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Remarks on the Possible Blow-Up Conditions via One Velocity Component for the 3D Navier–Stokes Equations

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Abstract

In this paper, we study some blow-up conditions via one velocity component for the 3D incompressible Navier–Stokes equations in the framework of scaling invariant anisotropic Besov spaces. In particular, we prove that if one component of the velocity remains small enough in the space $\dot{H}^{\frac{1}{2}}$, then there is no blow-up. This result improves the previous ones by Chemin et al. (Commun Partial Differ Equ 44:1387-1405, 2019) and Houamed (J Differ Equ 275:116–138, 2021).

Keywords Navier–Stokes equations · Blow-up criteria · One velocity component · Anisotropic Besov spaces

Mathematics Subject Classification 35Q30 · 76D05

1 Introduction and Main Results

In this paper, we consider possible blow-up behavior of a regular solutions to the 3D incompressible Navier–Stokes equations:

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$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \nabla \cdot u = 0 & \text{in } (0, \infty) \times \mathbb{R}^3, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ and $p = p(t, x)$ are the unknown velocity and pressure, respectively, and $u_0 = u_0(x)$ is a given initial velocity.

It was proved in [19] that for $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, (1.1) has at least one global weak solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))$ which satisfies the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \le \frac{1}{2}\|u_0\|_2^2, \quad \text{for every } t \in [0, \infty). \tag{1.2}$$

It was known in [12] that given $u_0 \in H^s(\mathbb{R}^3)$ with $s > 1/2$, there exist $T^* = T^*(\|u_0\|_{H^s}) > 0$ and a unique local strong solution u to (1.1) on $[0, T^*)$ satisfying

$$u \in C([0, T^*); H^s(\mathbb{R}^3)) \cap C^1((0, T^*); H^s(\mathbb{R}^3)) \cap C((0, T^*); H^{s+2}(\mathbb{R}^3)). \tag{1.3}$$

It is a challenging problem whether such local strong solution blows up at T^* or can be smoothly extended beyond T^* up to infinity. This problem still remains unsolved in spite of tremendous efforts by many researchers over the years. Nevertheless, there is a vast literature providing sufficient conditions to guarantee the regularity of weak solution, or equivalently to ensure the smooth extension of maximal solution. (see [8, 10, 27] and references therein). For instance, it was known that if the weak solution satisfies so called Prodi–Serrin condition

$$u \in L^p(0, T; L^q), \ \text{for } q \in (3, \infty] \text{ and } 2/p + 3/q = 1, \tag{1.4}$$

then u is regular on $(0, T]$ (see [23, 25]). The limiting case where $q = 3$ was proved by Escauriaza, Seregin, and Šverák in [11]. Note that (1.1) is invariant under the natural scaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \ p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x), \ u_{0\lambda} = \lambda u_0(\lambda x), \ \lambda > 0, \tag{1.5}$$

and if $2/p + 3/q = 1$, then we have $\|u\|_{L^p(0,T;L^q)} = \|u_\lambda\|_{L^p(0,\lambda^{-2}T;L^q)}$. Therefore, the Prodi-Serrin criterion (1.4) is optimal from the point of view of scaling invariance. We refer the readers to [4, 8–10, 13] and references therein for the most up-to-date results.

We mainly focus on some important regularity results involving one velocity component. Neustupa and Penel [21] first established a regularity criterion

$$u_3 \in L^p(0, T; L^q), \quad \text{for } q \in (6, \infty) \text{ and } 2/p + 3/q = 1/2. \tag{1.6}$$

He [17] showed the regularity criterion

$$\nabla u_3 \in L^p(0, T; L^q), \quad \text{for } q \in [3, \infty] \text{ and } 2/p + 3/q = 1. \tag{1.7}$$

After that, some improvements to [21] and [17] were made during the last decades (c.f. [2, 3, 22, 30] and [31]), but most of which are not scaling invariant under the natural scaling.

Chemin and Zhang [5] proved a scaling invariant blow-up criterion. They showed that for any initial data with gradient in $L^{\frac{3}{2}}$ and for any unit vector $e \in \mathbb{S}^2$, if its lifespan T^* of unique maximal solution associated with the initial data is finite, then there holds

$$\int_0^{T^*} \|u(t) \cdot e\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} dt = \infty, \tag{1.8}$$

for $p \in (4, 6)$. This criterion was extended to the case $p \in (4, \infty)$ in [6]. Han, Lei, Li, and Zhao [16] further extended (1.8) to $[2, \infty)$. Wolf [28] proved a scaling invariant criterion

$$\nabla u_3 \in L^4(0, T; L^2). \tag{1.9}$$

Chemin et al. [7] studied what happens to the endpoint criterion when $p = \infty$ in (1.8). As mentioned in [7], such a result in the case of $p = \infty$, assuming it is true, seems to be out of reach for the time being. They proved "almost" scaling invariant blow-up criterion reinforcing slightly the $\dot{H}^{\frac{1}{2}}$ norm in the horizontal direction. Precisely, they proved the following:

