The Journal of Geometric Analysis (2024) 34:221
https://doi.org/10.1007/s12220-024-01656-z

Ground State Solutions of Nehari-Pohozaev Type for Schrödinger–Poisson–Slater Equation with Zero Mass and Critical Growth

Yu Gu[1] · Fangfang Liao[1]

Received: 8 January 2024 / Accepted: 4 April 2024 / Published online: 9 May 2024

Abstract

In this article, we study the Schrödinger–Poisson–Slater type equation with the critical growth and zero mass:

$$\begin{cases} -\Delta u + \phi u = \mu |u|^{p-2}u + u^5, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $3 < p < 6$ and $\mu > 0$. By combining a new perturbation method and the mountain pass theorem, Liu et al. [J. Diff. Eq., 266 (2019), 5912–5941] prove that the above equation has at least one positive ground state solution for $p \in (4, 6)$ and $\mu > 0$ or $p \in (3, 4]$ if μ is sufficiently large. By using a much simpler method than the ones used in the above mentioned paper, together with subtle estimates and analyses, we obtain better results on the existence for a ground state solution of Nehari-Pohozaev type.

Keywords Schrödinger–Poisson–Slater type equation · Ground state solution of Nehari-Pohozaev type · Critical growth · Zero mass

Mathematics Subject Classification 35J10 · 35J20

This work was supported by the NNSF (12071395), Scientific Research Fund of Hunan Provincial Education Department(22A0588), and the Natural Science Foundation of Hunan Province(2022JJ30550), Aid Program for Science and Technology Inbovative Research Team in Higher Educational Institutions of Hunan Province(2023), and Chenzhou Applied Mathematics Achievement Transformation Technology Research and Development Center(2022).

✉ Fangfang Liao
liaofangfang1981@126.com

[1] School of Mathematics and Information, Xiangnan University, Chenzhou 423000, Hunan, People's Republic of China

1 Introduction

In this paper, we are concerned with the existence of ground state solutions for the Schrödinger–Poisson–Slater problem with critical growth and zero mass

$$\begin{cases} -\Delta u + \phi u = \mu |u|^{p-2}u + u^5, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mu > 0$ and $3 < p < 6$.

The interest on this system stems from the Schrödinger-Poisson-Slater problem

$$\begin{cases} -\Delta u + \omega u + \phi u = \mu |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\omega > 0$, which is the Slater approximation of the exchange term in the Hartree-Fock model, see [22]. The local term $|u|^{p-2}u$ was introduced by Slater, with $p = \frac{8}{3}$ and μ is the so-called Slater constant (up to renormalization), see [24]. Of course, other exponents have been employed in various approximations. In recent years, problem (1.2) has been the object of intensive research, a lot of attention has been focused on the study of the existence of solutions, sign-changing solutions, ground states, radial and semiclassical states, see [2–9, 11–13, 17, 24, 26, 28–30, 32, 34–37] and the references therein. From a mathematical point of view, this model presents an interesting competition between local and nonlocal nonlinearities.This interaction yields to some non expected situations, as has been shown in the literature.

For problem (1.2), the parameter ω corresponds to the phase of the standing wave for the time-dependent equation. In the case $\omega = 0$, i.e. the Schrödinger-Poisson-Slater problem with zero mass

$$\begin{cases} -\Delta u + \phi u = \mu |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

one could only search the static solutions (not periodic ones). The static case has been motivated and studied in [14, 27] when $p < 3$ and $p \geq 3$, respectively. The absence of a phase term ωu makes the usual Sobolev space $H^1(\mathbb{R}^3)$ not to be a good framework for the problem (1.3). In [27], the following working space and the norm are introduced:

$$E := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y < \infty \right\} \tag{1.4}$$

and

$$\|u\|_E := \left[\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}.$$

The double integral expression is the so-called Coulomb energy of the wave. In that paper, Ruiz proved that $(E, \|\cdot\|_E)$ is a uniformly convex Banach space, and $E \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in [3, 6]$. Moreover, the author gave also the equivalence characterizations of the convergences in the space E.

Based on the above information, Ianni and Ruiz [14] proved that (1.3) has a positive solution with minimal energy among all nontrivial solutions provided $3 < p < 6$. In the arguments, they used a technique that dates back to Struwe and is usually named "monotonicity trick" (see [15, 16]), well-known arguments of concentration-compactness of Lions ([33]) and "Pohozaev identity". Lei and Lei [20] used variational methods obtained existence of ground state solution of the Nehari–Pohozaev type. By the new variational approach, there is a series of analytical results on the Schrödinger-Poisson systems in the literature (see [16, 23] and the references therein).

Further, Liu, Zhang and Huang [24] studied the existence of ground state solutions for (1.1) by combining a new perturbation method and the mountain pass theorem, the authors obtained the existence of positive ground state solutions. To be specific, they proved that (1.1) has at least one positive ground state solution for $p \in (4, 6)$ and $\mu > 0$ or $p \in (3, 4]$ if μ is sufficiently large. Via a truncation technique and Krasnoselskii genus theory, Yang and Liu [34] obtained infinitely many solutions for (1.1) provided $\mu \in (0, \mu^*)$ with some $\mu^* > 0$. Zheng, Lei and Liao [35] discussed the existence of positive ground-state solutions and the multiplicity of positive solutions for a more general Schrödinger-Poisson-Slater-type equation with critical growth. Recently, Lei, Lei and Suo [21] obtained a ground state solution for (1.1) with the Coulomb-Sobolev critical growth by employing compactness arguments.

In this paper, inspired by [14, 24, 27, 30], we obtain ground state solutions of (1.1) under weaker assumptions on μ by using a much simpler method than the ones used in [24]. In particular, we introduce some new test functions, which, together with subtle estimates and analyses, to obtain a good energy estimate of the mountain pass level such that the compactness of (PS) sequences at the energy level still holds, see Lemmas 3.7 and 3.8.

Since $E \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in [3, 6]$, so, we have that the associated energy functional to (1.1)

$$\begin{aligned}\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\ - \int_{\mathbb{R}^3}\left(\frac{\mu}{p}|u|^p + \frac{1}{6}|u|^6\right)\mathrm{d}x \qquad (1.5)\end{aligned}$$

is well-defined and $\mathcal{C}^1$. Our main result is the following:

Theorem 1.1 *Assume that one of the following conditions holds:*

(i) $p \in (4, 6)$ *and* $\mu > 0$;

(ii) $p = 4$ *and* $\mu > \frac{7\sqrt{3}}{\pi}$;

(iii) $p \in (3, 4)$ *and* $\mu > \frac{3[6\pi^2(p-3)]^{2(p-3)/3}p^4}{16(2p-3)^{(2p-3)/3}\mathcal{S}^{(5p-12)/6}}\left[\frac{839803\mathcal{S}^{\frac{3}{2}}}{468750\sqrt[3]{2}\pi^3}\sqrt{\frac{2}{5\sqrt[3]{2}\pi}}\right]^{\frac{6-p}{9}}$.

