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Weighted Strong-Type Estimates on Classical Lorentz Spaces

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Abstract
We present new estimates in the setting of weighted classical Lorentz spaces for important operators in Harmonic Analysis such as Calderón-Zygmund operators, sparse operators and the Bochner-Riesz operator among others.

Keywords Weighted classical Lorentz spaces · Muckenhoupt weights · Extrapolation theory · Calderón-Zygmund operators · Sparse operators · Bochner-Riesz operator

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1 Introduction

Given positive locally integrable functions (called weights)

$$w : (0, \infty) \to (0, \infty) \qquad \text{and} \qquad u : \mathbb{R}^n \to (0, \infty),$$

the main goal of this paper is to prove boundedness of important operators in harmonic analysis on weighted classical Lorentz spaces $\Lambda_u^p(w)$, $p > 0$, defined by those measurable functions f such that

$$||f||_{\Lambda_u^p(w)} := \left(\int_0^\infty f_u^*(t)^p w(t)\, dt \right)^{\frac{1}{p}} < \infty,$$

(see [18]) where f_u^* is the decreasing rearrangement of f defined by

$$f_u^*(t) := \inf\{y > 0 : \lambda_f^u(y) \le t\}, \qquad t > 0,$$

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with

$$\lambda_f^u(y) := u(\{|f| > y\}) = \int_{\{|f|>y\}} u(x)\,dx, \qquad y > 0.$$

This will lead to the unification of classical results about the boundedness of operators on different weighted settings, like the Lebesgue spaces $L^p(u) := \Lambda_u^p(1)$ or the classical Lorentz spaces $\Lambda^p(w) := \Lambda_1^p(w)$, resulting into a general framework involving both theories.

Boundedness on weighted classical Lorentz spaces have been already studied for some operators. For instance, let us just mention [3], including the case of the Hilbert transform and the Hilbert maximal operator, and [17] that contains the case of Calderón-Zygmund operators and commutators. Further, we should also consider [18], where the authors study inter alia the characterization of the Hardy-Littlewood maximal operator.

Here, by means of the theory of weighted extrapolation, we present new estimates for operators such as Calderón-Zygmund operators T_K, sparse operators $A_{\mathcal{S}}$ or the Bochner-Riesz operator at the critical index $B_{\frac{n-1}{2}}$, among many others.

Now, the examples mentioned above share one important property: they all satisfy that, for some (and hence for all) $p_0 > 1$,

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v), \qquad \forall v \in A_{p_0}, \tag{1}$$

is bounded, where for a given exponent $p > 1$, A_p is the class of Muckenhoupt weights defined by those weights v that satisfy

$$||v||_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q v(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{\frac{1}{1-p}}dx\right)^{p-1} < \infty,$$

with the supremum being taken overall cubes $Q \subseteq \mathbb{R}^n$. Further, those A_p weights characterize the boundedness on $L^p(v)$ of the Hardy-Littlewood maximal operator M defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|dy, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ x \in \mathbb{R}^n,$$

where the supremum is taken overall cubes $Q \subseteq \mathbb{R}^n$ containing x. Moreover, the definition of A_p can be extended to $p = 1$, and we say that $v \in A_1$ if there exists a positive constant C such that

$$Mv(x) \le Cv(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

and $\|v\|_{A_1}$ is set to be the infimum of such constants C. We shall point here that the A_p class of weights increases with the exponent (that is, $A_p \subseteq A_q$, for $p \le q$) so it is natural to define the class of weights

$$A_\infty := \bigcup_{p\geq 1} A_p.$$

Throughout this paper, an operator T satisfying (1) for some p_0 and with norm constant less than or equal to $\varphi(\|v\|_{A_{p_0}})$, and $\varphi : [1,\infty) \to (0,\infty)$ being a nondecreasing function, will be called a Rubio de Francia operator [42]. Those operators satisfy the well-known Rubio de Francia extrapolation theorem.

Theorem 1.1 [29] *Given a Rubio de Francia operator T for some exponent $p_0 \geq 1$. Then, for all $p_1 > 1$,*

$$T : L^{p_1}(v) \to L^{p_1}(v), \qquad \forall v \in A_{p_1},$$

with norm constant less than or equal to $\Phi(\|v\|_{A_{p_1}})$ where

$$\Phi(r) = C_1\varphi\left(C_2 r^{\max\left(1, \frac{p_0-1}{p_1-1}\right)}\right), \qquad r \geq 1.$$

A Banach function norm ρ is a mapping $\rho : \mathcal{M}^+ \to [0,\infty]$, with $\mathcal{M}^+$ being the set of positive measurable functions, such that the following properties hold:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.;
(ii) $\rho(af) = a\rho(f)$, for $a \geq 0$;
(iii) $\rho(f+g) \leq \rho(f) + \rho(g)$;
(iv) if $0 \leq f \leq g$ a.e., then $\rho(f) \leq \rho(g)$;
(v) if $0 \leq f_n \nearrow f$ a.e., then $\rho(f_n) \nearrow \rho(f)$;
(vi) if E is a measurable set such that $|E| < \infty$, then $\rho(\chi_E) < \infty$ and $\int_E f\,dx \leq C_E\rho(f)$ for some constant $0 < C_E < \infty$, depending on E and ρ, but independent of f, where χ_E represents the characteristic function of E.

The collection $\mathbb{X} = \mathbb{X}(\rho)$ defined by

$$\mathbb{X} = \{f \in \mathcal{M} : \|f\|_{\mathbb{X}} := \rho(|f|) < \infty\}$$

is called a Banach function space. Besides, by means of a function norm ρ, we can define its associate norm $\rho' : \mathcal{M}^+ \to [0,\infty]$ by

$$\rho'(f) = \sup\left\{\int_{\mathbb{R}^n} f(x)g(x)\,dx : g \in \mathcal{M}^+, \rho(g) \leq 1\right\},$$

which is itself a function norm. This allows us to define the associate space of $\mathbb{X} = \mathbb{X}(\rho)$ to be the Banach function space $\mathbb{X}' = \mathbb{X}(\rho')$ (see [10, Ch. 1-Theorem 2.2]).

