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Lipschitz-Volume Rigidity and Sobolev Coarea Inequality for Metric Surfaces

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Abstract

We prove that every 1-Lipschitz map from a closed metric surface onto a closed Riemannian surface that has the same area is an isometry. If we replace the target space with a non-smooth surface, then the statement is not true and we study the regularity properties of such a map under different geometric assumptions. Our proof relies on a coarea inequality for continuous Sobolev functions on metric surfaces that we establish, and which generalizes a recent result of Esmayli–Ikonen–Rajala.

Keywords Lipschitz-volume rigidity · Metric surfaces · Reciprocal · Quasiconformal · Ahlfors regular · Coarea inequality · Sobolev

Mathematics Subject Classification 53C23 · 53C45 · 30C65 · 53A05

1 Introduction

The Lipschitz-volume rigidity problem in its general formulation asks whether every 1-Lipschitz and surjective map between metric spaces that have the same volume (e.g. arising from Hausdorff measure) is necessarily an isometry. It is well-known that the answer to this problem is affirmative for maps between smooth manifolds.

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Let X, Y *be closed Riemanniann-manifolds, where* $n \geq 1$*. If* $\mathrm{Vol}(X) = \mathrm{Vol}(Y)$*, then every* 1*-Lipschitz map from* X *onto* Y *is an isometric homeomorphism.*

See [6, Sect. 9] or [4, Appendix C] for a proof of this fact. Moreover, this statement has been generalized to singular settings of Alexandrov and limit RCD spaces by Storm [37], Li [23], and Li and Wang [25]. See also [24] for an overview of the Lipschitz-volume rigidity problem. The problem in the setting of integral current spaces has been recently studied by Basso et al. [2], Del Nin and Perales [9], and Züst [42].

The recent developments in the uniformization of non-smooth metric surfaces by Rajala, Romney, Wenger, and the current authors [29, 31, 32, 35], allow us to establish the above rigidity statement in the two-dimensional setting under no geometric, smoothness, or curvature assumptions on X.

Theorem 1.1 *Let* X *be a closed metric surface and* Y *be a closed Riemannian surface. If* $\mathcal{H}^2(X) = \mathcal{H}^2(Y)$*, then every 1-Lipschitz map from* X *onto* Y *is an isometric homeomorphism.*

Here a closed metric surface is a compact topological 2-manifold without boundary, equipped with a metric that induces its topology. Also, an isometric map is a distance-preserving map. We state an immediate corollary.

Corollary 1.2 *Among all metrics* d *on* $\mathbb{S}^2$ *that are at least as large as the spherical metric, the map* $d \mapsto \mathcal{H}^2_d(\mathbb{S}^2)$ *has a unique minimum attained by the spherical metric.*

We note in the next example that the conclusion is not true in general if we replace the spherical metric with a non-smooth metric.

Example 1.3 Consider a non-constant rectifiable curve E in $\mathbb{S}^2$ and let d_0 be the length metric $\chi_{\mathbb{S}^2\setminus E}\, ds + (1/2)\chi_E\, ds$. Then there exist infinitely many distinct metrics $d \geq d_0$ having the same area as d_0. Namely, for each $\delta \in (1/2, 1]$, the metric $\chi_{\mathbb{S}^2\setminus E}\, ds + \delta\chi_E\, ds$ has this property.

One of the most technical difficulties of Theorem 1.1 is establishing the injectivity of the map in question; see Lemma 3.7. Since this issue is not present in Corollary 1.2, it is conceivable that the result can be obtained in higher dimensions as well by a modification of our argument.

1.1 Area-Preserving and Lipschitz Maps Between Surfaces

A map as in Theorem 1.1 preserves the Hausdorff 2-measure, or else area measure, of every measurable set. Theorem 1.1 is a consequence of Theorem 1.4, which provides several topological and regularity results for area-preserving and Lipschitz maps between surfaces of locally finite Hausdorff 2-measure.

We provide the necessary definitions. Let X and Y be metric surfaces of locally finite Hausdorff 2-measure. A map $f: X \to Y$ is *area-preserving* if $\mathcal{H}^2(A) = \mathcal{H}^2(f(A))$ for every measurable set $A \subset X$. A map $f: X \to Y$ is *Lipschitz* if there exists $L > 0$ such that for all $x_1, x_2 \in X$ we have

$$d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2).$$

In this case, we say that f is L-Lipschitz. A homeomorphism $f\colon X \to Y$ is *quasi-conformal* (abbr. QC) if there exists $K \geq 1$ such that

$$K^{-1} \operatorname{Mod} f(\Gamma) \leq \operatorname{Mod} \Gamma \leq K \operatorname{Mod} f(\Gamma)$$

for each path family Γ in X; here Mod refers to 2-modulus and the precise definition is given in Sect. 2.3. In this case we say that f is K-quasiconformal. A map $f\colon X \to Y$ is a map of *bounded length distortion* (abbr. BLD) if there exists a constant $K \geq 1$ such that

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X; this includes curves of infinite length. In this case we say that f is a map of K-bounded length distortion.

We say that the surface X is *reciprocal* if there exists a constant $\kappa > 0$ such that for every quadrilateral $Q \subset X$ and for the families $\Gamma(Q)$ and $\Gamma^*(Q)$ of curves joining opposite sides of Q we have

$$\operatorname{Mod} \Gamma(Q) \cdot \operatorname{Mod} \Gamma^*(Q) \leq \kappa.$$

By a result of Rajala [35, Sect. 14], if a surface is reciprocal then the above holds for some $\kappa \leq (\pi/2)^2$. Reciprocal surfaces are important because they are precisely the metric surfaces that admit quasiconformal parametrizations by Riemannian surfaces [19, 31, 35]. We say that X is *upper Ahlfors 2-regular* if there exists $K > 0$ such that

$$\mathcal{H}^2(B(x,r)) \leq K r^2$$

for every ball $B(x,r) \subset X$. If X is (locally) upper Ahlfors 2-regular, then it is also reciprocal [35]. See Sect. 2.5 for further details. We state our main theorem, which is also concisely presented in Table 1.

Theorem 1.4 *Let X, Y be metric surfaces without boundary and with locally finite Hausdorff 2-measure, and let $f\colon X \to Y$ be an area-preserving surjective map.*

(1) *If X is reciprocal and f is Lipschitz, then there exists a constant $K \geq 1$ such that*

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{Mod} \Gamma_0 = 0$. *Moreover, if f is 1-Lipschitz, then $K = 1$.*

(2) *If Y is reciprocal and f is Lipschitz, then there exists a constant $K \geq 1$ such that f is a K-quasiconformal homeomorphism and*

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{Mod} \Gamma_0 = 0$. *Moreover, if f is 1-Lipschitz, then $K = 1$.*

Table 1 The conclusions of Theorem 1.4. In all cases f is assumed to be area-preserving

Reference	X	Y	f	Conclusion about f
Question 1.5	–	–	Lip	BLD on a.e. curve?
Theorem 1.4 (1)	Reciprocal	–	(1-)Lip	(1-)BLD on a.e. curve
Example 4.1	Riemannian	–	1-Lip	*Not* homeomorphic
Theorem 1.4 (2)	–	Reciprocal	(1-)Lip	(1-)QC homeom. (1-)BLD on a.e. curve
Example 4.2	Riemannian	Reciprocal	1-Lip	*Not* BLD
Theorem 1.4 (3)	–	Upper regular	Lip	QC homeom., BLD
Example 1.3	Riemannian	Upper regular	1-Lip	*Not* isometric
Theorem 1.4 (4)	–	Riemannian	1-Lip	Isometry

(3) *If Y is upper Ahlfors 2-regular and f is Lipschitz, then there exists a constant $K \geq 1$ such that f is a homeomorphism of K-bounded length distortion.*

The constant K in (1)–(3) depends quantitatively on the assumptions.

(4) *If Y is Riemannian and f is 1-Lipschitz, then f is an isometric homeomorphism.*

We were neither able to show that part (1) holds without the assumption that X is reciprocal, nor were we able to find a counterexample. This raises the following question.

Question 1.5 *Suppose that X, Y are metric surfaces of locally finite Hausdorff 2-measure. If $f : X \to Y$ is an area-preserving and Lipschitz map, does it quasi-preserve the length of a.e. path in X?*

We note that an affirmative answer to the question has been provided by Creutz–Soultanis [7, Proposition 4.1] with the additional assumptions that X is 2-rectifiable and f is 1-Lipschitz. This result does not imply Theorem 1.4 (1) or vice versa.

In Sect. 4 we present examples illustrating the optimality of Theorem 1.4. We first note that area-preserving and 1-Lipschitz maps are not injective in general without any assumptions on Y; a sufficient condition is the reciprocity of Y in part (2). Moreover, one cannot expect in part (2) that the length of *all* curves (rather than a.e. curve) is quasi-preserved; a sufficient condition is upper Ahlfors 2-regularity of Y as in (3). Finally, in part (3) one cannot expect a 1-Lipschitz map f to be an isometry without further assumptions on Y, such as smoothness, as in (4); this has already been illustrated in Example 1.3.

1.2 Coarea Inequality

The proof of Theorem 1.4 relies on a coarea inequality for continuous Sobolev functions on metric surfaces. The following result is an improvement of the coarea inequality for *monotone* Sobolev functions that was established recently in [13]; here monotonicity means that the maximum and minimum of a function on a precompact open set are attained at the boundary. We direct the reader to [13] for further background on the coarea inequality in metric spaces.

Theorem 1.6 *Let X be a metric surface of locally finite Hausdorff 2-measure and $u\colon X \to \mathbb{R}$ be a continuous function with a 2-weak upper gradient $\rho_u \in L^2_{\mathrm{loc}}(X)$.*

(1) *If $\mathcal{A}_u$ denotes the union of all non-degenerate components of the level sets $u^{-1}(t)$, $t \in \mathbb{R}$, of u, then $\mathcal{A}_u$ is a Borel set.*
(2) *For every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^{*} \int_{u^{-1}(t)\cap\mathcal{A}_u} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int g\rho_u \, d\mathcal{H}^2.$$

(3) *If, in addition, u is Lipschitz, then for every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int g \cdot (\rho_u \chi_{\mathcal{A}_u} + \mathrm{Lip}(u)\chi_{X\setminus\mathcal{A}_u}) \, d\mathcal{H}^2.$$

Here $\mathrm{Lip}(u)$ denotes the pointwise Lipschitz constant of a Lipschitz function $u\colon X \to \mathbb{R}$, defined by

$$\mathrm{Lip}(u)(x) = \limsup_{y\to x} \frac{|u(y)-u(x)|}{d(x,y)}.$$

Also, $\int^*$ denotes the upper integral, which is equal to the Lebesgue integral for measurable functions. The main result of [13] (for $p \geq 2$) states that (2) holds with the additional assumption that u is monotone and with $u^{-1}(t)$ in place of $u^{-1}(t) \cap \mathcal{A}_u$. Since the level sets of monotone functions are always non-degenerate (see e.g. [33, Corollary 2.8]), we see that $\mathcal{A}_u = X$ when u is monotone; hence our theorem implies the main result of [13] for $p \geq 2$. Moreover, without the monotonicity assumption, we note that part (2) is optimal and does not hold for the full level sets $u^{-1}(t)$ if we do not restrict to $\mathcal{A}_u$, even if u is Lipschitz. A relevant example is provided in [13, Sect. 5].

