldots, e_s$ form a simple, null-homotopic closed path which is not the boundary of a triangle. We need to prove

$$-\sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A) < 0 \tag{4.11}$$

for each $A \subset V$. Let $A \subset V$, $A \neq \emptyset$. If $A = V$, then the above inequality degenerates to $\chi(M) < 0$, which is already guaranteed by the genus $g > 1$. For $A \subsetneq V$, we just need to prove $\sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) > 2\pi\chi(F_A)$ on each connected component of F_A. Hence we may assume that F_A is connected. In this case, it is easy to see $\chi(F_A) \leq 1$. If $\chi(F_A) \leq 0$, then (4.11) holds naturally. If $\chi(F_A) = 1$, all triangles in F_A (i.e. all triangles that has at least one boundary vertex in A) constitutes a simply-connected domain bounded by the edges e marked with a triangle f such that $(e, f) \in$

$Lk(A)$. Denote all such edges as $e_1, e_2, \ldots, e_s$, then $e_1, e_2, \ldots, e_s$ form a simple, null-homotopic closed path which is not the boundary of a triangle. It follows that

$$\sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) \geq \sum_{i=1}^{s} \big(\pi - \Phi(e_i)\big) > 2\pi = 2\pi\chi(F_A),$$

which completes the proof.

"$(H_2) \Rightarrow (H_4)$" is obvious. By Theorem 1.1, (H_2) implies that $(0, \ldots, 0)$ belongs to the image of the curvature map.

"$(H_4) \Rightarrow (H_5)$". Assume the origin $(0, \ldots, 0)$ belongs to the image of the curvature map, i.e., there exists a circle pattern $r^* \in \mathbb{R}^N_{>0}$ with zero curvature. Let $r(t)$ be the unique solution to the combinatorial Calabi flow on a maximal time interval $[0, T)$, we need to prove $T = +\infty$ and $r(t) \to r^*$ exponentially fast.

Let $u^* \in \mathbb{R}^N_{sps0}$ be the u-coordinate of r^*. Consider the combinatorial Ricci potential

$$F(u) \triangleq \int_{u^*}^{u} \sum_{i=1}^{N} K_i du_i, \quad u \in \mathbb{R}^N_{sps0}. \tag{4.12}$$

By Lemma 2.5, we can see

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i},$$

the smooth differential 1-form $\sum_{i=1}^{N} K_i du_i$ is closed, and hence then (4.12) is well defined and is independent on the choice of piecewise smooth paths in $\mathbb{R}^N_{sps0}$ from u^* to u. By Lemma B.1 in [16], there holds

$$\lim_{\|u\|\to+\infty, u\in\mathbb{R}^N_{sps0}} F(u) = +\infty. \tag{4.13}$$

By Lemma 2.5 and a direct calculation, we have

$$\frac{d}{dt}F(u(t)) = \sum_i K_i \Delta K_i = K^T \Delta K \leq 0,$$

which implies that $F(u(t))$ is non-increasing along the Calabi flow. By (4.13), there is a positive constant δ, depending only on the triangulation $\mathcal{T}$ and the initial circle pattern $r(0)$. such that $u_i(t) \geq -\delta$ for all i and t. It follows that

$$r_i(t) \geq \ln \frac{1 + e^{-\delta}}{1 - e^{-\delta}} > 0 \tag{4.14}$$

for all i and t. By (4.14) and Theorem 4.4, we know that $r(t)$ lies in a compact subset of $\mathbb{R}^N_{>0}$. Then, by Lemma 4.1 in [15], $r(t)$ exists for all time and converges exponentially fast to r^*.

"$(H_5) \Rightarrow (H_6)$". Suppose the solution $r(t)$ to the combinatorial Calabi flow (1.7) converges as $t \to +\infty$. Let u^* be the corresponding u-coordinate of r^*, then $u(t)$ converges to u^*.

$$u_i(n+1) - u_i(n) = u_i'(\xi_n) = \Delta K_i(\xi_n) \to 0, \quad \text{as } n \to +\infty.$$

As $u(t) \to u^*$, we have $K(t) \to K(u^*)$. Thus $\Delta K_i(u^*) = 0$ for each vertex $i \in V$. From Lemma 2.5, we know the matrix Δ is negative definite, which implies $K_i(u^*) = 0$ for each $i \in V$ and then u^* has zero curvature. This also shows the image of the curvature map contains at least one point with non-positive coordinates.