A function norm ρ is called rearrangement invariant (r.i. in short) if $\rho(f) = \rho(g)$ for every pair of functions f and g that satisfy $\lambda_f(y) = \lambda_g(y)$ for every $y > 0$. In this case, we say that $\mathbb{X} = \mathbb{X}(\rho)$ is a r.i. Banach function space (see [10, 27] for more details on those spaces). For instance, since the decreasing rearrangement on $(0,\infty)$ of f satisfy $\lambda_{f^*}(y) = \lambda_f(y)$ for every $y > 0$, in fact it can be obtained a representation of $\mathbb{X}$ on $(\mathbb{R}^+, dt)$ (see [10, Ch. 2-Theorem 4.10]) as follows: there exists a r.i. Banach function space $\overline{\mathbb{X}}$ over $(\mathbb{R}^+, dt)$ such that $f \in \mathbb{X}$ if and only if $f^* \in \overline{\mathbb{X}}$ with

$$\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}} := \sup_{\|g\|_{\mathbb{X}'} \leq 1} \int_0^\infty f^*(t)g^*(t)\,dt.$$

Moreover, when restricted to a r.i. Banach function space $\mathbb{X}$ it is possible to define a weighted version of $\mathbb{X}$ as

$$\mathbb{X}(u) = \{f \in \mathcal{M} : \|f\|_{\mathbb{X}(u)} := \|f_u^*\|_{\overline{\mathbb{X}}} < \infty\},$$

for some weight u.

Now, as a consequence of [26, Theorem 4.10], it is known that the Rubio de Francia operators also satisfy boundedness over the weighted version of a r.i. Banach function spaces $\mathbb{X}$ where, for the sake of simplicity, here we state a different version of that theorem consisting on introducing the maximal operator M and its dual induced by a weight $u \in A_\infty$ defined as

$$M'_u f(x) = \frac{M(fu)(x)}{u(x)}, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n),\ x \in \mathbb{R}^n, \tag{2}$$

and where we have kept track of the involved norm constants.

Theorem 1.2 *Given a Rubio de Francia operator T for some exponent $p_0 > 1$. Let $\mathbb{X}$ be a r.i. Banach function space and let $u \in A_\infty$ such that*

$$M : \mathbb{X}(u) \to \mathbb{X}(u) \qquad \text{and} \qquad M'_u : \mathbb{X}(u)' \to \mathbb{X}(u)', \tag{3}$$

where $\mathbb{X}(u)'$ is the associate space of $\mathbb{X}(u)$ defined by those functions f that satisfy

$$\|f\|_{\mathbb{X}(u)'} := \sup_{\|g\|_{\mathbb{X}(u)} \leq 1} \left| \int_{\mathbb{R}^n} f(x)g(x)u(x)\,dx \right| < \infty.$$

Then,

$$T : \mathbb{X}(u) \to \mathbb{X}(u),$$

with constant less than or equal to $C_1 \varphi\left(C_2 \|M'_u\|_{\mathbb{X}(u)'} \|M\|_{\mathbb{X}(u)}^{p_0-1}\right)$, where $\|M\|_{\mathbb{X}(u)}$ and $\|M'_u\|_{\mathbb{X}(u)'}$ represent the norm constants of (3) respectively.

The previous result is very useful to prove the boundedness of operators for which condition (1) has been widely studied while this is not the case in other contexts such as, for example, weighted classical Lorentz spaces or more generally r.i. Banach function spaces. However, in order to get estimates over $\mathbb{X}(u)$ we first must study whether $\mathbb{X}$ is a r.i. Banach function space and when (3) holds.

For our goal, we will consider $\mathbb{X} = \Lambda^p(w)$, for $p > 0$ and w being a weight in $\mathbb{R}^+$, so that $\mathbb{X}(u) = \Lambda_u^p(w)$ and $\mathbb{X}(u)' = (\Lambda_u^p(w))'$ is defined by

$$\|f\|_{(\Lambda_u^p(w))'} := \sup_{\|g\|_{\Lambda_u^p(w)} \leq 1} \left| \int_{\mathbb{R}^n} f(x)g(x)u(x)\, dx \right| < \infty.$$

As a first approach, let us consider the particular case $u = 1$:

i) Ariño and Muckenhoupt [6] proved that for $p > 1$,

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \qquad \Longleftrightarrow \qquad w \in B_p, \tag{4}$$

where

$$w \in B_p \qquad \Longleftrightarrow \qquad ||w||_{B_p} := \sup_{t>0} \frac{\int_0^\infty w(r) \min\left(1, \frac{t^p}{r^p}\right) dr}{W(t)} < \infty,$$

with $W(t) = \int_0^t w(r)\, dr$. This class of weights has been widely studied (see for instance [7, 8, 13, 39, 40]) and now it is known that the same result in (4) holds for every $p > 0$ (see [19]). Further, although for $0 < p < 1$, $\Lambda^p(w)$ is never a Banach function space (see [45, Remark 3.2]) for $p \geq 1$, $\Lambda^p(w)$ is a Banach function space when $w \in B_p$, and the reciprocal is also true whenever $p > 1$ (see [15, 43]).

ii) The characterization of the boundedness of M on $(\Lambda^p(w))'$ (see [8]) is a consequence of the Lorentz-Shimogaki theorem (see, for instance, [10, Ch.3 p. 154]). In particular, if $w \in B_p$ then

$$M : (\Lambda^p(w))' \longrightarrow (\Lambda^p(w))' \qquad \Longleftrightarrow \qquad w \in B_\infty^*, \tag{5}$$

where [39]

$$w \in B_\infty^* \qquad \Longleftrightarrow \qquad ||w||_{B_\infty^*} := \sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(r)}{r}\, dr < \infty.$$

From (i) and (ii), we can conclude that for every Rubio de Francia operator T and $p \geq 1$,

$$w \in B_p \cap B_\infty^* \qquad \Longrightarrow \qquad T : \Lambda^p(w) \longrightarrow \Lambda^p(w),$$

and it is sharp since it is known [43] that

$$w \in B_p \cap B_\infty^* \qquad \Longleftrightarrow \qquad H : \Lambda^p(w) \longrightarrow \Lambda^p(w),$$

with H being the Hilbert transform, which is a Rubio de Francia operator defined as

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \qquad f \in \mathcal{C}_c^\infty(\mathbb{R}),\ x \in \mathbb{R}, \tag{6}$$

whenever this limit exists almost everywhere and where $\mathcal{C}_c^\infty(\mathbb{R})$ is the set of infinitely differentiable functions with compact support.

