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Exponential Decay for a Time-Varying Coefficients Wave Equation with Dynamic Boundary Conditions

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Abstract

We consider a wave equation with variable coefficients in time and space in a bounded domain Ω which has the smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. We study this system that has a homogeneous Dirichlet boundary on Γ_0 and a dynamic boundary on Γ_1. The innovation of the paper lies in the coefficients which depends on the time variable and the singularities generated by changing the boundary conditions along the interface, thus we need some special techniques to deal with these difficulties. Under some geometric assumptions, the exponential decay result of the system is established by the Riemannian geometry method and the energy perturbation method.

Keywords Wave equation · Time-varying coefficients · Dynamic boundary conditions · Exponential decay

Mathematics Subject Classification 35B37 · 35L55 · 74D05 · 93D15

1 Introduction

Let Ω be a bounded domain in R^n $(n \geq 2)$ which has a boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Here meas(Γ_0) and meas(Γ_1) are positive and $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. Let ω be an open neighborhood of the part Γ_1 of the boundary that is supposed to be connected and meas$(\overline{\omega} \cap \Gamma_0) > 0$ (see Fig. 1).

Here $\nu = (\nu_1, \cdots, \nu_n)$ represents the outward unit vector normal to Γ. We denote the gradient and the divergence by ∇ and div respectively, and the tangential-gradient and the tangential-divergence by ∇_T and div$_T$ respectively in the Eucliden metric. In

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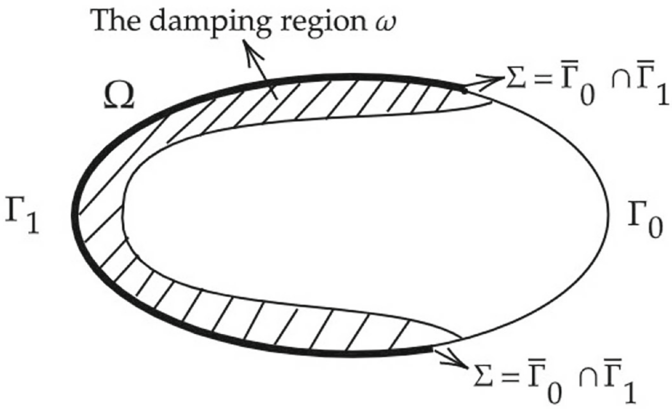

Fig. 1 An example of domain Ω satisfying the required geometrical assumptions for $n = 2$

this paper, we study the following problem

$$\begin{cases} u_{tt} + \mathcal{A}(x,t)u + b(x)u_t = 0 & \text{in } \Omega \times R^+, \\ u = 0 & \text{on } \Gamma_0 \times R^+, \\ u_{tt} + \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)u + u_t = 0 & \text{on } \Gamma_1 \times R^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$b(x) = b_0, \quad x \in \omega,$$

and $b_0 > 0$ is a constant. The initial data u_0 and u_1 are in suitable function spaces. The second-order differential operators $\mathcal{A}(x,t)$ and $\mathcal{A}_T(x,t)$ are given by

$$\begin{aligned} \mathcal{A}(x,t)u &:= -\beta(t)\operatorname{div}(A(x)\nabla u), \\ \mathcal{A}_T(x,t)u &:= -\beta(t)\operatorname{div}_T\left(A(x)|_{\Gamma_1}\nabla_T u\right), \end{aligned}$$

in which $\beta \in W^{1,\infty}(0,\infty)$ is a given function and $A(x) = (a_{ij}(x))_{n\times n}$ are symmetric and positive definite matrices functions with $a_{ij}(x) \in C^\infty(R^n)$, and the operators also satisfy the uniform ellipticity conditions

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^{n} \xi_i^2, \quad x \in \overline{\Omega},\ 0 \neq (\xi_1, \xi_2, ..., \xi_n)^T \in R^n,$$

for some constant $\lambda > 0$. And

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \beta(t) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i = \beta(t) A(x) \nabla u \cdot \nu$$

is the outer normal derivative.

Since the boundaries Γ_0 and Γ_1 satisfy $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, the singularities occur when the boundary conditions change from Γ_0 to Γ_1. We cannot get the regularity of solution by the elliptic results, so we have to use some technicalities to overcome this difficulty. For the two-dimensional case, we can refer to the method in [1] to deal with the problem of lack of regularity, and for more on this can be seen in [2, 3]. The main idea in these papers is to divide the weak solution u corresponding to the elliptic problem into two parts that the regular part and the singular part. More precisely, they decompose the solution into

$$u := u_1 + u_2,$$

where $u_1 \in H^2(\Omega)$ and u_2 is given by

$$u_2 := \sum_{x \in \Sigma} \rho(r,\theta)\sqrt{r} \sin\left(\frac{\theta}{2}\right),$$

here (r,θ) is a coordinate system centered on $x \in \Sigma$ and ρ is an appropriately smooth function with a compact support satisfying $0 \leq \rho \leq 1$. This decomposition of u allows us to estimate some integrals that resulting from the existence of singularities. For the case of higher dimensions ($n \geq 3$), Bey et al. [4] extended the above results, and [4, Theorem 4] is very important and helpful for the proof of our stability. Later, Cornilleau et al. [5] further developed the results of [4] and considered the possible singularities in which they changed the boundary conditions along the interface $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$. They assumed a partition (Γ_0, Γ_1) of $\Gamma = \partial\Omega$ such that

- $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is a C^3-manifold of dimension $n-2$,
- $(x - x_0) \cdot \nu = 0$ on Σ, where $x_0 \in R^n$ is a fixed point,
- $\Gamma \cap \varpi$ is a C^3-manifold of dimension $n-1$,
- $\mathcal{H}^{n-1}(\Gamma_0) > 0$,

where $n \geq 2$ is the dimension of Ω, ϖ is a suitable neighborhood of Σ and $\mathcal{H}^{n-1}$ denotes the usual $(n-1)$-dimensional Hausdorff measure. Under a simple geometrical condition concerning the orientation of the boundary, they obtained the stability results for systems with linear or nonlinear Neumann feedbacks.

It is worth noting that [4] and [5] mentioned here are the literature for the constant coefficients. For the variable coefficients case, we can refer to [6] which extended the stability results of [5] to a time-dependent coefficients case. In [6] Cavalcanti et al. concerned the following hyperbolic equation with boundary damping

$$\begin{cases} K(x,t)u_{tt} - A(t)u + F(x,t,u,\nabla u) = 0 & \text{in } \Omega \times R^+, \\ u = 0 & \text{on } \Gamma_0 \times R^+, \\ \dfrac{\partial u}{\partial \nu_A} + \beta(x)u_t = 0 & \text{on } \Gamma_1 \times R^+, \end{cases}$$

where $\Omega \subset R^n$ $(n \geq 2)$ is a bounded open set with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. Let $x_0 \in R^n$ be a fixed point, the sets Γ_0 and Γ_1 given by

$$\Gamma_0 = \{x \in \Gamma : (x - x_0) \cdot \nu < 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma : (x - x_0) \cdot \nu \geq 0\}.$$

Under some assumptions about the functions F, K and A, the authors obtained the exponential decay result by using the energy method.

Our work changes part of the boundary conditions in the above reference, that is, we study the influence of the dynamic boundary on the stability of system. This type of boundary condition takes acceleration into account on the boundary to affect the stability and the exact controllability of elastic structures. In [7], Li et al. considered the following one-dimensional system

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0, 1),\ t \geq 0, \\ u(0, t) = 0, & t \geq 0, \\ \kappa u_{tt}(1, t) + \dfrac{\partial u}{\partial \nu}(1, t) + u_t(1, t) = 0, & t \geq 0. \end{cases}$$

At $\kappa = 1$, they got the optimal polynomial decay result, even though the system is exponentially stable if $\kappa = 0$. Thus, for the study of dynamic boundary, it not only is very important for theoretical significance, but also is a good reference for some practical applications. This kind of boundary is suitable for dynamic vibration modeling of linear viscoelastic rods and beams with attached masses at their free ends, we refer to the reference [8–13]. These questions are common in analyzing the mechanical behavior of any structure with elongated members attached to smaller, heavier objects, for example, a structure consisting of robotic arms attached to satellites. For early studies of the system with dynamic boundary conditions we can refer to [14–16]. [16] was devoted to study of the following damped Cauchy-Ventcel problem

$$\begin{cases} u_{tt} + \mathcal{A}(x)u + a(x)g_1(u_t) = 0 & \text{in } \Omega \times R^+, \\ u = v & \text{on } \Gamma \times R^+, \\ u = 0 & \text{on } \Gamma_0 \times R^+, \\ v_{tt} + \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x)v + g_2(v_t) = 0 & \text{on } \Gamma_1 \times R^+, \end{cases}$$

where there exists a vector field H such that

$$\Gamma_1 = \{x \in \Gamma : H \cdot \nu > 0\} \quad \text{and} \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset.$$

The uniform energy decay rate for the above problem was established by Riemannian geometry method which was first introduced by Yao [17] to study the exact controllability of wave equation with variable coefficients. For more information about the variable coefficients that are only related to the space variable, we can refer to [18–23] and the references in them.