Then Problem (1.1) *has a solution* $\bar{u} \in E$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{M}} \Phi > 0$, *where*

$$\mathcal{M} := \{u \in E \setminus \{0\} : J(u) = 0\} \tag{1.6}$$

and

$$J(u) = \frac{3}{2}\|\nabla u\|_2^2 + \frac{3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y - \frac{(2p-3)\mu}{p}\|u\|_p^p - \frac{3}{2}\|u\|_6^6. \tag{1.7}$$

The set $\mathcal{M}$ was introduced by Ruiz [26], is usually named "Nehari-Pohozaev" manifold.

Throughout this paper, we let $u_t(x) := u(tx)$ for $t > 0$, and denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{1/s}$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \cdots$.

2 Variational Framework and Preliminaries

In this section we establish some notations that will be used throughout the paper. Let E be defined by (1.4) and study some basic properties of it.

Set

$$N(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y$$

and

$$\|u\|_E = \left[\|\nabla u\|_2^2 + \sqrt{N(u)}\right]^{1/2}.$$

Lemma 2.1 [27] $\|\cdot\|_E$ *is a norm, and* $(E, \|\cdot\|_E)$ *is a uniformly convex Banach space. Moreover,* $\mathcal{C}_0^\infty(\mathbb{R}^3)$ *is dense in* E.

Lemma 2.2 [31] *Assume that* $a, b > 0$. *Then there holds*

$$a\|\nabla u\|_2^2 + bN(u) \geq 2\sqrt{ab}\|u\|_3^3, \quad \forall\, u \in E. \tag{2.1}$$

Let E_0 denote the Banach space equipped with the norm defined by

$$\|u\|_0 = \left(\|\nabla u\|_2^2 + \|u\|_3^2\right)^{1/2}.$$

Then Lemma 2.2 shows that $E \hookrightarrow E_0$.

Let us define

$$\phi_u(x) := \frac{1}{|x|} * u^2 = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^3, \tag{2.2}$$

then, $u \in E$ if and only if both $u, \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. In such a case, $-\triangle\phi = u^2$ in a weak sense, and

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x, \quad \forall\, v \in E, \tag{2.3}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_u(x)u^2\mathrm{d}x. \tag{2.4}$$

Moreover, $\phi_u(x) > 0$ when $u \neq 0$. By using Hardy-Littlewood-Sobolev inequality (see [18] or [19, page 98]), we have the following inequality:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{6/5}\|v\|_{6/5}, \quad u, v \in L^{6/5}(\mathbb{R}^3). \tag{2.5}$$

Lemma 2.3 [27] *Suppose that $\{u_n\} \subset E$. Then*

(i) $u_n \to \bar{u}$ in E if and only if $u_n \to \bar{u}$ and $\phi_{u_n} \to \phi_{\bar{u}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$;
(ii) $u_n \rightharpoonup \bar{u}$ in E if and only if $u_n \rightharpoonup \bar{u}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\sup N(u_n) < +\infty$. *In such case, $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.*

As in [14, 27], we define

$$T : E^4 \to \mathbb{R},\ T(u, v, w, z) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(x)w(y)z(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \tag{2.6}$$

and

$$D : E^2 \to \mathbb{R},\ D(u, v) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y. \tag{2.7}$$

Lemma 2.4 [14] *Suppose that $\{u_n\}, \{v_n\}, \{w_n\} \subset E$, $z \in E$. If $u_n \rightharpoonup \bar{u}$, $v_n \rightharpoonup \bar{v}$, $w_n \rightharpoonup \bar{w}$ in E, then*

$$T(u_n, v_n, w_n, z) \to T(\bar{u}, \bar{v}, \bar{w}, z).$$

In view of Lemmas 2.1-2.4, (F1) implies that Φ defined by (1.5) is a well-defined of classes $\mathcal{C}^1$ functional in E, and that

$$\begin{aligned}\langle \Phi'(u), v\rangle = &\int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^3} \phi_u(x)uv\mathrm{d}x \\ &- \int_{\mathbb{R}^3} \left(|u|^{p-2} + u^4\right) uv\mathrm{d}x, \quad u, v \in E.\end{aligned} \tag{2.8}$$

Therefore, the solutions of (1.5) are then the critical points of the reduced functional (1.5).

In view of the Gagliardo-Nirenberg inequality [1, 25] and Sobolev inequality [33], one has

$$\|u\|_s^s \le K_{GN}^s \|u\|_3^{6-s} \|\nabla u\|_2^{2s-6}, \quad \forall\, u \in D^{1,3}(\mathbb{R}^3), \;\; s \in (3,6) \tag{2.9}$$

and

$$\mathcal{S}\|u\|_6^2 \le \|\nabla u\|_2^2, \quad \forall\, u \in D^{1,3}(\mathbb{R}^3). \tag{2.10}$$

where $K_{GN} > 0$ is a constant and $\mathcal{S}$ is the best embedding constant.

We also state here, for convenience of the reader, an adaptation to the space E of a result due to P.-L. Lions, see [22, Lemma I.1]:

Lemma 2.5 *If $u_n \rightharpoonup \bar{u}$ in E_0, and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^3 \mathrm{d}x = 0, \tag{2.11}$$

then

$$\|u_n\|_s \to 0, \quad \forall\, s \in (3,6). \tag{2.12}$$

3 Ground State Solutions

Set

$$g(t) := \frac{2(p-3) - (2p-3)t^3 + 3t^{2p-3}}{3p}, \quad t > 0. \tag{3.1}$$

Then we have the following lemma by a simple computation.

Lemma 3.1 *Assume that $p \in (3,6)$. Then $g(t) > g(1) = 0$ for all $t \in (0,1) \cup (1,+\infty)$.*

Lemma 3.2 *Assume that $p \in (3,6)$ and $\mu > 0$. Then*

$$\Phi(u) \ge \Phi(t^2 u_t) + \frac{1-t^3}{3} J(u) + \frac{(1-t^3)^2(2+t^3)}{6}\|u\|_6^6, \quad \forall\, u \in E, \;\; t \ge 0. \tag{3.2}$$

Proof Note that

$$\Phi(t^2 u_t) = \frac{t^3}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{t^3}{4}\int_{\mathbb{R}^3} \phi_u(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \left[\frac{\mu t^{2p-3}}{p}|u|^p + \frac{t^9}{6}|u|^6\right] \mathrm{d}x. \tag{3.3}$$

Thus, by (1.5), (1.7), (3.1) and (3.3), one has

$$
\begin{aligned}
\Phi(u)-\Phi(t^2u_t) &= \frac{1-t^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{1-t^3}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\left[\frac{\mu(t^{2p-3}-1)}{p}|u|^p+\frac{t^9-1)}{6}|u|^6\right]\mathrm{d}x\\
&=\frac{1-t^3}{3}J(u)+\int_{\mathbb{R}^3}\left[\mu g(t)|u|^p+\frac{(1-t^3)^2(2+t^3)}{6}|u|^6\right]\mathrm{d}x\\
&=\frac{1-t^3}{3}J(u)+\mu g(t)\|u\|_p^p+\frac{(1-t^3)^2(2+t^3)}{6}\|u\|_6^6.
\end{aligned}
$$

This shows that (3.2) holds. □

From Lemma 3.2, we have the following corollary immediately.