Theorem 1.1 *There exists a positive constant c_0 such that if u is a maximal solution of (1.1) in $C([0, T^*), H^1)$, then for all positive real number E and for any $e \in \mathbb{S}^2$, we have*

$$T^* < \infty \Longrightarrow \limsup_{t \to T^*} \|u(t) \cdot e\|_{\dot{H}^{\frac{1}{2}}_{logh,E}} \geq c_0, \tag{1.10}$$

where

$$\|a\|^2_{\dot{H}^{\frac{1}{2}}_{logh,E}} \overset{def}{=} \int_{\mathbb{R}^3} |\xi| log(E|\xi_h| + e) |\hat{a}(\xi)|^2 d\xi < \infty.$$

Later on, Houamed [18] proved the same "almost" scaling invariant blow-up criterion in the case of $p = \infty$, by slightly reinforcing the $\dot{H}^{\frac{1}{2}}$ norm in the vertical direction instead of the horizontal one. That is, it was proved that

Theorem 1.2 *There exists a positive constant c_0 such that if u is a maximal solution of (1.1) in $C([0, T^*), H^1)$, then for all positive real number E and for any $e \in \mathbb{S}^2$,* *we have*

$$T^* < \infty \Longrightarrow \limsup_{t \to T^*} \|u(t) \cdot e\|_{\dot{H}^{\frac{1}{2}}_{logv,E}} \geq c_0, \tag{1.11}$$

where

$$\|a\|^2_{\dot{H}^{\frac{1}{2}}_{logv,E}} \overset{def}{=} \int_{\mathbb{R}^3} |\xi| log(E|\xi_v| + e)|\hat{a}(\xi)|^2 d\xi < \infty.$$

Notice that all the spaces $\dot{H}^{\frac{1}{2}}_{logh,E}$ and $\dot{H}^{\frac{1}{2}}_{logv,E}$ are smaller than $\dot{H}^{\frac{1}{2}}$. Motivated by these two results, we first aim to show that if one component of the velocity remains small enough in space $\dot{H}^{\frac{1}{2}}$ itself, then there is no blow-up. Now we state the first result as follows.

Theorem 1.3 *There exists a positive constant c_0 such that if u is a maximal solution of* (1.1) *in $C([0, T^*), H^1)$, then for any $e \in \mathbb{S}^2$, we have*

$$T^* < \infty \Longrightarrow \limsup_{t \to T^*} \|u(t) \cdot e\|_{\dot{H}^{\frac{1}{2}}} \geq c_0. \tag{1.12}$$

Remark 1.1 Theorem 1.3 tells us that as long as the $\dot{H}^{\frac{1}{2}}$ norm of one component to the velocity field is less than c_0, the blow-up cannot happen. The introduction of spaces $\dot{H}^{\frac{1}{2}}_{logh,E}$ and $\dot{H}^{\frac{1}{2}}_{logv,E}$ in both Theorems 1.1 and 1.2 was due to the estimate for the term J given by (3.5) below using the suitable anisotropic Bony decomposition, while our proof is based on a new trilinear estimate involving scaling invariant anisotropic Besov norm (see Lemma 2.7) and the proof here is relatively simple.

Next, we are concerned with the anisotropic scaling invariant blow-up criteria. It is interesting to study the blow-up criterion in the framework of anisotropic Lebesgue spaces. It becomes most useful when one considers conditional regularity in terms of only one velocity component or its gradient. In general, the gradient of a function is more informative than the function itself, and hence the results are better. The anisotropic Lebesgue spaces make it possible to obtain almost scaling invariant blow-up criteria which is not the case for corresponding result formulated in the framework of standard Lebesgue spaces (see [14, 15, 24, 26, 29]).

Although the anisotropic Lebesgue spaces are convenient to obtain almost scaling invariant blow-up criteria involving one velocity component, it is hardly reachable the optimal Prodi-Serrin level. Recently, the second author [9] reached the Prodi-Serrin level in the endpoint anisotropic Lebesgue space for the first time, which states that u is regular if ∇u_3 satisfies an anisotropic scaling invariant condition

$$\nabla u_3 \in L^2(0, T; L^\infty_v L^2_h), \tag{1.13}$$

where h and v denote the horizontal and vertical components, respectively. Very recently, the authors [13] proved more general anisotropic scaling invariant regularity criterion, which covers (1.9) and (1.13), simultaneously.

In order to reduce the differentiability order, Liu and Zhang [20] proved a scaling invariant one component anisotropic regularity criterion, which states that u is regular if

$$u_3 \in L^p(0,T; L^{\frac{3p}{p-2}}) \cap L^p(0,T; (\dot{B}^{\mu+\frac{2}{p}+\frac{2}{q_1}-1}_{q_1,\kappa})_h(\dot{B}^{\frac{2}{q_2}-\mu}_{q_2,\kappa})_v), \tag{1.14}$$

where $p \in (4,\infty)$, $q_1 \in [1,2)$, $\mu > 0$, $q_2 \in [2,(1/p+\mu)^{-1})$, and $\kappa \in (1,\infty)$.

Motivated by the above cited results, the second aim of this paper is to establish new blow-up criteria involving the gradient of one velocity component in the framework of scaling invariant anisotropic Besov spaces. More precisely, we prove the following blow-up criteria.