While for the time-varying coefficients case, we can refer to [24–27]. In [25], Liu considered the mixed problem

$$u_{tt} + \mathcal{A}(x,t)u = 0,$$

with Neumann boundary control

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = v \quad \text{on } \Gamma,$$

or Dirichlet boundary control

$$u = v \quad \text{on } \Gamma_0 \quad \text{and} \quad u = 0 \quad \text{on } \Gamma_1,$$

where $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and v is a suitable control function. The observability inequalities were established by the Riemannian geometry method under some geometric conditions. For more on time-dependent linear operators, evolution families, and evolution equations and their applications, we refer the reader to [28, 29] and the references therein.

Inspired by the above literature, in this paper we mainly study the system (1.1) with the assumption

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset.$$

In fact, there are two main difficulties in our work. First, because the coefficients are related to the time variable, we cannot accurately estimate the positive and negative properties of the derivative of the energy functional, which makes it impossible to get the stability results of the system by traditional methods. Second, we lose the regularity of solution due to the existence of singularities. So for these two main difficulties, we need more skills to deal with system (1.1) to get the decay result that we want.

The paper is organized as follows. Section 2 presents the notations and some assumptions that we need to follow. By using the semigroup method, we give the well-posedness result in Sect. 3. In Sect. 4, we obtain the exponential decay estimate of the energy. Finally, a concluding remark is stated in Sect. 5.

2 Preliminaries

In this section, we present some materials and assumptions used in this paper. $L^2(\cdot)$ and $H^1(\cdot)$ denote the usual Sobolev spaces. $\|\cdot\|_2$ and $\|\cdot\|_{2,\Gamma_1}$ are the norms in the $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively. For simplicity, we write $\|\cdot\|$ and $\|\cdot\|_{\Gamma_1}$ instead of $\|\cdot\|_2$ and $\|\cdot\|_{2,\Gamma_1}$, respectively. Let C denote various positive constants which may be different at different occurrences.

Denote

$$H^1_{\Gamma_0}(\Omega) := \left\{ u \in H^1(\Omega) : u|_{\Gamma_0} = 0 \right\}.$$

Since the Poincaré inequality holds in $H^1_{\Gamma_0}(\Omega)$, then the apace $H^1_{\Gamma_0}(\Omega)$ can be endowed with the norm $\|\nabla \cdot\|$, which is equivalent to usual norm of $H^1(\Omega)$.

2.1 Riemannian Notations

We define

$$g(x) = (g_{ij}(x)) = A^{-1}(x), \quad x \in R^n,$$

as a Riemannian metric on R^n and consider the couple (R^n, g) as a Riemannian manifold with the inner product and the norm

$$\langle X, Y\rangle_g = (A^{-1}(x)X, Y) = A^{-1}(x)X \cdot Y, \quad |X|_g^2 = \langle X, X\rangle_g, \quad X, Y \in R^n_x,$$

where $(\cdot, \cdot)$ is the Euclidean product of R^n. For any C^1 function w, we define

$$\nabla_g w = A(x)\nabla w, \quad |\nabla_g w|_g^2 = \sum_{i,j=1}^{n} a_{ij}(x) w_{x_i} w_{x_j}, \quad x \in R^n,$$

where ∇_g is the gradient of the metric g. Denote by D the Levi-Civita connection in the Riemannian metric g and let H be a vector field on R^n, then for each $x \in R^n$, the covariant differential DH of H determines a bilinear form on $R^n \times R^n$:

$$DH(X, Y) = \langle D_Y H, X\rangle_g, \quad X, Y \in R^n_x,$$

where $D_Y H$ is the covariant derivative of the vector field H with respect to Y. Next we give the following lemma that provides some further relationships between the Riemannian metric g and the Euclidean metric.

Lemma 2.1 [17] *Let $f \in C^2\left(\overline{\Omega}\right)$ and H be vector field. Then, with the references to the above notations, we have*

(i) $H(f) = \left\langle \nabla_g f, H\right\rangle_g = \nabla f \cdot H,$

(ii) $\left\langle \nabla_g f, \nabla_g (H(f))\right\rangle_g = DH\left(\nabla_g f, \nabla_g f\right) + \frac{1}{2}\mathrm{div}\left(|\nabla_g f|_g^2 H\right) - \frac{1}{2}|\nabla_g f|_g^2 \mathrm{div} H,$

where div$H$ *is the divergence of the vector field H in the Euclidean metric.*

2.2 Assumptions

(**H1**) Assume that $\beta(t) \in W^{1,\infty}_{loc}(0, \infty)$, $\beta'(t) \in L^1(0, \infty)$ and

$$\beta(t) \geq \beta_0 > 0, \quad t \geq 0,$$

where β_0 is some positive constant.

(**H2**) [30] Let H be a vector field on Riemannian manifold (R^n, g), there exists a continuous function $\sigma(x)$ such that

$$DH(X, X) = \sigma(x)|X|_g^2, \quad X \in R_x^n, \quad x \in \overline{\Omega},$$

and denote $\sigma_1 = \min_{x\in\overline{\Omega}} \sigma(x) > 0$ and $\sigma_2 = \max_{x\in\overline{\Omega}} \sigma(x)$. Moreover, assuming that the vector field H satisfies

$$H \cdot \nu \leq 0 \quad \text{for } x \in \Gamma_0,$$

and

$$H \cdot \nu > 0 \quad \text{for } x \in \Gamma_1.$$

Remark 2.2 The vector field H in assumption (**H2**), which called the escape vector field and firstly introduced by [31] as a checkable assumption. The existence of the vector field H depends on the Riemannian curvature of the metric g. In [32], we know that if assumption (**H2**) holds, then GCC (Geometric Control Condition) holds. And for the constant coefficients case i.e. considering $\mathcal{A}(x, t) = -\Delta$, many papers always take $H = x - x_0$ where $x_0 \in R^n$ is a fixed point and $x \in \Gamma$.

2.3 Main Results

Consider the phase space

$$\mathcal{H} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1(\Gamma_1) \times L^2(\Gamma_1),$$

endowed with the inner product

$$\begin{aligned}&\left\langle (w_1, w_2, w_3, w_4)^T, (v_1, v_2, v_3, v_4)^T \right\rangle_{\mathcal{H}} \\ &= \int_\Omega \left[\beta(t)\nabla_g w_1 \nabla v_1 + w_2 v_2\right] dx + \int_{\Gamma_1} \left[\beta(t)(\nabla_T)_g w_3 (\nabla_T) v_3 + w_4 v_4\right] d\Gamma.\end{aligned} \tag{2.1}$$

where $(\nabla_T)_g w = A(x)|_{\Gamma_1}(\nabla_T)w$ for any C^1 function w. Taking $U(t) = (u, u_t, \gamma_1(u), \gamma_1(u_t))^T$ with the trace operator $\gamma_1(\cdot) = \cdot|_{\Gamma_1}$, the system (1.1) can be rewritten by

$$\begin{cases} \dfrac{dU}{dt} = \mathbb{A}(t)U, \qquad t > 0, \\ U(0) = U_0 = (u_0, u_1, \gamma_1(u_0), \gamma_1(u_1))^T, \end{cases} \tag{2.2}$$

where the time-dependent linear operators $\mathbb{A}(t)$ have the form

$$\mathbb{A}(t)\begin{pmatrix} u \\ \varphi \\ v \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ -\mathcal{A}(x,t)u - b(x)\varphi \\ \psi \\ -\frac{\partial v}{\partial \nu_{\mathcal{A}}} - \mathcal{A}_T(x,t)v - \psi \end{pmatrix},$$

with domain

$$D(\mathbb{A}(t)) = D(\mathbb{A}(0)) = \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \\ \times \left(H^2(\Gamma_1) \cap H^1(\Gamma_1)\right) \times H^1(\Gamma_1), \quad t \geq 0,$$

which means $D(A(t))$ do not depend on t.

Now, we state the well-posedness result for the Cauchy problem (2.2), which ensures that the system (1.1) is globally well-posed.

Theorem 2.3 *Suppose that* (***H1***) *and* (***H2***) *hold. Then for any* $U_0 \in \mathcal{H}$*, the problem* (2.2) *admits a unique solution* $U(t)$ *such that*

$$U(t) \in C(R_+; \mathcal{H}).$$

Further, assuming

$$\left(A(x)|_{\Gamma_1} H, \tau\right) \leq 0, \quad x \in \Sigma, \tag{2.3}$$

where τ is the unit tangent vector pointing towards the exterior of Γ_1, from Γ_1 to Γ_0. In fact, for $A(x) = I$, this geometric assumption (2.3) is the same as in [4, 5] when taking the vector field $H = x - x_0$ where $x_0 \in R^n$ is a fixed point and $x \in \Sigma$.

Define the associated energy of system (1.1) by

$$E(t) := \frac{1}{2}\left\{\|u_t\|^2 + \beta(t)\int_\Omega |\nabla_g u|_g^2 dx + \|u_t\|^2_{\Gamma_1} + \beta(t)\int_{\Gamma_1} |(\nabla_T)_g u|_g^2 d\Gamma\right\}, \tag{2.4}$$

according to the inner product of state space $\mathcal{H}$. Our main decay result can be given as follows

Theorem 2.4 *Assume that* (**H1**), (**H2**) *and* (2.3) *hold and there exist positive constants* C_2*,* α *and* m *such that for all* t *sufficiently large, it holds*

$$\int_0^t e^{C_2 s}|\beta'(s)|ds \leq \alpha t^m. \tag{2.5}$$

Then the energy decay exponentially, i.e.,

$$E(t) \leq C\left(E(0) + \alpha t^m\right)e^{-C_2 t}.$$

Remark 2.5 For assumption (2.5), we give the following two examples for the function $\beta(s)$.