Now, take a general $u \in A_\infty$. In [18, Theorem 3.3.5] it was characterized the weighted strong-type boundedness on $\Lambda_u^p(w)$ of the Hardy-Littlewood maximal operator for every $p > 0$ by

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w) \qquad \Longleftrightarrow \qquad w \in B_p(u), \tag{7}$$

where we say that $w \in B_p(u)$ if there exists some $\varepsilon > 0$ such that

$$\sup_{E_j \subseteq Q_j,\, \forall 1 \le j \le J} \left(\inf_{1 \le j \le J} \frac{|E_j|}{|Q_j|} \right) \frac{W\left(u\left(\bigcup_{j=1}^J Q_j\right)\right)^{\frac{1}{p-\varepsilon}}}{W\left(u\left(\bigcup_{j=1}^J E_j\right)\right)^{\frac{1}{p-\varepsilon}}} < \infty, \tag{8}$$

where the supremum is taken over every finite family of cubes $\{Q_j\}_{j=1}^J \subseteq \mathbb{R}^n$. (We refer the reader to [1, 18] for more details on this class of weights.)

(i) If $u = 1$, due to the B_p class of weights satisfy the $p - \varepsilon$ property [18, 40] (that is, for $w \in B_p$ there exists some $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$) we have that (8) is equivalent to $w \in B_p$.

(ii) If $w = 1$ and $p > 1$, then (8) is equivalent to that there exists some $0 < q < p$ such that

$$\frac{u\left(\bigcup_{j=1}^J Q_j\right)}{u\left(\bigcup_{j=1}^J E_j\right)} \lesssim \max_{1 \le j \le J} \left(\frac{|Q_j|}{|E_j|} \right)^q,$$

which agrees with $u \in A_p$ (see, for instance, [22, 33]).

Moreover, $B_p(u) \subseteq B_p$ for every $p > 0$ (see [18, Corollary 3.3.4]) so, for $p \ge 1$ and $w \in B_p(u)$ then $\Lambda^p(w)$ is a r.i. Banach function space.

Therefore, together with (7), if we were able to see when M_u' is bounded over $(\Lambda_u^p(w))'$, we could make use of Theorem 1.2 to obtain estimates for a Rubio de Francia operator T on $\Lambda_u^p(w)$, at least for $p \ge 1$ and $w \in B_p(u)$. Indeed, our main result on this paper says more:

Theorem 1.3 *Given a Rubio de Francia operator T for some exponent $p_0 > 1$. Let $u \in A_\infty$ and $p > 0$. Then,*

$$T : \Lambda_u^p(w) \to \Lambda_u^p(w), \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

Now, in [4, Theorem 5.5] it was shown that for $p > 1$,

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w) \qquad \Longleftrightarrow \qquad w \in B_p(u) \cap B_\infty^*, \tag{9}$$

while for $0 < p \leq 1$ it is just known that

$$w \in B_p(u) \cap B_\infty^* \quad \Longrightarrow \quad H : \Lambda_u^p(w) \to \Lambda_u^p(w).$$

Remark 1.4 At least for $p > 1$, by means of the Hilbert transform (see (9)) the condition $B_p(u) \cap B_\infty^*$ on the weight w of Theorem 1.3 is sharp in the sense that it can not be found a greater class for w.

Now, let $\Lambda_u^{p,\infty}(w)$, $p > 0$, be the space defined by those measurable functions f such that

$$\|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} t W(\lambda_f^u(t))^{\frac{1}{p}} < \infty,$$

and let $L^{p,\infty}(u) := \Lambda_u^{p,\infty}(1)$, $p \geq 1$. In [20] the authors consider as hypothesis weighted weak-type estimates instead of (1); that is, for some $p_0 \geq 1$,

$$T : L^{p_0}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A_{p_0}, \tag{10}$$

and then try to find conditions on p and w for which

$$T : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$$

holds, so it could be interesting to study the previous to weighted classical Lorentz spaces for a weight $u \in A_\infty$. However, this is just a consequence of the weighted strong-type extrapolation settled in Theorem 1.3, and the precise result is the following.

Corollary 1.5 *If an operator T satisfies (10) for some $p_0 \geq 1$ and with constant less than or equal to $\varphi(\|v\|_{A_{p_0}})$, where φ is a nonnegative nondecreasing function on $[1, \infty)$, then, for $p > 0$ and $u \in A_\infty$,*

$$T : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w), \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

Again, for $p > 1$ and by means of the Hilbert transform (see [2, Theorem 1.1]) the condition $B_p(u) \cap B_\infty^*$ on the weight w of Corollary 1.5 is sharp in the sense that it can not be found a greater class for w.

The paper is organized as follows: in Sect. 2 we study the boundedness of M_u' over the associate space of $\Lambda_u^p(w)$. The proofs of Theorem 1.3 and Corollary 1.5 will be given in Sect. 3, and Sect. 4 will be devoted to applying our results to the boundedness of Calderón-Zygmund operators, sparse operators, the Bochner-Riesz operator and intrinsic square functions.

As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. When $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

2 Boundedness on the Associate Space of $\Lambda_u^p(w)$

Given $u \in A_\infty$, our goal is to see for which conditions on p and the weight w

$$M_u' : (\Lambda_u^p(w))' \to (\Lambda_u^p(w))' \tag{11}$$

holds, where M_u' is defined in (2).

(I) If $u = 1$ and $w \in B_p$, then (11) is equivalent to $w \in B_\infty^*$ (see (5)).
(II) If $w = 1$ and $p > 1$, then (11) is equivalent to $M_u' : L^{\frac{p}{p-1}}(u) \to L^{\frac{p}{p-1}}(u)$, which in turn remains true whenever $u \in A_p$.

In order to see the general setting (see Theorem 2.6), it is necessary to establish a few technical findings initially. In an effort to ensure that the paper is self-contained, we include these results within its contents. First, for a nonnegative measurable function h let

$$Qh(t) := \int_t^\infty h(r)\,\frac{dr}{r}, \qquad t > 0.$$

Therefore, by carefully monitoring the constants involved in [5, Theorem 4] (or even [40, Theorem 3.3]), the following result holds.

Proposition 2.1 *For every $p > 0$,*

$$Q : L^p_{dec}(w) \to L^p(w) \qquad \Longleftrightarrow \qquad w \in B_\infty^*$$

with $\|Q\|_{L^p_{dec}(w)\to L^p(w)} \leq c_{n,p}\,\|w\|_{B_\infty^*}$ and where $L^p_{dec}(w)$ is the set of measurable decreasing functions f satisfying $\|f\|_{L^p(w)} < \infty$.