Corollary 3.3 *Assume that* $p\in(3,6)$ *and* $\mu>0$*. Then for* $u\in\mathcal{M}$*,*

$$
\Phi(u)=\max_{t\geq 0}\Phi(t^2u_t). \tag{3.4}
$$

Lemma 3.4 *Assume that* $p\in(3,6)$ *and* $\mu>0$*. Then for any* $u\in E\setminus\{0\}$*, there exists a unique* $t(u)>0$ *such that* $t(u)^2u_{t(u)}\in\mathcal{M}$*.*

Proof Let $u\in E\setminus\{0\}$ be fixed and define a function $\zeta(t):=\Phi(t^2u_t)$ on $[0,\infty)$. Clearly, by (3.3), we have

$$
\begin{aligned}
\zeta'(t)=0 \Leftrightarrow\ & \frac{3t^2}{2}\|\nabla u\|_2^2+\frac{3t^2}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\mathrm{d}x-\frac{(2p-3)\mu t^{2p-2}}{p}\|u\|_p^p-\frac{3t^8}{2}\|u\|_6^6=0\\
\Leftrightarrow\ & J(t^2u_t)=0 \ \Leftrightarrow\ t^2u_t\in\mathcal{M}.
\end{aligned}
$$

It is easy to verify that $\zeta(0)=0$, $\zeta(t)>0$ for $t>0$ small and $\zeta(t)<0$ for t large. Therefore $\max_{t\in[0,\infty)}\zeta(t)$ is achieved at a $t_0=t(u)>0$ so that $\zeta'(t_0)=0$ and $t_0^2u_{t_0}\in\mathcal{M}$.

Next we claim that $t(u)$ is unique for any $u\in E\setminus\{0\}$. In fact, for any given $u\in E\setminus\{0\}$, let $t_1,t_2>0$ such that $\zeta'(t_1)=\zeta'(t_2)=0$. Then $J(t_1^2u_{t_1})=J(t_2^2u_{t_2})=0$. Jointly with (3.2), we have

$$
\begin{aligned}
\Phi(t_1^2u_{t_1}) &\geq \Phi(t_2^2u_{t_2})+\frac{t_1^3-t_2^3}{3t_1^3}J(t_1^2u_{t_1})+\frac{(t_1^3-t_2^3)^2(2t_1^3+t_2^3)}{6t_1^9}\|u\|_6^6\\
&=\Phi(t_2^2u_{t_2})+\frac{(t_1^3-t_2^3)^2(2t_1^3+t_2^3)}{6t_1^9}\|u\|_6^6
\end{aligned} \tag{3.5}
$$

and

$$
\Phi(t_2^2u_{t_2}) \geq \Phi(t_1^2u_{t_1})+\frac{t_2^3-t_1^3}{3t_2^3}J(t_2^2u_{t_2})+\frac{(t_2^3-t_1^3)^2(2t_2^3+t_1^3)}{6t_2^9}\|u\|_6^6
$$

$$= \Phi(t_1^2 u_{t_1}) + \frac{(t_2^3 - t_1^3)^2(2t_2^3 + t_1^3)}{6t_2^9}\|u\|_6^6. \tag{3.6}$$

(3.5) and (3.6) imply $t_1 = t_2$. Therefore, $t(u) > 0$ is unique for any $u \in E \setminus \{0\}$. □

Both Corollary 3.3 and Lemma 3.4 imply the following lemma.

Lemma 3.5 *Assume that* $p \in (3, 6)$ *and* $\mu > 0$. *Then*

$$\inf_{u\in\mathcal{M}} \Phi(u) := m_0 = \inf_{u\in E\setminus\{0\}} \max_{t\geq 0} \Phi(t^2 u_t).$$

Lemma 3.6 *Assume that* $p \in (3, 6)$ *and* $\mu > 0$. *Then*

(i) there exists $\rho_0 > 0$ *such that* $\|\nabla u\|_2^2 \geq \rho_0$, $\forall\, u \in \mathcal{M}$;
(ii) $m_0 = \inf_{u\in\mathcal{M}} \Phi(u) > 0$.

Proof Since $J(u) = 0$, $\forall u \in \mathcal{M}$, by (1.7), (2.1), (2.9), (2.10) and the Young inequality, it has

$$\begin{aligned}
\frac{3}{4}\|\nabla u\|_2^2 + \frac{3}{2}\|u\|_3^3 &\leq \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{3}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x \\
&= \frac{(2p-3)\mu}{p}\|u\|_p^p + \frac{3}{2}\|u\|_6^6 \\
&\leq \frac{3}{2}\|u\|_3^3 + C_1\|u\|_6^6 \qquad (3.7)\\
&\leq \frac{3}{2}\|u\|_3^3 + \frac{C_1}{\mathcal{S}^3}\|\nabla u\|_2^6, \qquad (3.8)
\end{aligned}$$

where C_1 is a positive constant. This implies

$$\|\nabla u\|_2^2 \geq \rho_0 := \frac{\sqrt{3}\mathcal{S}^{\frac{3}{2}}}{2\sqrt{C_1}}, \quad \forall\, u \in \mathcal{M}. \tag{3.9}$$

From (1.5), (1.7), (3.7) and (3.9), we have

$$\begin{aligned}
\Phi(u) &= \Phi(u) - \frac{1}{3}J(u) \\
&= \frac{2(p-3)\mu}{3p}\|u\|_p^p + \frac{1}{3}\|u\|_6^6 \\
&\geq \frac{3}{4C_1}\|\nabla u\|_2^2 \\
&\geq \frac{3\sqrt{3}\mathcal{S}^{\frac{3}{2}}}{8C_1\sqrt{C_1}}, \quad \forall\, u \in \mathcal{M}.
\end{aligned}$$

This shows that $m_0 = \inf_{u\in\mathcal{M}} \Phi(u) > 0$. □

Now as in [10], we define functions $U_n(x) := \Theta_n(|x|)$, where

$$\Theta_n(r) = \sqrt[4]{3}\begin{cases}\sqrt{\frac{n}{1+n^2r^2}}, & 0 \le r < 1;\\ \sqrt{\frac{n}{1+n^2}}(2-r), & 1 \le r < 2;\\ 0, & r \ge 2.\end{cases} \tag{3.10}$$

