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A General Schwarz Lemma for Strongly Pseudoconvex Complex Finsler Manifolds

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Abstract
In this paper, we generalize a Schwarz lemma to strongly pseudoconvex complex Finsler manifolds and prove a Schwarz lemma between two strongly pseudoconvex complex Finsler manifolds. As an application, we give a rigidity result.

Keywords Schwarz lemma · Complex Finsler manifold · Holomorphic sectional curvature · Holomorphic bisectional curvature · Flag curvature

Mathematics Subject Classification 53C60 · 53C56 · 32H02

1 Introduction and Main Results

It is well known that the classical Schwarz lemma plays an important role in proving the Riemannian mapping theorem and Liouville theorem in complex analysis ([7, 36, 38]). In 1915, Pick [36] re-interpreted this lemma in terms of the Poincaré metric and distance, now known as the Schwarz-Pick lemma which states that any holomorphic function from a unit disk into itself decreases the Poincaré metric and distance. It establishes bridges the differential geometric ideas with the Schwarz lemma. Generalizations of the classical Schwarz lemma to higher dimensional spaces began in

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1921, with results for domains in $\mathbb{C}^2$ by Reinhardt and were followed some years later by Carathódory [8] and Cartan [9]. In 1938, Ahlfors [3] extended the Schwarz-Pick lemma to the Riemannian surface from the viewpoint of differential geometry, which states that any holomorphic map from a unit disk into a Riemannian surface equipped with a Hermitian metric with the Gauss curvature bounded from above by -4 decreases the Poincaré metric of the unit disk, and applied it to give an elementary new proof of the Bloch theorem with an explicit lower bound for Bloch's constant B, namely $B \geq \frac{\sqrt{3}}{4}$ ([3]). Later, there are various generalizations and their applications of the classical Schwarz lemma and Schwarz-Pick lemma from the viewpoints of both function theory and differential geometry (Bochner and Martin [6], Korányi [24], Kobayashi [21–23], Chern [12], Lu [29], Lu [27, 28], Yau [49], Greene and Wu [17], Royden [37], Chen, Cheng and Lu [11], Dineen [15], Siu and Yeung [41, 42], Hidetaka and Takashi [18], Takashi [43], Osserman [35], Kim and Lee [19], Mateljević [30], Yang and Chen [48], Tosatti [44], Zuo [53], Ni [31, 32] etc.).

In 1978, Yau [49] used the almost maximum principle to generalize the Schwarz lemma to a complete Kähler manifold and obtained the following Schwarz lemma. The Schwarz lemma has become a powerful tool in complex geometric analysis since then.

Theorem 1.1 *([49]) Let M be a complete Kähler manifold with Ricci curvature bounded from below by K_1. Let N be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant K_2. Then if there is a non-constant holomorphic mapping f from M into N, we have $K_1 \leq 0$ and*

$$f^* dS_N^2 \leq \frac{K_1}{K_2} dS_M^2.$$

In 1980, by using the special type of exhaustion functions of the manifolds, which is different from the method of Yau, Royden [37] gave the following Schwarz lemma for a Hermitian manifold.

For a Hermitian manifold (M, g), assume that M satisfies the following condition (C).

(C) There exists a continuous proper non-negative function u on (M, g) with the property that at each point p it has a smooth upper supporting function w with $||\nabla w|| \leq 1$ and $w_{\alpha\bar{\beta}} \leq g_{\alpha\bar{\beta}}$ at p.

Theorem 1.2 *([37]) Let (M, g) and (N, h) be two Hermitian manifolds, and holomorphic sectional curvature of M bounded from below by a constant $K_1 \leq 0$ and the holomorphic sectional curvature of N bounded from above by a constant $K_2 < 0$. Assume that M satisfies condition (C). Then any holomorphic map $f : M \to N$ satisfies*

$$f^* h \leq \frac{K_1}{K_2} g.$$

Remark 1.1 It follows from Propositions 2 and 3 in [37] by Royden that M satisfies condition (C) if it is a complete Hermitian manifold with the Riemannian sectional curvature bounded from below or it is a complete Kähler manifold with the holomorphic bisectional curvature bounded from below.

Chern [13] pointed out that Finsler geometry is just Riemannian geometry without the quadratic restriction, and that it is possible that Finsler geometry will be most useful in the complex domain, because every complex manifold, with or without boundary, has a Carathéodory pseudo-metric ([8]) and a Kobayashi pseudo-metric ([20]). Under favorable (though somewhat stringent) conditions they are C^2 metrics, and most importantly, they are naturally Finsler. He also pointed out that complex Finsler geometry is extremely beautiful ([13]). Lempert [25] proved that any bounded strongly convex domain $D \subset\subset C^n$ with smooth boundary, the Carathéodory metric and Kobayashi metric coincide, furthermore, they are weakly Kähler-Finsler metrics with constant holomorphic sectional curvature -4. Therefore, a natural and interesting equation is to generalize the Schwarz lemma in complex Finsler setting.

In 2013, Shen and Shen [39] generalized the Schwarz lemma to compact complex Finsler manifolds. In 2019, Wan [45] gave the Schwarz lemma from a complete Riemannian surface into a complex Finsler manifold. In 2022, Nie and Zhong [33, 34] generalized the Schwarz lemma for strongly convex weakly Kähler Finsler manifolds. However, the Schwarz lemma for general strongly pseudoconvex complex Finsler manifolds is still open.

In this paper, by generalizing Royden's method to find a proper function and its upper supporting function, we obtain the Schwarz lemma for strongly pseudoconvex complex Finsler manifolds.

For a strongly pseudoconvex complex Finsler manifold (M, G), we assume that M satisfies the following condition (A).

(A) There exists a continuous proper non-negative function u on (M, G) with the property that at each point p it has a smooth upper supporting function w with $|\partial w(\xi)|^2 \leq 1$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 w}{\partial z^\alpha \partial \overline{z^\beta}} \leq 1$ for any unit vector $\xi \in T_p^{1,0}M$.

Remark 1.2 For a given complex Finsler metric G on M in condition (A), $|\partial w(\xi)|^2 \leq G(\xi)$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 w}{\partial z^\alpha \partial \overline{z^\beta}} \leq G(\xi)$ always hold for any vector $\xi \in T_p^{1,0}M$. When $G(v) = g_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta$ comes from a Hermitian metric, condition (A) is equivalent to condition (C). Hence, condition (A) is a generalization of condition (C) in Royden [37].

We first give some strongly pseudoconvex complex Finsler manifolds satisfying condition (A).

Theorem 1.3 *Let (M, G) be a complete strongly convex uniform complex Finsler manifold, the first holomorphic bisectional curvature bounded from below by $-k^2 (k > 0)$, the mixed part of the $(1, 1)$-torsion bounded from above by $\tau (\tau > 0)$, and the horizontal part of the $(2, 0)$-torsion bounded from above by $\theta (\theta > 0)$. Then M satisfies condition (A).*

In particular, if M is a Kähler-Finsler manifold, then the horizontal part of $(2, 0)$-torsion term vanishes.

Corollary 1.1 *Let (M, G) be a complete strongly convex uniform Kähler-Finsler manifold, the first holomorphic bisectional curvature bounded from below by $-k^2(k > 0)$, and the mixed part of the $(1, 1)$-torsion bounded from above by $\tau(\tau > 0)$. Then M satisfies condition (A).*

For a Kähler-Finsler manifold M, we use the horizontal flag curvature instead of bisectional curvature.

Theorem 1.4 *Let (M, G) be a complete strongly convex uniform Kähler-Finsler manifold, the horizontal flag curvature bounded from below by $-k^2(k > 0)$. Then M satisfies condition (A).*

Remark 1.3 From Theorems 1.3, 1.4, Corollary 1.1 and the results in [34] by Nie and Zhong, it follows that M satisfies condition (A) if it is a complete strongly convex uniform complex Finsler manifold with the first holomorphic bisectional curvature bounded from below, the horizontal part of (2, 0)-torsion bounded from above and the mixed part of the (1, 1)-torsion bounded from above; or if it is a complete strongly convex uniform Kähler-Finsler manifold with the first holomorphic bisectional curvature bounded from below, and the mixed part of the (1, 1)-torsion bounded from above; or if it is a complete strongly convex uniform Kähler-Finsler manifold with the horizontal flag curvature bounded from below; or if is a complete strongly convex uniform weakly Kähler-Finsler manifold with the flag curvature bounded from below.

Now we give the following main theorem in this paper.

Theorem 1.5 *Let (M, G) be a strongly pseudoconvex complex Finsler manifold with the holomorphic sectional curvature bounded from below by a constant $K_1 \leq 0$, and let (N, H) be another strongly pseudoconvex complex Finsler manifold with the holomorphic sectional curvature bounded from above by a constant $K_2 < 0$. Suppose that M satisfies condition (A), then for any holomorphic map $f : M \to N$, we have*

$$f^*H \leq \frac{K_1}{K_2} G.$$

A rigidity result can be obtained directly from Theorem 1.5.