The proof of Theorem 1.6 relies on recent developments in the theory of uniformization of metric surfaces. Specifically, we use a result of Romney and the second-named author [31], which states that every metric surface of locally finite Hausdorff 2-measure admits a weakly quasiconformal parametrization by a Riemannian surface of the same topological type.

After the completion and distribution of a first version of the manuscript it was communicated to us by Wenger that in the case of orientable surfaces Theorem 1.1 may also be obtained via a combination of recent results on the geometric structure of metric surfaces due to Basso et al. [3] and of a rigidity theorem of Züst for integral current spaces [42]; see [3, Theorem 1.4]. The uniformization result of [31] is an important ingredient of this alternative approach too.

2 Preliminaries

2.1 Hausdorff Measures

For a metric space X and $s > 0$, the *Hausdorff s-measure* of a set $A \subset X$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A), \quad \text{where} \quad \mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} \frac{\omega_s}{2^s} \operatorname{diam}(A_j)^s \right\}$$

and the infimum is taken over all collections of sets $\{A_j\}_{j=1}^\infty$ such that $A \subset \bigcup_{j=1}^\infty A_j$ and $\operatorname{diam}(A_j) < \delta$ for each j. Here ω_s is a positive normalization constant, chosen so that the Hausdorff n-measure coincides with Lebesgue measure in $\mathbb{R}^n$. Note that $\omega_1 = 2$ and $\omega_2 = \pi$. If we need to emphasize the metric d being used for the Hausdorff s-measure, we write $\mathcal{H}^s_d$ instead of $\mathcal{H}^s$.

We state the coarea inequality for Lipschitz functions and the classical coarea formula for Sobolev functions.

Theorem 2.1 (Coarea inequality and formula) *Let X be a metric space, $u: X \to \mathbb{R}$ be a continuous function, and $g: X \to [0, \infty]$ be a Borel function.*

(1) *If u is Lipschitz, then for $K = 4/\pi$ we have*

$$\int^* \int_{u^{-1}(t)} g \, d\mathcal{H}^1 dt \leq K \int_X g \cdot \operatorname{Lip}(u) \, d\mathcal{H}^2.$$

If X is a Riemannian surface, we may take $K = 1$.

(2) *If X is an open subset of $\mathbb{R}^2$ and $u \in W^{1,1}_{\text{loc}}(X)$, then*

$$\int \int_{u^{-1}(t)} g \, d\mathcal{H}^1 dt = \int_X g \cdot |\nabla u| \, d\mathcal{H}^2.$$

Part (1) is a consequence of [16, Theorem 2.10.25] for general metric spaces X and of [15, Theorem 3.1] for Riemannian manifolds with $K = 1$. Part (2) is stated in [28] and attributed to Federer. See also [12] for a more general statement than (1) and [13, Lemma 5.2].

We will make use of the following area formula. Below, $N(f, y)$ denotes the number of preimages of a point y under a map f.

Theorem 2.2 [16, Theorem 2.10.10] *Let X, Y be metric spaces such that X is separable. Consider a map $f: X \to Y$ such that for every Borel set $A \subset X$ the set $f(A)$ is $\mathcal{H}^2$-measurable. For $S \subset X$ define $\zeta(S) = \mathcal{H}^2(f(S))$ and denote by ψ the measure on X resulting by Carathéodory's construction from ζ on the family of all Borel subsets of X. Then, for each Borel set $A \subset X$ we have*

$$\psi(A) = \int_Y N(f|_A, y) \, d\mathcal{H}^2.$$

2.2 Topological Preliminaries

Let X be a metric space. A *path* or *curve* is a continuous map $\gamma\colon [a,b] \to X$. The *trace* of γ is the set $|\gamma| = \gamma([a,b])$. The *length* of γ is its total variation and is denoted by $\ell(\gamma)$. The following theorem is a consequence of Theorem 2.2 and provides an area formula for length.

Theorem 2.3 [16, Theorem 2.10.13] *Let X be a metric space and $\gamma\colon [a,b] \to X$ be a curve. Then*

$$\ell(\gamma) = \int_X N(\gamma, x)\, d\mathcal{H}^1.$$

We say that a curve $\gamma\colon [a,b] \to X$ is a *Jordan arc* if γ is injective. Here we allow the possibility $a = b$, in which case γ is a *degenerate* Jordan arc. We say that γ is a *Jordan curve* if $\gamma|_{[a,b)}$ is injective and $\gamma(a) = \gamma(b)$. We also say that a set $K \subset X$ is a Jordan arc (resp. Jordan curve) if there exists a Jordan arc (resp. Jordan curve) γ with $|\gamma| = K$. A *continuum* is a compact and connected metric space. A *Peano continuum* is a locally connected continuum.

Lemma 2.4 *Let $\{K_i\}_{i\in I}$ be a collection of pairwise disjoint Peano continua in $\mathbb{R}^2$. Then, with the exception of countably many $i \in I$, each K_i is a Jordan arc or a Jordan curve.*

Proof A *triod* is the union of three non-degenerate Jordan arcs that have a common endpoint, the *junction point*, but are otherwise disjoint. A theorem of Moore [30] (see also [34, Proposition 2.18]) states that there is no uncountable collection of pairwise disjoint triods in the plane. On the other hand, if a Peano continuum is not a Jordan arc or Jordan curve, then it contains a triod [33, Lemma 2.4]. This completes the proof. □

Lemma 2.5 *Let K be a continuum with $\mathcal{H}^1(K) < \infty$. Then K is a Peano continuum.*

Proof If $\mathcal{H}^1(K) < \infty$, a result of Eilenberg–Harrold [10, Theorem 2] states that there exists a continuous and surjective mapping $\gamma\colon [0,1] \to K$ (with $\ell(\gamma) \leq 2\mathcal{H}^1(K) - \operatorname{diam}(K)$). By the Hahn–Mazurkiewicz theorem [40, Theorem 31.5], Peano continua are characterized as continuous images of the unit interval. □

Lemma 2.6 [5, Theorem 2.6.2] *Let X be a metric space and let $\gamma\colon [a,b] \to X$ be a curve. Then $\ell(\gamma) \geq \mathcal{H}^1(|\gamma|)$. Moreover, if γ is a Jordan arc or Jordan curve, then $\ell(\gamma) = \mathcal{H}^1(|\gamma|)$.*

We state a consequence of Lemmas 2.4, 2.5, and 2.6, and of the existence of arclength parametrizations of rectifiable curves [18, Sect. 5.1].

Corollary 2.7 *Let X be a metric space homeomorphic to a subset of $\mathbb{R}^2$. Let $\{K_i\}_{i\in I}$ be a collection of pairwise disjoint continua in X with $\mathcal{H}^1(K_i) < \infty$ for each $i \in I$. Then, with the exception of countably many $i \in I$, each K_i is a Jordan arc or a Jordan curve and there exists a Lipschitz parametrization $\gamma\colon [a_i, b_i] \to K_i$ that is injective in $[a_i, b_i)$.*

Lemma 2.8 *Let X be a topological space homeomorphic to $\mathbb{S}^2$ or to a closed disk. Let $K \subset X$ be a compact set separating two points $a, b \in X$. Then there exists a connected component of K that also separates a and b.*

In $\mathbb{S}^2$ this is a consequence of [39, Lemma II.5.20, p. 61]. For topological disks the conclusion follows from [27, Lemma 7.1].

Throughout the paper $\text{int}(X)$ denotes the manifold interior of a surface X. The topological interior of a set A in a topological space is denoted by $\text{int}_{\text{top}}(A)$. Similar notation is adopted for the notion of boundary.

2.3 Metric Sobolev Spaces

Let X be a metric space and Γ be a family of curves in X. A Borel function $\rho\colon X \to [0, \infty]$ is *admissible* for Γ if $\int_\gamma \rho\, ds \geq 1$ for all rectifiable paths $\gamma \in \Gamma$. We define the *2-modulus* of Γ as

$$\text{Mod}\,\Gamma = \inf_\rho \int_X \rho^2\, d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions ρ for Γ. By convention, $\text{Mod}\,\Gamma = \infty$ if there are no admissible functions for Γ. Observe that we consider X to be equipped with the Hausdorff 2-measure. This definition may be generalized by allowing for an exponent different from 2 or a different measure, though this generality is not needed for this paper.

Let $h\colon X \to Y$ be a map between metric spaces. We say that a Borel function $g\colon X \to [0, \infty]$ is an *upper gradient* of h if

$$d_Y(h(x), h(y)) \leq \int_\gamma g\, ds \tag{2.1}$$

for all $x, y \in X$ and every rectifiable path γ in X joining x and y. This is called the *upper gradient inequality*. If, instead the above inequality holds for all curves γ outside a curve family of 2-modulus zero, then we say that g is a *(2-)weak upper gradient* of h. In this case, there exists a curve family Γ_0 with $\text{Mod}\,\Gamma_0 = 0$ such that all paths outside Γ_0 and all subpaths of such paths satisfy the upper gradient inequality.

We equip the space X with the Hausdorff 2-measure $\mathcal{H}^2$. Let $L^2(X)$ denote the space of 2-integrable Borel functions from X to the extended real line $\widehat{\mathbb{R}}$, where two functions are identified if they agree $\mathcal{H}^2$-almost everywhere. The Sobolev space $N^{1,2}(X, Y)$ is defined as the space of Borel maps $h\colon X \to Y$ with a 2-weak upper gradient g in $L^2(X)$ such that the function $x \mapsto d_Y(y, h(x))$ is in $L^2(X)$ for some $y \in Y$. If $Y = \mathbb{R}$, we simply write $N^{1,2}(X)$. The spaces $L^2_{\text{loc}}(X)$ and $N^{1,2}_{\text{loc}}(X, Y)$ are defined in the obvious manner. Each map $h \in N^{1,2}_{\text{loc}}(X, Y)$ has a *minimal* 2-weak upper gradient g_h, in the sense that for any other 2-weak upper gradient g we have $g_h \leq g$ a.e. See the monograph [18] for background on metric Sobolev spaces.

We state a consequence of the coarea inequality for Lipschitz functions.

Lemma 2.9 [13, Lemma 2.13] *Let X be a metric surface of finite Hausdorff 2-measure and $u\colon X \to \mathbb{R}$ be a Lipschitz function. If Γ_0 is a curve family in X with* $\operatorname{Mod}\Gamma_0 = 0$*, then for a.e. $t \in \mathbb{R}$, every Lipschitz curve $\gamma\colon [a,b] \to u^{-1}(t)$ that is injective on $[a,b)$ lies outside Γ_0.*

2.4 Quasiconformal Maps

Let X, Y be metric surfaces of locally finite Hausdorff 2-measure. Recall that a homeomorphism $h\colon X \to Y$ is quasiconformal if there exists $K \geq 1$ such that

$$K^{-1}\operatorname{Mod}\Gamma \leq \operatorname{Mod} h(\Gamma) \leq K \operatorname{Mod}\Gamma$$

for every curve family Γ in X. A continuous map between topological spaces is *cell-like* if the preimage of each point is a continuum that is contractible in each of its open neighborhoods. A continuous, surjective, proper, and cell-like map $h\colon X \to Y$ is *weakly quasiconformal* if there exists $K > 0$ such that for every curve family Γ in X we have

$$\operatorname{Mod}\Gamma \leq K \operatorname{Mod} h(\Gamma).$$

In this case, we say that h is weakly K-quasiconformal.