Computing directly, we have

$$\begin{aligned}
\|\nabla U_n\|_2^2 &= \int_{\mathbb{R}^3} |\nabla U_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2|\Theta_n'(r)|^2 \mathrm{d}r\\
&= 4\sqrt{3}\pi\left[\int_0^1 \frac{n^5r^4}{(1+n^2r^2)^3}\mathrm{d}r + \frac{n}{1+n^2}\int_1^2 r^2\mathrm{d}r\right]\\
&= 4\sqrt{3}\pi\left[\int_0^n \frac{s^4}{(1+s^2)^3}\mathrm{d}s + \frac{7n}{3(1+n^2)}\right]\\
&= \mathcal{S}^{\frac{3}{2}} + 4\sqrt{3}\pi\left[-\int_n^{+\infty} \frac{s^4}{(1+s^2)^3}\mathrm{d}s + \frac{7n}{3(1+n^2)}\right], \qquad (3.11)\\
\|U_n\|_6^6 &= \int_{\mathbb{R}^3} |U_n|^6 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2|\Theta_n(r)|^6 \mathrm{d}r\\
&= 12\sqrt{3}\pi\left[\int_0^1 \frac{n^3r^2}{(1+n^2r^2)^3}\mathrm{d}r + \left(\frac{n}{1+n^2}\right)^3\int_1^2 r^2(2-r)^6\mathrm{d}r\right]\\
&= 12\sqrt{3}\pi\left[\int_0^n \frac{s^2}{(1+s^2)^3}\mathrm{d}s + \left(\frac{n}{1+n^2}\right)^3\int_0^1 s^6(2-s)^2\mathrm{d}s\right]\\
&= \mathcal{S}^{\frac{3}{2}} + 12\sqrt{3}\pi\left[-\int_n^{+\infty} \frac{s^2}{(1+s^2)^3}\mathrm{d}s + \frac{23}{126}\left(\frac{n}{1+n^2}\right)^3\right], \qquad (3.12)\\
\|U_n\|_q^q &= \int_{\mathbb{R}^3} |U_n|^q \mathrm{d}x = 4\pi \int_0^{+\infty} r^2|\Theta_n(r)|^q \mathrm{d}r\\
&= 4(\sqrt[4]{3})^q\pi\left[\int_0^1 \frac{n^{q/2}r^2}{(1+n^2r^2)^{q/2}}\mathrm{d}r + \left(\frac{n}{1+n^2}\right)^{q/2}\int_1^2 r^2(2-r)^q\mathrm{d}r\right]\\
&= 4(\sqrt[4]{3})^q\pi\left[\frac{1}{n^{(6-q)/2}}\int_0^n \frac{s^2}{(1+s^2)^{q/2}}\mathrm{d}s + \left(\frac{n}{1+n^2}\right)^{q/2}\int_0^1 s^q(2-s)^2\mathrm{d}s\right]\\
&= 4(\sqrt[4]{3})^q\pi\left[\frac{1}{n^{(6-q)/2}}\int_0^n \frac{s^2\mathrm{d}s}{(1+s^2)^{q/2}} + \frac{q^2+7q+14}{(q+1)(q+2)(q+3)}\left(\frac{n}{1+n^2}\right)^{\frac{q}{2}}\right] \qquad (3.13)
\end{aligned}$$

and

$$\|U_n\|_{12/5}^{12/5} = 4(\sqrt[4]{3})^{12/5}\pi\left[\frac{1}{n^{9/5}}\int_0^n \frac{s^2}{(1+s^2)^{6/5}}\mathrm{d}s + \frac{2285}{5049}\left(\frac{n}{1+n^2}\right)^{6/5}\right]. \tag{3.14}$$

Both (2.5), (3.11) and (3.14) imply that $U_n \in E$ for all $n \in \mathbb{N}$.

Lemma 3.7 *Assume that condition* (i) *or* (ii) *in Theorem 1.1 holds. Then there exists a positive integer* $\hat{n}$ *such that*

$$m_0 \le \sup_{t>0} \Phi\left(t^2(U_{\hat{n}})_t\right) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}. \tag{3.15}$$

Proof By (2.5), (3.3), (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
&\Phi\left((t^2(U_n)_t\right) \\
&= \frac{t^3}{2}\|\nabla U_n\|_2^2 + \frac{t^3}{4}N(U_n) - \frac{\mu t^{2p-3}}{p}\|U_n\|_p^p - \frac{t^9}{6}\|U_n\|_6^6 \\
&< \frac{t^3}{2}\left[\mathcal{S}^{\frac{3}{2}} + \frac{28\sqrt{3}\pi n}{3(1+n^2)}\right] \\
&\quad +4\sqrt[3]{4\pi}t^3\left[\frac{1}{n^{9/5}}\int_0^n \frac{s^2}{(1+s^2)^{6/5}}\mathrm{d}s + \frac{2285}{5049}\left(\frac{n}{1+n^2}\right)^{\frac{6}{5}}\right]^{\frac{5}{3}} \\
&\quad -\frac{4(\sqrt[4]{3})^p\pi\mu t^{2p-3}}{pn^{(6-p)/2}}\int_0^n \frac{s^2}{(1+s^2)^{p/2}}\mathrm{d}s \\
&\quad -\frac{t^9}{6}\left[\mathcal{S}^{\frac{3}{2}} + 12\sqrt{3}\pi\left(-\frac{1}{3n^3} + \frac{23}{126}\left(\frac{n}{1+n^2}\right)^3\right)\right] \\
&< \mathcal{S}^{\frac{3}{2}}\left(\frac{t^3}{2} - \frac{t^9}{6}\right) + \frac{\sqrt{3}\pi}{n^3}t^9 + \frac{29\sqrt{3}\pi}{6n}t^3 \\
&\quad -\frac{4(\sqrt[4]{3})^p\pi\mu t^{2p-3}}{pn^{(6-p)/2}}\int_0^n \frac{s^2}{(1+s^2)^{p/2}}\mathrm{d}s, \qquad \forall\, n \ge 100.
\end{aligned} \tag{3.16}$$

Under condition (i) or (ii) of Theorem 1.1, there are three cases to distinguish. Csae 1. $t \in [2, +\infty)$, $p \in (3, 6)$ and $\mu > 0$. It follows from (3.16) that

$$\begin{aligned}
\Phi\left((t^2(U_n)_t\right) &< \mathcal{S}^{\frac{3}{2}}\left(\frac{t^3}{2} - \frac{t^9}{6}\right) + O\left(\frac{1}{n^3}\right)t^9 + O\left(\frac{1}{n}\right)t^3 - O\left(\frac{1}{n^{(6-p)/2}}\right)t^{2p-3} \\
&< 0, \quad n \to \infty.
\end{aligned} \tag{3.17}$$

Csae 2. $t \in (0, 2)$, $p \in (4, 6)$ and $\mu > 0$. It follows from (3.16) that

$$\begin{aligned}
\Phi\left((t^2(U_n)_t\right) &< \mathcal{S}^{\frac{3}{2}}\left(\frac{t^3}{2} - \frac{t^9}{6}\right) + O\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n}\right) - \mu\left[O\left(\frac{1}{n^{(6-p)/2}}\right)\right] \\
&\le \frac{1}{3}\mathcal{S}^{\frac{3}{2}} - \mu\left[O\left(\frac{1}{n^{(6-p)/2}}\right)\right], \quad n \to \infty.
\end{aligned} \tag{3.18}$$