We also need some technical results related with the Fefferman-Stein maximal operator, which is defined, for every $h \in L^1_{\text{loc}}(\mathbb{R}^n)$, by

$$M^\# h(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| h(y) - \frac{1}{|Q|}\int_Q h(z)\,dz \right| dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing the point x (see [30, 41]). Hence, by carefully considering the constants involved in [9, Corollary 4.3 (a)], the following result can be established.

Proposition 2.2 *There exists a nondecreasing function φ_q on $[1, \infty)$ such that for every $t > 0$,*

$$f_u^*(t) \leq \varphi_q\left(\|u\|_{A_q}\right) Q\left(\left(M^\# f\right)_u^*\right)(t) + \lim_{r\to\infty} f_u^*(r).$$

And, as a consequence, the next result follows.

Proposition 2.3 *Given $u \in A_q$, $q > 1$, and $w \in B_\infty^*$. If $\lim_{r\to\infty} f_u^*(r) = 0$, then*

$$\|f\|_{\Lambda_u^p(w)} \le c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B_\infty^*}\left\|M^\# f\right\|_{\Lambda_u^p(w)},$$

where φ_q is a nonnegative nondecreasing function on $[1,\infty)$.

Proof By means of Proposition 2.2, there exists a nondecreasing function φ_q on $[1,\infty)$ such that for every $t > 0$,

$$f_u^*(t) \le \varphi_q\left(\|u\|_{A_q}\right) Q\left(\left(M^\# f\right)_u^*\right)(t).$$

Therefore, taking into account Proposition 2.1,

$$\begin{aligned}\|f\|_{\Lambda_u^p(w)} &\le \varphi_q\left(\|u\|_{A_q}\right)\left\|Q\left(\left(M^\# f\right)_u^*\right)\right\|_{L^p(w)}\\ &\le c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B_\infty^*}\left\|M^\# f\right\|_{\Lambda_u^p(w)}.\end{aligned}$$

□

Furthermore, the Fefferman-Stein maximal operator also satisfy the following statement.

Proposition 2.4 [28] *For every $f \in L^1_{loc}(\mathbb{R}^n)$, there exists a linear operator L_f such that*

$$Mf(x) \approx L_f(|f|)(x), \qquad a.e.\ x \in \mathbb{R}^n.$$

Moreover, the adjoint of L_f, $\widetilde{L_f}$, satisfies that for every $g \in L^1_{loc}(\mathbb{R}^n)$,

$$M^\#\left(\widetilde{L_f}(|g|)\right)(x) \lesssim Mg(x), \qquad a.e.\ x \in \mathbb{R}^n. \tag{12}$$

Additionally, we will utilize the following result related with weights properties.

Proposition 2.5 *Given $u \in A_q$, $q > 1$, and $w \in B_p(u)$, $p > 0$. Let, for $N \in \mathbb{N}$, $u_N = \min(u, N)$. Then,*

(i) $u_N \in A_q$ with

$$\|u_N\|_{A_q} \le 2^{q+1}\|u\|_{A_q},$$

(ii) and there exists some constant $C_{u,w} > 0$ independent of N such that

$$\|M\|_{\Lambda_{u_N}^p(w)} \le C_{u,w}.$$

Proof First, taking an arbitrary cube $Q \subseteq \mathbb{R}^n$,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q u_N(x)dx\right)\left(\frac{1}{|Q|}\int_Q u_N(x)^{\frac{1}{1-q}}dx\right)^{q-1}\\&=\frac{1}{|Q|^q}\left(\int_{Q\cap\{x:u(x)<N\}}u(x)dx+N\int_{Q\cap\{x:u(x)\geq N\}}dx\right)\\&\times\left(\int_{Q\cap\{x:u(x)<N\}}u(x)^{\frac{1}{1-q}}dx+N^{\frac{1}{1-q}}\int_{Q\cap\{x:u(x)\geq N\}}dx\right)^{q-1}\\&\leq 2^q\,\|u\|_{A_q}+2\leq 2^{q+1}\,\|u\|_{A_q},\end{aligned}$$

from which (i) follows by taking the supremum overall cubes Q.

To see (ii), let $f \in \Lambda^p_{u_N}(w)$ and write

$$f = f\chi_{\{x\,:\,u(x)<N\}} + f\chi_{\{x\,:\,u(x)\geq N\}} := f_1 + f_2.$$

Then,

$$\begin{aligned}\|Mf\|_{\Lambda^p_{u_N}(w)} &\leq C\left(\|Mf_1\|_{\Lambda^p_{u_N}(w)}+\|Mf_2\|_{\Lambda^p_{u_N}(w)}\right)\\&\leq C\left(\|Mf_1\|_{\Lambda^p_u(w)}+N\,\|Mf_2\|_{\Lambda^p(w)}\right)\\&\leq C\left(\|M\|_{\Lambda^p_u(w)}\,\|f_1\|_{\Lambda^p_u(w)}+N\,\|M\|_{\Lambda^p(w)}\,\|f_2\|_{\Lambda^p(w)}\right),\end{aligned}$$

where in the last estimate we are using that $w \in B_p(u) \subset B_p$. Now observe that for every $y > 0$,

$$u(\{x\,:\,f_1(x) > y\}) \leq u_N(\{x\,:\,f(x) > y\})$$

and

$$N|\{x\,:\,f_2(x) > y\}| \leq u_N(\{x\,:\,f(x) > y\}),$$

so that $\|f_1\|_{\Lambda^p_u(w)} \leq \|f\|_{\Lambda^p_{u_N}(w)}$ and $N\,\|f_2\|_{\Lambda^p(w)} \leq \|f\|_{\Lambda^p_{u_N}(w)}$. Therefore, the results follows by taking $C_{u,w} = C\left(\|M\|_{\Lambda^p_u(w)} + \|M\|_{\Lambda^p(w)}\right)$. □

Finally, with the previous results at hand, we are able to find conditions so that (11) holds.