Corollary 1.2 *Let (M, G) be a strongly pseudoconvex complex Finsler manifold with non-negative holomorphic sectional curvature, and let (N, H) be another strongly pseudoconvex complex Finsler manifold with negative holomorphic sectional curvature. Suppose that M satisfies condition (A), then for any holomorphic map $f : M \to N$ is a constant.*

2 Preliminaries

2.1 Real Finsler Geometry

Definition 2.1 [1] A real Finsler metric on a manifold M is a continuous function $G : TM \to [0, +\infty)$ with the following properties:

(i) G is smooth on $\tilde{M} = TM\backslash\{0\}$;
(ii) $G(u) > 0$ for all $u \in \tilde{M}$;
(iii) $G(\lambda u) = |\lambda|^2 G(u)$ for all $u \in TM$ and $\lambda \in \mathbb{R}$;
(iv) The fundamental tensor g, defined locally by its components

$$g_{ab} := \frac{1}{2}G_{ab} = \frac{1}{2}\frac{\partial^2 G}{\partial u^a \partial u^b},$$

is positive definite.

The pair (M, G) is called a real Finsler manifold. A real Finsler metric G comes from a Riemannian metric iff it is smooth on the whole tangent bundle TM.

Let $\pi : \tilde{M} \to M$ be the natural projective map, and denote the vertical bundle $\mathcal{V} = \ker d\pi$. We can introduce a Riemannian structure $\langle\cdot|\cdot\rangle$ on $\mathcal{V}$ by setting

$$\forall V, W \in \mathcal{V}_u, \langle V|W\rangle_u = g_{ij}(u)V^i W^j.$$

A local frame of $\mathcal{V}$ is $\{\dot{\partial}_1, \ldots, \dot{\partial}_n\}$, where $\dot{\partial}_a = \frac{\partial}{\partial u^a}$. One can define the Cartan connection $D : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*\tilde{M} \otimes \mathcal{V})$ that is compatible with $\langle\cdot|\cdot\rangle$ on $\mathcal{V}$. The connection form is given by

$$\omega_b^a = \Gamma_{b;i}^a dx^i + \Gamma_{bc}^a \psi^c,$$

where

$$\Gamma_{b;i}^a = \frac{1}{2}G^{ac}\left[\delta_i(G_{cb}) + \delta_b(G_{ci}) - \delta_c(G_{bi})\right], \Gamma_{bc}^a = \frac{1}{2}G^{ak}G_{bck},$$

and $\delta_i = \partial_i - \Gamma_i^b \dot{\partial}_b$, $\Gamma_i^b = \Gamma_{k;i}^b u^k$. $\{\delta_1, \ldots \delta_n\}$ forms a local frame of the horizontal bundle $\mathcal{H}$, and $\{dx^i, \psi^a = du^a + \Gamma_i^a dx^i\}$ is the dual frame of $T\tilde{M}$ with respect to $\{\delta_i, \dot{\partial}_a\}$. By defining the horizontal map $\Theta : \mathcal{V} \to \mathcal{H}$ corresponding to $\mathcal{H}$, locally given by $\Theta(\dot{\partial}_i) = \delta_i$, we can induce a Riemannian structure on $\mathcal{H}$ by setting

$$\forall H, K \in \mathcal{H}, \langle H|K\rangle = \left\langle \Theta^{-1}(H)|\Theta^{-1}(K)\right\rangle,$$

and define a linear connection on $\mathcal{H}$ (still denoted by D) by setting

$$\forall H \in \mathcal{X}(\mathcal{H}), DH = \Theta(D\Theta^{-1}(H)).$$

Hence we obtain a good linear connection on $T\tilde{M}$, still called Cartan connection, that is, compatible with a Riemannian structure $\langle\cdot|\cdot\rangle$ on $T\tilde{M}$,

$$\forall X, Y, Z \in T\tilde{M}, X\langle Y|Z\rangle = \langle D_X Y|Z\rangle + \langle Y|D_X Z\rangle.$$

Let $\nabla : \mathcal{X}(T\tilde{M}) \times \mathcal{X}(T\tilde{M}) \to \mathcal{X}(T\tilde{M})$ be a covariant differentiation associated to the Cartan connection D. The torsion θ and the curvature R are given by

$$\theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R_Z(X,Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathcal{X}(T\tilde{M})$. Let Ω be the curvature operator of D, we have $\Omega(X,Y)Z = R_Z(X,Y)$. The horizontal flag curvature in real Finsler geometry is the extension of the sectional curvature in Riemannian geometry. For a fixed point $x \in M$, taking a pair $(P; u)$, where $u \in T_x M$ and $P \subset T_x M$ is a two-plane such that $u \in P$, we call it a flag at x and define the flag curvature $K^F(P,u)$ by

$$\begin{aligned} K^F(P,u) = K^F(u,X) &= \frac{\left\langle \Omega(X^H, \chi(u))\chi(u) | X^H \right\rangle_u}{\langle \chi(u)|\chi(u)\rangle_u \left\langle X^H | X^H \right\rangle_u - \left\langle \chi(u) | X^H \right\rangle_u^2} \\ &= \frac{\left\langle \Omega(\chi(u), X^H) X^H | \chi(u) \right\rangle_u}{\langle \chi(u)|\chi(u)\rangle_u \left\langle X^H | X^H \right\rangle_u - \left\langle \chi(u) | X^H \right\rangle_u^2}, \end{aligned}$$

where $P = \mathrm{span}\{u, X\}$ and X^H is the horizontal lifting of X.

2.2 Complex Finsler Geometry

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, and let $\{z^1, \ldots, z^n\}$ be the local complex coordinates, with $z^\alpha = x^\alpha + i x^{n+\alpha}$, such that $\{x^1, \ldots, x^{2n}\}$ is the local real coordinates. Let $T_{\mathbb{R}} M$ be the real tangent bundle of M, which is a real bundle of rank $2n$ equipped with a complex structure J, and let $T_{\mathbb{C}} M = T_{\mathbb{R}} M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle. Set

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial x^{\alpha+n}}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z^\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial x^{\alpha+n}}\right).$$

Then $\left\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z^1}}, \ldots, \frac{\partial}{\partial \overline{z^n}}\right\}$ is a local frame of $T_{\mathbb{C}} M$. $T_{\mathbb{C}} M$ splits as the sum of two eigenbundles

$$T_{\mathbb{C}} M = T^{1,0} M \oplus T^{0,1} M,$$

where $T^{1,0}M = \{v \in T_{\mathbb{C}} M | Jv = iv\}$ and $T^{0,1}M = \{v \in T_{\mathbb{C}} M | Jv = -iv\}$. The local frames of $T^{1,0}M$ and $T^{0,1}M$ are $\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\}$ and $\{\frac{\partial}{\partial \overline{z^1}}, \ldots, \frac{\partial}{\partial \overline{z^n}}\}$, respectively. $T^{1,0}M$ is called the holomorphic tangent bundle of M.

Since the holomorphic tangent bundle $T^{1,0}M$ is isomorphic to $T_{\mathbb{R}} M$, we can take a bundle isomorphism $^\circ : T^{1,0}M \to T_{\mathbb{R}} M$ by

$$\forall v \in T^{1,0}M, \quad v^\circ = v + \bar{v}.$$

It preserves the complex structure J, that is $Jv^\circ = (Jv)^\circ$. And we denote the inverse $_\circ : T_{\mathbb{R}}M \to T^{1,0}$ by

$$\forall u \in T_{\mathbb{R}}M, \quad u_\circ = \frac{1}{2}(u - iJu).$$

From the definition, we know that if

$$v = v^\alpha \frac{\partial}{\partial z^\alpha},$$

then

$$u = v^\circ = u^a \frac{\partial}{\partial x^a},$$

where $v^\alpha = u^\alpha + iu^{\alpha+n}$, and the Roman indices run from 1 to $2n$, while the Greek indices run from 1 to n. Conversely, $u_\circ = (u^\alpha + iu^{\alpha+n})\frac{\partial}{\partial z^\alpha}$. In particular, we denote $\tilde{M}$ either $T^{1,0}M$ or $T_{\mathbb{R}}M$ minus the zero section. The local coordinates of $T_{\mathbb{R}}M$ and $T^{1,0}M$ are $\{x^a, u^a\}$ and $\{z^\alpha, v^\alpha\}$, respectively.

Definition 2.2 ([1]) A complex Finsler metric on a complex manifold M is a continuous function $G : T^{1,0}M \to [0, +\infty)$ with the following properties:

(i) G is smooth on $\tilde{M} = T^{1,0}M \backslash \{0\}$;
(ii) $G(v) > 0$ for all $v \in \tilde{M}$;
(iii) $G(\zeta v) = |\zeta|^2 G(v)$ for all $v \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

The pair (M, G) is called a complex Finsler manifold.

Definition 2.3 ([1]) A complex Finsler manifold (M, G) is called strongly pseudoconvex if the Levi matrix

$$\left(G_{\alpha\bar{\beta}}\right) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right)$$

is positive definite on $\tilde{M}$.

This is equivalent to requiring that the indicatrix $I_G(p) = \{v \in T_p^{1,0}M | G(v) < 1\}$ is strongly pseudoconvex. Note that it is important to ask for the smoothness of G only on $\tilde{M}$, and if G is smooth on the whole of $T^{1,0}M$, we shall say that F comes from a Hermitian metric.