Example (i). Let $\beta(s) = ae^{-bs} + \beta_0$ with $a > 0$ and $b \geq C_2 > 0$. It is obviously that taking β to satisfy (**H1**). By direct calculation, we have

$$|\beta'(s)| = abe^{-bs},$$

and

$$\int_0^t e^{C_2 s}|\beta'(s)|ds \leq C\int_0^t e^{-(b-C_2)s}ds.$$

Therefore, there exist positive constants α and m such that for all t sufficiently large, (2.5) holds.

Example (ii). Let $\beta(s) = se^{-as} + \beta_0$ with $a \geq C_2 > 0$. It is obviously that taking β to satisfy (**H1**). By direct calculation, we have

$$\beta'(s) = (1 - as)e^{-as},$$

and

$$\int_0^t e^{C_2 s}|\beta'(s)|ds \leq C\int_0^t (1 + |s|)e^{-(a-C_2)s}ds.$$

Therefore, there exist positive constants α and m such that for all t sufficiently large, (2.5) holds.

3 Well-Posedness

In this section, we study the existence and uniqueness of the solution to system (1.1), that is, using the semigroup method to prove Theorem 2.3.

Proof of Theorem 2.3 This proof is divided into four main steps.

Step 1. The first step is to prove that the linear operators $\mathbb{A}(t)$ are dissipative. Indeed, let $U = (v_1, v_2, v_3, v_4)^T \in D(\mathbb{A}(0))$, using (2.1) and the fact of $v_1 = v_3,\ v_2 = v_4$ on Γ_1, we have

$$\begin{aligned}
\langle \mathbb{A}(t)U, U\rangle_{\mathcal{H}} = &\int_\Omega \left[\beta(t)\nabla_g v_2 \nabla v_1 - (\mathcal{A}(x,t)v_1 + b(x)v_2)v_2\right]dx \\
&+ \int_{\Gamma_1}\left[\beta(t)(\nabla_T)_g v_4(\nabla_T)v_3 - \left(\frac{\partial v_3}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)v_3 + v_4\right)v_4\right]d\Gamma \\
= &- b_0\int_\omega |v_2|^2 dx - \int_{\Gamma_1}|v_4|^2 d\Gamma \\
\leq &\,0,
\end{aligned}$$

which yields the operators $\mathbb{A}(t)$ are dissipative.
Step 2. In this step, we prove the surjection of the operators $I - \mathbb{A}(t)$, where I stands for the identity operator. In fact, set $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we will prove that there exists $U = (v_1, v_2, v_3, v_4)^T$ such that

$$(I - \mathbb{A}(t))\, U = F,$$

which is equivalent to

$$\begin{cases} v_1 - v_2 = f_1 & \text{in } H^1_{\Gamma_0}(\Omega), \\ v_2 + \mathcal{A}(x,t)v_1 + b(x)v_2 = f_2 & \text{in } L^2(\Omega), \\ v_3 - v_4 = f_3 & \text{in } H^1(\Gamma_1), \\ v_4 + \dfrac{\partial v_3}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)v_3 + v_4 = f_4 & \text{in } L^2(\Gamma_1). \end{cases} \tag{3.1}$$

From the first and third equations of (3.1), we obtain

$$\begin{cases} v_2 = v_1 - f_1, \\ v_4 = v_3 - f_3. \end{cases} \tag{3.2}$$

Substituting the equations of (3.2) into the second and the forth equations of (3.1), we have

$$\begin{cases} v_1 + \mathcal{A}(x,t)v_1 + b(x)v_1 = f_1 + f_2 + b(x)f_1, \\ 2v_3 + \dfrac{\partial v_3}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)v_3 = 2f_3 + f_4. \end{cases} \tag{3.3}$$

This is an elliptic system of two equations. For $(\varphi_1, \psi_1),\ (\varphi_2, \psi_2) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$, we introduce the following bilinear form

$$\begin{aligned} B((\varphi_1, \psi_1), (\varphi_2, \psi_2)) = & \int_\Omega \left[\varphi_1\varphi_2 + \beta(t)\nabla_g\varphi_1\nabla\varphi_2 + b(x)\varphi_1\varphi_2\right] dx \\ & + \int_{\Gamma_1} \left[2\psi_1\psi_2 + \beta(t)(\nabla_T)_g\psi_1(\nabla_T)\psi_2\right] d\Gamma. \end{aligned}$$

It is easy to show that $B((\varphi_1, \psi_1), (\varphi_2, \psi_2))$ is a bounded bilinear form and

$$\begin{aligned} B((\varphi_1, \psi_1), (\varphi_1, \psi_1)) = & \int_\Omega \left[|\varphi_1|^2 + \beta(t)|\nabla_g\varphi_1|_g^2 + b(x)|\varphi_1|^2\right] dx \\ & + \int_{\Gamma_1} \left[2|\psi_1|^2 + \beta(t)|(\nabla_T)_g\psi_1|_g^2\right] d\Gamma, \end{aligned}$$

is coercive. Then by using the conditions of $v_1 = v_3,\ \varphi = \psi$ on Γ_1, we can find $(v_1, v_3) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$, such that for all $(\varphi, \psi) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$, the following holds

$$B((v_1, v_3), (\varphi, \psi)) = \int_\Omega [f_1 + f_2 + b(x) f_1]\varphi dx + \int_{\Gamma_1} (2f_3 + f_4)\psi d\Gamma.$$

Therefore, the system (3.3) admits a unique weak solution $(v_1, v_3) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$ by the well-known Lax-Milgram theorem. And we deduce from (3.2) that $v_2 \in H^1_{\Gamma_0}(\Omega)(\hookrightarrow L^2(\Omega))$ and $v_4 \in H^1(\Gamma_1)(\hookrightarrow L^2(\Gamma_1))$. This implies that $U \in \mathcal{H}$ which gives us the desired solution.

Step 3. Define a vector valued function $h : R_+ \to \mathcal{H}$ with $h(t) = \mathbb{A}(t)U$. We will prove in this step that h is differentiable and its Frechet derivative is the vector valued function

$$h'(t) = \begin{pmatrix} 0 \\ -\mathcal{A}'(x,t)u \\ 0 \\ -\mathcal{A}'_T(x,t)v \end{pmatrix},$$

where

$$\mathcal{A}'(x,t)u = -\beta'(t)\mathrm{div}(A(x)\nabla u),$$

and

$$\mathcal{A}'_T(x,t)v = -\beta'(t)\mathrm{div}_T(A(x)|_{\Gamma_1}\nabla_T v).$$

Indeed, it is quite obvious that $h'(t) \in \mathcal{H}$ for $t \geq 0$. And for any $t, \tau \geq 0$ with $t \neq \tau$, we have

$$\frac{h(t) - h(\tau)}{t - \tau} = \frac{1}{t - \tau} \begin{pmatrix} 0 \\ -[\mathcal{A}(x,t) - \mathcal{A}(x,\tau)]u \\ 0 \\ -[\mathcal{A}_T(x,t) - \mathcal{A}_T(x,\tau)]v \end{pmatrix}.$$

Then

$$\frac{h(t) - h(\tau)}{t - \tau} - h'(t) = \begin{pmatrix} 0 \\ -\left[\frac{\mathcal{A}(x,t) - \mathcal{A}(x,\tau)}{t - \tau} - \mathcal{A}'(x,t)\right]u \\ 0 \\ -\left[\frac{\mathcal{A}_T(x,t) - \mathcal{A}_T(x,\tau)}{t - \tau} - \mathcal{A}'_T(x,t)\right]v \end{pmatrix},$$

which yields

$$\left\| \frac{h(t) - h(\tau)}{t - \tau} - h'(t) \right\|_{\mathcal{H}} = \left\| -\left[\frac{\mathcal{A}(x,t) - \mathcal{A}(x,\tau)}{t - \tau} - \mathcal{A}'(x,t)\right]u \right\|$$

$$
\begin{aligned}
&+\left\| -\left[\frac{\mathcal{A}_T(x,t)-\mathcal{A}_T(x,\tau)}{t-\tau}-\mathcal{A}_T'(x,t)\right]v\right\|_{\Gamma_1}\\
&=\left\|\left(\frac{\beta(t)-\beta(\tau)}{t-\tau}-\beta'(t)\right)\mathrm{div}\nabla_g u\right\|\\
&+\left\|\left(\frac{\beta(t)-\beta(\tau)}{t-\tau}-\beta'(t)\right)\mathrm{div}_T(\nabla_T)_g v\right\|_{\Gamma_1}.
\end{aligned}
$$

We have

$$
\lim_{t\to\tau}\left\|\frac{h(t)-h(\tau)}{t-\tau}-h'(t)\right\|_{\mathcal{H}}=0.
$$