Theorem 2.6 *Given $u \in A_\infty$. For every $p > 0$,*

$$M'_u : \left(\Lambda^p_u(w)\right)' \to \left(\Lambda^p_u(w)\right)', \qquad \forall w \in B_p(u) \cap B^*_\infty. \tag{13}$$

Proof Since $w \in B_p(u) \subseteq B_p$ we have that $(\Lambda_u^p(w))' \neq \{0\}$ (see [18]) and, by definition of associate space,

$$\left\| \frac{M(fu)}{u} \right\|_{(\Lambda_u^p(w))'} = \sup_{\|h\|_{\Lambda_u^p(w)} \leq 1} \int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx, \tag{14}$$

where we can assume that the supremum is taken over all nonnegative functions h satisfying that $\|h\|_{\Lambda_u^p(w)} \leq 1$. We will see that, for such a function h,

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx \leq C_{n,p,q,u,w} \|f\|_{(\Lambda_u^p(w))'} \|h\|_{\Lambda_u^p(w)}, \tag{15}$$

from which together with (14), the boundedness of (13) will follow.

First, we observe that h can be chosen to be in $L^1(\mathbb{R}^n)$. Otherwise, we can take $h_k = \chi_{\mathbb{B}(0,k)} h \in L^1(\mathbb{R}^n)$ (with $\mathbb{B}(0,k)$ being the ball of center 0 and radius k) and by the monotone convergence theorem,

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} M(fu)(x)h_k(x)\,dx,$$

so together with $\|h_k\|_{\Lambda_u^p(w)} \leq \|h\|_{\Lambda_u^p(w)}$, we would obtain (15).

Hence, take such a nonnegative function h in (14) satisfying $h \in L^1(\mathbb{R}^n)$. Further, take $q > 1$ such that $u \in A_q$ (since $u \in A_\infty$). Then, by Proposition 2.4,

$$\begin{aligned}\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx &\leq C_n \int_{\mathbb{R}^n} |f(x)| \widetilde{L_{fu}}(h)(x)u(x)\,dx \\ &\leq C_n \|f\|_{(\Lambda_u^p(w))'} \left\| \widetilde{L_{fu}}(h) \right\|_{\Lambda_u^p(w)},\end{aligned}$$

and if we are able to see that

$$\lim_{t\to\infty} (\widetilde{L_{fu}}(h))_u^*(t) = 0, \tag{16}$$

by virtue of Proposition 2.3 and estimate (12) we will deduce that

$$\begin{aligned}\left\| \widetilde{L_{fu}}(h) \right\|_{\Lambda_u^p(w)} &\leq c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B_\infty^*} \left\| M^\# \left(\widetilde{L_{fu}}(h)\right) \right\|_{\Lambda_u^p(w)} \\ &\leq \tilde{c}_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B_\infty^*} \|Mh\|_{\Lambda_u^p(w)} \\ &\leq \tilde{c}_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B_\infty^*} \|M\|_{\Lambda_u^p(w)} \|h\|_{\Lambda_u^p(w)},\end{aligned}$$

where $\|M\|_{\Lambda_u^p(w)} < \infty$ since $w \in B_p(u)$.

Thus, it all reduces to show that (16) holds. Now, if we assume that u is bounded, that is $u \leq C_u$ for some positive constant C_u, then it is just a consequence of that by construction it is known that $\widetilde{L_{fu}}(h)$ is bounded in $L^1(\mathbb{R}^n)$ (see [28]) so for every $t > 0$,

$$u\left(\left\{x \in \mathbb{R}^n : |\widetilde{L_{fu}}(h)| > t\right\}\right) \leq C_u \left|\left\{x \in \mathbb{R}^n : |\widetilde{L_{fu}}(h)| > t\right\}\right| \\ \leq \frac{C_u}{t} \left\|\widetilde{L_{fu}}\right\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} \|h\|_{L^1(\mathbb{R}^n)},$$

and hence,

$$(\widetilde{L_{fu}}(h))_u^*(t) \leq \frac{C_u}{t} \left\|\widetilde{L_{fu}}\right\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} \|h\|_{L^1(\mathbb{R}^n)} \xrightarrow[t\to\infty]{} 0.$$

Finally, if u is not bounded, taking $N \in \mathbb{N}$, we just have to observe that $u_N = \min(u, N)$ is a bounded weight that satisfy (see Proposition 2.5)

$$\|u_N\|_{A_q} \leq 2^{q+1} \|u\|_{A_q}, \qquad \|M\|_{\Lambda^p_{u_N}(w)} \leq C_{u,w}$$

and $\|h\|_{\Lambda^p_{u_N}(w)} \leq 1$, so that by the monotone convergence theorem,

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx = \lim_{N\to\infty} \int_{\mathbb{R}^n} M(fu_N)(x)h(x)\,dx \\ \leq \tilde{c}_{n,p} C_{u,w} \varphi_q\left(2^{q+1}\|u\|_{A_q}\right) \|w\|_{B^*_\infty} \|f\|_{(\Lambda^p_u(w))'} \|h\|_{\Lambda^p_u(w)}.$$

□

3 Proof of Main Results

Proof of Theorem 1.3 Suppose first that $p \geq 1$. Since $w \in B_p(u) \subseteq B_p$, then $\Lambda^p(w)$ is a r.i. Banach function space, and so the result follows by means of Theorems 1.2 and 2.6, together with (7).

Now, assume that $0 < p < 1$. Since $w \in B_p(u)$, there exists some $0 < \varepsilon < p$ such that $w \in B_{p-\varepsilon}(u)$, and we can define

$$Rf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{\left(2\|M\|_{\Lambda_u^{p-\varepsilon}(w)}\right)^k}$$

to be a version of the function resulting from the Rubio de Francia Algorithm [42], where $M^0 = Id$ is the identity operator and $M^k = M \underbrace{\circ \cdots \circ}_{k} M$ is the k-fold composition of M with itself. Then,

(1) $f \leq \left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right]^{\frac{p-\varepsilon}{p}}$,

(2) $R\left(|f|^{\frac{p}{p-\varepsilon}}\right) \in A_1$ with $\left\|R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right\|_{A_1} \leq 2\|M\|_{\Lambda_u^{p-\varepsilon}(w)}$,

(3) and

$$\left\| \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right) \right]^{\frac{p-\varepsilon}{p}} \right\|_{\Lambda_u^p(w)} \leq \left(\frac{2^{p-\varepsilon}}{2^{p-\varepsilon}-1} \right)^{\frac{1}{p}} \|f\|_{\Lambda_u^p(w)} .$$