Different from the Hermitian case, a complex Finsler metric is not necessary a real Finsler metric, even if it is strongly pseudoconvex.

Definition 2.4 ([1]) A complex Finsler metric G is called strongly convex if G° is a real Finsler metric, where $G^\circ(u) = G(u_\circ)$ for $u \in T_{\mathbb{R}}M$ and $u_\circ = \frac{1}{2}(u - iJu) \in T^{1,0}M$.

Let $\pi : \tilde{M} \to M$ be the natural projective map. The differential map $d\pi : T_{\mathbb{C}}\tilde{M} \to T_{\mathbb{C}}M$ defines the (complex) vertical bundle $\mathcal{V}$ over $T^{1,0}\tilde{M}$ by

$$\mathcal{V} = \ker d\pi \cap T^{1,0}\tilde{M}.$$

$T^{1,0}\tilde{M}$ splits as $T^{1,0}\tilde{M} = \mathcal{V} \oplus \mathcal{H}$, here $\mathcal{H}$ is the horizontal bundle. A local frame of $\mathcal{V}$ is given by $\{\dot{\partial}_1, \dots, \dot{\partial}_n\}$ and a local frame of $\mathcal{H}$ is given by $\{\dot{\delta}_1, \dots, \dot{\delta}_n\}$, where

$$\delta_\alpha = \partial_\alpha - \Gamma^\beta_{;\alpha}\dot{\partial}_\beta, \quad \Gamma^\beta_{;\alpha} = G^{\beta\bar{\tau}}G_{\bar{\tau};\alpha}.$$

Clearly $\{\delta_\mu, \dot{\partial}_\alpha\}$ gives a local frame field of $T^{1,0}\tilde{M}$. We denote the dual frame by $\{dz^\mu, \delta v^\alpha\}$.

Let (M, G) be a strongly pseudoconvex complex Finsler manifold. Then G defines a Hermitian metric $\langle\ ,\ \rangle$ on the vertical bundle $\mathcal{V}$. Indeed, if $v \in \tilde{M}$ and $Z, W \in \mathcal{V}_v$ with $Z = Z^\alpha\dot{\partial}_\alpha$ and $W = W^\beta\dot{\partial}_\beta$, we set

$$\langle Z, W\rangle_v = G_{\alpha\bar{\beta}}(v)Z^\alpha\overline{W^\beta}.$$

There exists a unique complex vertical connection $D : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes \mathcal{V})$ such that

$$X\langle Z, W\rangle = \langle\nabla_X Z, W\rangle + \langle X, \nabla_{\overline{X}}W\rangle$$

for all $X \in T^{1,0}\tilde{M}$ and $Z, W \in \mathcal{X}(\mathcal{V})$. Furthermore, this connection is good. The unique good complex vertical connection D is called Chern-Finsler connection. Its connection (1,0)-forms are given by

$$\omega^\alpha_\beta = G^{\bar{\tau}\alpha}\partial G_{\beta\bar{\tau}} = \Gamma^\alpha_{\beta;\mu}dz^\mu + \Gamma^\alpha_{\beta\gamma}\delta v^\gamma, \tag{2.1}$$

where

$$\Gamma^\alpha_{\beta;\mu} = G^{\bar{\tau}\alpha}\delta_\mu(G_{\beta\bar{\tau}}), \quad \Gamma^\alpha_{\beta\gamma} = G^{\bar{\tau}\alpha}G_{\beta\bar{\tau}\gamma}, \tag{2.2}$$

and $(G^{\bar{\beta}\alpha}) = (G_{\alpha\bar{\beta}})^{-1}$. In particular,

$$\Gamma^\alpha_{;\mu} = \Gamma^\alpha_{\beta;\mu}v^\beta = G^{\bar{\tau}\alpha}G_{\bar{\tau};\mu}. \tag{2.3}$$

Definition 2.5 ([1]) A complex Finsler metric G is called strongly Kähler if $\Gamma^\alpha_{\beta;\mu} = \Gamma^\alpha_{\mu;\beta}$; called Kähler if $(\Gamma^\alpha_{\beta;\mu} - \Gamma^\alpha_{\mu;\beta})v^\beta = 0$; called weakly Kähler if $G_\alpha(\Gamma^\alpha_{\beta;\mu} - \Gamma^\alpha_{\mu;\beta})v^\beta = 0$.

By Chen-Shen's observation in [10], a Kähler-Finsler metric is actually strongly Kähler. Then G is Kähler-Finsler metric if and only if $\Gamma^\alpha_{\beta;\mu} = \Gamma^\alpha_{\mu;\beta}$, if and only if

$$\delta_\mu(G_{\beta\bar{\gamma}}) = \delta_\beta(G_{\mu\bar{\gamma}}) \left(\text{or } \delta_{\bar{\mu}}(G_{\gamma\bar{\beta}}) = \delta_{\bar{\beta}}(G_{\gamma\bar{\mu}})\right), \quad 1 \le \gamma \le n.$$

Remark 2.1 There are lots of strongly convex Kähler-Finsler metrics which are not Hermitian quadratic ([46]); there are also weakly Kähler-Finsler metrics which are not Kähler-Finsler metrics ([14]); there are also holomorphic invariant Kähler-Finsler metrics on a polydisk in $\mathbb{C}^n (n \geq 2)$ which are non-Hermitian quadratic ([52]).

The curvature tensor Ω of the Chern-Finsler connection is given by $\Omega^\alpha_\beta = \bar{\partial}\omega^\alpha_\beta$. In local coordinates, it can be decomposed as

$$\Omega^\alpha_\beta = R^\alpha_{\beta;\mu\bar{\nu}} dz^\mu \wedge d\overline{z^\nu} + R^\alpha_{\beta\delta;\bar{\nu}} \delta v^\delta \wedge d\overline{z^\nu} + R^\alpha_{\beta\bar{\gamma};\mu} dz^\mu \wedge \delta\overline{v^\gamma} + R^\alpha_{\beta\delta\bar{\gamma}} \delta v^\delta \wedge \delta\overline{v^\gamma}, \tag{2.4}$$

where

$$\begin{aligned} R^\alpha_{\beta;\mu\bar{\nu}} &= -\delta_{\bar{\nu}}(\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta\sigma}\delta_{\bar{\nu}}(\Gamma^\sigma_{;\mu}), \quad R^\alpha_{\beta\delta;\bar{\nu}} = -\delta_{\bar{\nu}}(\Gamma^\alpha_{\beta\delta}), \\ R^\alpha_{\beta\bar{\gamma};\mu} &= -\dot{\partial}_{\bar{\gamma}}(\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta\sigma}\dot{\partial}_{\bar{\gamma}}(\Gamma^\sigma_{;\mu}), \quad R^\alpha_{\beta\delta\bar{\gamma}} = -\dot{\partial}_{\bar{\gamma}}(\Gamma^\alpha_{\beta\delta}). \end{aligned} \tag{2.5}$$

For a strongly pseudoconvex complex Finsler metric G, one can define a complex Rund connection ∇^R on $\tilde{M}$, with its connection form $\tilde{\omega}^\alpha_\beta$ given by

$$\tilde{\omega}^\alpha_\beta = \Gamma^\alpha_{\beta;\mu} dz^\mu,$$

where $\Gamma^\alpha_{\beta;\mu} = G^{\bar{\tau}\alpha}\delta_\mu G_{\beta\bar{\tau}}$. The $(2,0)$-torsion θ and $(1,1)$-torsion τ for the Rund connection are given by ([4, 5, 51])

$$\theta = \theta^\sigma \otimes \delta_\sigma \quad \text{and} \quad \tau = \tau^\alpha \otimes \dot{\partial}_\alpha, \tag{2.6}$$

, respectively, where $\theta^\sigma = \frac{1}{2}[\Gamma^\sigma_{\nu;\mu} - \Gamma^\sigma_{\mu;\nu}]dz^\mu \wedge dz^\nu$, and $\tau^\alpha = -\delta_{\bar{\nu}}(\Gamma^\alpha_{;\mu})dz^\mu \wedge d\bar{z}^\nu - \dot{\partial}_{\bar{\beta}}(\Gamma^\alpha_{;\mu})dz^\mu \wedge \delta\bar{v}^\beta$. Moreover, we denote by $\tau_{\mathcal{H}} = -\delta_{\bar{\nu}}(\Gamma^\alpha_{;\mu})dz^\mu \wedge d\bar{z}^\nu \otimes \dot{\partial}_\alpha$ and $\tau_{\mathcal{M}} = \tau - \tau_{\mathcal{H}}$ the horizontal part and mixed part of τ, respectively.