Theorem 1.4 *There exists a positive constant c_0 such that if u is a maximal solution of (1.1) in $C([0,T^*),H^1)$, then for any $e \in \mathbb{S}^2$, $p \in [1,2]$ and $q \in [2,\infty)$, we have*

$$T^* < \infty \Longrightarrow \limsup_{t\to T^*} \|\nabla(u(t)\cdot e)\|_{(\dot{B}^{\frac{2}{p}+\frac{1}{q}-2}_{p,2})_h(L^q)_v} \geq c_0. \tag{1.15}$$

Theorem 1.5 *Let u be a maximal solution of (1.1) in $C([0,T^*),H^1)$. if $T^* < \infty$, then for any $e \in \mathbb{S}^2$, $p \in [1,2]$, $q \in [2,\infty)$ and $\alpha \in (2,\infty)$ such that*

$$\frac{2}{\alpha} + \frac{1}{q} < 1,$$

we have

$$\int_0^{T^*} \|\nabla(u(t)\cdot e)\|^{\alpha}_{(\dot{B}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2}_{p,\infty})_h(L^q)_v} dt = \infty. \tag{1.16}$$

Remark 1.2 The space-time norms in (1.15) and (1.16) are scaling invariant quantities under the natural scaling (1.5). We remark that the Besov spaces for horizontal variables in both Theorems 1.4 and 1.5 have the negative indices, even the limiting value -1 by proper choice of p, q, and α.

The rest of this paper is organized as follows. In Sect. 2, we introduce the anisotropic Besov spaces and useful lemmas, and establish two trilinear estimates involving the anisotropic Besov norm. Section 3 is devoted to the proof of the main results.

2 Preliminaries

Throughout this paper, we will use the following notations. We denote by C the positive constant which may vary from line to line. For simplicity, we omit $\mathbb{R}^3$ in all function spaces $X(\mathbb{R}^3)$ over $\mathbb{R}^3$ as long as no confusion arises. For a normed space X, we denote by $\|\cdot\|_X$ the X-norm. L^p denotes the standard Lebesgue space.

The anisotropic Lebesgue space $L_h^p L_v^q$ consists of all measurable functions f over $\mathbb{R}^3$ such that

$$\|f\|_{L_h^p L_v^q} := \Big\| \|f\|_{L^p_{x_1 x_2}(\mathbb{R}^2)} \Big\|_{L^q_{x_3}(\mathbb{R})} < \infty,$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Let us introduce the anisotropic Littlewood-Paley theory (see [1]). Let B be the ball $B = \{\xi \in \mathbb{R}^3 \mid |\xi| \le 4/3\}$ and $\mathcal{C}$ be the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^3 \mid 3/4 \le |\xi| \le 8/3\}$. Then, there exist radial smooth functions χ and φ with their values in the interval $[0, 1]$, and supports, respectively, in B and $\mathcal{C}$ such that

$$\begin{aligned} \chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^3, \\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \ \forall \xi \in \mathbb{R}^3 \backslash \{0\}. \end{aligned} \tag{2.1}$$

Moreover, $\text{supp}\varphi(2^{-j}\cdot) \cap \text{supp}\varphi(2^{-k}\cdot) = \emptyset$ if $|j-k| > 1$ and $\text{supp}\varphi(2^{-j}\cdot) \cap \text{supp}\chi = \emptyset$ if $j > 0$. Let $\mathcal{S}'$ be the space of tempered distributions, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively.

For $u \in \mathcal{S}'$ and $(j, k, \ell) \in \mathbb{Z}^3$, the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{u}), \qquad \dot{S}_j u = \sum_{j' \le j-1} \dot{\Delta}_{j'} u.$$

We have the anisotropic version of the dyadic decomposition:

$$\begin{aligned} \dot{\Delta}_k^h u = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{u}), \qquad \dot{S}_k^h u = \sum_{k' \le k-1} \dot{\Delta}_{k'}^h u, \\ \dot{\Delta}_\ell^v u = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_v|)\widehat{u}), \qquad \dot{S}_\ell^v u = \sum_{\ell' \le \ell-1} \dot{\Delta}_{\ell'}^v u, \end{aligned}$$

where $\xi = (\xi_h, \xi_v)$, $\xi_h = (\xi_1, \xi_2)$ and $\widehat{u} = \mathcal{F}u$.

We denote by $\mathcal{S}_h'$ the space of tempered distributions u such that

$$\lim_{j\to-\infty} \|\dot{S}_j u\|_\infty = 0.$$

Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as follows:

$$\dot{B}^s_{p,q} = \{u \in \mathcal{S}_h' \mid \|u\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\dot{\Delta}_j u\|_p^q\right)^{\frac{1}{q}}, & 1 \le p \le \infty,\ 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_p, & 1 \le p \le \infty,\ q = \infty. \end{cases}$$

We recall the definition of anisotropic Besov spaces (see [1] for more details).

Definition 2.1 Let s_1, s_2 be two real numbers and let p, q_1, q_2 be in $[1, \infty]$, we define the space $(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v$ as the space of tempered distributions u such that

$$\|u\|_{(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v} = \left(\sum_{k\in\mathbb{Z}} 2^{q_1 k s_1}\left(\sum_{l\in\mathbb{Z}} 2^{q_2 l s_2}\|\Delta_k^h\Delta_l^v u\|_{L^p}^{q_2}\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} < \infty, \tag{2.2}$$

with the usual modification if $q_1 = \infty$ and/or $q_2 = \infty$.