Hence according to the chapters 5 of Pazy's book [33], we define the solution operators of the initial value problem

$$
\begin{cases}
\dfrac{d\widetilde{U}}{dt}=\mathbb{A}(t)\widetilde{U} & 0\le s<t\le T,\\
\widetilde{U}(s)=\widetilde{U}_1,
\end{cases}
\tag{3.4}
$$

by

$$
W(t,s)\widetilde{U}_1=\widetilde{U}(t),\quad 0\le s<t\le T,
$$

where $\widetilde{U}(t)$ is the solution of (3.4) and $W(t,s)$ is a two parameter family of operators. Then, the evolution equation (2.2) has a unique mild solution

$$
U(t)=W(t,0)U_0,\quad t\in[0,T_{\max}).
$$

Step 4. Let us show that $T_{\max}=\infty$. From the definition of energy (2.4), we have

$$
E'(t)=\frac{1}{2}\beta'(t)\left\{\int_\Omega|\nabla_g u|_g^2dx+\int_{\Gamma_1}|(\nabla_T)_g u|_g^2d\Gamma\right\}-b_0\int_\omega u_t^2dx-\int_{\Gamma_1}u_t^2d\Gamma,
$$

and

$$
\begin{aligned}
|E'(t)|&\le\frac{|\beta'(t)|}{2}\left\{\int_\Omega|\nabla_g u|_g^2dx+\int_{\Gamma_1}|(\nabla_T)_g u|_g^2d\Gamma\right\}\\
&\le\frac{|\beta'(t)|\beta(t)}{2\beta_0}\left\{\int_\Omega|\nabla_g u|_g^2dx+\int_{\Gamma_1}|(\nabla_T)_g u|_g^2d\Gamma\right\}\\
&\le\frac{|\beta'(t)|}{\beta_0}E(t).
\end{aligned}
$$

The above inequality gives

$$
-\frac{|\beta'(t)|}{\beta_0}E(t)\le E'(t)\le\frac{|\beta'(t)|}{\beta_0}E(t).
\tag{3.5}
$$

Let

$$\beta_1 = \int_0^\infty \frac{|\beta'(s)|}{\beta_0} ds,$$

we have

$$e^{-\beta_1} E(0) \le E(t) \le e^{\beta_1} E(0), \quad t \in [0, T_{\max}). \tag{3.6}$$

Then using (2.1), (2.4) and (3.6), we have

$$\|U\|_{\mathcal{H}}^2 = 2E(t) \le 2e^{\beta_1} E(0), \quad t \in [0, T_{\max}),$$

where $U = (u, u_t, \gamma_1(u), \gamma_1(u_t))^T \in \mathcal{H}$. Therefore, the local solution cannot blow-up in finite time and it follows that $T_{\max} = \infty$. □

Remark 3.1 It is worth noting that in this paper we assumes that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, so we cannot use the elliptic regularity argument to get $u \in H^2(\Omega)$ and $\gamma_1(u) \in H^2(\Gamma_1)$ from

$$\mathcal{A}(x,t)u \in L^2(\Omega) \quad \text{and} \quad \mathcal{A}_T(x,t)\gamma_1(u) \in L^2(\Gamma_1).$$

Then, for the more regular initial data $U_0 \in D(\mathbb{A}(0))$, we do not have a more regular solution $U \in C(R_+; D(\mathbb{A}(0))) \cap C^1(R_+; \mathcal{H})$.

4 Decay Result

Because of the existence of singularities, we need to avoid them in the following work. As in [6], let $\delta > 0$ be a small and fixed constant and consider

$$B_\delta = \bigcup_{x \in \Sigma} B(x, \delta),$$

where $B(x, \delta) = \{y \in \Omega : |x - y| < \delta\}$. Denote

$$\Omega_\delta = \Omega \setminus B_\delta.$$

Next, we will study the stability result of the corresponding system in Ω_δ (see Fig. 2), whose boundary is defined as

$$\partial\Omega_\delta = \widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1 \cup \Gamma_\delta,$$

where

$$\widetilde{\Gamma}_0 = \partial\Omega_\delta \cap \Gamma_0, \quad \widetilde{\Gamma}_1 = \partial\Omega_\delta \cap \Gamma_1,$$

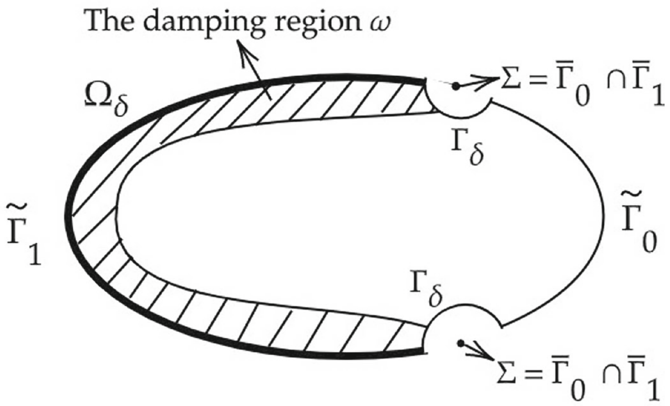

Fig. 2 The new domain Ω_δ for $n = 2$

and

$$\Gamma_\delta = \partial B_\delta \cap \Omega.$$

Then the system (1.1) is transformed into the following system

$$\begin{cases} u_{tt} + \mathcal{A}(x,t)u + b(x)u_t = 0 & \text{in } \Omega_\delta \times R^+, \\ u = 0 & \text{on } \widetilde{\Gamma}_0 \times R^+, \\ u_{tt} + \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)u + u_t = 0 & \text{on } \widetilde{\Gamma}_1 \times R^+. \end{cases} \tag{4.1}$$

Define the energy associated with the problem (4.1) in Ω_δ by

$$E_\delta(t) := \frac{1}{2}\left\{\int_{\Omega_\delta} |u_t|^2 dx + \beta(t)\int_{\Omega_\delta} |\nabla_g u|_g^2 dx + \int_{\widetilde{\Gamma}_1} |u_t|^2 d\Gamma + \beta(t)\int_{\widetilde{\Gamma}_1} |(\nabla_T)_g u|_g^2 d\Gamma\right\}, \tag{4.2}$$

then by a simple calculation, we have

$$E'_\delta(t) = \frac{1}{2}\beta'(t)\left\{\int_{\Omega_\delta} |\nabla_g u|_g^2 dx + \int_{\widetilde{\Gamma}_1} |(\nabla_T)_g u|_g^2 d\Gamma\right\} - b_0\int_\omega u_t^2 dx - \int_{\widetilde{\Gamma}_1} u_t^2 d\Gamma + \int_{\Gamma_\delta} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u_t d\Gamma. \tag{4.3}$$

Lemma 4.1 *The functional*

$$\Phi_\delta(t) := \int_{\Omega_\delta} u_t u dx + \int_{\widetilde{\Gamma}_1} u_t u d\Gamma \tag{4.4}$$

satisfies

$$\begin{aligned}\Phi_\delta'(t) \le &-\left(1-\frac{b_0C_p\varepsilon_1}{\lambda\beta_0}-\frac{C_t\varepsilon_2}{\lambda\beta_0}\right)\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx-\int_{\widetilde{\Gamma}_1}\beta(t)|(\nabla_T)_g u|_g^2d\Gamma\\ &+\left(1+\frac{b_0}{4\varepsilon_1}\right)\int_{\Omega_\delta}|u_t|^2dx+\left(1+\frac{1}{4\varepsilon_2}\right)\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma+\int_{\Gamma_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}ud\Gamma,\end{aligned} \tag{4.5}$$

where ε_1 and ε_2 are some positive constants, C_t is the smallest possible positive constant produced by the trace theorem and Poincaré inequality and $C_p > 0$ is the Poincaré optimal embedding constant.

Proof Taking the derivative of (4.4) with respect to the variable t and using (4.1) and Green formula yield

$$\begin{aligned}\Phi_\delta'(t) &= \int_{\Omega_\delta}|u_t|^2dx+\int_{\Omega_\delta}u_{tt}udx+\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma+\int_{\widetilde{\Gamma}_1}u_{tt}ud\Gamma\\ &=\int_{\Omega_\delta}|u_t|^2dx+\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma\\ &\quad-\int_{\Omega_\delta}[\mathcal{A}(x,t)u+b(x)u_t]\,udx-\int_{\widetilde{\Gamma}_1}\left[\frac{\partial u}{\partial\nu_{\mathcal{A}}}+\mathcal{A}_T(x,t)u+u_t\right]ud\Gamma\\ &=\int_{\Omega_\delta}|u_t|^2dx+\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma-\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx-\int_{\widetilde{\Gamma}_1}\beta(t)|(\nabla_T)_g u|_g^2d\Gamma\\ &\quad-\int_{\Omega_\delta}b(x)u_tudx-\int_{\widetilde{\Gamma}_1}u_tud\Gamma+\int_{\Gamma_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}ud\Gamma.\end{aligned} \tag{4.6}$$

Making use of Young's inequality, Poincaré inequality, the trace theorem and (**H1**), we have

$$\begin{aligned}\left|\int_{\Omega_\delta}b(x)u_tudx\right| &\le b_0\int_{\Omega_\delta}|u_tu|dx\\ &\le\frac{b_0}{4\varepsilon_1}\int_{\Omega_\delta}|u_t|^2dx+b_0\varepsilon_1\int_{\Omega_\delta}|u|^2dx\\ &\le\frac{b_0}{4\varepsilon_1}\int_{\Omega_\delta}|u_t|^2dx+\frac{b_0C_p\varepsilon_1}{\lambda\beta_0}\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\left|\int_{\widetilde{\Gamma}_1}u_tud\Gamma\right| &\le\int_{\widetilde{\Gamma}_1}|u_tu|d\Gamma\\ &\le\frac{1}{4\varepsilon_2}\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma+\frac{C_t\varepsilon_2}{\lambda\beta_0}\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx.\end{aligned} \tag{4.8}$$