The curvature operator $\widetilde{\Omega} = \widetilde{\Omega}^\alpha_\beta \delta v^\beta \otimes \dot{\partial}_\alpha$ associated to the Rund connection satisfies ([47])

$$\widetilde{\Omega}^\alpha_\beta = \widetilde{R}^\alpha_{\beta;\mu\bar{\nu}} dz^\mu \wedge d\overline{z^\nu} + \widetilde{P}^\alpha_{\beta\bar{\gamma};\mu} dz^\mu \wedge \delta\overline{v^\gamma} + \widetilde{S}^\alpha_{\beta\gamma;\mu} dz^\mu \wedge \delta v^\gamma,$$

where

$$\widetilde{R}^\alpha_{\beta;\mu\bar{\nu}} = -\delta_{\bar{\nu}}(\Gamma^\alpha_{\beta;\mu}), \quad \widetilde{P}^\alpha_{\beta\bar{\gamma};\mu} = -\dot{\partial}_{\bar{\gamma}}(\Gamma^\alpha_{\beta;\mu}), \quad \widetilde{S}^\alpha_{\beta\gamma;\mu} = -\dot{\partial}_\gamma(\Gamma^\alpha_{\beta;\mu}). \tag{2.7}$$

Definition 2.6 ([1]) The (horizontal) holomorphic sectional curvature of a strongly pseudoconvex complex Finsler metric G along v is given by

$$K_G(v) = \frac{2}{G(v)^2}\langle \Omega(\chi, \overline{\chi})\chi, \chi\rangle_v,$$

here $v \in T^{1,0}M \backslash \{0\}$ and $\chi : T^{1,0}M \to \mathcal{H}$ is the horizontal lifting.

The holomorphic sectional curvature is indeed independent of the length of v, for any $\zeta \in \mathbb{C}^*$, we have $K_G(v) = K_G(\zeta v)$. Hence we sometimes denote it by $K_G([v])$, such notation can be found in [45], in which the holomorphic sectional curvature is directly defined on $PT^{1,0}M$. Abate and Patrizio [1] proved an important theorem that the holomorphic sectional curvature of a complex Finsler metric G is the supremum of the Gauss curvature of the induced metric through a family of holomorphic maps.

Lemma 2.1 *([1]) Let (M, G) be a strongly pseudoconvex complex Finsler manifold, and take $p \in M$ and $v \in \tilde{M}_p$. Then*

$$K_G(v) = \sup\{K(\varphi^* G)(0)\}, \tag{2.8}$$

where the supremum is taken with respect to the family of all holomorphic maps $\varphi : \Delta \to M$ with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$, and $K(\varphi^* G)(0)$ is the Gauss curvature of $\varphi^* G$ at the origin 0.

In more detail, Abate and Patrizio [1] gave the following formula between holomorphic sectional curvature and the induced Gauss curvature.

Lemma 2.2 *([1]) Let (M, G) be a strongly pseudoconvex complex Finsler manifold, and let $\varphi : \Delta \to M$ be a holomorphic map. Set $p = \varphi(0)$ and $v = \varphi'(0)$, with $v \neq o_p$, then*

$$K(\varphi^* G)(0) = K_G(v) - \frac{2}{G(v)^2} \left\| \nabla_{(\varphi')^H} (\varphi')^H - \frac{\left\langle \nabla_{(\varphi')^H} (\varphi')^H, \chi \right\rangle_v}{\langle \chi, \chi \rangle_v} \chi \right\|_v^2. \tag{2.9}$$

The supremum in (2.8) is achieved by the maps φ such that

$$\nabla_{(\varphi')^H} (\varphi')^H \left(\varphi'(0)\right) = a\chi \left(\varphi'(0)\right), \tag{2.10}$$

for some $a \in \mathbb{C}$.

Definition 2.7 ([2, 47]) The first and second (horizonal) holomorphic bisectional curvatures of a strongly pseudoconvex complex Finsler metric G are defined by ([47])

$$\mathcal{B}_G^1(v, X) = \frac{\left\langle \Omega(\chi(X), \overline{\chi(X)})\chi(v), \chi(v) \right\rangle_v}{G(v)\langle X^H, X^H \rangle_v}, \tag{2.11}$$

$$\mathcal{B}_G^2(v, X) = \frac{\left\langle \Omega(\chi(v), \overline{\chi(v)})\chi(X), \chi(X) \right\rangle_v}{G(v)\langle X^H, X^H \rangle_v}, \tag{2.12}$$

here $v, X \in T^{1,0}M \backslash \{0\}$, and $\chi : T^{1,0}M \to \mathcal{H}$ is the horizontal lifting.

The (horizontal) holomorphic bisectional curvature of a strongly pseudoconvex complex Finsler metric G is defined by ([2])

$$\mathcal{B}_G(v, X) = \mathcal{B}_G^1(v, X) + \mathcal{B}_G^2(v, X). \tag{2.13}$$

Here we use the curvature operator associated to the Chern-Finsler connection, and it is the same to use the one associated to the complex Rund connection. Note that by definition of the Hermitian structure $\langle\ ,\ \rangle$ on $\mathcal{H}$ and $\mathcal{V}$, for any $v \in T_p^{1,0}M$, the horizontal lifting of v through χ and the vertical lifting of v through ι satisfies

$$\langle \chi(v), \chi(v) \rangle_v = \langle \iota(v), \iota(v) \rangle_v \,,$$

while we choose X to be equal to v, $\mathcal{B}_G^1(v, v)$ is indeed $\frac{1}{2} K_G(v)$.

Definition 2.8 ([26]) The horizontal flag curvature R_v^G of a strongly pseudoconvex complex Finsler metric G at v is given by

$$\begin{aligned} R_v^G(H, K) :&= \mathrm{Re}\big\langle \widetilde{\Omega}(\chi + \overline{\chi}, H + \overline{H})K, \chi \big\rangle_v \\ &= \frac{1}{2}\big\langle \widetilde{\Omega}(\chi + \overline{\chi}, H + \overline{H})(K + \overline{K}), \chi + \overline{\chi} \big\rangle_v \end{aligned} \tag{2.14}$$

for any $H, K \in \mathcal{H}_v$. And for a flag (v, X), the flag curvature of a strongly pseudoconvex complex Finsler metric G can be defined by

$$K^G(v, X) = \frac{R_v^G(X^H, X^H)}{\langle \chi(v), \chi(v) \rangle_v \big\langle X^H, X^H \big\rangle_v - [\mathrm{Re}\big\langle \chi(v), X^H \big\rangle_v]^2}, \tag{2.15}$$

where $X^H = \chi(X)$ is the horizontal lifting of X.

Definition 2.9 ([26]) For any $V, W \in \mathcal{X}(T^{1,0}M)$, the tangent curvature of a strongly pseudoconvex complex Finsler metric G is defined by

$$\mathcal{T}_V(W) = 2\,\mathrm{Re}\left[\left\langle \nabla_{W^H + \overline{W^H}} W^H|_W, V^H \right\rangle_v - \left\langle \nabla_{W^H + \overline{W^H}} W^H, V^H \right\rangle_v\right].$$

Without specification, the curvature operator we consider in the rest of the paper is the one that associated to the Rund connection ∇^R which has been defined above. In [26], Li and Qiu expressed the complex second variation formula with the horizontal flag curvature term for a Kähler-Finsler metric G.

Theorem 2.1 *([26]) Let G be a Kähler-Finsler metric on M. Take a geodesic $\sigma_0 : [a, b] \to M$ with $G(\dot{\sigma}_0) = 1$, and a regular variation $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ of σ_0, then*

$$\begin{aligned} \frac{d^2 \ell_\Sigma}{ds^2}(0) &= Re \left\langle \nabla^R_{U^H + \overline{U^H}} U^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_a^b \\ &+ \int_a^b \left\{ \left\| \nabla^R_{T^H + \overline{T^H}} U^H \right\|^2_{\dot{\sigma}_0} - \left| \frac{\partial}{\partial t} Re\big\langle U^H, T^H \big\rangle_{\dot{\sigma}_0} \right|^2 - R^G_{\dot{\sigma}_0}(U^H, U^H) \right\} dt, \end{aligned} \tag{2.16}$$

where $\ell_\Sigma(s)$ is the length of the curve $\sigma_s(t)$, $T = \frac{\partial \Sigma^\alpha}{\partial t} \frac{\partial}{\partial z^\alpha}$ and $U = \frac{\partial \Sigma^\alpha}{\partial s} \frac{\partial}{\partial z^\alpha}$. The connection ∇^R here is the complex Rund connection.

We can define a complex gradient of a smooth function in complex manifolds ([50]). Let G be a strongly convex complex Finsler metric on M and we denote by G^* the dual metric of G, that is, for any given $(1,0)$-form $\omega = \omega_\alpha dz^\alpha \in (T_z^{1,0}M)^*$,

$$G^*(\omega) := \sup_{v \in \widetilde{M}} \frac{|\omega(v)|^2}{G(v)},$$

then $G^*(\omega) = G^{*\bar{\beta}\alpha}(\omega)\omega_\alpha\omega_\beta$. The Legendre transform $\mathcal{L}_1 : T_z^{1,0}M \to (T_z^{1,0}M)^*$ is defined by

$$\mathcal{L}_1(v) := \begin{cases} G_\alpha(v)dz^\beta, & \text{if} \quad v \neq 0; \\ 0, & \text{if} \quad v = 0. \end{cases}$$

This is equivalent to $\mathcal{L}_1(v) = \langle \cdot, \iota(v) \rangle_v$, we can see that for any $\lambda \in \mathbb{C}$, $\mathcal{L}_1(\lambda v) = \bar{\lambda}\mathcal{L}_1(v)$. We should note that $\mathcal{L}_1$ is a norm preserving diffeomorphism for non-zero vectors.