Certainly, since $B_p(u) \subseteq B_p \subseteq B_1$ then $\Lambda^1(w)$ is also a r.i. Banach function space and property (3) can be established as follows:

$$\begin{aligned}
\left\| \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right) \right]^{\frac{p-\varepsilon}{p}} \right\|_{\Lambda_u^p(w)} &= \left\| \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right) \right]^{p-\varepsilon} \right\|_{\Lambda_u^1(w)}^{\frac{1}{p}} \\
&\leq \left\| \sum_{k=0}^{\infty} \frac{M^k\left(|f|^{\frac{p}{p-\varepsilon}} \right)(x)^{p-\varepsilon}}{\left(2\, \|M\|_{\Lambda_u^{p-\varepsilon}(w)} \right)^{k(p-\varepsilon)}} \right\|_{\Lambda_u^1(w)}^{\frac{1}{p}} \\
&\leq \left(\sum_{k=0}^{\infty} \frac{\left\| M^k\left(|f|^{\frac{p}{p-\varepsilon}} \right) \right\|_{\Lambda_u^{p-\varepsilon}(w)}^{p-\varepsilon}}{\left(2\, \|M\|_{\Lambda_u^{p-\varepsilon}(w)} \right)^{k(p-\varepsilon)}} \right)^{\frac{1}{p}} \\
&\leq \|f\|_{\Lambda_u^p(w)} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(p-\varepsilon)}} \right)^{\frac{1}{p}} = \left(\frac{2^{p-\varepsilon}}{2^{p-\varepsilon}-1} \right)^{\frac{1}{p}} \|f\|_{\Lambda_u^p(w)} .
\end{aligned}$$

What's more, by means of Theorem 2.6 we can define for an arbitrary nonnegative function $h \in (\Lambda_u^1(w))'$,

$$Sh(x) = \sum_{k=0}^{\infty} \frac{(M_u')^k h(x)}{\left(2\, \left\| M_u' \right\|_{(\Lambda_u^1(w))'} \right)^k}$$

so that

(1)' $h \leq Sh$,
(2)' $(Sh)u \in A_1$ with $\|(Sh)u\|_{A_1} \leq 2\, \left\| M_u' \right\|_{(\Lambda_u^1(w))'}$,
(3)' and

$$\|Sh\|_{(\Lambda_u^1(w))'} \leq 2\, \|h\|_{(\Lambda_u^1(w))'} .$$

Now, $\Lambda^1(w)$ is a Banach function space so that for any nonnegative function g (see [10, Ch. 1 Theorem 2.7]),

$$\|g\|_{\Lambda_u^p(w)}^p = \left\| g^p \right\|_{\Lambda_u^1(w)} = \sup_{\|h\|_{(\Lambda_u^1(w))'} \leq 1} \int_{\mathbb{R}^n} g(y)^p h(y) u(y)\, dy. \tag{17}$$

Then, taking such a nonnegative function h in (17), by Fubini,

$$\begin{aligned}\int_{\mathbb{R}^n} g(y)^p h(y)u(y)\,dy &= \int_0^\infty \int_{\{g(y)^p>x\}} h(y)u(y)\,dy\,dx\\ &= \int_0^\infty \int_{\left\{g(y)^p>x,\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)(y)\right]^{p-\varepsilon}>\gamma x\right\}} h(y)u(y)\,dy\,dx\\ &+ \int_0^\infty \int_{\left\{g(y)^p>x,\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)(y)\right]^{p-\varepsilon}\le\gamma x\right\}} h(y)u(y)\,dy\,dx\\ &:= I_1(h)+I_2(h).\end{aligned}$$

Let's work with $I_1(h)$ and $I_2(h)$ separately. First, by Fubini again,

$$\begin{aligned}I_1(h) &\le \int_0^\infty \int_{\left\{\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)(y)\right]^{p-\varepsilon}>\gamma x\right\}} h(y)u(y)\,dy\,dx\\ &= \frac{1}{\gamma}\int_{\mathbb{R}^n}\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)(y)\right]^{p-\varepsilon} h(y)u(y)\,dy,\end{aligned}$$

and taking the supremum in $0 \le h \in \left(\Lambda_u^1(w)\right)'$ we have using property (3) that

$$\begin{aligned}\sup_{\|h\|_{\left(\Lambda_u^1(w)\right)'}\le 1} \frac{1}{\gamma}\int_{\mathbb{R}^n}\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)(y)\right]^{p-\varepsilon} h(y)u(y)\,dy &= \frac{1}{\gamma}\left\|\left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right]^{p-\varepsilon}\right\|_{\Lambda_u^1(w)}\\ &\le \frac{1}{\gamma}\left(\frac{2^{p-\varepsilon}}{2^{p-\varepsilon}-1}\right)\|f\|^p_{\Lambda_u^p(w)}.\end{aligned}$$

Now let's work with $I_2(h)$. Observe that since $p_0 \ge 1$, then $\frac{p(1-p)}{p_0-p} \le p$. Hence, if we take

$$\varepsilon < \frac{p(1-p)}{p_0-p} \qquad \text{and} \qquad p_1 := \frac{p(1-p+\varepsilon)}{\varepsilon},$$

then, $p_1 \ge p_0$, $p_1 > 1$ and $1-p_1 = (p-\varepsilon)(1-p_1/p)$. Therefore, by [29, Lemma 2.1]

$$v = \left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right]^{(p-\varepsilon)(1-p_1/p)}(Sh)u = \left[R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right]^{1-p_1}(Sh)u \in A_{p_1}$$

with

$$\|v\|_{A_{p_1}} \le \|(Sh)u\|_{A_1}\left\|R\left(|f|^{\frac{p}{p-\varepsilon}}\right)\right\|_{A_1}^{p_1-1} \le 2^{p_1}\left\|M_u'\right\|_{(\Lambda_u^1(w))'}\|M\|^{p_1-1}_{\Lambda_u^{\frac{p}{p-\varepsilon}}(w)}$$

and, due Theorem 1.1, if we let $g = |Tf|$,

$$I_2(h) \le \gamma^{p_1/p-1} \int_0^\infty x^{p_1/p-1} \int_{\{|Tf(y)|^p > x\}} \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right)(y) \right]^{(p-\varepsilon)(1-p_1/p)}$$
$$Sh(y)u(y)\,dy\,dx$$
$$= \gamma^{p_1/p-1} \frac{p}{p_1} \int_{\mathbb{R}^n} |Tf(y)|^{p_1} \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right)(y) \right]^{1-p_1} Sh(y)u(y)\,dy$$
$$\le \gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \int_{\mathbb{R}^n} f(y)^{p_1} \left[R\left(|f|^{\frac{p}{p-\varepsilon}} \right)(y) \right]^{1-p_1} Sh(y)u(y)\,dy$$
$$\le \gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \int_{\mathbb{R}^n} f(y)^p Sh(y)u(y)\,dy$$
$$\le \gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \left\| f^p \right\|_{\Lambda_u^1(w)} \|Sh\|_{(\Lambda_u^1(w))'}$$
$$\le 2\gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \|f\|_{\Lambda_u^p(w)}^p \|h\|_{(\Lambda_u^1(w))'}.$$

