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Some Optimal Inequalities for Anti-invariant Submanifolds of the Unit Sphere

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Abstract

In this paper, we study the rigidity phenomena on the $(n+1)$-dimensional anti-invariant submanifolds of the unit sphere of dimension $(2n+1)$ from the intrinsic and extrinsic aspects, respectively. First of all, we establish a basic inequality for such submanifolds relative to the norm of the covariant differentiation of both the second fundamental form h and mean curvature vector field H. Secondly, the lower bound of the norm of H is further derived by means of a general inequality. Finally, in dealing with those minimal anti-invariant submanifolds with η-Einstein induced metrics, we obtain an inequality in terms of the Weyl curvature tensor, squared norm S of h, and scalar curvature. In particular, these inequalities above are optimal in the sense that all the submanifolds attaining the equalities are completely determined.

Keywords Unit sphere · Anti-invariant submanifold · Optimal inequality · Rigidity theorem

Mathematics Subject Classification Primary 53C24; Secondary 53C25 · 53C40 · 53C42

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1 Introduction

The study of rigidity phenomena relative to the submanifolds of the unit sphere under appropriate geometric properties is always an interesting topic and has attracted many geometers. It is well known that, as a real hypersurface of the complex Euclidean space $\mathbb{C}^{n+1}$, the $(2n+1)$-dimensional unit sphere $\mathbb{S}^{2n+1}(1)$ admits a natural Sasakian structure (φ, ξ, η, g) (cf. [33]). Moreover, we call an m-dimensional submanifold M^m of $\mathbb{S}^{2n+1}(1)$ *C-totally real* (or equivalently, *integral*) if the contact form η of $\mathbb{S}^{2n+1}(1)$ vanishes when it is restricted to M^m, i.e., $\eta(X) = 0$ for any $X \in TM^m$. In particular, a C-totally real submanifold M^m is said to be *Legendrian* if it meets the smallest possible codimension, that is, $m = n$ (cf. [34]). Related to the study of rigidity phenomena on such Legendrian submanifolds of $\mathbb{S}^{2n+1}(1)$, many results have been established in the last decades, see e.g., [9, 10] for the sectional curvature, [13, 15] for the Ricci curvature, [20, 21, 24–28, 35] for the scalar curvature, or the monograph [6] and references therein.

In addition to those above, there exists another class of special submanifolds of the unit sphere $\mathbb{S}^{2n+1}(1)$ called *anti-invariant*. As for an anti-invariant submanifold M^m of $\mathbb{S}^{2n+1}(1)$, it satisfies that $T_xM^m \perp \varphi(T_xM^m)$ for each point $x \in M^m$, where T_xM^m is the tangent space of M^m in $\mathbb{S}^{2n+1}(1)$ at x. It is noted that, for anti-invariant submanifolds, we have $m \leq n+1$ for the reason that φ is necessarily of rank $2n$, and such submanifolds with $m = n+1$ differ from Legendrian submanifolds which are in fact anti-invariant, since the structure vector field ξ of $\mathbb{S}^{2n+1}(1)$ is tangent to M^{n+1} (cf. [36]), whereas ξ is normal to M^n. Although the above anti-invariant submanifolds with $m = n+1$ have been investigated in [19, 36, 37, 39] and so on, very little is known about their rigidity phenomena, compared with the Legendrian submanifolds of $\mathbb{S}^{2n+1}(1)$.

In this paper, inspired by the statement above, we will study rigidity phenomena on the anti-invariant submanifold M^{n+1} of $\mathbb{S}^{2n+1}(1)$ from both the intrinsic and extrinsic aspects. Before introducing our main results, we should remark that M^{n+1} has *η-Einstein* induced metric g if its Ricci tensor Ric satisfies that $\mathrm{Ric}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where a, b are smooth functions on M^{n+1}, and M^{n+1} is *totally contact geodesic* if and only if $h(X, Y) = \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi)$ holds for the second fundamental form h of $M^{n+1} \to \mathbb{S}^{2n+1}(1)$ and arbitrary vectors X, Y tangent to M^{n+1}.

In order to better state our main results, we recall the following canonical Lagrangian submanifolds of the complex projective space $\mathbb{C}P^n(4)$.