For a real function f on M, the complex $(1,0)$-gradient $\nabla_1 f$ is defined by ([26])

$$\nabla_1 f := \mathcal{L}_1^{-1}(\partial f).$$

Since G^o is a real Finsler metric, there exits the Legendre transform $\mathcal{L}_\mathbb{R} : T_\mathbb{R}M \to T_\mathbb{R}^*M$ for G^o. Recall that for any $V \in \mathcal{X}(\mathcal{V})$, we have ([1])

$$\langle V^o | (\iota(v))^o \rangle_{v^o} = \mathrm{Re}\langle V, \iota(v) \rangle_v,$$

where ι is the vertical lifting. It follows that

$$\mathcal{L}_\mathbb{R}(v^o) = \langle \cdot | (\iota(v))^o \rangle_{v^o} = \mathrm{Re}\langle \cdot, \iota(v) \rangle_v = \mathrm{Re}(\mathcal{L}_1(v)) = \frac{1}{2}[\mathcal{L}_1(v)]^o. \tag{2.17}$$

Hence, we get from (2.17) that the real gradient of f can be given by

$$\nabla f := \mathcal{L}_\mathbb{R}^{-1}(df) = 2(\nabla_1 f)^o.$$

In fact, Yin and Zhang [50] have showed that $\nabla f = 2\mathcal{L}_1^{-1}(\partial f) + 2\mathcal{L}_1^{-1}(\bar{\partial} f)$. Hence, we have

$$G^o(\nabla f) = G((\nabla f)_o) = G(2\nabla_1 f) = 4G(\nabla_1 f).$$

We give a briefly introduction of two Hessian of a smooth function f. Let f be a smooth function on a strongly pseudoconvex complex Finsler manifold (M, G), the first Hessian of f is a map $D^2 f : T_z^{1,0}M \to \mathbb{R}$ defined by ([26])

$$D^2 f(v) := \frac{d^2}{ds^2}(f \circ c)\Big|_{s=0}, \quad v \in T_z^{1,0}M.$$

where $c : (-\varepsilon, \varepsilon) \to M$ is the geodesic with $\dot{c}(0) = v \in T_z^{1,0}M$.

Definition 2.10 ([26]) For any $X, Y \in T_z^{1,0}M$ and a smooth function f whose complex $(1,0)$-gradient $\nabla_1 f|_z \neq 0$, the second Hessian is defined by

$$Hf(X,Y) := (\widetilde{X} + \overline{\widetilde{X}})(\widetilde{Y} + \overline{\widetilde{Y}})f - \nabla^{\nabla_1 f}_{\widetilde{X}^H + \overline{\widetilde{X}^H}}(\widetilde{Y}^H + \overline{\widetilde{Y}^H})f, \tag{2.18}$$

where $\widetilde{X}, \widetilde{Y}$ are any extensions of X and Y, respectively, and $\nabla^{\nabla_1 f}_{\widetilde{X}^H + \overline{\widetilde{X}^H}}(\widetilde{Y}^H + \overline{\widetilde{Y}^H}) := \nabla_{\widetilde{X}^H + \overline{\widetilde{X}^H}}(\widetilde{Y}^H + \overline{\widetilde{Y}^H})\Big|_{\nabla_1 f}$.

We have the following relations among D^2, H and $\partial\bar{\partial}$ for a weakly Kähler-Finsler metric G.

Lemma 2.3 *([26]) Let f be a smooth real-valued function on a weakly Kähler-Finsler manifold (M, G). Then for every $z \in M$ and for every $X \in T_z^{1,0}M$, we have*

$$4\partial\bar{\partial} f(X, \overline{X}) = D^2 f(X) + D^2 f(iX).$$

Furthermore, if G is strongly convex, we have

$$D^2 f(X) = Hf(X,X) - \mathcal{T}_{\nabla_1 f}(X),$$
$$4\partial\bar{\partial} f(X, \overline{X}) = Hf(X,X) + Hf(iX, iX).$$

3 Strongly Pseudoconvex Complex Finsler Manifolds Satisfying Condition (A)

In this subsection, we will find the strongly pseudoconvex complex Finsler manifolds satisfying condition (A). Let (M, G) be a strongly pseudoconvex complex Finsler manifold of $\dim_{\mathbb{C}} M = n$. Fixing $o \in M$, we denote by $\rho(p) = d(o, p)$ the distance function from o to p. The Levi form of distance function is defined by

$$L(\rho)(\xi, \bar{\xi}) = \frac{\partial^2 \rho}{\partial z^\alpha \partial \overline{z^\beta}} \xi^\alpha \overline{\xi^\beta}, \quad \xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in T^{1,0}M, \tag{3.1}$$

where ξ is a unit vector. Let $\sigma : [a,b] \to M$ be a regular curve, we define $\dot{\sigma} : [a,b] \to \tilde{M}$ by setting

$$\dot{\sigma}(t) = \frac{d\sigma^\alpha}{dt}(t) \frac{\partial}{\partial z^\alpha}\Big|_{\sigma(t)}.$$

Definition 3.1 ([47]) Let $\sigma_0 : [a,b] \to M$ be a regular curve with $G(\dot{\sigma}_0) \equiv 1$ and $\Delta_\varepsilon = \{z \in \mathbb{C} | \, |z| < \varepsilon\}$. A holomorphic variation of σ_0 is a map $\Sigma : [a,b] \times \Delta_\varepsilon \to M$ such that

(1) $\sigma_0(t) = \Sigma(t, 0)$ for all $t \in [a,b]$;
(2) for every $\omega \in \Delta_\varepsilon$, $\sigma_\omega(t) = \Sigma(t, \omega)$ is a curve in M;

(3) for all fixed t, the map $\Sigma(t,\omega):\Delta_\varepsilon\to M$ is a holomorphic map for all $\omega\in\Delta_\varepsilon$.

By using the holomorphic variation, Xiao, Qiu, He, and Chen [47] proved a complex second variation formula for a strongly pseudoconvex complex Finsler metric.

Theorem 3.1 *([47]) Let (M,G) be a strongly pseudoconvex complex Finsler manifold, and $\Delta_\varepsilon=\{z\in\mathbb{C}|\,|z|<\varepsilon\}$. Take a regular geodesic $\sigma_0:[a,b]\to M$ with $G(\dot\sigma_0)\equiv 1$ (select arc length as parameter) and a holomorphic variation $\Sigma:[a,b]\times\Delta_\varepsilon\to M$ of σ_0. Then*

$$\begin{aligned}\frac{\partial^2\ell_\Sigma}{\partial\omega\partial\bar\omega}\bigg|_{\omega=0}=&\frac{1}{2}\int_a^b\left\{-\frac{1}{2}\left|\left\langle\psi^{\mathcal{H}},T^H\right\rangle_{\dot\sigma_0}\right|^2+\left\langle\psi^{\mathcal{H}},\psi^{\mathcal{H}}\right\rangle_{\dot\sigma_0}\right.\\&-\left\langle\Omega\left(U^H,\overline{U^H}\right)\iota(T),\iota(T)\right\rangle_{\dot\sigma_0}\\&\left.-\left\langle\tau^H_{\mathcal{M}}\left(U^H,\overline{\psi^{\mathcal{V}}}\right),T^H\right\rangle_{\dot\sigma_0}\right\}dt,\end{aligned}\tag{3.2}$$

where $\ell_\Sigma(\omega)$ is the length of the curve $\sigma_\omega(t)$, and $T=\dot\sigma_0(t)$, $\psi^{\mathcal{H}}=\nabla^R_{T^H+\overline{T^H}}U^H+\theta\left(U^H,T^H\right)=\left(\Gamma^\alpha_{\gamma;\mu}\frac{\partial z^\mu}{\partial\omega}v^\gamma+\frac{\partial v^\alpha}{\partial\omega}\right)\Big|_{\omega=0}\delta_\alpha$ and $\psi^{\mathcal{V}}=\left(\Gamma^\alpha_{\gamma;\mu}\frac{\partial z^\mu}{\partial\omega}v^\gamma+\frac{\partial v^\alpha}{\partial\omega}\right)\Big|_{\omega=0}\dot\partial_\alpha$.

By taking a special holomorphic variation, one can obtain an estimate of the Levi form of ρ through this variation formula.

Theorem 3.2 *([47]) Let (M,G) be a complete strongly pseudoconvex complex Finsler manifold with the first holomorphic bisectional curvature bounded from below by $-k^2(k>0)$, the horizontal part of $(2,0)$-torsion bounded from above by $\left(\sqrt{2}-1\right)\theta(\theta>0)$ and the mixed part of the $(1,1)$-torsion bounded from above by $\tau(\tau>0)$. Then*

$$\frac{\partial^2\rho}{\partial z^\alpha\partial\overline{z^\beta}}\xi^\alpha\overline{\xi^\beta}\le\frac{1}{2}\left(\frac{1}{\rho}+\theta+k+\tau\right)\tag{3.3}$$

for any vector $\xi\in T^{1,0}_pM$ with $\langle\xi^H,\xi^H\rangle_{\nabla\rho}=1$.