If X and Y are compact surfaces that are homeomorphic to each other, then we may replace cell-likeness with the weaker requirement that h is monotone; that is, the preimage of every point is a continuum. In that case, continuous, surjective, and monotone maps from X to Y coincide with uniform limits of homeomorphisms; see [31, Theorem 6.3] and the references therein. Alternatively, if X, Y have empty boundary, then continuous, proper, and cell-like maps from X to Y also coincide with uniform limits of homeomorphisms, see [8, Corollary 25.1A].

We note that a weakly K-quasiconformal map between planar domains is a K-quasiconformal homeomorphism. Indeed, by [32, Theorem 7.4], such a map is a homeomorphism. Also, note that a quasiconformal homeomorphism between planar domains is a priori required to satisfy only one modulus inequality, as in the definition of a weakly quasiconformal map; see [22, Sect. I.3].

The next theorem of Williams ([41, Theorem 1.1 and Corollary 3.9]) relates the above definitions of quasiconformality with the "analytic" definition that relies on upper gradients; see also [32, Sect. 2.4].

Theorem 2.10 (Definitions of quasiconformality) *Let X, Y be metric surfaces of locally finite Hausdorff 2-measure, $h\colon X \to Y$ be a continuous map, and $K > 0$. The following are equivalent.*

(i) *$h \in N^{1,2}_{\mathrm{loc}}(X,Y)$ and there exists a 2-weak upper gradient g of h such that for every Borel set $E \subset Y$ we have*

$$\int_{h^{-1}(E)} g^2\, d\mathcal{H}^2 \leq K\mathcal{H}^2(E).$$

(i') *Each point of X has a neighborhood U such that $h|_U \in N^{1,2}(U, Y)$ and there exists a 2-weak upper gradient g_U of $h|_U$ such that for every Borel set $E \subset Y$ we have*

$$\int_{(h|_U)^{-1}(E)} g_U^2 \, d\mathcal{H}^2 \leq K\mathcal{H}^2(E).$$

(ii) *For every curve family Γ in X we have*

$$\operatorname{Mod}\Gamma \leq K \operatorname{Mod} h(\Gamma).$$

Theorem 2.11 [32, Theorem 7.1 and Remark 7.2] *Let X, Y be metric surfaces of locally finite Hausdorff 2-measure and $h\colon X \to Y$ be a weakly K-quasiconformal map for some $K > 0$.*

(1) *The set function $\nu(E) = \mathcal{H}^2(h(E))$ is a locally finite Borel measure on X. Moreover, for a.e. $x \in X$ we have*

$$g_h(x)^2 \leq K J_h(x), \quad \textit{where} \quad J_h = \frac{d\nu}{d\mathcal{H}^2}.$$

(2) *$N(h, y) = 1$ for a.e. $y \in Y$.*

Recall that $N(h, y)$ denotes the number of preimages of y under h.

2.5 Reciprocal Surfaces

Let X be a metric surface of locally finite Hausdorff 2-measure. For a set $G \subset X$ and disjoint sets $E, F \subset G$ we define $\Gamma(E, F; G)$ to be the family of curves in G joining E and F. A *quadrilateral* in X is a closed Jordan region Q together with a partition of ∂Q into four non-overlapping edges $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \subset \partial Q$ in cyclic order. When we refer to a quadrilateral Q, it will be implicitly understood that there exists such a marking on its boundary. We define $\Gamma(Q) = \Gamma(\zeta_1, \zeta_3; Q)$ and $\Gamma^*(Q) = \Gamma(\zeta_2, \zeta_4; Q)$. According to the definition of Rajala [35], the metric surface X is reciprocal if there exist constants $\kappa, \kappa' \geq 1$ such that

$$\kappa^{-1} \leq \operatorname{Mod}\Gamma(Q) \cdot \operatorname{Mod}\Gamma^*(Q) \leq \kappa' \quad \text{for each quadrilateral} \, Q \subset X \tag{2.2}$$

and

$$\lim_{r\to 0} \operatorname{Mod}\Gamma(\overline{B}(a, r), X \setminus B(a, R); X) = 0 \quad \text{for each ball} \, B(a, R). \tag{2.3}$$

By work of Rajala and Romney [36] it is now known that the lower bound in (2.2) is always satisfied for some uniform constant κ. In fact, the optimal constant was shown to be $\kappa = (4/\pi)^2$ [11]. Moreover, (2.3) follows from the upper bound in (2.2), as was shown by Romney and the second-named author [31]. Therefore, we may only require the upper inequality of (2.2) in the definition of a reciprocal surface.

Rajala [35] proved that a metric surface X of locally finite Hausdorff 2-measure that is homeomorphic to $\mathbb{R}^2$ is 2-quasiconformally equivalent to an open subset of $\mathbb{R}^2$ if and only if X is reciprocal. This result was generalized to all metric surfaces (with or without boundary) of locally finite Hausdorff 2-measure, where $\mathbb{R}^2$ is replaced with a Riemannian surface [19, 31].

More generally, it was shown in [31] that any metric surface of locally finite Hausdorff 2-measure admits a weakly quasiconformal parametrization by a Riemannian surface of the same topological type. The following special case is sufficient for our purposes.

Theorem 2.12 [31, Theorem 1.2] *Let X be a metric surface of finite Hausdorff 2-measure that is homeomorphic to a topological closed disk. Then there exists a weakly $(4/\pi)$-quasiconformal map from $\overline{\mathbb{D}}$ onto X.*

Here $\mathbb{D}$ denotes the open unit disk in the plane. We show that weakly quasiconformal maps can be upgraded to quasiconformal homeomorphisms under certain conditions.

Lemma 2.13 *Let X, Y be metric surfaces without boundary and with locally finite Hausdorff 2-measure such that Y is reciprocal. Then every weakly quasiconformal map $f: X \to Y$ is a quasiconformal homeomorphism, quantitatively.*

Proof Let $f: X \to Y$ be a weakly K-quasiconformal map for some $K > 0$. Since Y is reciprocal, condition (2.3) implies that the modulus of the family of non-constant curves passing through any point of Y is zero. By [32, Theorem 7.4] we conclude that f is a homeomorphism. Now, the reciprocity of Y implies that the upper bound in (2.2) is satisfied for X as well. Therefore, X is reciprocal.

Consider a domain $V' \subset Y$ that is homeomorphic to $\mathbb{R}^2$. By Rajala's theorem, there exists a 2-quasiconformal homeomorphism ϕ from V' onto a domain $V \subset \mathbb{R}^2$. The set $U' = f^{-1}(V')$ is homeomorphic to $\mathbb{R}^2$, so by Rajala's theorem there exists a 2-quasiconformal homeomorphism ψ from U' onto a domain $U \subset \mathbb{R}^2$. The composition $g = \phi \circ f \circ \psi^{-1}$ is a weakly $4K$-quasiconformal map from U onto V. Since the domains are planar, g is a $4K$-quasiconformal homeomorphism. Therefore, f is a $16K$-quasiconformal homeomorphism from U' onto V'. By Theorem 2.10, quasiconformality is a local condition, so $f: X \to Y$ is $16K$-quasiconformal. □

2.6 Metric Differentiability

Throughout the section we let $U \subset \mathbb{R}^2$ be a domain and Y be a metric space. We say that a map $h: U \to Y$ is *approximately metrically differentiable* at a point $x \in U$ if there exists a seminorm N_x on $\mathbb{R}^2$ for which

$$\operatorname{ap}\lim_{y \to x} \frac{d(h(y), h(x)) - N_x(y-x)}{y-x} = 0.$$

Here, ap lim denotes the approximate limit as defined in [14, Sect. 1.7.2]. In this case, the seminorm N_x is unique, is denoted by $\operatorname{ap}\operatorname{md} h_x$, and we call it the *approximate metric derivative* of h at x.

Proposition 2.14 [26, Proposition 4.3] *If $h \in N^{1,2}(U, Y)$ then there exist countably many pairwise disjoint compact sets $K_i \subset U$, $i \in \mathbb{N}$, such that $\mathcal{H}^2(U \setminus \bigcup_{i\in\mathbb{N}} K_i) = 0$ with the following property. For every $i \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $r_i(\varepsilon) > 0$ such that h is approximately metrically differentiable at every $x \in K_i$ and*

$$|d(h(x), h(x+v)) - \operatorname{ap}\operatorname{md} h_x(v)| \leq \varepsilon|v|$$

for all $x \in K_i$ and all $v \in \mathbb{R}^2$ with $|v| \leq r_i(\varepsilon)$ and $x + v \in K_i$.

In particular, every map $h \in N^{1,2}(U, Y)$ is approximately metrically differentiable at a.e. $x \in U$.

Lemma 2.15 [27, Lemma 3.1] *If $h \in N^{1,2}(U, Y)$ then*

$$\ell(h \circ \gamma) = \int_a^b \operatorname{ap}\operatorname{md} h_{\gamma(t)}(\dot{\gamma}(t))\, dt$$

for every curve $\gamma \colon [a, b] \to U$ parametrized by arclength outside a family Γ_0 with $\operatorname{Mod}\Gamma_0 = 0$.

Lemma 2.16 *If $h \in N^{1,2}(U, X)$ then the function $L \colon U \to [0, \infty]$ defined by $L(x) =$* $\max\{\operatorname{ap}\operatorname{md} h_x(v) : |v| = 1\}$ *is a representative of the minimal 2-weak upper gradient of h.*

Proof It is an immediate consequence of Lemma 2.15 that L is a 2-weak upper gradient of h. It remains to show that if g is an upper gradient of h in $L^2(U)$, then $L(x) \leq g(x)$ for a.e. $x \in U$; this will imply that the same conclusion is true for the minimal 2-weak upper gradient. Let $g \in L^2(U)$ be an upper gradient of h. It can be deduced from Fubini's theorem that for each $v \in \mathbb{S}^1$ and for a.e. $x \in U$ we have

$$g(x) = \lim_{\delta\to 0} \frac{1}{\delta} \int_0^\delta g(x + tv)\, dt = \lim_{\delta\to 0} \frac{1}{\delta} \int_{\gamma_v|_{[0,\delta]}} g\, ds, \tag{2.4}$$

where $\gamma_v \colon [0, 1] \to \mathbb{R}^2$ is the curve $\gamma_v(t) = x + tv$. Consider a set K_i as in Proposition 2.14. An application of Fubini's theorem shows that for each $v \in \mathbb{S}^1$ and for a.e. $x \in K_i$ we have $x + \delta v \in K_i$ for arbitrarily small values of $\delta > 0$. Let $\varepsilon > 0$, $v \in \mathbb{S}^1$, and $x \in K_i$ such that (2.4) is true and $x + \delta_n v \in K_i$ for a sequence $\delta_n \to 0$. By Proposition 2.14, whenever $|\delta_n v| \leq r_i(\varepsilon)$, we have

$$\operatorname{ap}\operatorname{md} h_x(v) \leq \frac{1}{\delta_n} d(h(x), h(x + \delta_n v)) + \varepsilon|v| \leq \frac{1}{\delta_n} \int_{\gamma_v|_{[0,\delta_n]}} g + \varepsilon.$$

We let $n \to \infty$ and then $\varepsilon \to 0$ to obtain $\operatorname{ap}\operatorname{md} h_x(v) \leq g(x)$. Since this is true for every $v \in \mathbb{S}^1$, we obtain $L(x) \leq g(x)$ for a.e. $x \in K_i$. The sets K_i, $i \in \mathbb{N}$, cover U up to a set of measure zero, so the conclusion follows. □

Before providing the definition of the Jacobian of a Sobolev map, we state the following version of John's theorem; see [1, Theorem 3.1].