Thus, taking (4.7) and (4.8) into (4.6), (4.5) is obtained. □

Lemma 4.2 *Suppose that H is the vector field defined in assumption (**H2**), and define $H_T = H - (H \cdot \nu)\nu$. Then, the functional*

$$\Psi_\delta(t) := 2\int_{\Omega_\delta} u_t H(u)dx + 2\int_{\widetilde{\Gamma}_1} u_t H_T(u)d\Gamma \tag{4.9}$$

satisfies

$$\begin{aligned}
\Psi_\delta'(t) \leq & -(n\sigma_1 - b_0\epsilon_1)\int_{\Omega_\delta} |u_t|^2 dx - (c_\sigma - \epsilon_2)\int_{\widetilde{\Gamma}_1} |u_t|^2 d\Gamma \\
& + \left(n\sigma_2 - 2\sigma_1 + \frac{\max_{x\in\bar{\Omega}_\delta} |H|^2 b_0}{2\lambda\beta_0\epsilon_1}\right)\int_{\Omega_\delta} \beta(t)|\nabla_g u|_g^2 dx \\
& + \left(\frac{\|\nabla_T A\|_{L^\infty(\widetilde{\Gamma}_1)} \max_{x\in\bar{\widetilde{\Gamma}}_1} |H_T|}{\lambda} + \frac{\max_{x\in\bar{\widetilde{\Gamma}}_1} |H_T|^2}{2\lambda\beta_0\epsilon_2} - c_\sigma\right)\int_{\widetilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 d\Gamma \\
& + \int_{\widetilde{\Gamma}_1}\left[|u_t|^2 + \frac{\beta(t)}{|\nu_{\mathcal{A}}|_g^2}\left|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\right|^2\right] H\cdot\nu d\Gamma \\
& + \int_{\Gamma_\delta} [|u_t|^2 - \beta(t)|\nabla_g u|_g^2] H\cdot\nu d\Gamma + 2\int_{\Gamma_\delta} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u)d\Gamma,
\end{aligned} \tag{4.10}$$

where ϵ_1, ϵ_2 and c_σ are some positive constants.

Proof Taking the derivative of (4.9) with respect to the variable t and using (4.1) and Green formula yield

$$\begin{aligned}
\Psi_\delta'(t) = & 2\int_{\Omega_\delta} u_t H(u_t)dx + 2\int_{\Omega_\delta} u_{tt} H(u)dx \\
& + 2\int_{\widetilde{\Gamma}_1} u_t H_T(u_t)d\Gamma + 2\int_{\widetilde{\Gamma}_1} u_{tt} H_T(u)d\Gamma \\
= & \int_{\Omega_\delta} H(u_t^2)dx - 2\int_{\Omega_\delta} [\mathcal{A}(x,t)u + b(x)u_t]\, H(u)dx \\
& + \int_{\widetilde{\Gamma}_1} H_T(u_t^2)d\Gamma - 2\int_{\widetilde{\Gamma}_1}\left[\frac{\partial u}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T(x,t)u + u_t\right] H_T(u)d\Gamma \\
= & \int_{\Omega_\delta} H(u_t^2)dx - 2\int_{\Omega_\delta} \beta(t)\nabla_g u \nabla(H(u))dx + 2\int_{\partial\Omega_\delta} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u)d\Gamma \\
& - 2\int_{\Omega_\delta} b(x)u_t H(u)dx + \int_{\widetilde{\Gamma}_1} H_T(u_t^2)d\Gamma - 2\int_{\widetilde{\Gamma}_1} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H_T(u)d\Gamma \\
& - 2\int_{\widetilde{\Gamma}_1} \beta(t)(\nabla_T)_g u \nabla_T(H_T(u))d\Gamma - 2\int_{\widetilde{\Gamma}_1} u_t H_T(u)d\Gamma.
\end{aligned} \tag{4.11}$$

We begin to deal with the terms on the right of (4.11). Since the assumption (**H2**), we have

$$n\sigma_1 \leq \text{div} H \leq n\sigma_2,$$

then there are positive constants c_σ and C_σ such that

$$c_\sigma \le \operatorname{div}_T H_T \le C_\sigma.$$

Using the divergence theorem, (**H2**) and the fact of $H_T \cdot \nu = 0$, we have

$$\begin{aligned}
&\int_{\Omega_\delta} H(u_t^2)dx + \int_{\widetilde{\Gamma}_1} H_T(u_t^2)d\Gamma \\
&= \int_{\Omega_\delta} \operatorname{div}(u_t^2 H)dx - \int_{\Omega_\delta} |u_t|^2 \operatorname{div} H dx - \int_{\widetilde{\Gamma}_1} |u_t|^2 \operatorname{div}_T H_T d\Gamma \\
&= \int_{\partial\Omega_\delta} |u_t|^2 H \cdot \nu d\Gamma - \int_{\Omega_\delta} |u_t|^2 \operatorname{div} H dx - \int_{\widetilde{\Gamma}_1} |u_t|^2 \operatorname{div}_T H_T d\Gamma \\
&\le \int_{\widetilde{\Gamma}_1} |u_t|^2 H \cdot \nu d\Gamma + \int_{\Gamma_\delta} |u_t|^2 H \cdot \nu d\Gamma \\
&\quad - n\sigma_1 \int_{\Omega_\delta} |u_t|^2 dx - c_\sigma \int_{\widetilde{\Gamma}_1} |u_t|^2 d\Gamma. \qquad (4.12)
\end{aligned}$$

Using Lemma 2.1 and (**H2**), we get

$$\begin{aligned}
&-2\int_{\Omega_\delta} \beta(t)\nabla_g u \nabla(H(u))dx - 2\int_{\widetilde{\Gamma}_1} \beta(t)(\nabla_T)_g u \nabla_T(H_T(u))d\Gamma \\
&= -2\int_{\Omega_\delta} \beta(t)\langle \nabla_g u, \nabla_g(H(u))\rangle_g dx \\
&\quad - \int_{\widetilde{\Gamma}_1} \beta(t)A(x)|_{\Gamma_1} \nabla_T \left(|\nabla_T u|^2\right) \cdot H_T d\Gamma - 2\int_{\widetilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 \operatorname{div}_T H_T d\Gamma \\
&= -2\int_{\Omega_\delta} \beta(t) DH(\nabla_g u, \nabla_g u)dx - \int_{\partial\Omega_\delta} \beta(t)|\nabla_g u|_g^2 H \cdot \nu d\Gamma \\
&\quad + \int_{\Omega_\delta} \beta(t)|\nabla_g u|_g^2 \operatorname{div} H dx \\
&\quad + \int_{\widetilde{\Gamma}_1} \beta(t)\left(\nabla_T A(x)|_{\Gamma_1}\right)|\nabla_T u|^2 \cdot H_T d\Gamma - \int_{\widetilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 \operatorname{div}_T H_T d\Gamma \\
&\le (n\sigma_2 - 2\sigma_1)\int_{\Omega_\delta} \beta(t)|\nabla_g u|_g^2 dx - \int_{\partial\Omega_\delta} \beta(t)|\nabla_g u|_g^2 H \cdot \nu d\Gamma \\
&\quad + \left(\frac{\|\nabla_T A(x)|_{\Gamma_1}\|_{L^\infty(\widetilde{\Gamma}_1)} \max_{x\in\tilde{\widetilde{\Gamma}}_1} |H_T|}{\lambda} - c_\sigma\right)\int_{\widetilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 d\Gamma. \qquad (4.13)
\end{aligned}$$

And using Young's inequality, we have

$$-2\int_{\Omega_\delta} b(x)u_t H(u)dx \le 2b_0\int_{\Omega_\delta}|u_t H(u)|dx \le b_0\epsilon_1\int_{\Omega_\delta}|u_t|^2dx + \frac{\max_{x\in\bar{\Omega}_\delta}|H|^2 b_0}{2\lambda\beta_0\epsilon_1}\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx, \tag{4.14}$$

and

$$-2\int_{\widetilde{\Gamma}_1} u_t H_T(u)dx \le \epsilon_2\int_{\widetilde{\Gamma}_1}|u_t|^2dx + \frac{\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|^2}{2\lambda\beta_0\epsilon_2}\int_{\widetilde{\Gamma}_1}\beta(t)|(\nabla_T)_g u|_g^2 d\Gamma, \tag{4.15}$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are some constants. Substituting (4.12)–(4.15) into (4.11), we obtain

$$\begin{aligned}\Psi_\delta'(t) \le & -(n\sigma_1 - b_0\epsilon_1)\int_{\Omega_\delta}|u_t|^2dx - (c_\sigma - \epsilon_2)\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma \\ & + \left(n\sigma_2 - 2\sigma_1 + \frac{\max_{x\in\bar{\Omega}_\delta}|H|^2 b_0}{2\lambda\beta_0\epsilon_1}\right)\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx \\ & + \left(\frac{\|\nabla_T A(x)|_{\Gamma_1}\|_{L^\infty(\widetilde{\Gamma}_1)}\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|}{\lambda}\right. \\ & \left. + \frac{\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|^2}{2\lambda\beta_0\epsilon_2} - c_\sigma\right)\int_{\widetilde{\Gamma}_1}\beta(t)|(\nabla_T)_g u|_g^2 d\Gamma \\ & + \int_{\widetilde{\Gamma}_1}|u_t|^2 H\cdot\nu d\Gamma - 2\int_{\widetilde{\Gamma}_1}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H_T(u)d\Gamma + \int_{\Gamma_\delta}|u_t|^2H\cdot\nu d\Gamma \\ & - \int_{\partial\Omega_\delta}\beta(t)|\nabla_g u|_g^2 H\cdot\nu d\Gamma + 2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\Gamma. \end{aligned} \tag{4.16}$$