Example 1.1 (cf. [11]) Those Lagrangian submanifolds of $\mathbb{C}P^n(4)$ with parallel second fundamental form are exactly one of the following:

(a) totally geodesic submanifolds;
(b) locally a finite Riemannian covering of the unique flat torus, minimally embedded into $\mathbb{C}P^2(4)$ with parallel second fundamental form;

(c) locally the submanifolds which are congruent to one of the standard embeddings from the following compact symmetric spaces into $\mathbb{C}P^n(4)$:

$$\begin{aligned} \mathrm{SU}(k)/\mathrm{SO}(k), &\quad n=(k-1)(k+2)/2, \quad k\geq 3, \\ \mathrm{SU}(k), &\quad n=k^2-1, \quad k\geq 3, \\ \mathrm{SU}(2k)/\mathrm{Sp(k)}, &\quad n=2k^2-k-1, \quad k\geq 3, \\ \mathrm{E}_6/\mathrm{F}_4, &\quad n=26; \end{aligned}$$

(d) locally the Calabi product of a point with a lower dimensional Lagrangian submanifold with parallel second fundamental form;
(e) locally the Calabi product of two lower dimensional Lagrangian submanifolds with parallel second fundamental form.

Remark 1.1 It should be pointed out that all the submanifolds (a)-(c) in Example 1.1 are minimal Lagrangian ones, whereas the submanifolds (d) and (e) include both minimal and non-minimal Lagrangian ones. Moreover, according to the classification results of Ejiri [12] and Li-Zhao [22], the minimal Lagrangian submanifold of $\mathbb{C}P^n(4)$ with constant sectional curvature is either totally geodesic, i.e., the real projective space $\mathbb{R}P^n$, or the flat Clifford torus T^n. For the latter one, it appears in (b) for $n=2$, and in (d) or (e) for $n\geq 3$. In particular, the submanifolds (c) are of Einstein induced metrics.

Example 1.2 (cf. [3, 4, 7]) The Whitney spheres of $\mathbb{C}P^n(4)$ are a one-parameter family of Lagrangian sphere immersions, given by $\Psi_\theta:\mathbb{S}^n\to\mathbb{C}P^n(4)$ for $\theta>0$ with

$$\Psi_\theta(u_1,\dots,u_{n+1})=\pi\left(\frac{(u_1,\dots,u_n)}{\cosh\theta+i\sinh\theta u_{n+1}};\frac{\sinh\theta\cosh\theta(1+u_{n+1}^2)+iu_{n+1}}{\cosh^2\theta+\sinh^2\theta u_{n+1}^2}\right), \tag{1.1}$$

where $\pi:\mathbb{S}^{2n+1}(1)\to\mathbb{C}P^n(4)$ is the Hopf projection. Note that Ψ_θ are embeddings except at the poles of $\mathbb{S}^n$ where it has a double points, and Ψ_0 is the totally geodesic Lagrangian immersion of $\mathbb{S}^n$ into $\mathbb{C}P^n(4)$.

Now, with all previous preparations being completed, our main theorems can be stated in the sense of (5.2) as follows:

Theorem 1.1 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ with the second fundamental form h and mean curvature vector field H. Then it holds the following inequality:*

$$\|\bar\nabla h\|^2\geq\frac{3(n+1)^2}{n+2}\|\nabla^\perp H\|^2, \tag{1.2}$$

where $\bar\nabla$ denotes the Levi–Civita connection of $\mathbb{S}^{2n+1}(1)$, $\nabla^\perp$ and $\|\cdot\|$ are, respectively, the normal connection and the tensorial norm with respect to the contact metric on $\mathbb{S}^{2n+1}(1)$. *Moreover, the equality in* (1.2) *holds identically if and only if one of the two cases occurs:*

(i) *M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Lagrangian submanifolds (a)-(e) of $\mathbb{C}P^n(4)$, as described in Example 1.1;*
(ii) *M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Whitney spheres of $\mathbb{C}P^n(4)$, as described in Example 1.2.*

Theorem 1.2 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ with the scalar curvature R. Then it holds the following inequality:*

$$\|H\|^2 \geq \tfrac{n+2}{(n+1)^2(n-1)}R - \tfrac{n(n+2)}{(n+1)^2}. \tag{1.3}$$

Moreover, the equality in (1.3) holds identically if and only if either M^{n+1} is a totally contact geodesic submanifold, or it is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Whitney spheres of $\mathbb{C}P^n(4)$, as described in Example 1.2.