Theorem 2.17 (John's theorem) *Each symmetric convex body $K \subset \mathbb{R}^2$ contains a unique ellipse E of maximal area, called the John ellipse of K. Moreover,*

$$E \subset K \subset \sqrt{2}E.$$

If $\operatorname{ap\,md} h_x$ is a norm, let $B_x = \{y \in \mathbb{R}^2 : \operatorname{ap\,md} h_x(y) \leq 1\}$ be the closed unit ball in $(\mathbb{R}^2, \operatorname{ap\,md} h_x)$. The *Jacobian* of $\operatorname{ap\,md} h_x$ is defined to be $J(\operatorname{ap\,md} h_x) = \pi/|B_x|$, where $|B_x|$ is the Lebesgue measure of B_x. Since B_x is a symmetric convex body, by John's theorem there exists a unique ellipse $E_x \subset B_x$ of maximal area. When $\operatorname{ap\,md} h_x$ is not a norm, the closed unit ball B_x has infinite area and we define $J(\operatorname{ap\,md} h_x) = 0$.

Theorem 2.18 (Area formula) *If $h \in N^{1,2}(U, Y)$, then there exists a set $G_0 \subset U$ with $\mathcal{H}^2(G_0) = 0$ such that for every measurable set $A \subset U \setminus G_0$ we have*

$$\int_A J(\operatorname{ap\,md} h_x)\, d\mathcal{H}^2 = \int_Y N(h|_A, y)\, d\mathcal{H}^2.$$

Proof It is a consequence of [18, Theorem 8.1.49] that U can be covered up to a set of measure zero by countably many disjoint measurable sets G_j, $j \in \mathbb{N}$, such that $h|_{G_j}$ is Lipschitz. This implies that outside a set of measure zero $G_0 \subset U$, h satisfies Lusin's condition (N). The statement now follows from [20, Theorem 3.2] □

Lemma 2.19 *Let Y be a metric surface of locally finite Hausdorff 2-measure and $h \colon U \to Y$ be a weakly K-quasiconformal map for some $K > 0$. Then*

$$J(\operatorname{ap\,md} h_x) \leq \max\{(\operatorname{ap\,md} h_x(v))^2 : |v| = 1\} \leq K\, J(\operatorname{ap\,md} h_x)$$

for a.e. $x \in U$. In particular, for a.e. $x \in U$ we have $J(\operatorname{ap\,md} h_x) = 0$ if and only if $\operatorname{ap\,md} h_x \equiv 0$.

Proof By Theorem 2.10, $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$, so h is approximately metrically differentiable at a.e. $x \in U$. We set $N_x = \operatorname{ap\,md} h_x$ and $J_x = J(\operatorname{ap\,md} h_x)$ for a.e. $x \in U$. By Lemma 2.16, the quantity $L_x = \max\{N_x(v) : |v| = 1\}$ is a representative of the minimal 2-weak upper gradient of h, so $L_x = g_h(x)$ for a.e. $x \in U$. By the area formula of Theorem 2.18, there exists a set $G_0 \subset U$ of measure zero such that for each measurable set $A \subset U \setminus G_0$ we have

$$\int_A J_x = \int_Y N(h|_A, y)\, d\mathcal{H}^2 = \mathcal{H}^2(h(A)),$$

where the latter equality follows from Theorem 2.11. This implies that J_x is the Radon–Nikodym derivative of the measure $A \mapsto \mathcal{H}^2(h(A))$, so $J_x = J_h(x)$ for a.e. $x \in U$, again by Theorem 2.11. Finally, since $g_h(x)^2 \leq K J_h(x)$, we conclude that $L_x^2 \leq K J_x$ for a.e. $x \in U$. The inequality $J_x \leq L_x^2$ follows by the fact that the unit ball $B_x = \{y \in \mathbb{R}^2 : N_x(y) \leq 1\}$ contains a Euclidean ball of radius $1/L_x$. □

Remark 2.20 It is a consequence of Lemma 2.19 that if f is a weakly K-quasiconformal map from a planar (or Riemannian) domain U onto a metric surface Y, then we necessarily have $K \geq 1$. It is unclear how to show this for maps between arbitrary metric surfaces.

3 Proof of Main Theorem

This section is devoted to the proof of Theorem 1.4. Throughout the section we assume that X, Y are metric surfaces without boundary and with locally finite Hausdorff 2-measure.

3.1 Preservation of Length

In this section we establish Theorem 1.4 (1).

Lemma 3.1 *Let $f: X \to Y$ be a map that is area-preserving and L-Lipschitz for some $L > 0$. Then* $\operatorname{Mod} \Gamma \leq L^2 \operatorname{Mod} f(\Gamma)$ *for each curve family Γ in X.*

Proof Since f is L-Lipschitz, the constant function L is an upper gradient of f. Moreover, for every Borel set $A \subset Y$ we have

$$\int_{f^{-1}(A)} L^2 \, d\mathcal{H}^2 = L^2 \mathcal{H}^2(f^{-1}(A)) = L^2 \mathcal{H}^2(f(f^{-1}(A))) \leq L^2 \mathcal{H}^2(A).$$

The conclusion now follows from Theorem 2.10. □

Lemma 3.2 *Let $f: X \to Y$ be a map that is area-preserving and continuous. Then $N(f, y) = 1$ for a.e. $y \in f(X)$.*

Proof For each Borel set $A \subset X$ the set $f(A)$ is analytic [21, Proposition 14.4] and thus $\mathcal{H}^2$-measurable [21, Theorem 29.7]. Define $\zeta(S) = \mathcal{H}^2(f(S))$, where $S \subset X$ is a Borel set. By assumption, $\zeta(S) = \mathcal{H}^2(S)$. The measure on X resulting by Carathéodory's construction from ζ is precisely $\mathcal{H}^2$. By Theorem 2.2, for each Borel set $A \subset X$ we have

$$\mathcal{H}^2(A) = \int_Y N(f|_A, y) \, d\mathcal{H}^2.$$

In particular, since f is area-preserving we have

$$\mathcal{H}^2(A) = \int_{f(A)} N(f|_A, y) \, d\mathcal{H}^2 \geq \mathcal{H}^2(f(A)) = \mathcal{H}^2(A).$$

If $\mathcal{H}^2(A) < \infty$, we conclude that $N(f|_A, y) = 1$ for a.e. $y \in f(A)$. Since X has σ-finite Hausdorff 2-measure, we have $N(f, y) = 1$ for a.e. $y \in f(X)$. □

Lemma 3.3 *Let $U \subset \mathbb{R}^2$ be a domain and $\phi \colon U \to X$ be a weakly quasiconformal map. Let $f \colon X \to Y$ be a map that is area-preserving and L-Lipschitz for some $L > 0$. Then there exists a constant $C(L) > 0$ such that*

$$C(L)\ell(\phi \circ \beta) \leq \ell(f \circ \phi \circ \beta) \leq L\ell(\phi \circ \beta)$$

for all curves β in U outside a curve family Γ_0 with $\operatorname{Mod}\Gamma_0 = 0$. *Moreover, if $L = 1$, then we can choose $C(1) = 1$.*

Proof By Lemma 3.1 and the weak quasiconformality of ϕ, there exists a constant $K \geq 1$ such that for each curve family Γ in U we have

$$\operatorname{Mod}\Gamma \leq K \operatorname{Mod} f(\phi(\Gamma)).$$

By Theorem 2.10, $f \circ \phi \in N^{1,2}_{\text{loc}}(U, Y)$ and $\phi \in N^{1,2}_{\text{loc}}(U, X)$. In particular, both maps are approximately metrically differentiable almost everywhere.

Set $N_x = \operatorname{ap}\operatorname{md}\phi_x$ and $\widetilde{N}_x = \operatorname{ap}\operatorname{md}(f \circ \phi)_x$ for a.e. $x \in U$. We use the notation B_x, $\widetilde{B}_x$ for the corresponding unit balls, and J_x, $\widetilde{J}_x$ for the corresponding Jacobians. By Lemma 2.15 we have

$$\ell(\phi \circ \beta) = \int_a^b N_{\beta(t)}(\dot{\beta}(t))\, dt \tag{3.1}$$

for every curve $\beta \colon [a, b] \to U$ parametrized by arclength outside a family Γ_1 with $\operatorname{Mod}\Gamma_1 = 0$. Analogously, we get

$$\ell((f \circ \phi) \circ \beta) = \int_a^b \widetilde{N}_{\beta(t)}(\dot{\beta}(t))\, dt \tag{3.2}$$

for every curve $\beta \colon [a, b] \to U$ parametrized by arclength outside a family Γ_2 with $\operatorname{Mod}\Gamma_2 = 0$.

Next, we claim that for a.e. $x \in U$ and all $v \in \mathbb{R}^2$ we have,

$$C(L)N_x(v) \leq \widetilde{N}_x(v) \leq LN_x(v)$$

for some constant $C(L) > 0$ with $C(1) = 1$. This implies that there exists a curve family Γ_3 in U with $\operatorname{Mod}\Gamma_3 = 0$ such that for all curves $\beta \colon [a, b] \to U$ parametrized by arclength that are outside Γ_3 we have

$$C(L)\int_a^b N_{\beta(t)}(\dot{\beta}(t))\, dt \leq \int_a^b \widetilde{N}_{\beta(t)}(\dot{\beta}(t))\, dt \leq L\int_a^b N_{\beta(t)}(\dot{\beta}(t))\, dt. \tag{3.3}$$

Let Γ_0 be the family of curves that have a reparametrization in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Then $\operatorname{Mod}\Gamma_0 = 0$. By combining (3.1), (3.2), and (3.3), we see that the conclusions of the lemma are true for the family Γ_0.