"$(H_6) \Rightarrow (H_1)$". We deform the metric $r(t)$ according to Ricci flow (1.9), beginning from a initial metric $r(0)$ which is exactly the special metric with non-positive curvatures. Set $M(t) = \max\{K_1(t), \ldots, K_{|V|}(t), 0\}$, $m(t) = \min\{K_1(t), \ldots, K_{|V|}(t), 0\}$. Using the maximum principle, Chow-Luo (Corollary 3.3, [5]) proved that $M(t)$ is non-increasing while $m(t)$ is non-decreasing. So $M(t) \leq M(0) \leq 0$, and hence all $K_i(t) \leq 0$. Thus $\frac{dr_i}{dt} \geq 0$ and every $r_i(t)$ is increasing, which implies that all $r_i(t)$ are uniformly bounded below from a positive constant. By Corollary 3.6 in [5], all $r_i(t)$ are uniformly bounded from above. Thus the solution $\{r(t)\}$ lies in a compact region in $\mathbb{R}_N > 0$. Using Proposition 3.7 in [5], we get (H_1).

Moreover, from the step "$(H_4) \Rightarrow (H_5)$", we can see if one of the above properties holds, then the combinatorial Ricci/Calabi flow converges exponentially fast to a circle packing which produces a complete hyperbolic metric on M (with no cone points) □

5 Appendix 1: Proof of Lemma 2.1

Proof Assume the background geometry is Euclidean. Let $l_{ij} > 0$ such that

$$l_{ij}^2 = r_i^2 + r_j^2 + 2r_i r_j \cos\Phi_{ij}.$$

Define l_{ik}, l_{jk} similarly. The objective is to check that l_{ij}, l_{ik}, l_{jk} satisfy the triangle inequalities. Namely,

$$l_{ij} + l_{ik} > l_{jk},$$

and

$$|l_{ij} - l_{ik}| < l_{jk}.$$

Combining the above two relations, we have

$$(l_{ij}^2 + l_{ik}^2 - l_{jk}^2)^2 < 4l_{ij}^2 l_{ik}^2. \tag{5.1}$$

To simplify the notations, we set $I_{st} = \cos\Phi_{st}$ for $st = ij, jk, ik$. And taking I_{st} into (5.1), we need to prove that

$$r_i^2 r_j^2(1 - I_{ij}^2) + r_i^2 r_k^2(1 - I_{ik}^2) + r_j^2 r_k^2(1 - I_{jk}^2) + 2r_i^2 r_j r_k(I_{jk} + I_{ij} I_{ik}) \\ + 2r_i r_j^2 r_k(I_{ik} + I_{ij} I_{jk}) + 2r_i r_j r_k^2(I_{ij} + I_{ik} I_{jk}) > 0. \tag{5.2}$$

Now there are two cases to distinguish.

If $\Phi_{ij} + \Phi_{jk} + \Phi_{ik} \leq \pi$, then

$$\begin{aligned} I_{jk} + I_{ij} I_{ik} &= \cos \Phi_{jk} + \cos \Phi_{ij} \cos \Phi_{ik} \\ &= \cos \Phi_{jk} + \cos(\Phi_{ij} + \Phi_{ik}) + \sin \Phi_{ij} \sin \Phi_{ik} \\ &\geq \cos \Phi_{jk} + \cos(\pi - \Phi_{jk}) + \sin \Phi_{ij} \sin \Phi_{ik} \\ &\geq 0. \end{aligned}$$

Similarly, $I_{ik} + I_{ij} I_{jk} \geq 0$ and $I_{ij} + I_{ik} I_{jk} \geq 0$. Note that $1 - I_{ij}^2 \geq 0$, $1 - I_{ik}^2 \geq 0$, $1 - I_{jk}^2 \geq 0$. Thus we deduce (5.2), by these six inequalities can not obtain "=" at the same time.

If $\Phi_{ij} + \Phi_{jk} + \Phi_{ik} > \pi$. Considering that

$$\Phi_{ij} + \Phi_{jk} < \pi + \Phi_{ik}, \ \Phi_{ik} + \Phi_{jk} < \pi + \Phi_{ij}, \ \Phi_{ij} + \Phi_{ik} < \pi + \Phi_{jk},$$

there is a spherical triangle with inner angles $\Phi_{ij}, \ \Phi_{jk}, \ \Phi_{ik}$. Let $\phi_{ij}, \ \phi_{jk}, \ \phi_{ik}$ denote the corresponding lengths of the three sides. By the second cosine law of spherical triangles,

$$\cos \phi_{jk} = \frac{\cos \Phi_{jk} + \cos \Phi_{ij} \cos \Phi_{ik}}{\sin \Phi_{ij} \sin \Phi_{ik}}.$$

So

$$I_{jk} + I_{ij} I_{ik} = \cos \Phi_{jk} + \cos \Phi_{ij} \cos \Phi_{ik} = \cos \phi_{jk} \sin \Phi_{ij} \sin \Phi_{ik},$$

and

$$I_{ik} + I_{ij} I_{jk} = \cos \phi_{ik} \sin \Phi_{ij} \sin \Phi_{jk}, \ I_{ij} + I_{ik} I_{jk} = \cos \phi_{ij} \sin \Phi_{ik} \sin \Phi_{jk}.$$