We remark here that

$$\begin{aligned} \|u\|_{(\dot{B}^s_{p,q})_h} &= \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} = \left\|(2^{ks}\|\Delta_k^h u\|_{L^p(\mathbb{R}^2)})\right\|_{\ell^q(\mathbb{Z})}, \\ \|u\|_{\dot{H}^s_h} &= \|u\|_{\dot{H}^s_h(\mathbb{R}^2)} \approx \left\|(2^{ks}\|\Delta_k^h u\|_{L^2(\mathbb{R}^2)})\right\|_{\ell^2(\mathbb{Z})}. \end{aligned} \tag{2.3}$$

We review some useful lemmas from the literature.

Lemma 2.2 ([1], Proposition 2.20) *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}(\mathbb{R}^d)$ is continuously embedded in $\dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}(\mathbb{R}^d)$.*

Lemma 2.3 ([1], Proposition 2.22) *A constant C exists which satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0, 1)$, then we have, for any $(p, r) \in [1, \infty]^2$ and $u \in \mathcal{S}'_h$,*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{2.4}$$

Lemma 2.4 ([1], Proposition 2.39) *For any $p, q \in [1, \infty]^2$ such that $p \le q$, the space $\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1}$ is continuously embedded in L^q. In addition, if p is finite, then $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in the space of $\mathcal{C}_0$ of continuous functions vanishing at infinity.*

Lemma 2.5 ([5], Lemma 4.3) *For any s positive and any $\theta \in (0, s)$, we have*

$$\|u\|_{(\dot{B}^{s-\theta}_{p,q})_h(\dot{B}^{\theta}_{p,1})_v} \lesssim \|u\|_{\dot{B}^s_{p,q}}. \tag{2.5}$$

The following law of product in $\mathbb{R}^2$ which is the slight generalization of Lemma A.3 in [7] is also useful.

Lemma 2.6 *A constant* $C > 0$ *exists such that if* $u \in L^\infty \cap \dot{H}^1(\mathbb{R}^2)$ *and* $v \in \dot{B}^\theta_{2,1}(\mathbb{R}^2)$ *with* $0 < \theta < 1$, *then*

$$\|uv\|_{\dot{B}^\theta_{2,1}} \leq C(\|u\|_{L^\infty} + \|u\|_{\dot{H}^1})\|v\|_{\dot{B}^\theta_{2,1}}, \tag{2.6}$$

and if $u \in L^\infty \cap \dot{H}^1(\mathbb{R}^2)$ *and* $v \in \dot{H}^\theta(\mathbb{R}^2)$ *with* $0 < \theta < 1$, *then*

$$\|uv\|_{\dot{H}^\theta} \leq C(\|u\|_{L^\infty} + \|u\|_{\dot{H}^1})\|v\|_{\dot{H}^\theta}. \tag{2.7}$$

Proof We only prove the first assertion since the second one is similarly proved up to a slight modification. We use the Bony decomposition in the horizontal variables:

$$uv = T^h_u v + T^h_v u + R^h(u, v), \tag{2.8}$$

where

$$T^h_u v = \sum_j S^h_{j-1} u \Delta^h_j v \quad \text{and} \quad R^h(u, v) = \sum_{|j-k|\leq 1} \Delta^h_j u \Delta^h_k v.$$

According to Theorem 2.47 of [1], we have

$$\|T^h_u v\|_{\dot{B}^\theta_{2,1}} \leq C\|u\|_{L^\infty}\|v\|_{\dot{B}^\theta_{2,1}}. \tag{2.9}$$

As to $T^h_v u$, it follows from (2.3) that

$$\begin{aligned}
\|T^h_v u\|_{\dot{B}^\theta_{2,1}} &= \sum_{k\in\mathbb{Z}} 2^{k\theta} \left\| \Delta^h_k \left(\sum_{l\in\mathbb{Z}} S^h_{l-1} v \Delta^h_l u \right) \right\|_{L^2} \\
&= \sum_{k\in\mathbb{Z}} 2^{k\theta} \left\| \Delta^h_k \left(\sum_{|l-k|\leq 4} S^h_{l-1} v \Delta^h_l u \right) \right\|_{L^2} \\
&\leq C \sum_{k\in\mathbb{Z}} 2^{k\theta} \left\| \left(\sum_{|\nu|\leq 4} S^h_{k+\nu-1} v \Delta^h_{k+\nu} u \right) \right\|_{L^2} \\
&\leq C \sum_{|\nu|\leq 4} \sum_{k\in\mathbb{Z}} 2^{k\theta} \|S^h_{k+\nu-1} v\|_{L^\infty} \|\Delta^h_{k+\nu} u\|_{L^2} \\
&\leq C \sum_{|\nu|\leq 4} \sum_{k\in\mathbb{Z}} 2^{(k+\nu-1)(\theta-1)} \|S^h_{k+\nu-1} v\|_{L^\infty} 2^{k+\nu} \|\Delta^h_{k+\nu} u\|_{L^2} \\
&\leq C\|v\|_{\dot{B}^{\theta-1}_{\infty,2}} \|u\|_{\dot{B}^1_{2,2}} \leq C\|u\|_{\dot{H}^1}\|v\|_{\dot{B}^\theta_{2,1}},
\end{aligned} \tag{2.10}$$

where we have used the equivalence of Besov norms

$$\left(\sum_{l\in\mathbb{Z}} 2^{2(k+\nu-1)(\theta-1)}\|S^h_{k+\nu-1}v\|^2_{L^\infty}\right)^{\frac{1}{2}} \sim \|v\|_{\dot{B}^{\theta-1}_{\infty,2}}$$

for negative index $\theta - 1 < 0$ and the embedding $\dot{B}^{\theta}_{2,1} \hookrightarrow \dot{B}^{\theta-1}_{\infty,2}$ in $\mathbb{R}^2$.