Csae 3. $t \in (0, 2)$, $p = 4$ and $\mu > \frac{7\sqrt{3}}{\pi}$. It follows from (3.16) that

$$
\begin{aligned}
\Phi\left((t^2(U_n)_t\right) &< \mathcal{S}^{\frac{3}{2}}\left(\frac{t^3}{2} - \frac{t^9}{6}\right) + \frac{5\sqrt{3}\pi t^3}{n} - \frac{3\pi\mu t^5}{n}\int_0^n \frac{s^2}{(1+s^2)^2}\mathrm{d}s \\
&= \mathcal{S}^{\frac{3}{2}}\left(\frac{t^3}{2} - \frac{t^9}{6}\right) + \frac{5\sqrt{3}\pi}{n}t^3 - \frac{3\pi^2\mu}{4n}t^5 + O\left(\frac{1}{n^3}\right) \\
&\leq \frac{1}{3}\mathcal{S}^{\frac{3}{2}} - O\left(\frac{1}{n}\right), \quad n \to \infty.
\end{aligned} \tag{3.19}
$$

Case 1-Case 3 imply that there exists a positive integer $\hat{n} > 100$ such that (3.15) holds. □

Set

$$
\kappa^3 := \frac{839803\mathcal{S}^{\frac{3}{2}}}{468750\sqrt[3]{2}\pi^3}\sqrt{\frac{2}{5\sqrt[3]{2}\pi}} \tag{3.20}
$$

and

$$
w = \kappa e^{-|x|}. \tag{3.21}
$$

Then $w \in H^1(\mathbb{R}^3)$, and

$$
\|\nabla w\|_2^2 = \int_{\mathbb{R}^3} |\nabla w|^2 \mathrm{d}x = 4\pi^2\kappa^2\int_0^{+\infty} r^2 e^{-2r}\mathrm{d}r = \pi^2\kappa^2, \tag{3.22}
$$

$$
\|w\|_s^s = \int_{\mathbb{R}^3} |w|^s \mathrm{d}x = 4\pi^2\kappa^s\int_0^{+\infty} r^2 e^{-sr}\mathrm{d}r = \frac{8\pi^2\kappa^s}{s^3}, \quad \forall\, s \in [2, 6] \tag{3.23}
$$

and

$$
\|w\|_{12/5}^4 = \left(\int_{\mathbb{R}^3} |w|^{12/5}\mathrm{d}x\right)^{\frac{5}{3}} = \left[8\pi^2\kappa^{12/5}\left(\frac{5}{12}\right)^3\right]^{\frac{5}{3}} = \left(\frac{5}{6}\right)^5 \pi^3\sqrt[3]{\pi}\kappa^4. \tag{3.24}
$$

Lemma 3.8 *Assume that condition* (iii) *in Theorem 1.1 holds. Then*

$$
m_0 \leq \sup_{t>0} \Phi\left(t^2 w_t\right) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}. \tag{3.25}
$$

Proof Both (2.5) and (3.24) imply

$$
N(w) = \int_{\mathbb{R}^3} \phi_w(x) w^2 \mathrm{d}x \leq \frac{2\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|w\|_{12/5}^4 = \frac{2\sqrt[3]{2}}{3}\left(\frac{5}{6}\right)^5 \pi^3\kappa^4. \tag{3.26}
$$

From (2.5), (3.20), (3.22), (3.23), (3.26) and condition (iii) in Theorem 1.1, we have

$$\begin{aligned}\Phi(t^2 w_t) &= \frac{t^3}{2}\|\nabla w\|_2^2 + \frac{t^3}{4}N(w) - \frac{\mu t^{2p-3}}{p}\|w\|_p^p - \frac{t^9}{6}\|w\|_6^6 \\ &\le \frac{\pi^2\kappa^2 t^3}{2} + \frac{\sqrt[3]{2}t^3}{6}\left(\frac{5}{6}\right)^5 \pi^3\kappa^4 - \frac{8\pi^2\kappa^p \mu t^{2p-3}}{p^4} - \frac{8\pi^2\kappa^6 t^9}{6^4} \\ &= \left[\frac{\pi^2\kappa^2 t^3}{2} - \frac{8\pi^2\kappa^p\mu t^{2p-3}}{p^4}\right] + \left[\frac{\sqrt[3]{2}t^3}{6}\left(\frac{5}{6}\right)^5\pi^3\kappa^4 - \frac{8\pi^2\kappa^6 t^9}{6^4}\right] \\ &\le \frac{(p-3)\pi^2}{2p-3}\kappa^{(p-6)/2(p-3)}\left[\frac{3p^4}{16(2p-3)\mu}\right]^{\frac{3}{2(p-3)}} \\ &\quad + \frac{78125\sqrt[3]{2}\pi^3\kappa^3}{839803}\sqrt{\frac{5\sqrt[3]{2}\pi}{2}} \\ &= \frac{(p-3)\pi^2}{2p-3}\kappa^{(p-6)/2(p-3)}\left[\frac{3p^4}{16(2p-3)\mu}\right]^{\frac{3}{2(p-3)}} + \frac{1}{6}\mathcal{S}^{\frac{3}{2}} \\ &< \frac{1}{3}\mathcal{S}^{\frac{3}{2}}. \end{aligned} \tag{3.27}$$

This shows that (3.25) holds. □

Lemma 3.9 *Assume that $p \in (3, 6)$ and $\mu > 0$. If $u_n \rightharpoonup \bar{u}$ in E, then*

$$\Phi(u_n) = \Phi(\bar{u}) + \Phi(u_n - \bar{u}) + o(1), \tag{3.28}$$

$$\langle \Phi'(u_n), u_n\rangle = \langle \Phi'(\bar{u}), \bar{u}\rangle + \langle \Phi'(u_n - \bar{u}), u_n - \bar{u}\rangle + o(1) \tag{3.29}$$

and

$$J(u_n) = J(\bar{u}) + J(u_n - \bar{u}) + o(1). \tag{3.30}$$

Proof Set

$$\begin{aligned} I_1(u) &:= \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x, \quad I_2(u) := \int_{\mathbb{R}^3} \phi_u(x) u^2 \mathrm{d}x, \\ I_3(u) &:= \int_{\mathbb{R}^3}\left(\frac{\mu}{p}|u|^p + \frac{1}{6}|u|^6\right)\mathrm{d}x. \end{aligned} \tag{3.31}$$

Let $v_n = u_n - \bar{u}$. Then $u_n \rightharpoonup \bar{u}$ and $v_n \rightharpoonup 0$ in E. From (2.4), (2.6), (2.7), (3.31) and Lemma 2.4, we have

$$I_2(u_n) = D((\bar{u} + v_n)^2, (\bar{u} + v_n)^2)$$

$$
\begin{aligned}
&= D(\bar{u}^2, \bar{u}^2) + D(v_n^2, v_n^2) + 4D(\bar{u}^2, \bar{u}v_n) + 4D(v_n^2, \bar{u}v_n) \\
&\quad +4D(\bar{u}v_n, \bar{u}v_n) + 2D(\bar{u}^2, v_n^2) \\
&= D(\bar{u}^2, \bar{u}^2) + D(v_n^2, v_n^2) + o(1) \\
&= I_2(\bar{u}) + I_2(v_n) + o(1) \qquad (3.32)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle I_2'(u_n), u_n\rangle &= 4D((\bar{u} + v_n)^2, (\bar{u} + v_n)^2) \\
&= 4D(\bar{u}^2, \bar{u}^2) + 4D(v_n^2, v_n^2) + o(1) \\
&= \langle I_2'(\bar{u}, \bar{u}\rangle + \langle I_2'(v_n), v_n\rangle + o(1). \qquad (3.33)
\end{aligned}
$$