Now, we prove the claim. Theorem 2.18 applied to ϕ provides a set of measure zero $G_1 \subset U$ such that for any measurable set $A \subset U \setminus G_1$ we have

$$\int_A J_x = \int_X N(\phi|_A, x)\, d\mathcal{H}^2 = \mathcal{H}^2(\phi(A)), \tag{3.4}$$

where the last equality follows from Theorem 2.11. Similarly, there exists a set $G_2 \subset U$ of measure zero such that for any measurable set $A \subset U \setminus G_2$,

$$\int_A \widetilde{J}_x = \int_Y N(f \circ \phi|_A, y)\, d\mathcal{H}^2.$$

From Lemma 3.2 we know that $N(f, y) = 1$ for a.e. $y \in f(X)$. By Theorem 2.11, for a.e. $x \in X$, $\phi^{-1}(x)$ is a singleton. Since f is area-preserving and in particular has the Lusin (N) property, we conclude that for a.e. $y \in f(X)$ the set $\phi^{-1}(x)$ is a singleton whenever $f(x) = y$. In summary, $N(f \circ \phi|_A, y) = 1$ for a.e. $y \in f(\phi(A))$. In particular, for any measurable set $A \subset U \setminus G_2$,

$$\int_A \widetilde{J}_x = \int_Y N(f \circ \phi|_A, y)\, d\mathcal{H}^2 = \mathcal{H}^2(f(\phi(A))).$$

The area-preserving property of f and (3.4) now imply that $J_x = \widetilde{J}_x$ for a.e. $x \in U$ and hence

$$|B_x| = |\widetilde{B}_x| \tag{3.5}$$

for a.e. $x \in U$. This equality implies that N_x is not a norm if and only if $\widetilde{N}_x$ is also not a norm. By Lemma 2.19, if N_x is not a norm, then $N_x \equiv 0$.

Let $K_i, \widetilde{K}_j \subset U$, $i, j \in \mathbb{N}$, be the sets from Proposition 2.14 applied to ϕ, $f \circ \phi$, respectively. Let $\varepsilon > 0$. The Lipschitz property of f implies that

$$\widetilde{N}_x(v) \le L N_x(v) + (1 + L)\varepsilon |v|$$

for every $x \in K_{i,j} = K_i \cap \widetilde{K}_j$ and every $v \in \mathbb{R}^2$ with $|v| \le \min\{r_i(\varepsilon), \widetilde{r}_j(\varepsilon)\}$ and $x + v \in K_{i,j}$. This shows that

$$\widetilde{N}_x(v) \le L N_x(v) \quad \text{and thus} \quad B_x \subset L\widetilde{B}_x \tag{3.6}$$

for a.e. $x \in U$ and all $v \in \mathbb{R}^2$. Here $L\widetilde{B}_x$ denotes the closed ball $\{y \in \mathbb{R}^2 : \widetilde{N}_x(y) \le L\}$. In particular, if N_x is not a norm, then $\widetilde{N}_x \equiv N_x \equiv 0$.

If $L = 1$, then (3.6) implies that $B_x \subset \widetilde{B}_x$ for a.e. $x \in U$. By (3.5), we have $B_x = \widetilde{B}_x$ for a.e. $x \in U$, since N_x and $\widetilde{N}_x$ are either both norms or vanish identically. Hence, $N_x(v) = \widetilde{N}_x(v)$ for a.e. $x \in U$ and all $v \in \mathbb{R}^2$.

Denote by $E_x, \widetilde{E}_x$ the John ellipse of $B_x, \widetilde{B}_x$, respectively, whenever N_x and $\widetilde{N}_x$ are norms. John's theorem (Theorem 2.17) implies that

$$E_x \subset B_x \subset \sqrt{2} E_x \quad \text{and} \quad \widetilde{E}_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x. \tag{3.7}$$

Denote by $a_x, \widetilde{a}_x$ (resp. $b_x, \widetilde{b}_x$) the length of the major (resp. minor) axis of $E_x, \widetilde{E}_x$, respectively. By (3.6) and (3.7) we have that

$$L^{-1}E_x \subset L^{-1}B_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x,$$

which implies that $b_x \leq \sqrt{2}L\widetilde{b}_x$. Moreover, combining (3.5) and (3.7) gives

$$|\widetilde{E}_x| \leq |\widetilde{B}_x| = |B_x| \leq 2|E_x|.$$

Since $|E_x| = \pi a_x b_x$ and $|\widetilde{E}_x| = \pi \widetilde{a}_x \widetilde{b}_x$, we get

$$\widetilde{a}_x \leq 2\frac{a_x b_x}{\widetilde{b}_x} \leq 2\sqrt{2}La_x.$$

In particular, if we assume in addition that E_x is a geometric ball, then $\widetilde{E}_x \subset 2\sqrt{2}LE_x$. All in all we obtain that

$$L^{-1}B_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x \subset 4LE_x \subset 4LB_x, \tag{3.8}$$

with the additional assumption that E_x is a geometric ball. Note that (3.8) shows that the claim holds for $C(L) = (4L)^{-1}$.

For the general case that E_x is not a geometric ball, we consider a linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(E_x)$ is a round ball. Note that (3.5) remains true for the images of $B_x, \widetilde{B}_x$ under T. Since the John ellipse is preserved under linear maps, the above calculations are true for the images of the corresponding sets under T, and thus one obtains the inclusions (3.8) for the images. Therefore, the inclusions also hold for the original sets. □

Proof of Theorem 1.4 (1) We cover X with a countable collection of open sets $\{X_n\}_{n\in\mathbb{N}}$, each homeomorphic to $\mathbb{R}^2$. Every X_n is reciprocal and, by Rajala's theorem [35], there exists a quasiconformal homeomorphism $\phi_n : U_n \to X_n$, where $U_n \subset \mathbb{R}^2$ is an open set. By Lemma 3.3,

$$C(L)\ell(\phi_n \circ \beta) \leq \ell(f \circ \phi_n \circ \beta) \leq L\ell(\phi_n \circ \beta)$$

holds for every curve β in U_n outside a curve family Γ_n with $\operatorname{Mod}\Gamma_n = 0$, where $C(L) > 0$ is some constant with $C(1) = 1$. Since ϕ_n is quasiconformal, $\operatorname{Mod}\phi_n(\Gamma_n) = 0$ for each $n \in \mathbb{N}$. Note that if γ is a curve in X_n outside $\phi_n(\Gamma_n)$, then after setting $\beta = \phi_n^{-1} \circ \gamma$ we see that the statement of Theorem 1.4 (1) holds for γ. We define Γ_0 to be the family of curves in X that have a subcurve in some $\phi_n(\Gamma_n)$, $n \in \mathbb{N}$. Then $\operatorname{Mod}\Gamma_0 = 0$ and the conclusions of Theorem 1.4 (1) hold for all curves γ in X outside Γ_0. □

3.2 Injectivity

In this section we establish Theorem 1.4 (2). The main difficulty is to establish the injectivity of f. A map $f: X \to Y$ is *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

Lemma 3.4 *Suppose that Y is reciprocal. Let $f: X \to Y$ be a non-constant continuous map such that there exists $K > 0$ with the property that* $\operatorname{Mod}\Gamma \leq K \operatorname{Mod} f(\Gamma)$ *for each curve family Γ in X. Then f is a light map.*

Proof Let $y \in Y$ and suppose that $f^{-1}(y)$ contains a non-degenerate continuum E. Consider a non-degenerate continuum $F \subset X\setminus f^{-1}(y)$; note that the latter set is non-empty because f is non-constant. The family Γ of curves joining E and F has positive modulus [35, Proposition 3.5]. On the other hand, each curve of $f(\Gamma)$ joins the continuum $f(F)$ to y. Since Y is reciprocal, we have $\operatorname{Mod} f(\Gamma) = 0$ (see (2.3)). This is a contradiction. □

For $y_0 \in Y$ and $r > 0$ we denote by $S(y_0, r)$ the set $\{y \in Y : d(y, y_0) = r\}$.

Lemma 3.5 *Let $y_0 \in Y$ and $K \subset Y\setminus\{y_0\}$ be a closed set. There exists $\delta > 0$ such that for a.e. $r \in (0, \delta)$ there exists a component $E \subset S(y_0, r)$ that is a rectifiable Jordan curve separating y_0 and K.*

Proof Let $U \subset Y$ be the interior of a topological closed disk $\overline{U} \subset Y$ such that $Y \setminus U$ is connected, $y_0 \in U$, and $K \subset Y\setminus U$. Note that $\mathcal{H}^2(\overline{U}) < \infty$. Let $\delta > 0$ be sufficiently small such that $\overline{B}(y_0, \delta) \subset U$. Then for all $r \in (0, \delta)$ the set $S(y_0, r)$ is compact. By the coarea inequality for Lipschitz functions (Theorem 2.1), $\mathcal{H}^1(S(y_0, r)) < \infty$ for a.e. $r \in (0, \delta)$. By Corollary 2.7 (see also [33, Theorem 1.5]), for a.e. $r \in (0, \delta)$, each component of $S(y_0, r)$ is a rectifiable Jordan arc or Jordan curve. Fix such a parameter r. Since $S(y_0, r)$ separates y_0 from all points of ∂U, by Lemma 2.8 there exists a component E of $S(y_0, r)$ that separates y_0 from ∂U. In particular, E must be a Jordan curve and separates y_0 from K. □

Lemma 3.6 *Let $Z \subset X$ be homeomorphic to a topological closed disk and let $f: Z \to Y$ be a continuous map in $N^{1,2}(Z, Y)$. For every $y_0 \in Y$ and for a.e. $r \in (0, \infty)$, each component of $f^{-1}(S(y_0, r))$ is a Jordan arc or a Jordan curve.*

Proof Define $u(x) = d(f(x), y_0)$ on Z, which is continuous and lies in $N^{1,2}(Z)$. Observe that $u^{-1}(r) = f^{-1}(S(y_0, r))$ for every $r > 0$. By the coarea inequality of Theorem 1.6 we see that $\mathcal{H}^1(u^{-1}(r) \cap \mathcal{A}_u) < \infty$ for a.e. $r > 0$. In particular, for such values r, if E is a non-degenerate component of $u^{-1}(r)$, then $E \subset \mathcal{A}_u$, so $\mathcal{H}^1(E) < \infty$. By Corollary 2.7, for a.e. $r > 0$, every non-degenerate component of $u^{-1}(r)$ is a Jordan arc or a Jordan curve. □

Lemma 3.7 *Let $f: X \to Y$ be a continuous light map in $N^{1,2}(X, Y)$ such that $N(f, y) \leq 1$ for a.e. $y \in Y$. Then $N(f, y) \leq 1$ for every $y \in Y$. In particular, f is injective.*

Proof Let $y \in f(X)$ and $x \in f^{-1}(y)$. For the moment, we consider the restriction $g = f|_Z$ to a compact neighborhood Z of x that is homeomorphic to a closed disk and contains x in its interior. Since g is light, it is non-constant on $\mathrm{int}(Z)$ and there exists a point $z \in \mathrm{int}(Z) \setminus g^{-1}(y)$. Note that for each $r \in (0, d(y, g(z)))$ the set $S(y, r)$ separates y from $g(z)$. Therefore, the compact set $g^{-1}(S(y, r))$ separates x from z. By Lemma 3.6, for a.e. $r > 0$, each component of $g^{-1}(S(y, r))$ is a Jordan arc or a Jordan curve. Combining these facts with Lemma 2.8, we see that there exists a full measure subset I of $(0, d(y, g(z)))$ such that for each $r \in I$, there exists a component of $g^{-1}(S(y, r))$ that separates x from z and is a Jordan arc or a Jordan curve.