For the remaining term $R^h(u, v)$, by the use of Theorem 2.52 in [1]

$$\|R^h(u, v)\|_{\dot{B}^{\theta}_{2,1}} \le C\|u\|_{\dot{B}^0_{\infty,\infty}}\|v\|_{\dot{B}^{\theta}_{2,1}} \le C\|u\|_{L^\infty}\|v\|_{\dot{B}^{\theta}_{2,1}}. \tag{2.11}$$

Collecting (2.8), (2.9), (2.10), and (2.11) imply that (2.6) holds.

We need the following trilinear inequality for the proof of Theorems 1.3 and 1.4.

Lemma 2.7 *Let* $p \in [1, 2]$, $q \in [2, \infty)$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^3} fg\varphi\, dx \le C\|f\|_{(\dot{B}^{\frac{2}{p}+\frac{1}{q}-2}_{p,2})_h(L^q)_v}\|\nabla_h g\|_{L^2}\|\nabla\varphi\|_{L^2}, \tag{2.12}$$

for any $f \in (\dot{B}^{\frac{2}{p}+\frac{1}{q}-2}_{p,2})_h(L^q)_v$, $g \in H^1$ *and* $\varphi \in H^1$.

Proof By the density argument, it is sufficient to prove the inequality for $g \in C_0^\infty$. Applying the Hölder inequality and the duality argument between Besov spaces, we have

$$\begin{aligned}\int_{\mathbb{R}^3} fg\varphi\, dx &= \int_{\mathbb{R}_v}\left(\int_{\mathbb{R}_h} fg\varphi\, dx_h\right)dx_v \\ &\le \int_{\mathbb{R}_v}\|f\|_{(\dot{B}^{\frac{2}{p}+\frac{1}{q}-2}_{p,2})_h}\|g\varphi\|_{(\dot{B}^{2-\frac{2}{p}-\frac{1}{q}}_{\frac{p}{p-1},2})_h}\,dx_v \\ &\le \|f\|_{(\dot{B}^{\frac{2}{p}+\frac{1}{q}-2}_{p,2})_h(L^q)_v}\|g\varphi\|_{(\dot{B}^{2-\frac{2}{p}-\frac{1}{q}}_{\frac{p}{p-1},2})_h(L^{\frac{q}{q-1}})_v}.\end{aligned} \tag{2.13}$$

Then, by virtue of Lemma 2.2, we have

$$(\dot{B}^{1-\frac{1}{q}}_{2,2})_h \hookrightarrow (\dot{B}^{2-\frac{2}{p}-\frac{1}{q}}_{\frac{p}{p-1},2})_h,$$

and

$$(\dot{B}^{\frac{1}{q}}_{2,1})_v \hookrightarrow (L^{\frac{2q}{q-2}})_v.$$

Considering the above embedding and taking into account the conditions $q \in [2,\infty)$ and $1-\frac{1}{q} > 0$, we now use Lemma 2.6 and Lemma 2.5. Then, (2.13) reduces to

$$
\begin{aligned}
\int_{\mathbb{R}^3} fg\varphi\,dx &\le \|f\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|g\varphi\|_{(\dot{B}_{\frac{p}{p-1},2}^{2-\frac{2}{p}-\frac{1}{q}})_h(L^{\frac{q}{q-1}})_v} \\
&\le C\|f\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|g\varphi\|_{(\dot{B}_{2,2}^{1-\frac{1}{q}})_h(L^{\frac{q}{q-1}})_v} \\
&\le C\|f\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} (\|g\|_{L_h^\infty L_v^2} + \|g\|_{\dot{H}_h^1 L_v^2})\|\varphi\|_{(\dot{B}_{2,2}^{1-\frac{1}{q}})_h(L^{\frac{2q}{q-2}})_v} \\
&\le C\|f\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} (\|g\|_{L_h^\infty L_v^2} + \|\nabla_h g\|_{L^2})\|\varphi\|_{(\dot{B}_{2,2}^{1-\frac{1}{q}})_h(\dot{B}_{2,1}^{\frac{1}{q}})_v} \\
&\le C\|f\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} (\|g\|_{L_h^\infty L_v^2} + \|\nabla_h g\|_{L^2})\|\varphi\|_{\dot{B}_{2,2}^1}.
\end{aligned}
\tag{2.14}
$$

As argued in [13], since g has compact support, by virtue of the Newton-Leibniz formula, we have

$$
\begin{aligned}
g(x_h, x_v) &= \int_{-\infty}^{x_1} \partial_1 g(\xi, x_2, x_v)d\xi = \int_{-\infty}^{x_2} \partial_2 g(x_1, \eta, x_v)d\eta, \\
|g(x_h, x_v)|^2 &= 2\int_{-\infty}^{x_1} g(\xi, x_2, x_v)\partial_1 g(\xi, x_2, x_v)d\xi \\
&= 2\int_{-\infty}^{x_1} \left(\int_{-\infty}^{x_2} \partial_2 g(\xi, \eta, x_v)d\eta\right) \partial_1 g(\xi, x_2, x_v)d\xi.
\end{aligned}
$$