By (3.32), (3.33) and the Brezis-Lieb lemma, one can easily prove that

$$
\Phi(u_n) = \Phi(\bar{u}) + \Phi(v_n) + o(1) \tag{3.34}
$$

and

$$
\langle \Phi'(u_n), u_n\rangle = \langle \Phi'(v_n), v_n\rangle + \langle \Phi'(\bar{u}), \bar{u}\rangle + o(1).
$$

Note that

$$
J(u) = 2\langle \Phi'(u), u\rangle - 3\Phi(u) + \|\nabla u\|_2^2 - \frac{1}{2} I_2(u), \tag{3.35}
$$

then from (3.28), (3.29) and (3.35), we can prove that (3.30) holds. □

Lemma 3.10 *Assume that the conditions in Theorem 1.1 hold. Then m_0 is achieved.*

Proof We prove this lemma by using the strategy used in [30]. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m_0$. Since $J(u_n) = 0$, then it follows from (1.5) and (1.7) that

$$
m_0 + o(1) = \frac{2(p-3)\mu}{3p}\|u_n\|_p^p + \frac{1}{3}\|u_n\|_6^6 \tag{3.36}
$$

and

$$
m_0 + o(1) = \frac{1}{3}\|\nabla u_n\|_2^2 + \frac{1}{4}N(u_n) - \frac{2(6-p)\mu}{9p}\|u_n\|_p^p. \tag{3.37}
$$

By (1.7) and $J(u_n) = 0$, we have

$$
\frac{3}{2}\|\nabla u_n\|_2^2 + \frac{3}{4}N(u_n) = \frac{(2p-3)\mu}{p}\|u_n\|_p^p + \frac{3}{2}\|u_n\|_6^6. \tag{3.38}
$$

Hence, (3.36) (3.38) show that $\{u_n\}$ is bounded in E. From (3.38), one has

$$
\|\nabla u_n\|_2^2 \le \frac{2(2p-3)\mu}{3p}\|u_n\|_p^p + \|u_n\|_6^6. \tag{3.39}
$$

We claim that there exist a $\delta > 0$ and a sequence $y_n \in \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_1(y_n)} |u_n|^3 \mathrm{d}x > \delta. \tag{3.40}$$

Indeed, suppose that (3.40) does not hold. Then we have

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^3 \mathrm{d}x = 0. \tag{3.41}$$

By Lemma 2.5, we have

$$\|u_n\|_p^p \to 0. \tag{3.42}$$

Up to a subsequence, we assume that

$$\|\nabla u_n\|_2^2 \to l_1 \geq 0, \quad \|u_n\|_6^6 \to l_2 \geq 0. \tag{3.43}$$

Then it from (2.10), (3.39), (3.42) and (3.43) follows that

$$l_1 = \lim_{n\to\infty} \|\nabla u_n\|_2^2 \leq \lim_{n\to\infty} \|u_n\|_6^6 \leq \mathcal{S}^{-3} \lim_{n\to\infty} \|\nabla u_n\|_2^6 = \mathcal{S}^{-3} l_1^3. \tag{3.44}$$

If $l_1 > 0$, then (3.44) implies that $l_1 \geq \mathcal{S}^{\frac{3}{2}}$, which, together with (3.37) and (3.42), implies that $m_0 \geq \frac{1}{3}\mathcal{S}^{\frac{3}{2}}$. This contradicts with (3.15) and (3.25). Therefore, (3.40) holds.

Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\|_E = \|u_n\|_E$ and

$$J(\hat{u}_n) = 0, \quad \Phi(\hat{u}_n) \to m_0, \quad \liminf_{n\to\infty} \int_{B_1(0)} |\hat{u}_n|^3 \mathrm{d}x > \delta. \tag{3.45}$$

Therefore, there exists $\bar{u} \in E \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \bar{u}, & \text{in } E; \\ \hat{u}_n \rightharpoonup \bar{u}, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3), \ \forall\, s \in [1, 6); \\ \hat{u}_n \to \bar{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.46}$$

Let $w_n = \hat{u}_n - \bar{u}$. Then (3.46) and Lemma 3.9 yield

$$\Phi(\hat{u}_n) = \Phi(\bar{u}) + \Phi(w_n) + o(1) \tag{3.47}$$

and

$$J(\hat{u}_n) = J(\bar{u}) + J(w_n) + o(1). \tag{3.48}$$

From (1.5), (1.7), (3.45), (3.47) and (3.48), one has

$$\frac{2(p-3)\mu}{3p}\|w_n\|_p^p + \frac{1}{3}\|w_n\|_6^6 = m_0 - \frac{2(p-3)\mu}{3p}\|\bar{u}\|_p^p - \frac{1}{3}\|\bar{u}\|_6^6 + o(1) \quad (3.49)$$

and

$$J(w_n) = -J(\bar{u}) + o(1). \quad (3.50)$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\Phi(\bar{u}) = m_0, \quad J(\bar{u}) = 0, \quad (3.51)$$

which implies the conclusion of Lemma 3.10 holds. Next, we assume that $w_n \neq 0$. In view of Lemma 3.4, there exists $t_n > 0$ such that $t_n^2(w_n)_{t_n} \in \mathcal{M}$. We claim that $J(\bar{u}) \leq 0$. Otherwise, if $J(\bar{u}) > 0$, then (3.50) implies $J(w_n) < 0$ for large n. From (1.5), (1.7), (3.2) and (3.49), we obtain

$$\begin{aligned} m_0 - \frac{2(p-3)\mu}{3p}\|\bar{u}\|_p^p - \frac{1}{3}\|\bar{u}\|_6^6 + o(1) &= \frac{2(p-3)\mu}{3p}\|w_n\|_p^p + \frac{1}{3}\|w_n\|_6^6 \\ &= \Phi(w_n) - \frac{1}{3}J(w_n) \\ &\geq \Phi\left(t_n^2(w_n)_{t_n}\right) - \frac{t_n^3}{3}J(w_n) \\ &\geq m_0 - \frac{t_n^3}{3}J(w_n) \\ &\geq m_0, \end{aligned}$$

which implies $J(\bar{u}) \leq 0$ due to $\frac{2(p-3)\mu}{3p}\|\bar{u}\|_p^p + \frac{1}{3}\|\bar{u}\|_6^6 > 0$. Since $\bar{u} \in E \setminus \{0\}$, in view of Lemma 3.4, there exists $\bar{t} > 0$ such that $\bar{t}^2\bar{u}_{\bar{t}} \in \mathcal{M}$. From (1.5), (1.7), (3.2), (3.45) and Fatou's lemma, one has