Therefore,

$$\sup_{\|h\|_{\left(\Lambda_u^1(w)\right)'} \le 1} I_2(h) \le 2\gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \|f\|_{\Lambda_u^p(w)}^p.$$

Thus,

$$\|Tf\|_{\Lambda_u^p(w)}^p \le \max\left(\frac{1}{\gamma} \frac{2^{p-\varepsilon}}{2^{p-\varepsilon}-1}, 2\gamma^{p_1/p-1} \frac{p\Phi\left(\|v\|_{A_{p_1}}\right)^{p_1}}{p_1} \right) \|f\|_{\Lambda_u^p(w)}^p,$$

and taking the infimum in $\gamma > 0$,

$$\|Tf\|_{\Lambda_u^p(w)} \lesssim \varphi\left(C 2^{\frac{p(1-p+\varepsilon)}{\varepsilon}} \|M_u'\|_{(\Lambda_u^1(w))'} \|M\|_{\Lambda_u^{p-\varepsilon}(w)}^{\frac{(1-p)(p-\varepsilon)}{\varepsilon}} \right) \|f\|_{\Lambda_u^p(w)}.$$

□

Proof of Corollary 1.5 Observe that

$$\|Tf\|_{L^{p_0,\infty}(v)} \le \varphi\left(\|v\|_{A_{p_0}}\right) \|f\|_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0},$$

implies that for every $y > 0$,

$$\left\| y\chi_{\{|Tf|>y\}} \right\|_{L^{p_0}(v)} \le \varphi\left(\|v\|_{A_{p_0}}\right) \|f\|_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0}.$$

Fix $y > 0$ and define $T_y f := y\chi_{\{|Tf|>y\}}$. Hence, by means of Theorem 1.3 we obtain that, for every $p > 0$ and $u \in A_\infty$,

$$\left\| y\chi_{\{|Tf|>y\}} \right\|_{\Lambda_u^p(w)} = \left\| T_y f \right\|_{\Lambda_u^p(w)} \le C_{n,p_0,p,u,w} \left\| f \right\|_{\Lambda_u^p(w)}, \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

Therefore, taking the supremum over all $y > 0$, we obtain the desired result.

4 Examples and Applications

4.1 Calderón-Zygmund Operators and Commutators

A function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ is called a standard kernel if there exists $A, \delta > 0$ satisfying the size condition

$$|K(x, y)| \le \frac{A}{|x-y|^n},$$

and the regularity conditions

$$\left|K(x, y) - K(x', y)\right| \le \frac{A|x-x'|^\delta}{(|x-y| + |x'-y|)^{n+\delta}},$$

when $|x - x'| \le \frac{1}{2} \max(|x-y|, |x'-y|)$ and

$$\left|K(x, y) - K(x, y')\right| \le \frac{A|y-y'|^\delta}{(|x-y| + |x-y'|)^{n+\delta}},$$

when $|y - y'| \le \frac{1}{2} \max(|x-y|, |x-y'|)$. Now, given a standard Kernel K, T_K is called a Calderón-Zygmund operator associated with K if it is defined on the class of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ (that is, the space of functions in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ that *decreases rapidly*), which admits a bounded extension on $L^2(\mathbb{R}^n)$

$$\|T_K\|_{L^2(\mathbb{R}^n)} \le C \|f\|_{L^2(\mathbb{R}^n)}, \qquad \forall f \in L^2(\mathbb{R}^n),$$

and

$$T_K f(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy, \qquad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } x \notin \operatorname{supp}(f),$$

for $\mathcal{C}_c^\infty(\mathbb{R}^n)$ being the space of infinitely differentiable functions with compact support (see [31, Ch. 4] for more details on these operators). Also, for a locally integrable function b, the commutator of T_K and b is defined as

$$[b, T_K](f) = bT_K(f) - T_K(bf).$$

For a general Calderón-Zygmund operator T_K, it is well known that it is bounded on $L^p(R^n)$, for all $p > 1$ (see [31, Theorem 4.2.2]). Moreover, R.R. Coifman and C. Fefferman [24] proved that T_K is bounded on the weighted Lebesgue space $L^p(v)$ for $v \in A_p$ and $p > 1$. Indeed, T.P. Hytönen proved in [32] that

$$\|T_K f\|_{L^2(v)} \lesssim \|v\|_{A_2} \|f\|_{L^2(v)}, \qquad \forall v \in A_2. \tag{18}$$

For the commutator, R.R. Coifman, R. Rochberg and G. Weiss [25] proved that $[b, T_K]$ is bounded on $L^p(\mathbb{R}^n)$, $p > 1$, if b is a BMO($\mathbb{R}^n$) function, where the space of functions of bounded mean oscillation BMO($\mathbb{R}^n$) is defined by

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q \left| b(y) - b_Q \right| dy < \infty \right\},$$

where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ with sides parallel to the axes and $b_Q = \frac{1}{|Q|} \int_Q b(x)\,dx$. An analogue of (18) for the commutator $[b, T_K]$ with a BMO($\mathbb{R}^n$) function b is due to D. Chung, C. Pereyra and C. Pérez (see [21]), who proved that

$$\|[b, T_K] f\|_{L^2(v)} \lesssim \|b\|_{\mathrm{BMO}} \|v\|^2_{A_2} \|f\|_{L^2(v)}, \qquad \forall v \in A_2.$$

Therefore, as a consequence of Theorem 1.3:

Corollary 4.1 *Let K be a standard kernel. Given $u \in A_\infty$ and $p > 0$,*

$$T_K : \Lambda^p_u(w) \to \Lambda^p_u(w), \qquad \forall w \in B_p(u) \cap B^*_\infty \tag{19}$$

and, for $b \in BMO(\mathbb{R}^n)$,

$$[b, T_K] : \Lambda^p_u(w) \to \Lambda^p_u(w), \qquad \forall w \in B_p(u) \cap B^*_\infty. \tag{20}$$

Remark 4.2 This extends the results on [17, Theorems 1.1 and 1.2] where the authors saw (19) and (20) for $p > 1$. Moreover, we recall that an example of a Calderón-Zygmund operator is the Hilbert transform (see (6) for its definition).