Thanks to the Fubini theorem, we get

$$
\begin{aligned}
\|g\|_{L_h^\infty L_v^2}^2 &= \int_{\mathbb{R}_v} \max_{x_h} |g|^2 dx_v \\
&\le 2\int_{\mathbb{R}_v} \max_{x_h} \int_{-\infty}^{x_1} \left(\int_{-\infty}^{x_2} |\partial_2 g(\xi, \eta, x_v)|d\eta\right) |\partial_1 g(\xi, x_2, x_v)|d\xi dx_v \\
&\le 2\int_{\mathbb{R}^3} |\partial_1 g||\partial_2 g|\,dx \le 2\|\nabla_h g\|_{L^2}^2.
\end{aligned}
\tag{2.15}
$$

Thus, collecting the estimates (2.13), (2.14), and (2.15) gives the target inequality (2.12). Finally, the Lemma 2.7 is proved.

We establish the following trilinear estimate involving the anisotropic Besov norm for the proof of Theorem 1.5.

Lemma 2.8 *Let $p \in [1,2]$, $\alpha \in (2,\infty)$, $q \in [2,\infty)$ and $\frac{2}{\alpha}+\frac{1}{q} < 1$. Then, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} fg\varphi\, dx \le C\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v} \|\nabla_h g\|_{L^2}\|\varphi\|^{\frac{2}{\alpha}}_{L^2}\|\nabla\varphi\|^{1-\frac{2}{\alpha}}_{L^2}, \tag{2.16}$$

for any $f \in (\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v$, $g \in H^1$ and $\varphi \in H^1$.

Proof By the density argument, it is enough to prove the inequality for $g \in C_0^\infty$. Applying the Hölder inequality and the duality argument between Besov spaces, we have

$$\begin{aligned}\int_{\mathbb{R}^3} fg\varphi\, dx &= \int_{\mathbb{R}_v}\left(\int_{\mathbb{R}_h} fg\varphi\, dx_h\right)dx_v\\ &\le \int_{\mathbb{R}_v}\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h}\|g\varphi\|_{(\dot B^{2-\frac{2}{p}-\frac{2}{\alpha}-\frac{1}{q}}_{\frac{p}{p-1},1})_h}dx_v\\ &\le \|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v}\|g\varphi\|_{(\dot B^{2-\frac{2}{p}-\frac{2}{\alpha}-\frac{1}{q}}_{\frac{p}{p-1},1})_h(L^{\frac{q}{q-1}})_v}.\end{aligned} \tag{2.17}$$

On the one hand, by virtue of Lemma 2.2 and Lemma 2.4, we have

$$(\dot B^{1-\frac{2}{\alpha}-\frac{1}{q}}_{2,1})_h \hookrightarrow (\dot B^{2-\frac{2}{p}-\frac{2}{\alpha}-\frac{1}{q}}_{\frac{p}{p-1},1})_h,$$

and

$$(\dot B^{\frac{1}{q}}_{2,1})_v \hookrightarrow (L^{\frac{2q}{q-2}})_v.$$

Considering the above embeddings and taking into account the conditions $\alpha \in (2,\infty)$, $q \in [2,\infty)$, and $1-\frac{2}{\alpha}-\frac{1}{q} > 0$, we now use Lemma 2.6 and Lemma 2.5. Then, (2.17) reduces to

$$\begin{aligned}\int_{\mathbb{R}^3} fg\varphi\, dx &\le C\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v}\|g\varphi\|_{(\dot B^{1-\frac{2}{\alpha}-\frac{1}{q}}_{2,1})_h(L^{\frac{q}{q-1}})_v}\\ &\le C\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v}(\|g\|_{L^\infty_h L^2_v}+\|g\|_{\dot H^1_h L^2_v})\|\varphi\|_{(\dot B^{1-\frac{2}{\alpha}-\frac{1}{q}}_{2,1})_h(L^{\frac{2q}{q-2}})_v}\\ &\le C\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v}(\|g\|_{L^\infty_h L^2_v}+\|\nabla_h g\|_{L^2})\|\varphi\|_{(\dot B^{1-\frac{2}{\alpha}-\frac{1}{q}}_{2,1})_h(\dot B^{\frac{1}{q}}_{2,1})_v}\\ &\le C\|f\|_{(\dot B^{\frac{2}{p}+\frac{2}{\alpha}+\frac{1}{q}-2}_{p,\infty})_h(L^q)_v}(\|g\|_{L^\infty_h L^2_v}+\|\nabla_h g\|_{L^2})\|\varphi\|_{\dot B^{1-\frac{2}{\alpha}}_{2,1}}.\end{aligned} \tag{2.18}$$

On the other hand, as in the proof of Lemma 2.7, we have

$$\|g\|^2_{L^\infty_h L^2_v} \le 2\|\nabla_h f\|^2_{L^2}. \tag{2.19}$$

On the another hand, it follows from Lemma 2.3 that

$$\begin{aligned}\|\varphi\|_{\dot{B}^{1-\frac{2}{\alpha}}_{2,1}} &\le C\|\varphi\|_{\dot{B}^{\frac{2}{\alpha}}_{2,\infty}}\|\nabla\varphi\|_{\dot{B}^{1-\frac{2}{\alpha}}_{2,\infty}} \\ &\le C\|\varphi\|_{L^2}^{\frac{2}{\alpha}}\|\nabla\varphi\|_{L^2}^{1-\frac{2}{\alpha}}.\end{aligned} \tag{2.20}$$

Therefore, collecting the estimates (2.17), (2.18), (2.19), and (2.20) gives the target inequality (2.16). Finally, the Lemma 2.8 is proved.