Now, we are going to estimate the integrals over $\partial\Omega_\delta$. On $\widetilde{\Gamma}_0\cup\widetilde{\Gamma}_1$, we have

$$|\nabla_g u|_g^2 = \frac{1}{|\nu_{\mathcal{A}}|_g^2}\left|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right|_g^2 + |(\nabla_T)_g u|_g^2,$$

and

$$\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u) = \frac{\partial u}{\partial\nu_{\mathcal{A}}}H_T(u) + \frac{\beta(t)}{|\nu_{\mathcal{A}}|_g^2}\left|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right|_g^2 H\cdot\nu.$$

Since $u = u_t = 0$ on $\widetilde{\Gamma}_0$, it hold that

$$\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u) = \beta(t)|\nabla_g u|_g^2 H\cdot\nu, \quad x\in\widetilde{\Gamma}_0,$$

then

$$\begin{aligned}
&-\int_{\partial\Omega_\delta}\beta(t)|\nabla_g u|_g^2 H\cdot\nu d\Gamma+2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\Gamma\\
&=\int_{\widetilde{\Gamma}_0}\beta(t)|\nabla_g u|_g^2 H\cdot\nu d\Gamma\\
&\quad+\int_{\widetilde{\Gamma}_1}\left[2\frac{\partial u}{\partial\nu_{\mathcal{A}}}H_T(u)-\beta(t)|(\nabla_T)_g u|_g^2 H\cdot\nu+\frac{\beta(t)}{|\nu_{\mathcal{A}}|_g^2}\left|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right|^2 H\cdot\nu\right]d\Gamma\\
&\quad-\int_{\Gamma_\delta}\beta(t)|\nabla_g u|_g^2 H\cdot\nu d\Gamma+2\int_{\Gamma_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\Gamma.
\end{aligned}\tag{4.17}$$

Combining (4.16) and (4.17), we infer that

$$\begin{aligned}
\Psi_\delta'(t)\le&-(n\sigma_1-b_0\epsilon_1)\int_{\Omega_\delta}|u_t|^2dx-(c_\sigma-\epsilon_2)\int_{\widetilde{\Gamma}_1}|u_t|^2d\Gamma\\
&+\left(n\sigma_2-2\sigma_1+\frac{\max_{x\in\bar{\Omega}_\delta}|H|^2b_0}{2\lambda\beta_0\epsilon_1}\right)\int_{\Omega_\delta}\beta(t)|\nabla_g u|_g^2dx\\
&+\left(\frac{\|\nabla_T A(x)|_{\Gamma_1}\|_{L^\infty(\widetilde{\Gamma}_1)}\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|}{\lambda}\right.\\
&\left.+\frac{\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|^2}{2\lambda\beta_0\epsilon_2}-c_\sigma\right)\int_{\widetilde{\Gamma}_1}\beta(t)|(\nabla_T)_g u|_g^2d\Gamma+\int_{\widetilde{\Gamma}_0}\beta(t)|\nabla_g u|_g^2H\cdot\nu d\Gamma\\
&+\int_{\widetilde{\Gamma}_1}\left[|u_t|^2-\beta(t)|(\nabla_T)_g u|_g^2+\frac{\beta(t)}{|\nu_{\mathcal{A}}|_g^2}\left|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right|^2\right]H\cdot\nu d\Gamma\\
&+\int_{\Gamma_\delta}\left[|u_t|^2-\beta(t)|\nabla_g u|_g^2\right]H\cdot\nu d\Gamma+2\int_{\Gamma_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\Gamma.
\end{aligned}\tag{4.18}$$

Therefore, using (**H2**), we obtain (4.10). □

Next we define the perturbed energy associated with Ω_δ by

$$E_{\delta,\theta}(t):=NE_\delta(t)+\Phi_\delta(t)+\theta\Psi_\delta(t),\tag{4.19}$$

where N and θ are some positive constants.

Lemma 4.3 *If*

$$E_\delta\to E,\quad \Phi_\delta\to\Phi\quad\text{and}\quad\Psi_\delta\to\Psi,\quad\text{as }\delta\to 0,$$

then

$$E_{\delta,\theta}\to E_\theta,\quad\text{as }\delta\to 0,$$

where

$$\Phi(t) := \int_\Omega u_t u dx + \int_{\Gamma_1} u_t u d\Gamma,$$
$$\Psi(t) := 2\int_\Omega u_t H(u) dx + 2\int_{\Gamma_1} u_t H_T(u) d\Gamma,$$

and

$$E_\theta(t) := NE(t) + \Phi(t) + \theta\Psi(t).$$

Proof Using Lebesgue dominated convergence theorem, we can easily obtain Lemma 4.3. □

Obviously, for N large enough, we have

$$E_\delta \sim E_{\delta,\theta}.$$

Moreover, due to (3.6), there exists a positive constant β_2 such that

$$E_\delta(t) \le \beta_2, \quad t \ge 0. \tag{4.20}$$

Proof of Theorem 2.4 Differentiating (4.19) and taking (4.3), (4.5) and (4.10) into it, we have

$$\begin{aligned}
E'_{\delta,\theta}(t) &= NE'_\delta(t) + \Phi'_\delta(t) + \theta\Psi'_\delta(t) \\
&\le -\left[Nb_0 + (n\sigma_1 - b_0\epsilon_1)\theta - \left(1 + \frac{b_0}{4\varepsilon_1}\right)\right]\int_{\Omega_\delta} |u_t|^2 dx \\
&\quad - \left[N + (c_\sigma - \epsilon_2)\theta - \left(1 + \frac{1}{4\varepsilon_2}\right) - C_H\theta\right]\int_{\tilde{\Gamma}_1} |u_t|^2 d\Gamma \\
&\quad - \left[\left(1 - \frac{b_0 C_p \varepsilon_1}{\lambda\beta_0} - \frac{C_t\varepsilon_2}{\lambda\beta_0}\right)\right. \\
&\quad \left. - \left(n\sigma_2 - 2\sigma_1 + \frac{\max_{x\in\bar{\Omega}_\delta} |H|^2 b_0}{2\lambda\beta_0\epsilon_1} + C_H\right)\theta\right]\int_{\Omega_\delta} \beta(t)|\nabla_g u|_g^2 dx \\
&\quad - \left[1 - \left(\frac{\|\nabla_T A\|_{L^\infty(\tilde{\Gamma}_1)} \max_{x\in\bar{\tilde{\Gamma}}_1} |H_T|}{\lambda}\right.\right. \\
&\quad \left.\left. + \frac{\max_{x\in\bar{\tilde{\Gamma}}_1} |H_T|^2}{2\lambda\beta_0\epsilon_2} - c_\sigma\right)\theta\right]\int_{\tilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 d\Gamma \\
&\quad + \frac{N|\beta'(t)|}{2\beta_0}\int_{\Omega_\delta} \beta(t)|\nabla_g u|_g^2 dx + \frac{N|\beta'(t)|}{2\beta_0}\int_{\tilde{\Gamma}_1} \beta(t)|(\nabla_T)_g u|_g^2 d\Gamma \\
&\quad + N\int_{\Gamma_\delta} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u_t d\Gamma + \int_{\Gamma_\delta} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u d\Gamma
\end{aligned}$$

$$+\theta\int_{\Gamma_\delta}\left[|u_t|^2-\beta(t)|\nabla_g u|_g^2\right]H\cdot\nu d\Gamma+2\theta\int_{\Gamma_\delta}\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\Gamma, \quad (4.21)$$

where $C_H > 0$ is some constant depending on H. Thus, we choose ε_1, ε_2, ϵ_1 and ϵ_2 small enough such that

$$n\sigma_1 - b_0\epsilon_1 > 0,$$
$$c_\sigma - \epsilon_2 > 0,$$

and

$$1-\frac{b_0C_p\varepsilon_1}{\lambda\beta_0}-\frac{C_tC_p\varepsilon_2}{\lambda\beta_0}>0.$$

Fixing them, choosing θ sufficiently small and N lager enough such that

$$c_1 := \left(1-\frac{b_0C_p\varepsilon_1}{\lambda\beta_0}-\frac{C_t\varepsilon_2}{\lambda\beta_0}\right)-\left(n\sigma_2-2\sigma_1+\frac{\max_{x\in\bar{\Omega}_\delta}|H|^2b_0}{2\lambda\beta_0\epsilon_1}\right)\theta>0,$$
$$c_2 := 1-\left(\frac{\|\nabla_T A\|_{L^\infty(\widetilde{\Gamma}_1)}\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|}{\lambda}+\frac{\max_{x\in\bar{\widetilde{\Gamma}}_1}|H_T|^2}{2\lambda\beta_0\epsilon_2}-c_\sigma\right)\theta>0,$$
$$c_3 := Nb_0+(n\sigma_1-b_0\epsilon_1)\theta-\left(1+\frac{b_0}{4\varepsilon_1}\right)>0,$$

and

$$c_4 := N+(c_\sigma-\epsilon_2)\theta-\left(1+\frac{1}{4\varepsilon_2}\right)-C_H\theta>0.$$