4.2 Sparse Operators

These operators have become very popular due to their role in the often-called A_2 conjecture consisting in proving, for instance, (18). This result was first obtained by T.P. Hytönen [32] and then simplified by Lerner [34, 35], who proved that the norm of a Calderón-Zygmund operator in a Banach function space $\mathbb{X}$ is dominated by the supremum of the norm in $\mathbb{X}$ of all the possible sparse operators, and then proved that every sparse operator is bounded in $L^2(v)$ for every weight $v \in A_2$ with sharp constant.

Let us give the precise definition. First, a general dyadic grid $\mathcal{D}$ is a collection of cubes in $\mathbb{R}^n$ satisfying the following properties:

(i) For any cube $Q \in \mathcal{D}$, its side length is 2^k for some $k \in \mathbb{Z}$.
(ii) Every two cubes in $\mathcal{D}$ are either disjoint or one is wholly contained in the other.
(iii) If $\mathcal{D}_k \subseteq \mathcal{D}$ is the subfamily of cubes formed by the cubes of exactly side length 2^k, $k \in \mathbb{Z}$, then $\mathcal{D}_k$ form a partition of $\mathbb{R}^n$.

Hence, let $0 < \eta < 1$ and let $\mathcal{D}$ be a family of dyadic cubes. A collection of cubes $\mathcal{S} \subseteq \mathcal{D}$ is called η-sparse if one can choose pairwise disjoint measurable sets $E_Q \subseteq Q$ with $|E_Q| \geq \eta |Q|$, where $Q \in \mathcal{S}$ (see [30, 37] for more details). Hence, given a η-sparse family of cubes $\mathcal{S} \subseteq \mathcal{D}$, the sparse operator $\mathcal{A}_{\mathcal{S}}$ corresponding to the family $\mathcal{S}$ is defined by

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|} \int_Q |f(y)|\, dy\, \chi_Q(x), \qquad x \in \mathbb{R}^n.$$

Therefore, as a consequence of Theorem 1.3:

Corollary 4.3 *Given $u \in A_\infty$ and $p > 0$,*

$$A_{\mathcal{S}} : \Lambda_u^p(w) \to \Lambda_u^p(w), \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

4.3 Bochner-Riesz Operator

Let $n > 1$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx, \qquad \xi \in \mathbb{R}^n,$$

be the Fourier transform of a function $f \in L^2(\mathbb{R}^n)$. For $\lambda > 0$, the Bochner-Riesz operator is defined as

$$\widehat{B_\lambda f}(\xi) = \left(1 - |\xi|^2\right)_+^\lambda \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n,\ f \in L^2(\mathbb{R}^n).$$

These operators were first introduced by Bochner in [11] and, since then, they have been widely studied (see [12, 14, 23, 31, 46]).

When $\lambda > \frac{n-1}{2}$, it is well known that B_λ is controlled by the Hardy-Littlewood maximal operator M. As a consequence, all weighted inequalities for M are also satisfied by B_λ. The value $\lambda = \frac{n-1}{2}$ is called the critical index. In this case, Shi and Sun [44] proved that $B_{\frac{n-1}{2}}$ is bounded in $L^p(v)$ for every $p > 1$ and $v \in A_p$. Moreover, in [38, Theorem 1.6] the authors obtained the following quantitative result.

Proposition 4.4 *Let $n > 1$. Then,*

$$B_{\frac{n-1}{2}} : L^2(v) \to L^2(v), \qquad \forall v \in A_2,$$

with constant less than or equal to $C\, \|v\|_{A_2}^2$.

Therefore, as a consequence of Theorem 1.3:

Corollary 4.5 *Let* $\lambda \geq \frac{n-1}{2}$. *Given* $u \in A_\infty$ *and* $p > 0$,

$$B_\lambda : \Lambda_u^p(w) \to \Lambda_u^p(w), \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

4.4 Intrinsic Square Functions

For $0 < \alpha \leq 1$, let $\mathcal{C}_\alpha$ be the family of functions ϕ supported in $\mathbb{B}(0,1)$ (the n-th dimensional open ball of center 0 and radius 1) such that

$$\int_{\mathbb{B}(0,1)} \phi(x)\,dx = 0 \quad \text{and} \quad |\phi(x) - \phi(x')| < |x - x'|^\alpha, \quad \forall x, x' \in \mathbb{R}^n.$$

Then, given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(y,t) = \sup_{\phi \in \mathcal{C}^\alpha} |(\phi_t * f)(y)|, \qquad (y,t) \in \mathbb{R}^{n+1}_+,$$

where we are using ϕ_t to denote the usual $L^1(\mathbb{R}^n)$ dilatation of ϕ; that is $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$.

The intrinsic square function (of order α) introduced by M. Wilson in [47] is defined by

$$G_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |A_\alpha(f)(y,t)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^n,$$

with $\Gamma_\alpha(x) = \{(y,t) : |x - y| < \alpha t\}$. In [36] it was proved that

$$\|G_\alpha f\|_{L^3(v)} \lesssim \|v\|_{A_3}^{\frac{1}{2}} \|f\|_{L^3(v)}, \qquad \forall v \in A_3.$$

Therefore, as a consequence of Theorem 1.3:

Corollary 4.6 *Let* $0 < \alpha \leq 1$. *Given* $u \in A_\infty$ *and* $p > 0$,

$$G_\alpha : \Lambda_u^p(w) \to \Lambda_u^p(w), \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

Remark 4.7 In [47] was proved that G_α dominates pointwise (modulo constant) operators such as the Lusin area integral, the Littlewood-Paley g-function and the continuous square function (see also [16]). Therefore, analogous results as in Corollary 4.6 can be derived for those operators as well.

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