We claim for all sufficiently small $r \in I$, each such component must be a Jordan curve. To prove this, suppose that there exists a sequence of positive numbers $r_n \to 0$ and components F_{r_n} of $g^{-1}(S(y, r_n))$ that are Jordan arcs and separate x from z. Fix a continuum $K \subset \mathrm{int}(Z)$ connecting x and z. Since F_{r_n} separates x from z, it intersects K. Moreover, since F_{r_n} is a Jordan arc, it cannot be contained in $\mathrm{int}(Z)$, as $\mathrm{int}(Z) \setminus F_{r_n}$ would then be connected. Therefore, F_{r_n} intersects ∂Z and

$$\mathrm{diam}(F_{r_n}) \geq \mathrm{dist}(K, \partial Z) > 0$$

for all $n \in \mathbb{N}$. After passing to a subsequence, F_{r_n} converges in the Hausdorff sense to a non-degenerate continuum F. Since $r_n \to 0$, we have that $F \subset g^{-1}(y)$. This contradicts the lightness of g. The claim is proved.

By the assumption that $N(f, w) \leq 1$ for a.e. $w \in Y$ and the coarea inequality for Lipschitz functions (Theorem 2.1), we see that for a.e. $r > 0$, $\mathcal{H}^1$-a.e. point of $S(y, r)$ has at most one preimage under f. Also, given a closed set $K \subset Y \setminus \{y\}$, by Lemma 3.5, for a.e. sufficiently small $r > 0$ there exists a Jordan curve $E \subset S(y, r)$ separating y from K. Altogether, there exists $\delta' > 0$ and a set $I' \subset (0, \delta')$ of full measure so that for every $r \in I'$ the following statements are true.

(1) $\mathcal{H}^1$-a.e. point of $S(y, r)$ has at most one preimage under f.
(2) There exists a component of $S(y, r)$ that is a Jordan curve separating y and K.
(3) Each component of $g^{-1}(S(y, r))$ that separates x and z is a Jordan curve.

Let E be a component of $S(y, r)$, $r \in I'$, that is a Jordan curve and let $F \subset g^{-1}(E)$ be a Jordan curve. We claim that $g(F) = E$. By (1), $\mathcal{H}^1$-a.e. point of E has at most one preimage under g. The map $g|_F$ is conjugate to a continuous map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ with the property that a dense set of points of $\mathbb{S}^1$ have at most one preimage. Suppose that $g(F)$ is a strict subarc of E. Note that $g(F)$ cannot be a point since g is light. Then $\phi(\mathbb{S}^1)$ is a non-degenerate strict subarc of $\mathbb{S}^1$. This contradicts the fact that a dense set of points of $\mathbb{S}^1$ have at most one preimage. We have shown the following.

(4) If E is a component of $S(y, r)$ that is a Jordan curve and $F \subset g^{-1}(E)$ is a Jordan curve, then $g(F) = E$.

We have completed our preparation to show the injectivity of f. Suppose that $f^{-1}(y)$ contains two points x_1, x_2 for some $y \in f(X)$. We consider disjoint topological closed disks $Z_1, Z_2 \subset X$ such that $x_i \in \mathrm{int}(Z_i)$, $i = 1, 2$. We also fix $z_i \in \mathrm{int}(Z_i) \setminus f^{-1}(y)$. Consider the restrictions $g_i = f|_{Z_i}$, $i = 1, 2$. By the previous, for $i = 1, 2$, there exists a set I_i' of full measure in an interval $(0, \delta_i')$, such that

(1)–(4) are true for the map g_i; specifically, in (2) we use the set $K = \{f(z_1), f(z_2)\}$. Let $I' = I_1' \cap I_2'$, which has full measure in $(0, \delta')$, where $\delta' = \min\{\delta_1', \delta_2'\}$. By (2), for $r \in I'$ there exists a component E of $S(y, r)$ that is a Jordan curve separating each of the pairs $(y, f(z_1))$ and $(y, f(z_2))$. Let F_i be a component of $g_i^{-1}(E)$ that separates x_i and z_i, $i = 1, 2$; such components exist by Lemma 2.8. Note that F_i is also a component of $g_i^{-1}(S(y, r))$. By (3), F_i is a Jordan curve for $i = 1, 2$. By (4), we conclude that $g_i(F_i) = E$, $i = 1, 2$. Thus, each point of E has at least two preimages under f. This contradicts (1). □

Lemma 3.8 *Let $f: X \to Y$ be an area-preserving map that is a quasiconformal homeomorphism. Suppose that there exists $K \geq 1$ such that*

$$K^{-1/2}\ell(\gamma) \leq \ell(f \circ \gamma) \leq K^{1/2}\ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{Mod}\Gamma_0 = 0$. *Then f is K-quasiconformal.*

Proof The constant function $K^{1/2}$ is a 2-weak upper gradient of f and lies in $N^{1,2}_{\mathrm{loc}}(X)$. Moreover, by the preservation of area, for each Borel set $E \subset Y$ we have

$$\int_{f^{-1}(E)} K \, d\mathcal{H}^2 = K\mathcal{H}^2(f^{-1}(E)) = K\mathcal{H}^2(E).$$

In view of Theorem 2.10, we derive that f is weakly K-quasiconformal. Since f is quasiconformal, we have

$$\ell(f^{-1} \circ \gamma) \leq K^{1/2}\ell(\gamma)$$

for all curves γ in Y outside a curve family Γ_0' with $\operatorname{Mod}\Gamma_0' = 0$. Thus, the same argument applies to f^{-1} and shows that it is weakly K-quasiconformal. Altogether, f is K-quasiconformal. □

Proof of Theorem 1.4 (2) Suppose that f is L-Lipschitz and area-preserving. By Lemma 3.2, $N(f, y) = 1$ for a.e. $y \in f(X)$. Also, Lemmas 3.1 and 3.4 imply that f is a light map. Now, Lemma 3.7 implies that the restriction of f to any precompact open subset U of X (so that $f|_U \in N^{1,2}(U, Y)$) is injective. This implies that f is injective in all of X. The invariance of domain theorem implies that f is an embedding. Since f is surjective by assumption, we conclude that f is a homeomorphism. By Lemma 3.1, we see that f is a weakly L^2-quasiconformal homeomorphism. Since Y is reciprocal, Lemma 2.13 yields that f is K-quasiconformal for some $K = K(L) \geq 1$. In particular, this implies that X is also reciprocal.

The final inequality in Theorem 1.4 (2) involving the lengths follows from Theorem 1.4 (1). In the case that f is 1-Lipschitz, we obtain $\ell(\gamma) = \ell(f \circ \gamma)$ for all curves γ in X outside a curve family Γ_0 with $\operatorname{Mod}\Gamma_0 = 0$. By Lemma 3.8, we conclude that f is 1-quasiconformal. □

3.3 Bounded Length Distortion and Isometry

Here we prove Theorem 1.4 (3). Our goal is to upgrade the conclusion of Theorem 1.4 (2) so that the length of every path, rather than almost every path, is quasi-preserved. This is achieved with the aid of upper Ahlfors 2-regularity. We say that a space is locally upper Ahlfors 2-regular with constant $K > 0$ if each point has a neighborhood U such that $\mathcal{H}^2(B(x,r)) \leq Kr^2$ for all $x \in U$ and $r < \operatorname{diam}(U)$. We denote by $N_r(E)$ the open r-neighborhood of a set E.

Lemma 3.9 *Suppose that Y is locally upper Ahlfors 2-regular with constant $K > 0$ and γ is a curve in Y. Then for all sufficiently small $r > 0$ we have*

$$\mathcal{H}^2(N_r(|\gamma|)) \leq 2Kr\ell(\gamma) + 8Kr^2.$$

Proof Without loss of generality, $\gamma\colon [0, \ell(\gamma)] \to X$ is non-constant, rectifiable and parametrized by arclength. Assume that $0 < r < \ell(\gamma)/2$ and that for every $x \in |\gamma|$ we have

$$\mathcal{H}^2(B(x, 2r)) \leq 4Kr^2.$$

Consider a partition $\{t_0, \ldots, t_n\}$ of $[0, \ell(\gamma)]$ such that $|t_i - t_{i-1}| \leq 2r, i \in \{1, \ldots, n\}$, and $(n-1)2r < \ell(\gamma) \leq 2nr$. Then $\{B(\gamma(t_i), 2r)\}_{i=0}^{n}$ covers $N_r(|\gamma|)$ and we can compute

$$\mathcal{H}^2(N_r(|\gamma|)) \leq \sum_{i=0}^{n} \mathcal{H}^2(B(\gamma(t_i), 2r)) \leq (n+1)4Kr^2 \leq 2Kr\ell(\gamma) + 8Kr^2.$$

□

Lemma 3.10 *Suppose that Y is locally upper Ahlfors 2-regular with constant $K > 0$. Let Γ_0 be a curve family in Y with* $\operatorname{Mod}\Gamma_0 = 0$. *Then for each curve $\gamma\colon [a, b] \to Y$ and for each $\varepsilon > 0$ there exists a curve $\gamma_\varepsilon\colon [a, b] \to Y$ with the following properties.*

(1) $\gamma_\varepsilon \notin \Gamma_0$.
(2) $|\gamma(a) - \gamma_\varepsilon(a)| < \varepsilon$, $|\gamma(b) - \gamma_\varepsilon(b)| < \varepsilon$, *and* $|\gamma_\varepsilon| \subset N_\varepsilon(|\gamma|)$.
(3) $\ell(\gamma_\varepsilon) \leq 4\pi^{-1}K\ell(\gamma) + \varepsilon$.

Moreover, if Y is Riemannian, then

(3') $\ell(\gamma_\varepsilon) \leq \ell(\gamma) + \varepsilon$.

Proof Assume that γ is simple, otherwise we consider a simple curve with trace in $|\gamma|$ connecting $\gamma(a)$ and $\gamma(b)$. Let $\varepsilon > 0$. Consider the distance function $g(x) = d(x, |\gamma|)$. By the coarea inequality for Lipschitz functions (Theorem 2.1) and Lemma 3.9, there exists $r_1 > 0$ such that for all $0 < r < r_1$ we have

$$\int^{*} \chi_{(0,r)}(t)\mathcal{H}^1(g^{-1}(t))\, dt \leq \frac{4}{\pi}\mathcal{H}^2(N_r(|\gamma|)) < \frac{8}{\pi}Kr\ell(\gamma) + \varepsilon r. \tag{3.9}$$

Therefore, for all $0 < r < r_1$ we have

$$\operatorname*{essinf}_{t\in(0,r)} \mathcal{H}^1(g^{-1}(t)) < \frac{8}{\pi} K\ell(\gamma) + \varepsilon. \tag{3.10}$$

By Lemma 2.9, for a.e. $t \in (0, r_1)$, every Lipschitz and injective curve $\alpha\colon [a, b] \to g^{-1}(t)$ does not lie in Γ_0.