$$\begin{aligned} m_0 &= \lim_{n\to\infty}\left[\Phi(\hat{u}_n) - \frac{1}{3}J(\hat{u}_n)\right] \\ &= \lim_{n\to\infty}\left[\frac{2(p-3)\mu}{3p}\|u_n\|_p^p + \frac{1}{3}\|u_n\|_6^6\right] \\ &\geq \frac{2(p-3)\mu}{3p}\|\bar{u}\|_p^p + \frac{1}{3}\|\bar{u}\|_6^6 \\ &= \Phi(\bar{u}) - \frac{1}{3}J(\bar{u}) \\ &\geq \Phi\left(\bar{t}^2\bar{u}_{\bar{t}}\right) - \frac{\bar{t}^3}{3}J(\bar{u}) \\ &\geq m_0 - \frac{\bar{t}^3}{3}J(\bar{u}) \geq m_0, \end{aligned}$$

which implies (3.51) holds also. □

Lemma 3.11 *Assume that the conditions in Theorem 1.1 hold. If $\bar{u} \in \mathcal{M}$ and $\Phi(\bar{u}) = m_0$, then $\bar{u}$ is a critical point of Φ.*

Proof We prove this lemma by using the method introduced in [9]. Assume that $\Phi'(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \bar{u}\| \leq 3\delta \Rightarrow \|\Phi'(u)\| \geq \varrho. \tag{3.52}$$

Let $\{t_n\} \subset \mathbb{R}$ such that $t_n \to 1$. Since $t_n^2 \bar{u}_{t_n} \rightharpoonup \bar{u}$ in E, then it follows from (2.7) and Lemma 2.4 that

$$\begin{aligned}\left\|\nabla\left(t_n^2\bar{u}_{t_n}\right) - \nabla\bar{u}\right\|_2^2 &= \int_{\mathbb{R}^3}\left|\nabla\left(t_n^2\bar{u}_{t_n}\right) - \nabla\bar{u}\right|^2 \mathrm{d}x\\ &= (t_n^3 + 1)\int_{\mathbb{R}^3}|\nabla\bar{u}|^2\mathrm{d}x - 2\int_{\mathbb{R}^3}\nabla\left(t_n^2\bar{u}_{t_n}\right)\cdot\nabla\bar{u}\mathrm{d}x\\ &= o(1)\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}&N\left(t_n^2\bar{u}_{t_n} - \bar{u}\right)\\ &= D\left((t_n^2\bar{u}_{t_n} - \bar{u})^2, (t_n^2\bar{u}_{t_n} - \bar{u})^2\right)\\ &= D\left((t_n^2\bar{u}_{t_n})^2, (t_n^2\bar{u}_{t_n})^2\right) + D\left(\bar{u}^2, \bar{u}^2\right) - 4D\left((t_n^2\bar{u}_{t_n})^2, (t_n^2\bar{u}_{t_n})\bar{u}\right)\\ &\quad -4D\left(\bar{u}^2, (t_n^2\bar{u}_{t_n})\bar{u}\right)\\ &\quad +4D\left((t_n^2\bar{u}_{t_n})\bar{u}, (t_n^2\bar{u}_{t_n})\bar{u}\right) + 2D\left((t_n^2\bar{u}_{t_n})^2, \bar{u}^2\right)\\ &= D\left((t_n^2\bar{u}_{t_n})^2, (t_n^2\bar{u}_{t_n})^2\right) - D\left(\bar{u}^2, \bar{u}^2\right) + o(1)\\ &= (t_n^3 - 1)D\left(\bar{u}^2, \bar{u}^2\right) + o(1)\\ &= o(1).\end{aligned} \tag{3.54}$$

Combining (3.53) with (3.54), one has

$$\lim_{t\to 1}\left\|t^2\bar{u}_t - \bar{u}\right\|_E = 0. \tag{3.55}$$

Thus, there exists $\delta_1 > 0$ such that

$$|t - 1| < \delta_1 \Rightarrow \|t^2\bar{u}_t - \bar{u}\|_E < \delta. \tag{3.56}$$

In view of Lemma 3.1, one has

$$\Phi(t^2\bar{u}_t) \leq \Phi(\bar{u}) - \frac{(1 - t^3)^2(2 + t^3)}{6}\|\bar{u}\|_6^6$$

$$= m_0 - \frac{(1-t^3)^2(2+t^3)}{6}\|\bar{u}\|_6^6, \quad \forall\, t > 0. \tag{3.57}$$

It follows from (1.7) that there exist $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$J\left(T_1^2\bar{u}_{T_1}\right) > 0, \quad J\left(T_2^2\bar{u}_{T_2}\right) < 0. \tag{3.58}$$

Set $\Theta := \inf_{t\in(0,T_1]\cup[T_2,+\infty)} \frac{(1-t^3)^2(2+t^3)}{6}\|\bar{u}\|_6^6$. Let $S := B(\bar{u}, \delta)$ and $\varepsilon := \min\{\Theta/24, 1, \varrho\delta/8\}$. Then [33, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0, 1]\times E, E)$ such that

(i) $\eta(1, u) = u$ if $\Phi(u) < m_0 - 2\varepsilon$ or $\Phi(u) > m_0 + 2\varepsilon$;
(ii) $\eta\left(1, \Phi^{m_0+\varepsilon} \cap B(\bar{u}, \delta)\right) \subset \Phi^{m_0-\varepsilon}$;
(iii) $\Phi(\eta(1, u)) \le \Phi(u), \ \forall\, u \in E$;
(iv) $\eta(1, u)$ is a homeomorphism of E.

By Corollary 2.3, $\Phi(t^2\bar{u}_t) \le \Phi(\bar{u}) = m_0$ for $t > 0$, then it follows from (3.56) and ii) that

$$\Phi(\eta(1, t^2\bar{u}_t)) \le m_0 - \varepsilon, \quad \forall\, t > 0, \ |t-1| < \delta_1. \tag{3.59}$$

On the other hand, by iii) and (3.57), one has

$$\Phi(\eta(1, t^2\bar{u}_t)) \le \Phi(t^2\bar{u}_t) \le m_0 - \frac{(1-t^3)^2(2+t^3)}{6}\|\bar{u}\|_6^6, \\ \forall\, t > 0, \ |t-1| \ge \delta_1. \tag{3.60}$$

Combining (3.59) with (3.60), we have

$$\max_{t\in[T_1,T_2]} \Phi(\eta(1, t^2\bar{u}_t)) < m_0. \tag{3.61}$$

Define $\Psi_0(t) := J\left(\eta\left(1, t^2\bar{u}_t\right)\right)$ for $t > 0$. It follows from (3.60) and i) that $\eta(1, \bar{u}_t) = \bar{u}_t$ for $t = T_1$ and $t = T_2$, which, together with (3.58), implies

$$\Psi_0(T_1) = J\left(T_1^2\bar{u}_{T_1}\right) > 0, \quad \Psi_0(T_2) = J\left(T_2^2\bar{u}_{T_2}\right) < 0.$$

Since $\Psi_0(t)$ is continuous on $(0, \infty)$, then we have that $\eta\left(1, t^2\bar{u}_t\right) \cap \mathcal{M} \neq \emptyset$ for some $t_0 \in [T_1, T_2]$, contradicting to the definition of m_0. □

Theorem 1.1 is a direct corollary of Lemmas 3.6, 3.10 and 3.11.