In particular, if M is a Kähler-Finsler manifold, then the horizontal part of $(2,0)$-torsion term vanishes. So we have

Corollary 3.1 *[47] Let (M,G) be a complete Kähler-Finsler manifold with the first holomorphic bisectional curvature bounded from below by $-k^2(k\ge 0)$, and the mixed part of the $(1,1)$-torsion bounded from above by $\tau(\tau>0)$. Then*

$$\frac{\partial^2\rho}{\partial z^\alpha\partial\overline{z^\beta}}\xi^\alpha\overline{\xi^\beta}\le\frac{1}{2}\left(\frac{1}{\rho}+k+\tau\right)\tag{3.4}$$

for any vector $\xi\in T^{1,0}_pM$ with $\langle\xi^H,\xi^H\rangle_{\nabla\rho}=1$.

Definition 3.2 [37] A non-negative real-valued function u on M is said to be proper if the sets $\{p : u(p) \leq c\}$ are compact for each real constant c. A function w defined in a neighborhood U of p is called an upper supporting function for u at p if $w(p) = u(p)$ and $w(q) \geq u(q)$ for $q \in U$.

A complex Finsler manifold is called uniform with uniformity constant C_0 ([16]) if there exists a positive constant C_0 such that

$$\frac{1}{C_0}G(\xi) \leq \langle \xi^H, \xi^H \rangle_v \leq C_0 G(\xi)$$

for any $p \in M$ and $v, \xi \in T_p^{1,0}M$.

Now we consider the strongly convex complex Finsler metrics. We have the following complete strongly convex uniform complex Finsler manifold satisfying condition (A).

Theorem 3.3 *(i.e., Theorem 1.3) Let (M, G) be a complete strongly convex uniform complex Finsler manifold with the first holomorphic bisectional curvature bounded from below by $-k^2 (k > 0)$, the horizontal part of $(2, 0)$-torsion bounded from above by $\theta (\theta > 0)$ and the mixed part of the $(1, 1)$-torsion bounded from above by $\tau (\tau > 0)$. Then M satisfies condition (A).*

Proof Since we may always divide u (and w) by a given positive constant, it suffices to show that there is a constant C and a non-negative proper function u on M which has a smooth upper supporting function w at each point with $|\partial w(\xi)|^2 \leq C$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 w}{\partial z^\alpha \partial \overline{z^\beta}} \leq C$.

Since (M, G) is strongly convex and complete, the generalized Hopf-Rinow theorem for a real Finsler manifold ([1]) can be adapted here. Fixing $o \in M$, let the distance function $\rho(p) = d(o, p)$, then the closed sets $\{z \in M | \rho(z) \leq c\}$ are compact, hence ρ is proper. Moreover, $\rho(p)$ is smooth outside o and the cut points of o. Denote by $2a$ the distance from o to the nearest cut point. Then $B_a = \{p \in M | \rho(p) \leq a\}$ is a ball that contains no cut point of o.

Let u be a smooth non-negative function inside B_{2a} and equals to ρ outside B_a. For the points inside B_a, since B_a is compact, we can choose a constant C_1 to satisfy $|\partial u(\xi)|^2 \leq C_1$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z^\beta}} \leq C_1$ in B_a. For the points outside B_a and not the cut points, since u is just the distance function ρ, we have

$$\begin{aligned} |\partial u(\xi)|^2 &\leq G^*(\partial u) G(\xi) \\ &\leq G(\mathcal{L}_1^{-1}(\partial u)) \\ &= G(\nabla_1 u) \\ &= \frac{1}{4} G^o(\nabla u) = \frac{1}{4}, \end{aligned}$$

where we use the fact that the distance function $\rho = u$ satisfies $G^o(\nabla \rho) = 1$. On uniform complex Finsler manifolds, $\langle \xi^H, \xi^H \rangle_{\nabla \rho}$ is uniform bounded from above by

C_0 for any unit vector $\xi \in T^{1,0}M$. Hence, by the assumption and Theorem 3.2, we have

$$\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 \rho}{\partial z^\alpha \partial \overline{z^\beta}} \leq \frac{1}{2}\left(\frac{1}{\rho}+\theta_1+k+\tau\right)\langle \xi^H, \xi^H\rangle_{\nabla\rho} \leq \frac{C_0}{2}\left(\frac{1}{\rho}+\theta_1+k+\tau\right)$$

for any non-cut points outside B_a and any unit vector $\xi \in T_p^{1,0}M$, where $\theta_1 = (\sqrt{2}+1)\theta$. Now we set

$$C = \max\left\{\frac{1}{4}, C_1, \frac{C_0}{2}\left(\frac{1}{a}+\theta_1+k+\tau\right)\right\},$$

then $|\partial u(\xi)|^2 \leq C$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z^\beta}} \leq C$ at p for all ordinary points. In this case, we just let $w = u$.

If p is a cut point of o, take $o' \in B_a$ on the minimal geodesic joining o and p. Let $\rho'(p) = d(o', p)$, then ρ' is smooth in a neighborhood of p. And set $w = d(o, o') + \rho'$, then w is smooth in a neighborhood of p, $u(p) = w(p)$, $u(q) \leq w(q)$, $|\partial w(\xi)|^2 \leq 1$;moreover, at p, we have

$$\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 w}{\partial z^\alpha \partial \overline{z^\beta}} \leq \frac{C_0}{2}\left(\frac{1}{a}+\theta_1+k+\tau\right) \leq C$$

for any unit vector $\xi \in T_p^{1,0}M$. The function w is the one we desired. □

Now we consider the case that M is a Kähler-Finsler manifold, and we use the horizontal flag curvature instead of the holomorphic bisectional curvature. Let (M, G) be a complete strongly convex Kähler-Finsler manifold. Fixing $o \in M$, let $p \in M$ be a non-cutting point of o, and take a vector $X \in T_p^{1,0}M$, then there exists a geodesic $\gamma(s) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Taking a shortest geodesic $\sigma_0(t)$ connecting o and p, we consider two variations of this geodesic.

Denote by $\widetilde{\Sigma} : (-\varepsilon, \varepsilon) \times [0, r] \to M$ the geodesic variation of $\sigma_0(t)$ whose starting points are fixed and the ending points lie on the geodesic $\gamma(s)$. More specifically, for all $s \in (-\varepsilon, \varepsilon)$, the curve $\tilde{\sigma}_s(t) = \widetilde{\Sigma}(s, t)$ is a shortest geodesic joining $o = \tilde{\sigma}_s(0)$ and $\tilde{\sigma}_s(r) = \gamma(s)$. We denote $\ell_{\widetilde{\Sigma}}(s)$ by the length of $\tilde{\sigma}_s(t)$, and $\widetilde{T}(t) = \nabla_1\rho(t)$, $\widetilde{U}(t) = \frac{\partial \widetilde{\Sigma}^\alpha}{\partial s}\frac{\partial}{\partial z^\alpha}\Big|_{s=0}$. If we take $X \in T_p^{1,0}M$ with $\langle X, \widetilde{T}(r)\rangle_{\widetilde{T}(r)} = 0$, then by Theorem 2.1 and the definition of the Hessian H, we have ([26])

$$H\rho(X, X) = \int_0^r \left\{\left\|\nabla^R_{\widetilde{T}^H+\overline{\widetilde{T}^H}}\widetilde{U}^H\right\|^2_{\dot{\sigma}_0} - R^G_{\dot{\sigma}_0}(\widetilde{U}^H, \widetilde{U}^H)\right\} dt. \tag{3.5}$$

We choose a vector field $Y(t)$ along $\sigma_0(t)$ satisfying $Y(0) = 0$ and $Y(t) = X$, and associate it with another regular variation $\Sigma : (-\varepsilon, \varepsilon) \times [0, r] \to M$ such that

(i) $\sigma_0(t) = \Sigma(0, t)$ for all $t \in [0, r]$;
(ii) $\sigma_s(0) = \Sigma(s, 0) = o, \forall s \in (-\varepsilon, \varepsilon)$;

(iii) for all $s \in (-\varepsilon, \varepsilon)$, $\sigma_s(t)$ is a regular curve on M;
(iv) $\Sigma_*(\frac{\partial}{\partial s})|_{s=0} = Y(t)$;
(v) for $t = r$, $\sigma_s(r) = \gamma(s)$.