Let $U \subset Y$ be a neighborhood of $|\gamma|$ homeomorphic to $\mathbb{D}$. Since γ is simple, the space $Z := U/|\gamma|$ equipped with the quotient metric is homeomorphic to $\mathbb{D}$. The quotient map $\pi\colon U \to Z$ is a local isometry on $U \setminus |\gamma|$. This together with Lemma 3.5 provides the existence of $r_2 > 0$ such that for a.e. $t \in (0, r_2)$, the level set $g^{-1}(t)$ contains a rectifiable Jordan curve γ_t in U separating $|\gamma|$ from ∂U. Note that $|\gamma_t|$ converges to $|\gamma|$ in the Hausdorff sense as $t \to 0$. Thus, there exists $r_3 \in (0, r_2)$ such that for a.e. $t \in (0, r_3)$ we can find distinct points $a_t \in \overline{B}(\gamma(a), \varepsilon) \cap |\gamma_t|$ and $b_t \in \overline{B}(\gamma(b), \varepsilon) \cap |\gamma_t|$. Let γ_t' be a Lipschitz and injective parametrization of the closure of the shorter component of $|\gamma_t| \setminus \{a_t, b_t\}$. For $0 < r < \min\{r_1, r_2, r_3, \varepsilon\}$ we have

$$\operatorname*{essinf}_{t\in(0,r)} \ell(\gamma_t') < \frac{4}{\pi} K\ell(\gamma) + \frac{\varepsilon}{2}.$$

By the previous, $\gamma_t' \notin \Gamma_0$ for a.e. $t \in (0, r)$. Moreover, $|\gamma_t'| \subset g^{-1}(t) \subset N_\varepsilon(|\gamma|)$. Therefore, there exists $t \in (0, r)$ so that γ_t' satisfies (1)–(3).

If Y is Riemannian we have a local upper area bound of the form

$$\mathcal{H}^2(N_r(|\gamma|)) \leq 2r\ell(\gamma) + O(r^2);$$

see [17, Corollary 9.24]. By arguing as in (3.9) while applying the coarea inequality for Riemannian manifolds (Theorem 2.1), we obtain

$$\operatorname*{essinf}_{t\in(0,r)} \ell(\gamma_t') \leq \ell(\gamma) + \varepsilon,$$

for all sufficiently small $r > 0$. Hence, (3') follows. □

Lemma 3.11 *Suppose that Y is locally upper Ahlfors 2-regular with constant $K > 0$. Let $g\colon Y \to X$ be continuous map such that there exists $L > 0$ with the property that $\ell(g \circ \gamma) \leq L\ell(\gamma)$ for all curves γ in Y outside a curve family Γ_0 with* $\operatorname{Mod}\Gamma_0 = 0$. *Then*

$$\ell(g \circ \gamma) \leq \frac{4}{\pi} KL\ell(\gamma)$$

for every rectifiable curve γ in Y. Moreover, if Y is Riemannian then

$$\ell(g \circ \gamma) \leq L\ell(\gamma)$$

for every rectifiable curve γ in Y.

Proof Let γ be a rectifiable Jordan arc in Y. By Lemma 3.10, for each $n \in \mathbb{N}$ we can find a curve $\gamma_n \subset N_{1/n}(|\gamma|)$ whose endpoints are $(1/n)$-close to the endpoints of γ, $\gamma_n \notin \Gamma_0$, and

$$\ell(\gamma_n) \le 4\pi^{-1} K \ell(\gamma) + n^{-1}.$$

Suppose that γ_n is parametrized by $[0, 1]$ with constant speed. After passing to a subsequence, we may assume that γ_n converges uniformly to a path $\widetilde{\gamma} \colon [0, 1] \to |\gamma|$ with the same endpoints as γ. It follows that $\widetilde{\gamma}$ is surjective, but it is possibly not injective. Moreover, $g \circ \gamma_n$ converges uniformly to $g \circ \widetilde{\gamma}$. Since γ is a Jordan arc, we have $N(g \circ \widetilde{\gamma}, y) \ge N(g \circ \gamma, y)$ for each $y \in g(|\gamma|)$. The area formula for length (Theorem 2.3) and the lower semi-continuity of length imply that

$$\ell(g \circ \gamma) \le \ell(g \circ \widetilde{\gamma}) \le \liminf_{n\to\infty} \ell(g \circ \gamma_n).$$

Since $\gamma_n \notin \Gamma_0$, the latter is bounded by

$$L \liminf_{n\to\infty} \ell(\gamma_n) \le 4\pi^{-1} K L \ell(\gamma).$$

This completes the proof in the case of Jordan arcs.

Now, suppose that $\gamma \colon [a, b] \to Y$ is an arbitrary path. Let $\{t_0, \dots, t_n\}$ be a partition of $[a, b]$. For $i \in \{1, \dots, n\}$, let $\gamma_i \colon [t_{i-1}, t_i] \to \gamma([t_{i-1}, t_i])$ be a Jordan arc with endpoints $\gamma(t_{i-1}), \gamma(t_i)$. Then

$$\begin{aligned}\sum_{i=1}^{n} d(g(\gamma(t_{i-1})), g(\gamma(t_i))) &\le \sum_{i=1}^{n} \ell(g \circ \gamma_i) \le 4\pi^{-1} K L \sum_{i=1}^{n} \ell(\gamma_i)\\ &\le 4\pi^{-1} K L \sum_{i=1}^{n} \ell(\gamma|_{[t_{i-1}, t_i]}) = 4\pi^{-1} K L \ell(\gamma).\end{aligned}$$

This yields $\ell(g \circ \gamma) \le 4\pi^{-1} K L \ell(\gamma)$.

If Y is Riemannian, the statement follows after applying (3') from Lemma 3.10 instead of (3). □

Proof *(Proof of Theorem 1.4 (3))* The upper Ahlfors 2-regularity implies that Y is reciprocal [35, Theorem 1.6]. By Theorem 1.4 (2), we have that f is a quasiconformal homeomorphism and the length of a.e. path is quasi-preserved. We now apply Lemma 3.11 to $g = f^{-1}$, together with the fact that f is Lipschitz, and conclude that the length of every rectifiable path is quasi-preserved. It also follows that $\ell(\gamma) < \infty$ if and only if $\ell(f \circ \gamma) < \infty$. Therefore, f is a map of bounded length distortion. □

Proof *(Proof of Theorem 1.4 (4))* Since Y is reciprocal, by Theorem 1.4 (2), f is a 1-quasiconformal homeomorphism and preserves the length of all curves in X outside a curve family Γ_0 with $\operatorname{Mod} \Gamma_0 = 0$. It follows from Lemma 3.11 that $\ell(f^{-1} \circ \gamma) \le \ell(\gamma)$

for every rectifiable curve γ in Y. If $x, y \in X$ and γ is a rectifiable curve in Y joining $f(x)$ and $f(y)$ then

$$d(x, y) \leq \ell(f^{-1} \circ \gamma) \leq \ell(\gamma).$$

Infimizing over γ gives $d(x, y) \leq d(f(x), f(y))$. Equality follows from f being 1-Lipschitz. □

4 Examples

We present examples that show the optimality of Theorem 1.4. In all examples X, Y are metric surfaces of locally finite Hausdorff 2-measure and $f: X \to Y$ is an area-preserving and 1-Lipschitz map.

Example 4.1 This example shows that f is not a homeomorphism in general, even if X is Euclidean. Let I be the interval $[0, 1] \times \{0\}$ and $Y = \mathbb{R}^2/I$, equipped with the quotient metric. The natural projection map $f: \mathbb{R}^2 \to Y$ is area-preserving and 1-Lipschitz, but it is not a homeomorphism.

Example 4.2 This example shows that if Y is reciprocal as in Theorem 1.4 (2), then f is not BLD in general, even if X is Euclidean. Define the weight $\omega: \mathbb{R}^2 \to [0, 1]$ by $\omega(x) = x_1$ if $x = (x_1, 0) \in I{:=}(0, 1] \times \{0\}$ and $\omega(x) = 1$ otherwise. We define a metric d on $\mathbb{R}^2$ by

$$d(x, y) := \inf_{\gamma} \int_{\gamma} \omega \, ds,$$

where the infimum is taken over all rectifiable curves γ connecting $x, y \in \mathbb{R}^2$. Let $f: \mathbb{R}^2 \to Y := (\mathbb{R}^2, d)$ be the identity map, which is 1-Lipschitz, since $\omega \leq 1$, and a local isometry on $\mathbb{R}^2 \setminus I$, hence area-preserving. Moreover, f is a homeomorphism, and thus Y is a metric space homeomorphic to $\mathbb{R}^2$.

One can show that for each Borel set $E \subset \mathbb{R}^2$ we have $\mathcal{H}^1_d(E) = \int_E \omega \, d\mathcal{H}^1$; in fact, it suffices to show this for sets $E \subset I$. This fact and the area formula for length (Theorem 2.3) imply that if γ is a rectifiable curve with respect to the Euclidean metric, then $\ell_d(\gamma) = \int_\gamma \omega \, ds$. This implies that

$$\int_{\gamma} \rho \, ds_d = \int_{\gamma} \rho\omega \, ds$$

for every Borel function $\rho: \mathbb{R}^2 \to [0, \infty]$.

Let Γ be a family of curves in $\mathbb{R}^2$. Since f is 1-Lipschitz and area-preserving, by Lemma 3.1 we have $\operatorname{Mod} \Gamma \leq \operatorname{Mod} f(\Gamma)$; here the latter modulus is with respect to the metric d. We now show the reverse inequality. Let $\rho: \mathbb{R}^2 \to [0, \infty]$ be admissible for Γ. We set $\rho' = \rho\omega^{-1}$. If $\gamma \in \Gamma$, then

$$\int_{f\circ\gamma} \rho' \, ds_d = \int_\gamma \rho\omega^{-1}\omega \, ds = \int_\gamma \rho \, ds \geq 1.$$

Thus, ρ' is admissible for $f(\Gamma)$. Since $\mathcal{H}^2_d(I) = 0$, we conclude that

$$\operatorname{Mod} f(\Gamma) \leq \int \rho^2 \, \mathrm{d}\mathcal{H}^2$$

and thus $\operatorname{Mod} f(\Gamma) \leq \operatorname{Mod}\Gamma$. This shows that f is 1-quasiconformal and that Y is reciprocal.

By Theorem 1.4 (1), f preserves the length of a.e. curve with respect to 2-modulus; this can also be seen immediately here, since a.e. curve intersects I at a set of length zero. However, f does not preserve the length of *every* curve and is not BLD. Indeed, for $t \in (0, 1]$ denote by γ_t the straight line segment connecting $(0, 0)$ and $(t, 0)$. Then $\ell(\gamma_t) = t$, whereas

$$\ell_d(\gamma_t) = \int_{\gamma_t} \omega \, ds = t^2/2.$$

5 Coarea Inequality

In this section we establish the general coarea inequality of Theorem 1.6. First we prove the statement in the case that X is a topological closed disk. The proof follows the same strategy as in [13, Theorem 4.8].