3 Proofs of Main Results

Proof of Theorem 1.3 We adopt the proof procedure of [7]. Without loss of generality, we shall always take $\sigma = e_3$. The proof is divided into two steps. First, we estimate $\|\nabla_h u\|_{L^2}$. Multiplying (1.1) by $\Delta_h u$ and integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2 &= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j dx \\ &= \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j dx + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_j\partial_k u_j dx \\ &\quad + \sum_{i,k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_3\partial_k u_3 dx + \sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\partial_k u_3 dx \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.3}$$

It is obvious from the divergence free condition that

$$\begin{aligned}I_1 &= \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j dx \\ &= \int_{\mathbb{R}^3} -\partial_3 u_3\left(\sum_{i,j=1}^{2}(\partial_i u_j)^2 + \partial_1 u_2\partial_2 u_1 - \partial_1 u_1\partial_2 u_2\right)dx.\end{aligned}$$

The three terms I_1, I_3, and I_4 are sums of terms by the form

$$I = \int_{\mathbb{R}^3}\partial_i u_3\partial_j u_k\partial_l u_m dx, \tag{3.4}$$

with $(j, l) \in \{1, 2\}^2$ and $(i, k, m) \in \{1, 2, 3\}^3$. In order to estimate I_2, it is sufficient to study the following term by the form

$$J = \int_{\mathbb{R}^3}\partial_i u_3\partial_3 u_l\partial_i u_l dx, \tag{3.5}$$

with $(i, l) \in \{1, 2\}^2$. We now estimate I and J, separately. We use the duality argument and product law in three-dimensional Sobolev spaces to estimate I. Then, we have

$$
\begin{aligned}
I &\le C\|\nabla u_3\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla_h u \nabla_h u\|_{\dot{H}^{\frac{1}{2}}} \\
&\le C\|u_3\|_{\dot{H}^{\frac{1}{2}}}\|\nabla_h u\|^2_{\dot{H}^1}.
\end{aligned}
\tag{3.6}
$$

We now use Lemma 2.7 with $p = 2$ and $q = 2$ to estimate J. Then, we have

$$
\begin{aligned}
J &\le C\|\nabla_h u_3\|_{(\dot{B}^{-\frac{1}{2}}_{2,2})_h(L^2)_v}\|\nabla_h \partial_3 u\|_{L^2}\|\nabla_h u\|_{\dot{H}^1} \\
&\le C\|\nabla_h u_3\|_{(\dot{B}^{-\frac{1}{2}}_{2,2})_h(L^2)_v}\|\nabla_h u\|^2_{\dot{H}^1} \\
&\le C\|u_3\|_{(\dot{B}^{\frac{1}{2}}_{2,2})_h(L^2)_v}\|\nabla_h u\|^2_{\dot{H}^1} \\
&\le C\|u_3\|_{\dot{H}^{\frac{1}{2}}}\|\nabla_h u\|^2_{\dot{H}^1}.
\end{aligned}
\tag{3.7}
$$

Combining (3.3) with (3.6) and (3.7), we obtain

$$
\frac{d}{dt}\|\nabla_h u\|^2_{L^2} + 2\|\nabla\nabla_h u\|^2_{L^2} \le C\|u_3\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\nabla_h u\|^2_{L^2}. \tag{3.8}
$$

We set

$$
T_* = \sup\left\{T \in [0, T^*) | \sup_{[0,T]} \|u_3\|_{\dot{H}^{\frac{1}{2}}} \le \frac{1}{C}\right\}.
$$

Then, we have for all $t \le T_*$

$$
\|\nabla_h u(t)\|^2_{L^2} + \int_0^t \|\nabla\nabla_h u(s)\|^2_{L^2} ds \le \|\nabla_h u(0)\|^2_{L^2}. \tag{3.9}
$$

Second, we estimate $\|\partial_3 u\|_{L^2}$. Multiplying (1.1) by $\partial^2_{33} u$ and integrating by parts together with the divergence free condition give us

$$
\frac{1}{2}\frac{d}{dt}\|\partial_3 u\|^2_{L^2} + \|\nabla\partial_3 u\|^2_{L^2} \le C\|\nabla_h u\|_{L^2}\|\nabla\nabla_h u\|_{L^2}\|\partial_3 u\|^2_{L^2} + \frac{1}{2}\|\nabla\partial_3 u\|^2_{L^2}.
$$

The Gronwall lemma leads to

$$
\|\partial_3 u(t)\|^2_{L^2} \le \|\partial_3 u(0)\|^2_{L^2} \cdot \exp\left(C\int_0^t \|\nabla_h u(s)\|_{L^2}\|\nabla\nabla_h u(s)\|_{L^2} ds\right). \tag{3.10}
$$