Therefore, combining (4.20) and (4.21) we obtain

$$E'_{\delta,\theta}(t)\le -C_1E_\delta(t)+C|\beta'(t)|+\theta\Theta_\delta(t)+\Lambda_\delta(t), \quad (4.22)$$

where

$$C_1 := \min\{c_1,c_2,c_3,c_4\}>0,$$
$$\Theta_\delta(t) := \int_{\Gamma_\delta}\left[2\frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)-\beta(t)|\nabla_g u|_g^2H\cdot\nu\right]d\Gamma,$$

and

$$\Lambda_\delta(t) := \int_{\Gamma_\delta}\left[\frac{\partial u}{\partial\nu_{\mathcal{A}}}u+N\frac{\partial u}{\partial\nu_{\mathcal{A}}}u_t+\theta|u_t|^2H\cdot\nu\right]d\Gamma.$$

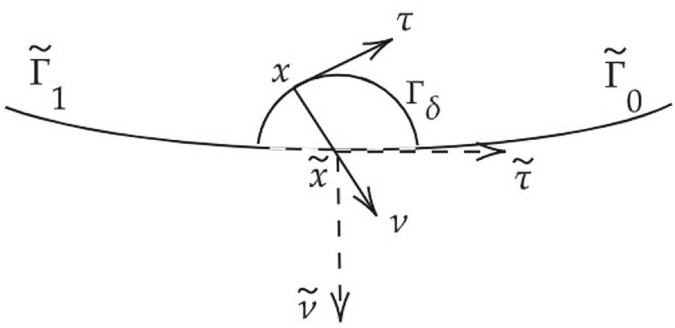

Fig. 3 The vectors ν, τ, $\tilde{\nu}$ and $\tilde{\tau}$ for $n = 2$

Then, integrating (4.22) from 0 to t, we obtain

$$E_{\delta,\theta}(t) \le C\left[E_{\delta,\theta}(0) + \int_0^t e^{C_2 s}|\beta'(s)|ds + \int_0^t e^{C_2 s}\left[\Theta_\delta(s) + \Lambda_\delta(s)\right]ds\right]e^{-C_2 t}, \tag{4.23}$$

where $C_2 > 0$ is some constant.

• **Estimate for** $L_\delta(t) := \displaystyle\int_0^t e^{C_2 s}\Theta_\delta(s)ds$

(i) For $n = 2$, at each point $\tilde{x} \in \Sigma$, we can build the vectors $\tilde{\nu} = \nu(\tilde{x})$ and $\tilde{\tau} = \tau(\tilde{x})$ which depend on $\tilde{x}$ and Ω. $\tilde{\nu}$ is the unit normal vector pointing towards the exterior and $\tilde{\tau}$ is the tangent vector pointing towards from Γ_1 to Γ_0. If δ is small enough, each point $x \in \Gamma_\delta$ belongs to one and only one coordinate system $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$ and x belongs to some arc of circle $\partial B(\tilde{x}, \delta)$ contained in this coordinate system (see Fig. 3).

We decompose the solution

$$u(x) = u_1(x) + u_2(x) := u_1(x) + \eta(\tilde{x})U_2(x - \tilde{x}),$$

where $u_1 \in H^2(\Omega)$, η is locally in $H^{\frac{1}{2}}(\Sigma)$ and U_2 is given by

$$U_2(x - \tilde{x}) = U_2(r, \theta) = \rho(r)\sqrt{r}\sin\left(\frac{\theta}{2}\right),$$

where ρ is a C^∞-function with compact support such that $\rho(r) = 1$ in some neighbourhood of 0 and $\mathrm{supp}(\rho) \subset [-\varrho, \varrho] \subset (-1, 1)$, where $\varrho > 0$ is as small as we want. Then, using coordinate in $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$, similar to [4], we have

$$2\frac{\partial U_2}{\partial \nu_{\mathcal{A}}}H(U_2) - \beta(t)|\nabla_g U_2|_g^2 H \cdot \nu = \frac{\beta(t)A(x)}{4}\left(\frac{1}{\delta}\tilde{H} \cdot \tilde{\tau} - \nu \cdot \tilde{\tau}\right), \tag{4.24}$$

where $\tilde{H} = H(\tilde{x})$ with $\tilde{H} \cdot \tilde{\nu} = 0$. When $\delta \to 0$, $\partial B(\tilde{x}, \delta)$ behaves as a half-circle, then we obtain

$$\frac{1}{\pi\delta} \int_{\partial B(\tilde{x},\delta)} A(x) d\Gamma \to A(\tilde{x}) = \tilde{A} \quad \text{as} \quad \delta \to 0.$$

Integrating (4.24) along $\partial B(\tilde{x}, \delta)$, we get

$$\int_{\partial B(\tilde{x},\delta)} \left[2 \frac{\partial U_2}{\partial \nu_{\mathcal{A}}} H(U_2) - \beta(t) |\nabla_g U_2|_g^2 H \cdot \nu \right] d\Gamma \to \frac{\pi \beta(t)}{4} \tilde{A} \tilde{H} \cdot \tilde{\tau} \quad \text{as} \quad \delta \to 0.$$

Therefore, integrating the regular part of solution u_1 on $\partial B(\tilde{x}, \delta)$ and using Cauchy-Schwarz inequality, Lebesgue dominated convergence theorem, (2.3) and the compactness of function ρ, we obtain that $L_\delta(t)$ has a limit of non-positive number as $\delta \to 0$.

(ii) For $n \geq 3$, when δ is small enough, we also know that each point $x \in \Gamma_\delta$ belongs to one and only one plane defined by $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$, and x belongs to some arc of circle $l(\tilde{x}, \delta)$ contained in this plane (the figure is similar to Fig. 3).

Firstly, just like in the case of $n = 2$, we write

$$u(x) = u_1(x) + u_2(x),$$

then

$$\begin{aligned}
&\int_{\partial B(\tilde{x},\delta)} \left[2 \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) - \beta(t) |\nabla_g u|_g^2 H \cdot \nu \right] d\Gamma \\
&= \int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) \left[2 \nabla u \cdot \nu \nabla u \cdot H - |\nabla u|^2 H \cdot \nu \right] d\Gamma \\
&= \int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) \left[2 \nabla (u_1 + u_2) \cdot \nu \nabla (u_1 + u_2) \cdot H - |\nabla (u_1 + u_2)|^2 H \cdot \nu \right] d\Gamma \\
&= \underbrace{\int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) \left[2 \nabla u_1 \cdot \nu \nabla u_1 \cdot H - |\nabla u_1|^2 H \cdot \nu \right] d\Gamma}_{I_\delta(\nabla u_1)} \\
&\quad + \underbrace{\int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) \left[2 \nabla u_2 \cdot \nu \nabla u_2 \cdot H - |\nabla u_2|^2 H \cdot \nu \right] d\Gamma}_{I_\delta(\nabla u_2)} \\
&\quad + \underbrace{2 \int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) \left[\nabla u_1 \cdot \nu \nabla u_2 \cdot H + \nabla u_2 \cdot \nu \nabla u_1 \cdot H - \nabla u_1 \cdot \nabla u_2 H \cdot \nu \right] d\Gamma}_{J_\delta(\nabla u_1, \nabla u_2)}.
\end{aligned}$$

Since $u_1 \in H^2(\Omega)$ and $\text{meas}(\partial B(\tilde{x}, \delta)) \to 0$ as $\delta \to 0$ and using the regularity of $\beta(t)$ and $A(x)$, it is easy to get

$$I_\delta(\nabla u_1) \to 0 \quad \text{as } \delta \to 0.$$

Further, let us do the decomposition of ∇u_2:

$$\nabla u_2 = \nabla_X u_2 + \nabla_2 u_2,$$

where $\nabla_2 u_2$ belongs to the plane $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$ defined above and $\nabla_X u_2$ is orthogonal to $\nabla_2 u_2$, i.e.,

$$\nabla_X u_2 \cdot \nabla_2 u_2 = 0 \quad \text{and} \quad |\nabla u_2|^2 = |\nabla_X u_2|^2 + |\nabla_2 u_2|^2.$$

Then

$$\int_{\partial B(\tilde{x},\delta)} |\nabla u_2|^2 d\Gamma = \int_{\partial B(\tilde{x},\delta)} |\nabla_X u_2|^2 d\Gamma + \int_{\partial B(\tilde{x},\delta)} |\nabla_2 u_2|^2 d\Gamma.$$

Thanks to the first part of Theorem 4 in [4] and the Lebesgue dominated convergence theorem, we get

$$\int_{\partial B(\tilde{x},\delta)} |\nabla_X u_2|^2 d\Gamma \to 0 \quad \text{as } \delta \to 0.$$

And as $u_2(x) = \eta(\tilde{x}) U_2(x - \tilde{x})$, using the Fubini's theorem, we have

$$\int_{\partial B(\tilde{x},\delta)} |\nabla_2 u_2|^2 d\Gamma = \int_{\Sigma} \eta^2 \int_{l(\tilde{x},\delta)} |\nabla_2 U_2|^2 dl d\Gamma(\tilde{x}),$$

and know that this integral is bounded using $\eta \in H^{\frac{1}{2}}(\Sigma)$ and the definition of U_2. So we can end up with

$$\int_{\partial B(\tilde{x},\delta)} |\nabla u_2|^2 d\Gamma \leq C.$$

Therefore, making use of Cauchy-Schwarz inequality, we obtain

$$|J_\delta(\nabla u_1, \nabla u_2)| \leq C \left(\int_{\partial B(\tilde{x},\delta)} |\nabla u_1|^2 d\Gamma \right)^{\frac{1}{2}} \left(\int_{\partial B(\tilde{x},\delta)} |\nabla u_2|^2 d\Gamma \right)^{\frac{1}{2}}.$$