Set $y_{st} = r_s r_t \sin \Phi_{st}$ for $st = ij, \ jk, \ ik$. Hence (5.2) is equivalent to

$$y_{ij}^2 + y_{jk}^2 + y_{ik}^2 + 2y_{ij} y_{jk} \cos \phi_{ik} + 2y_{ij} y_{ik} \cos \phi_{jk} + 2y_{ik} y_{jk} \cos \phi_{ij} > 0.$$

By the cosine law of spherical triangles, we obtain

$$\cos \phi_{ij} - \cos \phi_{ik} \cos \phi_{jk} = \cos \Phi_{ij} \sin \phi_{ik} \sin \phi_{jk}.$$

It follows that

$$\begin{aligned} & y_{ij}^2 + y_{jk}^2 + y_{ik}^2 + 2y_{ij} y_{jk} \cos \phi_{ik} + 2y_{ij} y_{ik} \cos \phi_{jk} + 2y_{ik} y_{jk} \cos \phi_{ij} \\ &= (y_{ij} + \cos \phi_{ik} y_{jk} + \cos \phi_{jk} y_{ik})^2 + y_{jk}^2 \sin^2 \phi_{ik} + y_{ik}^2 \sin^2 \phi_{jk} \\ &+ 2y_{jk} y_{ik}(\cos \phi_{ij} - \cos \phi_{ik} \cos \phi_{jk}) \end{aligned}$$

$$
\begin{aligned}
&\geq y_{jk}^2 \sin^2 \phi_{ik} + y_{ik}^2 \sin^2 \phi_{jk} + 2 y_{jk} y_{ik} (\cos \phi_{ij} - \cos \phi_{ik} \cos \phi_{jk}) \\
&= (y_{jk} \sin \phi_{ik} + y_{ik} \sin \phi_{jk} \cos \Phi_{ij})^2 + y_{ik}^2 \sin^2 \phi_{jk} \sin^2 \Phi_{ij} \\
&\geq y_{ik}^2 \sin^2 \phi_{jk} \sin^2 \Phi_{ij} = r_i^2 r_j^2 \sin^2 \phi_{jk} \sin^2 \Phi_{ij} \sin \Phi_{ik} \\
&> 0.
\end{aligned}
$$

Thus the lemma is proved. □

Remark 4 We refer a more geometric formulation of Lemma 2.1, see Ge-Jiang-Liu [7].

6 Appendix 2: Thurston's Existence Theorem

Following Thurston's formulation of Andreev's theorem and Marden-Rodin's original methods [1], we give the image of the curvature map $K = K(r)$. For Euclidean background geometry, the image of $K(r)$ were already obtained by Ge-Jiang in [11–13]. For hyperbolic background geometry, we give a complete proof here for reader's convenience. Firstly, by Lemma 2.5 in [36] and Lemma 2.2, we have

Lemma 6.1 *Let Φ_{ij}, Φ_{jk}, $\Phi_{ik} \in [0, \pi)$ be three intersection angles satisfying* **(Z)***, then we have*

$$
\begin{aligned}
&\lim_{r_i \to \infty} \theta_i^{jk} = 0, \\
&\lim_{(r_i, r_j, r_k) \to (0, a, b)} \theta_i^{jk} = \pi - \Phi_{jk}, \\
&\lim_{(r_i, r_j, r_k) \to (0, 0, c)} (\theta_i^{jk} + \theta_j^{ik}) = \pi, \\
&\lim_{(r_i, r_j, r_k) \to (0, 0, 0)} (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) = \pi.
\end{aligned}
$$

We use the above lemma to give the proof of Theorem 1.1.

Proof of Theorem 1.1 in hyperbolic background geometry. We can see that $K(u)$ is injective as a function of u. Obviously, $K(r)$ is also injective as a function of r. We next prove that the image of the curvature map $K(r)$, $r \in \mathbb{R}_{>0}^{|V|}$ is

$$
\mathcal{Z} = \bigcap_{A \subset V} \left\{ K \in (-\infty, 2\pi)^{|V|} : \sum_{i \in A} K_i > - \sum_{(e,v) \in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi \chi(F_A) \right\}.
$$

Obviously, all $K_i < 2\pi$ by definition. For each vertex subset $A \subset V$, we consider all the triangles in F having a vertex in A. These triangles can be classified into three types A_1, A_2 and A_3. For each $i \in \{1, 2, 3\}$, a triangle is in A_i if and only if it has exactly i vertices in A. Since $\frac{\partial \theta_i^{jk}}{\partial r_i} < 0$ and the second limited of Lemma 6.1, we have