Data Availability There are no relevant data in our paper.

References

1. Agueh, M.: Sharp Gagliardo-Nirenberg inequalities and mass transport theory. J. Dyn. Differ. Equ. **18**, 1069–1093 (2006)
2. Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger–Poisson problem. Commun. Contemp. Math. **10**, 391–404 (2008)
3. Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Schrödinger–Maxwell equations. J. Math. Anal. Appl. **345**, 90–108 (2008)
4. Cerami, G., Vaira, G.: Positive solutions for some non-autonomous Schrödinger–Poisson systems. J. Differ. Equ. **248**, 521–543 (2010)
5. Chen, S.T., Fiscella, A., Pucci, P., Tang, X.H.: Semiclassical ground state solutions for critical Schrödinger–Poisson systems with lower perturbations. J. Differ. Equ. **268**, 2672–2716 (2020)
6. Chen, S.T., Tang, X.H.: Ground state sign-changing solutions for a class of Schrödinger–Poisson type problems in $\mathbb{R}^3$. Z. Angew. Math. Phys. **67**, 1–18 (2016)
7. Chen, S.T., Tang, X.H.: Axially symmetric solutions for the planar Schrödinger–Poisson system with critical exponential growth. J. Differ. Equ. **269**, 9144–9174 (2020)
8. Chen, S.T., Tang, X.H.: On the planar Schrödinger–Poisson system with the axially symmetric potentials. J. Differ. Equ. **268**, 945–976 (2020)
9. Chen, S.T., Tang, X.H.: Berestycki-Lions conditions on ground state solutions for a nonlinear Schrödinger equation with variable potentials. Adv. Nonlinear Anal. **9**, 496–515 (2020)
10. Chen, S.T., Tang, X.H.: Another look at Schrödinger equations with prescribed mass. J. Differ. Equ. **386**, 435–479 (2024)
11. Coclite, G.M.: A multiplicity result for the nonlinear Schrödinger–Maxwell equations. Commun. Appl. Anal. **7**, 417–423 (2003)
12. Huang, L.R., Rocha, E.M., Chen, J.Q.: Two positive solutions of a class of Schrödinger–Poisson system with indefinite nonlinearity. J. Differ. Equ. **255**, 2463–2483 (2013)
13. Huang, W.N., Tang, X.H.: Semiclassical solutions for the nonlinear Schrödinger–Maxwell equations. J. Math. Anal. Appl. **415**, 791–802 (2014)
14. Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger–Poisson–Slater problem. Commun. Contemp. Math. **14**, 1250003 (2012)
15. Jeanjean, L.: On the existence of bounded Palais-Smale sequence and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$. Proc. Roy. Soc. Edinburgh Sect. A. **129**, 787–809 (1999)
16. Jeanjean, L., Toland, J.: Bounded Palais-Smale mountain-pass sequences, C. R. Acad. Sci. Paris Sér. I Math. **327** 23–28 (1998)
17. Lei, C.Y., Radulescu, V.D., Zhang, B.L.: Ground states of the Schrödinger–Poisson–Slater equation with critical growth. Racsam. Rev. R. Acad. A. **117**(3), 128 (2023)
18. Lieb, E.H.: Sharp constants in the Hardy–Littlewood–Sobolev inequality and related inequalities. Ann. Math. **118**, 349–374 (1983)
19. Lieb, E.H., Loss, M.: Analysis, graduate studies in mathematics, vol. 14. American Mathematical Society, Providence (2001)
20. Lei, C.Y., Lei, Y.T.: On the existence of ground states of an equation of Schrödinger–Poisson–Slater type. Comptes Rendus Mathématique **359**, 219–227 (2021)
21. Lei, C.Y., Lei, J., Suo, H.M.: Ground state for the Schrödinger–Poisson–Slater equation involving the Coulomb-Sobolev critical exponent. Adv. Nonlinear Anal. **12**, 1–17 (2023)
22. Lions, P.L.: Solutions of Hartree-Fock equations for Coulomb systems. Commun. Math. Phys. **109**, 33–97 (1984)
23. Liu, Z., Radulescu, V.D., Tang, C., Zhang, J.: Another look at planar Schrödinger–Newton systems. J. Differ. Equ. **328**, 65–104 (2022)
24. Liu, Z.S., Zhang, Z.T., Huang, S.B.: Existence and nonexistence of positive solutions for a static Schrödinger–Poisson–Slater equation. J. Differ. Equ. **266**, 5912–5941 (2019)
25. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.D.: Nonlinear analysis-theory and methods. Springer, Switzerland (2019)
26. Ruiz, D.: The Schrödinger–Poisson equation under the effect of a nonlinear local term. J. Funct. Anal. **237**, 655–674 (2006)
27. Ruiz, D.: On the Schrödinger–Poisson–Slater system: behavior of minimizers, radial and nonradial cases. Arch. Ration. Mech. Anal. **198**, 349–368 (2010)

28. Seok, J.: On nonlinear Schrödinger–Poisson equations with general potentials. J. Math. Anal. Appl. **401**, 672–681 (2013)
29. Sun, J.J., Ma, S.W.: Ground state solutions for some Schrödinger–Poisson systems with periodic potentials. J. Differ. Equ. **260**, 2119–2149 (2016)
30. Tang, X.H., Chen, S.T.: Ground state solutions of Nehari-Pohozaev type for Schrödinger–Poisson problems with general potentials. Disc. Contin. Dyn. Syst. **37**, 4973–5002 (2017)
31. Wang, X.P., Liao, F.F.: Existence and nonexistence of solutions for Schrödinger–Poisson problems. J. Geom. Anal. **33**, 56 (2023)
32. Wen, L., Chen, S., Radulescu, V.D.: Axially symmetric solutions of the Schrödinger–Poisson system with zero mass potential in $\mathbb{R}^N$. Appl. Math. Lett. **104**, 106244 (2020)
33. Willem, M.: Minimax theorems. Birkhäuser, Boston (1996)
34. Yang, L., Liu, Z.S.: Infinitely many solutions for a zero mass Schrödinger–Poisson–Slater problem with critical growthe. J. Appl. Anal. Comput. **5**, 1706–1718 (2019)
35. Zheng, T.T., Lei, C.Y., Liao, J.F.: Multiple positive solutions for a Schrödinger–Poisson–Slater equation with critical growth. J. Math. Anal. Appl. **525**, 127206 (2023)
36. Zhao, L.G., Zhao, F.K.: On the existence of solutions for the Schrödinger–Poisson equations. J. Math. Anal. Appl. **346**, 155–169 (2008)
37. Zhao, L.G., Zhao, F.K.: Positive solutions for Schrödinger–Poisson equations with a critical exponent. Nonlinear Anal. **70**, 2150–2164 (2009)