Theorem 5.1 *Let X be a metric surface of finite Hausdorff 2-measure that is homeomorphic to a topological closed disk and suppose that there exists a weakly K-quasiconformal map from $\overline{\mathbb{D}}$ onto X for some $K \geq 1$. Let $u\colon X \to \mathbb{R}$ be a continuous function with a 2-weak upper gradient $\rho_u \in L^2(X)$.*

(1) *If $\mathcal{A}_u$ denotes the union of all non-degenerate components of the level sets $u^{-1}(t)$, $t \in \mathbb{R}$, of u, then $\mathcal{A}_u$ is a Borel set.*
(2) *For every Borel function $g\colon X \to [0, \infty]$ we have*

$$\int^* \int_{u^{-1}(t)\cap\mathcal{A}_u} g \, d\mathcal{H}^1 \, dt \leq K \int g\rho_u \, d\mathcal{H}^2.$$

Proof First we show that $\mathcal{A}_u$ is a Borel set. We can write

$$\mathcal{A}_u = \bigcup_{k=1}^{\infty} A_k,$$

where A_k is the union of the components E of $u^{-1}(t)$, $t \in \mathbb{R}$, with $\operatorname{diam}(E) \geq 1/k$. We will show that A_k is closed for each $k \in \mathbb{N}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in A_k.

If $x_n \in E_n \subset u^{-1}(t_n)$, $n \in \mathbb{N}$, then after passing to a subsequence, the continua E_n converge in the Hausdorff sense to a continuum E with $\operatorname{diam}(E) \geq 1/k$. Moreover, after passing to a further subsequence, t_n converges to some $t \in \mathbb{R}$, so $E \subset u^{-1}(t)$. This shows that $E \subset A_k$. Therefore, all limit points of $\{x_n\}_{n\in\mathbb{N}}$ lie in A_k, as desired.

Let $f : \overline{\mathbb{D}} \to X$ be a weakly K-quasiconformal map. By Theorem 2.10 there exists a 2-weak upper gradient $\rho_f \in L^2(\overline{\mathbb{D}})$ such that

$$\int_{f^{-1}(E)} \rho_f^2 \, d\mathcal{H}^2 \leq K\mathcal{H}^2(E)$$

for each Borel set $E \subset X$. This implies that

$$\int (g \circ f) \cdot \rho_f^2 \, d\mathcal{H}^2 \leq K \int g \, d\mathcal{H}^2 \tag{5.1}$$

for each Borel function $g \colon X \to [0, \infty]$. Moreover, for all curves γ in $\overline{\mathbb{D}}$ outside a curve family Γ_0 with $\operatorname{Mod} \Gamma_0 = 0$ we have (see [18, Prop. 6.3.3])

$$\int_{f\circ\gamma} g \, ds \leq \int_\gamma (g \circ f) \cdot \rho_f \, ds. \tag{5.2}$$

Consider the function $v = u \circ f$ on $\overline{\mathbb{D}}$. Then by [13, Lemma 4.5], v has a 2-weak upper gradient ρ_v such that for a.e. $x \in \overline{\mathbb{D}}$ we have

$$\rho_v(x) \leq (\rho_u \circ f)(x) \cdot \rho_f(x).$$

In conjunction with (5.1), this implies that $\rho_v \in L^2(\overline{\mathbb{D}})$, so $v \in W^{1,2}(\mathbb{D})$, and

$$|\nabla v(x)| \leq (\rho_u \circ f)(x) \cdot \rho_f(x) \tag{5.3}$$

for a.e. $x \in \mathbb{D}$, because $|\nabla v|$ is the minimal 2-weak upper gradient of v (see [18, Theorem 7.4.5]). We can extend v by reflection to a continuous function $\widetilde{v} \in W^{1,2}(U)$ for some neighborhood U of $\overline{\mathbb{D}}$. By the classical coarea formula (Theorem 2.1), the set $v^{-1}(t) = \widetilde{v}^{-1}(t) \cap \overline{\mathbb{D}}$ has finite Hausdorff 1-measure for a.e. $t \in \mathbb{R}$. Corollary 2.7 implies that for a.e. $t \in \mathbb{R}$ each component E of $v^{-1}(t)$ is a Jordan arc or a Jordan curve and can be parametrized by a Lipschitz function $\gamma \colon [a, b] \to E$ that is injective on $[a, b)$. Moreover, using the classical coarea formula for $\widetilde{v}$ one can show that for a.e. $t \in \mathbb{R}$, each Lipschitz curve $\gamma \colon [a, b] \to v^{-1}(t)$ that is injective on $[a, b)$ lies outside the given curve family Γ_0 of 2-modulus zero (cf. Lemma 2.9); hence γ satisfies (5.2). Therefore, the following statements are true for a.e. $t \in \mathbb{R}$.

(1) $\mathcal{H}^1(v^{-1}(t)) < \infty$. (Consequence of classical coarea formula.)
(2) Each non-degenerate component E of $v^{-1}(t)$ is a Jordan arc or a Jordan curve and there exists a Lipschitz parametrization $\gamma \colon [a, b] \to E$ that is injective in $[a, b)$. (Consequence of Corollary 2.7.)

(3) For each Lipschitz curve $\gamma\colon [a,b] \to v^{-1}(t)$ that is injective on $[a,b)$ and for each Borel function $g\colon X \to [0,\infty]$, we have

$$\int_{f\circ\gamma} g\, ds \leq \int_{\gamma} (g\circ f)\cdot \rho_f\, ds.$$

(Consequence of classical coarea formula and (5.2).)

We fix a Borel function $g\colon X \to [0,\infty]$, a value $t \in \mathbb{R}$ satisfying the above statements, a non-degenerate component E of $v^{-1}(t)$, and a Lipschitz parametrization $\gamma\colon [a,b] \to E$ that is injective in $[a,b)$. We have

$$\begin{aligned}\int_{f(E)} g\, d\mathcal{H}^1 = \int_{f(|\gamma|)} g\, d\mathcal{H}^1 &\leq \int_{f\circ\gamma} g\, ds \\ &\leq \int_{\gamma} (g\circ f)\cdot \rho_f\, ds = \int_E (g\circ f)\cdot \rho_f\, d\mathcal{H}^1.\end{aligned}$$

Note that if G is a non-degenerate component of $u^{-1}(t)$, then by the monotonicity of f, $f^{-1}(G)$ is a non-degenerate component of $v^{-1}(t)$. Hence,

$$\int_G g\, d\mathcal{H}^1 \leq \int_{f^{-1}(G)} (g\circ f)\cdot \rho_f\, d\mathcal{H}^1.$$

The finiteness of the Hausdorff 1-measure of $v^{-1}(t)$ implies that it can have at most countably many non-degenerate components. Summing over all the non-degenerate components gives

$$\int_{u^{-1}(t)\cap\mathcal{A}_u} g\, d\mathcal{H}^1 \leq \int_{v^{-1}(t)} (g\circ f)\cdot \rho_f\, d\mathcal{H}^1.$$

We now integrate over $t \in \mathbb{R}$, use the classical coarea formula for $\widetilde{v}$, and inequalities (5.3) and (5.1), to obtain

$$\begin{aligned}\int^{*} \int_{u^{-1}(t)\cap\mathcal{A}_u} g\, d\mathcal{H}^1 dt &\leq \int \int_{v^{-1}(t)} (g\circ f)\cdot \rho_f\, d\mathcal{H}^1 dt \\ &= \int_{\overline{\mathbb{D}}} (g\circ f)\cdot \rho_f \cdot |\nabla \widetilde{v}|\, d\mathcal{H}^2 \\ &= \int_{\mathbb{D}} (g\circ f)\cdot \rho_f \cdot |\nabla v|\, d\mathcal{H}^2 \\ &\leq \int (g\circ f)\cdot (\rho_u\circ f)\cdot \rho_f^2\, d\mathcal{H}^2 \\ &\leq K \int g\rho_u\, d\mathcal{H}^2.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.6 We write X as the countable union of topological closed disks X_n with $\mathcal{H}^2(X_n) < \infty, n \in \mathbb{N}$. We also consider topological closed disks $Z_n \supset X_n$, so that the topological interior $\operatorname{int}_{\text{top}}(Z_n)$ contains X_n. We have $\operatorname{int}(Z_n) \subset \operatorname{int}_{\text{top}}(Z_n) \subset Z_n$, where $\operatorname{int}(Z_n)$ refers to the manifold interior. Therefore the topological closure of $\operatorname{int}_{\text{top}}(Z_n)$ is precisely the closed disk Z_n. Let $u_n = u|_{Z_n}$. We claim that

$$\mathcal{A}_u = \bigcup_{n=1}^{\infty} \mathcal{A}_{u_n}. \tag{5.4}$$

For this, it suffices to show that

$$\mathcal{A}_u \cap X_n \subset \mathcal{A}_{u_n} \tag{5.5}$$

for each $n \in \mathbb{N}$. Let $x \in \mathcal{A}_u \cap X_n$ and consider a non-degenerate component E of $u^{-1}(t)$ for some $t \in \mathbb{R}$ such that $x \in E$. Note that x lies in $\operatorname{int}_{\text{top}}(Z_n)$. If $E \subset Z_n$, then $E \subset \mathcal{A}_{u_n}$ and $x \in \mathcal{A}_{u_n}$. Suppose that E is not contained in Z_n; in this case $E \cap \partial_{\text{top}} Z_n \neq \emptyset$ by the connectedness of E. Since E is a generalized continuum (i.e., a locally compact connected set), by [38, (10.1), p. 16], we conclude that each component of $E \cap Z_n$ intersects $\partial_{\text{top}} Z_n$. In particular, the component E_x of $E \cap Z_n$ that contains x must intersect $\partial_{\text{top}} Z_n$, and thus E_x is non-degenerate. We conclude that $E_x \subset \mathcal{A}_{u_n}$, so $x \in \mathcal{A}_{u_n}$. The claim is proved. Now, each $\mathcal{A}_{u_n}$ is a Borel set by Theorem 5.1, so $\mathcal{A}_u$ is Borel measurable by (5.4) and we have established (1).

Let $g\colon X \to [0, \infty]$ be a Borel function. For $n \in \mathbb{N}$, let $g_n = g \cdot \chi_{X_n \setminus \bigcup_{i=1}^{n-1} X_i}$. Let $x \in \mathcal{A}_u \cap (X_n \setminus \bigcup_{i=1}^{n-1} X_i)$. Then $x \in \mathcal{A}_{u_n}$ by (5.5), so

$$g(x)\chi_{\mathcal{A}_u}(x) = g_n(x)\chi_{\mathcal{A}_u}(x) = g_n(x)\chi_{\mathcal{A}_{u_n}}(x).$$

We conclude that

$$g\chi_{\mathcal{A}_u} = \sum_{n \in \mathbb{N}} g_n \chi_{\mathcal{A}_{u_n}}.$$

By Theorem 5.1, applied to $u_n\colon Z_n \to \mathbb{R}$, and the existence of weakly $(4/\pi)$-quasiconformal parametrizations (Theorem 2.12), we have

$$\int^{*} \int_{u^{-1}(t)} g_n \chi_{\mathcal{A}_{u_n}} \, d\mathcal{H}^1 dt \leq \frac{4}{\pi} \int g_n \rho_u \, d\mathcal{H}^2$$

for each $n \in \mathbb{N}$. Thus, upon summing we obtain the claimed inequality (2).

Finally, part (3) follows from part (2) and the coarea inequality for Lipschitz functions. Namely, one applies (2) to the Borel function $g\chi_{\mathcal{A}_u}$ and Theorem 2.1 to $g\chi_{X \setminus \mathcal{A}_u}$. □

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