$\theta_i^{jk} < \pi - \Phi_{jk}$. Noting that $\theta_i^{jk} + \theta_j^{ik} < \theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij} < \pi$, it follows

$$\begin{aligned}\sum_{i\in A} K_i =&2\pi|A| - \sum_{i\in A}\sum_{\{ijk\}\in F}\theta_i^{jk}\\ =&2\pi|A| - \left(\sum_{i\in A,\{ijk\}\in A_1}\theta_i^{jk} + \sum_{i,j\in A,\{ijk\}\in A_2}(\theta_i^{jk}+\theta_j^{ik})\right.\\ &\left.+\sum_{\{ijk\}\in A_3}(\theta_i^{jk}+\theta_j^{ik}+\theta_k^{ij})\right)\\ >&2\pi|A| - \sum_{(e,v)\in Lk(A)}\big(\pi-\Phi(e)\big) - \pi|A_2| - \pi|A_3|\\ =&-\sum_{(e,v)\in Lk(A)}\big(\pi-\Phi(e)\big) + 2\pi\chi(F_A).\end{aligned}$$

Then it follows that $K(\mathbb{R}^{|V|}_{>0})$, the image of the curvature map, is contained in $\mathcal{Z}$. If we can further prove $K(r)$ is proper map (that is, the preimage of every compact set in $\mathcal{Z}$ is compact in $\mathbb{R}^{|V|}_{>0}$), then by the invariance of domain theorem, K is a diffeomorphism from $\mathbb{R}^{|V|}_{>0}$ to $\mathcal{Z}$. We just need to prove, if there is a sequence $r^{(n)}$ tends to the boundary of $\mathbb{R}^{|V|}_{>0}$, then $K(r^{(n)})$ contains a subsequence that tends to the boundary of $\mathcal{Z}$. To see this, assume $r^{(n)}$ tends to the boundary of $\mathbb{R}^{|V|}_{>0}$, then there is a subsequence, which is still denoted as $r^{(n)}$ itself, there is a vertex subset $A \subset V$, so that $r_i^{(n)} \to 0$ for each $i \in A$, while $r_j^{(n)} \to c_j \in (0, +\infty]$ for each $j \in V - A$.

In case $A = \emptyset$, which means that $r_i^{(n)} \to +\infty$ for all $i \in V$, all $\theta_i^{jk}(r^{(n)}) \to 0$ by the first limit in Lemma 6.1, hence all curvatures $K_i(r^{(n)}) \to 2\pi$. This implies that $K(r^{(n)})$ tends to the boundary of $\mathcal{Z}$.

In case $A \neq \emptyset$ and $A \neq V$, by Lemma 6.1, for $i \in A$ and $\{ijk\} \in A_1$, we have

$$\theta_i^{jk}(r^{(n)}) \to \pi - \Phi_{jk},$$

for $i, j \in A$ and $\{ijk\} \in A_2$, we have

$$(\theta_i^{jk} + \theta_j^{ik})(r^{(n)}) \to \pi,$$

while for $\{ijk\} \in A_3$, we have

$$(\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij})(r^{(n)}) \to \pi.$$

Then it follows

$$\sum_{i\in A} K_i =2\pi|A| - \left(\sum_{i\in A,\{ijk\}\in A_1}\theta_i^{jk} + \sum_{i,j\in A,\{ijk\}\in A_2}(\theta_i^{jk}+\theta_j^{ik})\right.$$

$$
\begin{aligned}
&+ \sum_{\{ijk\}\in A_3} (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij})\Bigg) \\
&\to 2\pi|A| - \sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) - \pi|A_2| - \pi|A_3| \\
&= - \sum_{(e,v)\in Lk(A)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_A).
\end{aligned}
$$

This implies that $K(r^{(n)})$ tends to the boundary of $\mathcal{Z}$.

In case $A = V$, by definition $A_1,\ A_2$, we have $A_1 = \emptyset,\ A_2 = \emptyset$, and

$$
\begin{aligned}
\sum_{i\in V} K_i(r^{(n)}) &= 2\pi|V| - \sum_{\{ijk\}\in A_3} (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) \\
&\to 2\pi|V| - \pi|F| = 2\pi\chi(F_V)
\end{aligned}
\tag{6.1}
$$

by the fourth limited of Lemma 6.1. This implies that $K(r^{(n)})$ tends to the boundary of $\mathcal{Z}$. We finish the proof. □

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Data availability As this paper focuses primarily on theoretical derivations and proofs, it does not involve the collection or use of experimental data. Therefore, there are no direct datasets available for public access or sharing. However, all theoretical derivations and proofs presented in this paper are based on widely accepted mathematical principles, and all referenced literature and equations are clearly indicated in the text. Readers can refer to these literature and equations to verify and reproduce the theoretical derivations and proofs in this paper.

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