The associated regular variation can always be taken. Since $\sigma_0(t)$ is a regular curve, which implies $\frac{d\sigma_0}{dt}(t) \neq 0$ for all $t \in [0, r]$, i.e., $\frac{\partial \sigma_s}{\partial t}\Big|_{s=0} \neq 0$, hence we can choose ε sufficiently small so that $\frac{\partial \sigma_s}{\partial t} \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. By Theorem 2.1, we have

$$\begin{aligned} \frac{d^2 \ell_\Sigma}{ds^2}(0) = & \operatorname{Re} \left\langle \nabla^R_{Y^H + \overline{Y^H}} Y^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_0^r \\ & + \int_0^r \left\{ \left\| \nabla^R_{T^H + \overline{T^H}} Y^H \right\|^2_{\dot{\sigma}_0} - \left| \frac{\partial}{\partial t} \operatorname{Re} \left\langle Y^H, T^H \right\rangle_{\dot{\sigma}_0} \right|^2 - R^G_{\dot{\sigma}_0}(Y^H, Y^H) \right\} dt \\ & \leq \operatorname{Re} \left\langle \nabla^R_{Y^H + \overline{Y^H}} Y^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_0^r + \int_0^r \left\{ \left\| \nabla^R_{T^H + \overline{T^H}} Y^H \right\|^2_{\dot{\sigma}_0} - R^G_{\dot{\sigma}_0}(Y^H, Y^H) \right\} dt. \end{aligned} \tag{3.6}$$

Since $\tilde{\sigma}_s(t) = \tilde{\Sigma}(s, t)$ is the shortest geodesic for all $s \in (-\varepsilon, \varepsilon)$, we have

$$\ell_{\tilde{\Sigma}}(s) - \ell_\Sigma(s) \leq 0, \ell_{\tilde{\Sigma}}(0) - \ell_\Sigma(0) = 0.$$

By the maximum principle, at $s = 0$,we have

$$\begin{aligned} \frac{d^2 \ell_{\tilde{\Sigma}}}{ds^2}(0) \leq & \frac{d^2 \ell_\Sigma}{ds^2}(0) \\ & \leq \operatorname{Re} \left\langle \nabla^R_{Y^H + \overline{Y^H}} Y^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_0^r + \int_0^r \left\{ \left\| \nabla^R_{T^H + \overline{T^H}} Y^H \right\|^2_{\dot{\sigma}_0} - R^G_{\dot{\sigma}_0}(Y^H, Y^H) \right\} dt. \end{aligned}$$

By Lemma 2.3, we conclude that

$$\begin{aligned} H\rho(X, X) = & \frac{d^2 \ell_{\tilde{\Sigma}}}{ds^2}(0) - \operatorname{Re} \left\langle \nabla^R_{Y^H + \overline{Y^H}} Y^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_0^r \\ & \leq \int_0^r \left\{ \left\| \nabla^R_{T^H + \overline{T^H}} Y^H \right\|^2_{\dot{\sigma}_0} - R^G_{\dot{\sigma}_0}(Y^H, Y^H) \right\} dt. \end{aligned} \tag{3.7}$$

We can obtain better estimate of $H\rho(X, X)$. Choose an adapt vector field $f(t)Y(t)$ instead of $Y(t)$ in (3.7), where $f : [0, r] \to R$ is a differentiable function satisfying $f(0) = 0$, $f(r) = 1$, and assume that $Y(t)$ is parallel along $\sigma_0(t)$, then we have

$$H\rho(X, X) \leq \int_0^r \left\{ \dot{f}^2 - f^2 R^G_{\dot{\sigma}_0}(Y^H, Y^H) \right\} dt.$$

Set $f(t) = \left(\frac{t}{r}\right)^{\alpha}$,$(\alpha > 1)$ and assume that the horizontal flag curvature is bounded from below by a negative constant $-k^2$, then we get

$$H\rho(X,X) \leq \int_0^r \frac{\alpha^2}{r^{2\alpha}} t^{2\alpha-2} + k^2 \left(\frac{t}{r}\right)^{2\alpha} dt$$
$$= \frac{1}{r} + \frac{(\alpha-1)^2}{(2\alpha-1)\,r} + \frac{k^2 r}{2\alpha+1}.$$

for any vector $X \in T_p^{1,0}M$ with $\langle X^H, X^H \rangle_{T(r)} = 1$. We choose an α that satisfies

$$\frac{(\alpha-1)^2}{(2\alpha-1)\,r} = \frac{k^2 r}{2\alpha+1},$$

then

$$H\rho(X,X) \leq \frac{1}{r} + 2\sqrt{\frac{(\alpha-1)^2}{(2\alpha-1)\,r} \cdot \frac{k^2 r}{2\alpha+1}} \leq \frac{1}{r} + k. \tag{3.8}$$

Take $\rho = r$, and use the vector ξ instead of X, then we obtain

Theorem 3.4 *Let (M, G) be a complete strongly convex Kähler-Finsler manifold with the horizontal flag curvature bounded from below by $-k^2 (k > 0)$, then*

$$\frac{\partial^2 \rho}{\partial z^\alpha \partial \overline{z^\beta}} \xi^\alpha \overline{\xi^\beta} \leq \frac{1}{2}\left(\frac{1}{\rho} + k\right) \tag{3.9}$$

for any vector $\xi \in T_p^{1,0}M$ with $\langle \xi^H, \xi^H \rangle_{\nabla\rho} = 1$.

Proof By Lemma 2.3, the Levi form of ρ at point z satisfies

$$\frac{\partial^2 \rho}{\partial z^\alpha \partial \overline{z^\beta}} \xi^\alpha \overline{\xi^\beta} = \frac{1}{4}\left[H\rho(\xi,\xi) + H\rho(i\xi, i\xi)\right]$$
$$\leq \frac{1}{2}\left(\frac{1}{\rho} + k\right).$$

□

Just like Theorem 3.3, from Theorem 3.4, we have

Theorem 3.5 *(i.e., Theorem 1.4) Let (M, G) be a complete strongly convex uniform Kähler-Finsler manifold with the horizontal flag curvature bounded from below by $-k^2 (k > 0)$, then M satisfies condition (A).*

Remark 3.1 We do not know if there is a strongly pseudoconvex complex Finsler manifold satisfying condition (A) which is not strongly convex.

4 Proof of the Schwarz Lemma

Theorem 4.1 *(i.e., Theorem 1.5) Let (M, G) be a strongly pseudoconvex complex Finsler manifold with the holomorphic sectional curvature bounded from below by a constant $K_1 \le 0$, and let (N, H) be another strongly pseudoconvex complex Finsler manifold with the holomorphic sectional curvature bounded from above by a constant $K_2 < 0$. Suppose that M satisfies condition (A), then for any holomorphic map $f : M \to N$, we have*

$$f^*H \le \frac{K_1}{K_2} G.$$

Proof Set $\mu = \frac{f^*H}{G}$, it is a non-negative function on $T^{1,0}M$. By the homogeneity of G and H, $\mu(p, \zeta v) = \mu(p, v)$ for $\zeta \in \mathbb{C}^*$, so it is indeed a function on $PT^{1,0}M$, we can denote it by $\mu(p, [v])$. By condition (A), there is a proper function u on M, set $D_\varepsilon := \left\{p \in M | u(p) < \frac{1}{\varepsilon}\right\}$, then $\overline{D_\varepsilon}$ is compact. We consider $PT^{1,0}M$ by restricting the base point p to $\overline{D_\varepsilon}$, i.e.,

$$PT^{1,0}M\Big|_{\overline{D_\varepsilon}} := \bigcup_{p \in \overline{D_\varepsilon}} \tilde{\pi}^{-1}(p),$$

where $\tilde{\pi} : PT^{1,0}M \to M$ is the natural projection map. Since $\overline{D_\varepsilon}$ is compact, the set $PT^{1,0}M|_{\overline{D_\varepsilon}}$ is also compact. Lifting the function u to $PT^{1,0}M$ by denoting

$$\tilde{u}(p, [v]) := u(\tilde{\pi}(p, [v])) = u(p).$$

Consider a special function $\mu(p, [v])^{\frac{1}{2}}(1 - \varepsilon\tilde{u}(p, [v]))$ on $PT^{1,0}M|_{\overline{D_\varepsilon}}$. It attains its local maximum at some point $(p_0, [v_0]) \in PTM|_{\overline{D_\varepsilon}}$, the representative element v_0 here satisfies $G(v_0) > 0$. It is necessary to note that p_0 is not the boundary point of $\overline{D_\varepsilon}$. Since $\mu(p, [v])^{\frac{1}{2}}(1 - \varepsilon\tilde{u}(p, [v]))$ goes to zero as $p \to \partial\overline{D_\varepsilon}$, $(p_0, [v_0])$ must be the interior point of $PT^{1,0}M|_{\overline{D_\varepsilon}}$. Furthermore, if the maximum value equals to zero, then $\mu(p_0, [v_0])$ must be zero, which means $f^*H(p_0, \zeta v_0) = 0$ for all $\zeta \in \mathbb{C}^*$. This makes a contradiction, thus we can set $\mu(p_0, [v_0])^{\frac{1}{2}}(1 - \varepsilon\tilde{u}(p_0, [v_0])) > 0$.

By condition (A), there exists an upper supporting function w of u at p_0 satisfying $u(p) \le w(p)$ and $u(p_0) = w(p_0)$, with $|\partial w(\xi)|^2 \le 1$ and $\xi^\alpha \overline{\xi^\beta} \frac{\partial^2 w}{\partial z^\alpha \partial \overline{z^\beta}} \le 1$ for any unit vector ξ. Lifting w to $PT^{1,0}M|_{\overline{D_\varepsilon}}$ just like u, denote by $\tilde{w}$, then $\mu(p, [v])^{\frac{1}{2}}(1 - \varepsilon\tilde{w}(p, [v]))$ is smooth in $PT^{1,0}M|_{\overline{D_\varepsilon}}$ and satisfies

$$\begin{aligned}
\mu(p, [v])^{\frac{1}{2}}(1 - \varepsilon\tilde{w}(p, [v])) &\le \mu(p, [v])^{\frac{1}{2}}(1 - \varepsilon\tilde{u}(p, [v])) \\
&\le \mu(p_0, [v_0])^{\frac{1}{2}}(1 - \varepsilon\tilde{u}(p_0, [v_0])) \\
&= \mu(p_0, [v_0])^{\frac{1}{2}}(1 - \varepsilon\tilde{w}(p_0, [v_0])). \qquad (4.1)
\end{aligned}$$

Hence $\mu(p,[v])^{\frac{1}{2}}(1-\varepsilon\tilde{w}(p,[v]))$ attains its maximum at $(p_0,[v_0])$.