As we have (3.9), we see that $\|\partial_3 u\|_{L^2}$ is also bounded on $(0, T_*)$. Thus, $\|\nabla u\|_{L^2}$ remains bounded on $(0, T_*)$. Thus, by contraposition, if $\|\nabla u\|_{L^2}$ blows up at finite $T^* > 0$, then

$$\forall t \in [0, T^*), \quad \sup_{s\in[0,t]} \|u_3\|_{\dot{H}^{\frac{1}{2}}} \geq \frac{1}{C} = c_0,$$

which proves the Theorem 1.3 by passing to the limit $t \to T^*$. □

Proof of Theorem 1.4 We proceed exactly in the same way as in the proof of Theorem 1.3 up to (3.5). Without loss of generality, we shall always take $\sigma = e_3$. We now use Lemma 2.7 to estimate I and J, separately. Then, we get

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \partial_i u_3 \partial_j u_k \partial_l u_m dx \\ &\leq C\|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla_h \nabla_h u\|_{L^2} \|\nabla \nabla_h u\|_{L^2} \\ &\leq C\|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla \nabla_h u\|_{L^2}^2. \end{aligned} \tag{3.11}$$

As to the term J, we have

$$\begin{aligned} J &\leq C\|\nabla_h u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla_h \partial_3 u\|_{L^2} \|\nabla \nabla_h u\|_{L^2} \\ &\leq C\|\nabla_h u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla \nabla_h u\|_{L^2}^2 \\ &\leq C\|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla \nabla_h u\|_{L^2}^2. \end{aligned} \tag{3.12}$$

Combining (3.3) with (3.11) and (3.12), we obtain

$$\frac{d}{dt}\|\nabla_h u\|_{L^2}^2 + 2\|\nabla \nabla_h u\|_{L^2}^2 \leq C\|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \|\nabla \nabla_h u\|_{L^2}^2. \tag{3.13}$$

We set

$$T_* = \sup\left\{T \in [0, T^*)/ \sup_{[0,T]} \|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \leq \frac{1}{C}\right\}.$$

Then, we have for all $t \leq T_*$

$$\|\nabla_h u(t)\|_{L^2}^2 + \int_0^t \|\nabla \nabla_h u(s)\|_{L^2}^2 ds \leq \|\nabla_h u(0)\|_{L^2}^2, \tag{3.14}$$

which together with (3.10) implies that $\|\nabla u\|_{L^2}$ remains bounded on $(0, T_*)$. Thus, by contraposition, if $\|\nabla u\|_{L^2}$ blows up at finite $T^* > 0$, then

$$\forall t \in [0, T^*), \quad \sup_{s\in[0,t]} \|\nabla u_3\|_{(\dot{B}_{p,2}^{\frac{2}{p}+\frac{1}{q}-2})_h(L^q)_v} \geq \frac{1}{C} = c_0,$$

which proves the Theorem 1.4 by passing to the limit $t \to T^*$. □

Proof of Theorem 1.5 Without loss of generality, we shall always take $\sigma = e_3$. We now use Lemma 2.8 to estimate I and J, separately. Then, we get

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \partial_i u_3 \partial_j u_k \partial_l u_m dx \\ &\le C\|\nabla u_3\|_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} \|\nabla_h \nabla_h u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{2}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{1-\frac{2}{\alpha}} \\ &\le C\|\nabla u_3\|_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} \|\nabla_h u\|_{L^2}^{\frac{2}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{2-\frac{2}{\alpha}}. \end{aligned} \tag{3.15}$$

The remaining term J is similarly estimated by

$$\begin{aligned} J &\le C\|\nabla_h u_3\|_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} \|\nabla_h \partial_3 u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{2}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{2-\frac{2}{\alpha}} \\ &\le C\|\nabla u_3\|_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} \|\nabla_h u\|_{L^2}^{\frac{2}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{2-\frac{2}{\alpha}}. \end{aligned} \tag{3.16}$$

Thus, combining (3.3) with (3.15) and (3.16), we obtain

$$\frac{d}{dt}\|\nabla_h u\|_{L^2}^2 + 2\|\nabla\nabla_h u\|_{L^2}^2 \le C\|\nabla u_3\|_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} \|\nabla_h u\|_{L^2}^{\frac{2}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{2-\frac{2}{\alpha}}. \tag{3.17}$$

Applying the Young inequality and Gronwall lemma leads then to

$$\begin{aligned} \|\nabla_h u(t)\|_{L^2}^2 + \int_0^t \|\nabla\nabla_h u(s)\|_{L^2}^2 ds \le \|\nabla_h u(0)\|_{L^2}^2 \\ \cdot \exp\left(\int_0^t C\|\nabla u_3\|^\alpha_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} ds\right), \end{aligned} \tag{3.18}$$

which together with (3.10) implies that $\|\nabla u\|_{L^2}$ remains bounded on $(0, T)$. By contraposition, if there is a blow-up of the $\dot{H}^1$ at finite $T^* > 0$, then we have

$$\int_0^{T^*} \|\nabla u_3(t)\|^\alpha_{(\dot{B}_{p,\infty}^{\frac{2}{p}+\frac{1}{q}+\frac{2}{\alpha}-2})_h (L^q)_v} dt = \infty.$$

Thus, the proof of Theorem 1.5 is complete. □

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