It tends to zero since the first term vanishes as $\delta \to 0$ and the second one is bounded. And finally we start to deal with the term $I_\delta(\nabla u_2)$. Similar to the above process, we decompose ∇u_2 and have

$$I_\delta(\nabla u_2) = \underbrace{\int_{\partial B(\tilde{x},\delta)} \beta(t) A(x) [2 \nabla_X u_2 \cdot \nu \nabla_X u_2 \cdot H - |\nabla_X u_2|^2] d\Gamma}_{I_\delta(\nabla_X u_2)}$$

$$+\underbrace{\int_{\partial B(\tilde{x},\delta)} \beta(t)A(x)[2\nabla_2 u_2 \cdot \nu \nabla_2 u_2 \cdot H - |\nabla_2 u_2|^2 H \cdot \nu] d\Gamma}_{I_\delta(\nabla_2 u_2)}$$

$$+2\underbrace{\int_{\partial B(\tilde{x},\delta)} \beta(t)A(x)[\nabla_X u_2 \cdot \nu \nabla_2 u_2 \cdot H + \nabla_2 u_2 \cdot \nu \nabla_X u_2 \cdot H] d\Gamma}_{J_\delta(\nabla_X u_2, \nabla_2 u_2)}.$$

As above, $I_\delta(\nabla_X u_2) \to 0$ as $\delta \to 0$. Since

$$|J_\delta(\nabla_X u_2, \nabla_2 u_2)| \le C \left(\int_{\partial B(\tilde{x},\delta)} |\nabla_X u_2|^2 d\Gamma \right)^{\frac{1}{2}} \left(\int_{\partial B(\tilde{x},\delta)} |\nabla_2 u_2|^2 d\Gamma \right)^{\frac{1}{2}},$$

we have the first term vanishes as $\delta \to 0$ and the second one is bounded, thus we obtain $J_\delta(\nabla_X u_2, \nabla_2 u_2)$ tends to zero. For $I_\delta(\nabla_2 u_2)$, as the case of $n = 2$, we also have

$$2\nabla_2 U_2 \cdot \nu \nabla_2 U_2 \cdot H - |\nabla_2 U_2|^2 H \cdot \nu = \frac{1}{4} \left(\frac{1}{\delta} \tilde{H} \cdot \tilde{\tau} - \nu \cdot \tilde{\tau} \right),$$

and

$$\int_{l(\tilde{x},\delta)} \beta(t)A(x)[2\nabla_2 U_2 \cdot \nu \nabla_2 U_2 \cdot H - |\nabla_2 U_2|^2 H \cdot \nu] dl \to \frac{\pi \beta(t)}{4} \tilde{A} \tilde{H} \cdot \tilde{\tau} \quad \text{as } \delta \to 0.$$

In other words, this integral term on $l(\tilde{x}, \delta)$ is bounded. Then, the dominated convergence theorem can be used to get

$$I_\delta(\nabla_2 u_2) \to \frac{\pi \beta(t)}{4} \int_\Sigma \eta^2 \tilde{A} \tilde{H} \cdot \tilde{\tau} d\Gamma(\tilde{x}) \quad \text{as } \delta \to 0.$$

Therefor, using the assumption (2.3), we know that $I_\delta(\nabla_2 u_2)$ converges to a non-positive number.

In conclusion, we infer

$$\Theta_\delta(t) \to \zeta \quad \text{as } \delta \to 0,$$

where $\zeta \le 0$ is a real number. When $\zeta < 0$, then there exists $\delta_1 > 0$ such that

$$\Theta_\delta(t) < 0, \quad \delta < \delta_1,$$

and

$$L_\delta(t) = \int_0^t e^{C_2 s} \Theta_\delta(t) ds < 0.$$

When $\zeta = 0$, we need to talk about two cases. One case is that there exists a positive constant δ_2 such that

$$\Theta_\delta(t) < 0, \quad \delta < \delta_2.$$

Then the process is the same as $\zeta < 0$. The other case is that there exists a positive constant δ_3 such that

$$\Theta_\delta(t) > 0, \quad \delta < \delta_3.$$

Then

$$0 \leq L_\delta(t)e^{-C_2 t} = \int_0^t e^{C_2 s}\Theta_\delta(s)ds e^{-C_2 t} \leq \int_0^t \Theta_\delta(s)ds \to 0 \quad \text{as} \quad \delta \to 0,$$

i.e.,

$$L_\delta(t) \equiv 0 \quad \text{as} \quad \delta \to 0.$$

Thus, we get

$$L_\delta(t) \leq 0 \quad \text{as} \quad \delta \to 0. \tag{4.25}$$

• **Estimate for** $M_\delta(t) := \int_0^t e^{C_2 s}\Lambda_\delta(s)ds$

Just like we did above, we split u into two parts, this is,

$$u(x) = u_1(x) + u_2(x),$$

where $u_1 \in H^2(\Omega)$ is the regular part and $u_2 = \eta \cdot U_2$ is the singular part, then

$$\begin{aligned}
&\int_{\partial B(\tilde{x},\delta)} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u d\Gamma \\
&\quad = \int_{\partial B(\tilde{x},\delta)} \beta(t)A(x)(\nabla u_1 + \nabla u_2)\cdot \nu(u_1 + u_2)d\Gamma \\
&\quad = \int_{\partial B(\tilde{x},\delta)} \beta(t)A(x)[\nabla u_1 \cdot \nu u_1 + \nabla u_2 \cdot \nu u_2 + \nabla u_1 \cdot \nu u_2 + \nabla u_2 \cdot \nu u_1]d\Gamma.
\end{aligned} \tag{4.26}$$

Using Cauchy-Schwarz inequality and the above results, we have

$$\int_{\partial B(\tilde{x},\delta)} \nabla u_1 \cdot \nu u_1 d\Gamma \to 0 \quad \text{as} \quad \delta \to 0, \tag{4.27}$$

$$\int_{\partial B(\tilde{x},\delta)} \nabla u_2 \cdot \nu u_2 d\Gamma \to 0 \quad \text{as} \quad \delta \to 0, \tag{4.28}$$

$$\int_{\partial B(\tilde{x},\delta)} \nabla u_1 \cdot \nu u_2 d\Gamma \to 0 \quad \text{as} \quad \delta \to 0, \tag{4.29}$$

and

$$\int_{\partial B(\tilde{x},\delta)} \nabla u_2 \cdot \nu u_1 d\Gamma \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.30}$$

Taking (4.27)–(4.30) into (4.26) and using Lebesgue dominated convergence theorem allow us to get

$$\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u d\Gamma ds \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.31}$$

Analogously,

$$\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u_t d\Gamma ds \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.32}$$

On the other hand, using the decomposition of u into a regular part and a singular part, we have

$$\begin{aligned} &\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_t|^2 H \cdot \nu d\Gamma ds \\ &\quad = \int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_{1,t}|^2 H \cdot \nu d\Gamma ds + \int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_{2,t}|^2 H \cdot \nu d\Gamma ds. \end{aligned}$$

From the dominated convergence theorem, we have

$$\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_{1,t}|^2 H \cdot \nu d\Gamma ds \to 0 \quad \text{as} \quad \delta \to 0.$$

As

$$\int_{l(\tilde{x},\delta)} |u_{2,t}|^2 H \cdot \nu dl \le C \int_0^{2\pi} \int_0^{\delta} r^{\frac{1}{2}} dr d\theta \le C \delta^{\frac{3}{2}},$$

we get

$$\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_{2,t}|^2 H \cdot \nu d\Gamma ds \to 0 \quad \text{as} \quad \delta \to 0.$$

Then it follows

$$\int_0^t e^{C_2 s} \int_{\partial B(\tilde{x},\delta)} |u_t|^2 H \cdot \nu d\Gamma ds \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.33}$$

Therefore, combining (4.31)–(4.33), we get

$$M_\delta(t) = \int_0^t e^{C_2 s} \Lambda_\delta(s) ds \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.34}$$

In conclusion, let $\delta \to 0$ and taking (4.25) and (4.34) into (4.23), we have

$$\mathcal{E}_\theta(t) \le C\left(\mathcal{E}_\theta(0) + \int_0^t e^{C_2 s}|\beta'(s)|ds\right)e^{-C_2 t}.$$

Then, using the energy equivalence relation and (2.5), we get

$$E(t) \le C\left(E(0) + \alpha t^m\right)e^{-C_2 t}.$$

Thus, we complete the proof of Theorem 4.4. □

5 Conclusions

In this paper, we present a study on the stability of a time-varying coefficients wave equation in the bounded domain Ω. The smooth boundary of Ω is $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. We consider that a homogeneous Dirichlet boundary on Γ_0 and a dynamic boundary with damping term on Γ_1. Since the coefficients depends on the time variable and the singularities are generated by changing the boundary conditions along the interface, these bring no small difficulty to our proof, so some special techniques are needed to deal with these problems. Under the appropriate geometric assumptions, the exponential decay result of the system is established by the Riemannian geometry method and the energy perturbation method.

There are many other issues associated with this type of problem, but we have not studied them here.

(i) The geometric conditions (**H2**) and (2.3) are essential in the proof of our exponential stability result, but their necessity leads us to exclude many mathematical models of interest that should also be uniformly stable.
(ii) We assume that there are no external forces acting on the system or its boundary other than friction. If there is thermal force in system, can the energy still be uniformly stable? What if there is a nonlinear negative source term on Γ_1?

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Declarations

Conflict of Interest The authors declare there is no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by the authors.

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