Now we choose an arbitrary holomorphic function $\varphi:\Delta_a\to M$ that satisfies $\varphi(0)=p_0$, $\varphi'(0)=v_0$, and Δ_a is taken sufficiently small such that $\varphi(\Delta_a)$ is contained in the domain of w, then $\zeta=0$ is a local maximum point of the function $\mu\left(\varphi(\zeta),\left[\varphi'(\zeta)\right]\right)^{\frac{1}{2}}(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]))$, here we should note that $\tilde{w}$ is independent of the direction $[\varphi'(\zeta)]$, i.e., $w(\varphi(\zeta))=\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])$. Notice that the pull back of a complex Finsler metric into a Riemannian surface (we use the disk $\Delta_a\subseteq\mathbb{C}$ here) is a Hermitian metric, we can set

$$\lambda^2(\zeta)d\zeta d\bar{\zeta}:=\varphi^*G(\zeta), \tag{4.2}$$

$$\sigma^2(\zeta)d\zeta d\bar{\zeta}:=(f\circ\varphi)^*H(\zeta), \tag{4.3}$$

where

$$\lambda^2(\zeta)=G\left(\varphi(\zeta),\varphi'(\zeta)\right),$$
$$\sigma^2(\zeta)=H\left(f\circ\varphi(\zeta),(f\circ\varphi)'(\zeta)\right).$$

Hence

$$\frac{\sigma^2(\zeta)}{\lambda^2(\zeta)}=\frac{H\left(f\circ\varphi(\zeta),(f\circ\varphi)'(\zeta)\right)}{G\left(\varphi(\zeta),\varphi'(\zeta)\right)}=\mu\left(\varphi(\zeta),\left[\varphi'(\zeta)\right]\right), \tag{4.4}$$

in particular,

$$\frac{\sigma^2(0)}{\lambda^2(0)}=\mu\left(p_0,[v_0]\right). \tag{4.5}$$

We apply the maximum principle to the function $\mu\left(\varphi(\zeta),\left[\varphi'(\zeta)\right]\right)^{\frac{1}{2}}(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]))$ at the maximum point $\zeta=0$, we have

$$0\geq i\,\partial\bar{\partial}\log\left(\mu(\varphi(\zeta),\left[\varphi'(\zeta)\right]\right)^{\frac{1}{2}}(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])))\Big|_{\zeta=0},$$

, i.e.,

$$0\geq i\,\partial\bar{\partial}\log\left(\mu(\varphi(\zeta),\left[\varphi'(\zeta)\right]\right)(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]))^2)\Big|_{\zeta=0},$$

for all holomorphic maps $\varphi:\Delta_a\to M$ with $\varphi(0)=p_0$, $\varphi'(0)=v_0$. Now taking the supremum for these family of φ, by Lemma 2.1, we can continue our progress.

$$\begin{aligned}
0&\geq\frac{2}{G(v_0)}\sup_{\varphi}\left\{\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\left[\mu\left(\varphi(\zeta),[\varphi'(\zeta)]\right)\left(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])\right)^2\right]\Big|_{\zeta=0}\right\}\\
&=\frac{2}{G(v_0)}\sup_{\varphi}\left\{\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\frac{\sigma^2(\zeta)}{\lambda^2(\zeta)}\Big|_{\zeta=0}+2\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\left(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])\right)\Big|_{\zeta=0}\right\}\\
&=\frac{2}{G(v_0)}\sup_{\varphi}\left\{-\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\lambda^2(\zeta)\Big|_{\zeta=0}+\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\sigma^2(\zeta)\Big|_{\zeta=0}\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +2\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\left(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])\right)\Big|_{\zeta=0}\Bigg\} \\
&= \sup_{\varphi}\left\{-\frac{2}{\lambda^2(0)}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\lambda^2(\zeta)\Big|_{\zeta=0}+\frac{\sigma^2(0)}{\lambda^2(0)}\cdot\frac{2}{\sigma^2(0)}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\sigma^2(\zeta)\Big|_{\zeta=0}\right. \\
&\qquad\qquad \left.+\frac{4}{\lambda^2(0)}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\left(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])\right)\Big|_{\zeta=0}\right\} \\
&\geq \sup_{\varphi}\left\{K\left(\varphi^*G\right)(0)\right\}-\mu(p_0,[v_0])\sup_{\varphi}\left\{K\left((f\circ\varphi)^*H\right)(0)\right\} \\
&\qquad -\sup_{\varphi}\left\{\frac{4}{\lambda^2(0)}\left(\frac{\varepsilon[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]_{\zeta\bar{\zeta}}}{1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])}+\frac{\varepsilon^2\left|[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]_{\zeta}\right|^2}{(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]))^2}\right)\Bigg|_{\zeta=0}\right\} \\
&= K_G(p_0,v_0)-\mu(p_0,[v_0])K_{f^*H}(p_0,v_0) \\
&\qquad -\sup_{\varphi}\left\{\frac{4}{\lambda^2(0)}\left(\frac{\varepsilon[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]_{\zeta\bar{\zeta}}}{1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])}+\frac{\varepsilon^2\left|[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]_{\zeta}\right|^2}{(1-\varepsilon\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]))^2}\right)\Bigg|_{\zeta=0}\right\}.
\end{aligned}
\tag{4.6}
$$

By the assumption and Lemma 2.1, it is clear that

$$K_{f^*H}(p_0,v_0)\leq K_H(f(p_0),f_*(v_0))\leq K_2$$

and

$$K_G(p_0,v_0)\geq K_1.$$

The rest just needs to deal with the last term in (4.6). Since $\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)]) = w(\varphi(\zeta))$, then by Remark 1.2, we have

$$\left|\frac{\partial[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]}{\partial\zeta}(0)\right|^2=\left|\partial w\left(\varphi'(0)\right)\right|^2\leq G(\varphi'(0))=\lambda^2(0) \tag{4.7}$$

and

$$\frac{\partial^2[\tilde{w}(\varphi(\zeta),[\varphi'(\zeta)])]}{\partial\zeta\partial\bar{\zeta}}(0)=\frac{\partial^2 w}{\partial z^\alpha\partial\overline{z^\beta}}(\varphi(0))\frac{\partial\varphi^\alpha}{\partial\zeta}(0)\overline{\frac{\partial\varphi^\beta}{\partial\zeta}}(0)\leq G(\varphi'(0))=\lambda^2(0). \tag{4.8}$$

Combining (4.6) $\sim$ (4.8), we get

$$
\begin{aligned}
0 &\geq K_1-\mu(p_0,[v_0])K_2-4\left(\frac{\varepsilon}{1-\varepsilon\tilde{w}(p_0,[v_0])}+\frac{\varepsilon^2}{(1-\varepsilon\tilde{w}(p_0,[v_0]))^2}\right) \\
&= K_1-\mu(p_0,[v_0])K_2-4\left(\frac{\varepsilon-\varepsilon^2\tilde{w}(p_0,[v_0])+\varepsilon^2}{(1-\varepsilon\tilde{w}(p_0,[v_0]))^2}\right)
\end{aligned}
$$

$$\geq K_1 - \mu(p_0, [v_0])K_2 - \frac{8\varepsilon}{(1-\varepsilon\tilde{w}(p_0, [v_0]))^2}.$$

Rearranging terms, we have

$$\begin{aligned}\mu(p_0, [v_0])(1-\varepsilon\tilde{w}(p_0, [v_0]))^2 &\leq \frac{K_1}{K_2}(1-\varepsilon\tilde{w}(p_0, [v_0]))^2 + \frac{8\varepsilon}{-K_2}\\ &\leq \frac{K_1}{K_2} + \frac{8\varepsilon}{-K_2}.\end{aligned}$$

Note that $(p_0, [v_0])$ is the maximum point of the function $\mu^{\frac{1}{2}}(1-\varepsilon\tilde{w})$, thus for any other $z \in D_\varepsilon$ and $v \in T_z^{1,0}M$,

$$\begin{aligned}\mu(z, [v])(1-\varepsilon\tilde{w}(z, [v]))^2 &\leq \mu(p_0, [v_0])(1-\varepsilon\tilde{w}(p_0, [v_0]))^2\\ &\leq \frac{K_1}{K_2} + \frac{8\varepsilon}{-K_2}\end{aligned}$$

always holds. By the construction of the proper function u and the definition of $\overline{D_\varepsilon}$, for any $z \in M$, when ε is small enough, we have $z \in \overline{D_\varepsilon}$. So letting $\varepsilon \to 0$, we obtain

$$\mu(z, [v]) \leq \frac{K_1}{K_2}, \tag{4.9}$$

for any $z \in M$ and $v \in T_z^{1,0}M$. This implies

$$f^*H \leq \frac{K_1}{K_2}G.$$

The main theorem is proved. □

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