Theorem 1.3 *Let M^{n+1} be an $(n+1)$-dimensional minimal anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ with the Weyl curvature tensor W and squared norm S of the second fundamental form h, $n \geq 2$. Assume that M^{n+1} has η-Einstein induced metric. Then it holds the following inequality:*

$$\|W\|^2 \geq \tfrac{n+1}{n(n-1)}SR - \tfrac{2(n+1)}{n-1}R. \tag{1.4}$$

Moreover, the equality in (1.4) holds identically if and only if either M^{n+1} is a totally contact geodesic submanifold, or one of the two cases occurs:

(i) *M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the standard embeddings of the compact symmetric spaces into $\mathbb{C}P^n(4)$, as described in Example 1.1(c);*
(ii) *M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of the standard embedding of the flat Clifford torus T^n into $\mathbb{C}P^n(4)$, as described in Example 1.1(b) for $n = 2$, and (d) or (e) for $n \geq 3$.*

Remark 1.2 The so-called totally contact geodesic submanifold of $\mathbb{S}^{2n+1}(1)$ is actually a Riemannian product of $\mathbb{R}$ and N^n, where the Lagrangian submanifold N^n of $\mathbb{C}P^n(4)$ is totally geodesic, and by Remark 1.1, it is the real projective space $\mathbb{R}P^n$.

Remark 1.3 It is worth mentioning that, compared with Theorem 1.1, an inequality for Legendrian submanifolds of the unit sphere $\mathbb{S}^{2n+1}(1)$, between the norm of the covariant differentiation of second fundamental form and that of mean curvature vector field, was established by Yin and his coauthor in [16]. Moreover, studying Legendrian submanifolds of the Sasakian space form $\mathbb{R}^{2n+1}(-3)$, Blair and Carriazo [2] obtained an inequality in terms of the scalar curvature and mean curvature vector field, similar to (1.3), which was later extended to Legendrian submanifolds in $\mathbb{S}^{2n+1}(1)$ (see Corollary 16.3 of [6]). Conversely, there is a question about whether an optimal inequality similar to (1.4) can be verified for Legendrian submanifolds of $\mathbb{S}^{2n+1}(1)$ under suitable geometric conditions.

Remark 1.4 As stated before, the difference between $(n + 1)$-dimensional anti-invariant submanifolds and Legendrian submanifolds of $\mathbb{S}^{2n+1}(1)$ lies only on the role of structure vector field ξ. On one hand, different from Legendrian submanifolds, by Lemma 2.1, the tangent vector field ξ always causes an $(n + 1)$-dimensional anti-invariant submanifold to be locally a Riemannian product manifold. On the other hand, given the property of second fundamental form with respect to ξ, the techniques used in these two cases are not exactly the same, for example, the proof of Lemma 2.2 and that of Lemma 3.6 in [16].

The rest of this paper is organized as follows. In Sect. 2, we briefly review some necessary material on Sasakian structure (φ, ξ, η, g) of the unit sphere $\mathbb{S}^{2n+1}(1)$ and the theory of anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$. In Sect. 3, before proving our main results, two crucial lemmas relative to the properties of anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$ shall be presented. Moreover, to prove Theorem 1.1, we obtain two propositions in Sect. 4 depending on whether the function μ is constant or not. Section 5 and Sect. 6 are finally dedicated to the completion of the proofs of Theorems 1.1–1.3.

2 Preliminaries

In this section, we begin with collecting some basic material of the Sasakian structure (φ, ξ, η, g) of the unit sphere $\mathbb{S}^{2n+1}(1)$ that can be regarded as a Sasakian space form with constant φ-sectional curvature 1. Moreover, we briefly review the theory of anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$ and some relevant results shall be presented here for later use. For more details, we refer to the references [16, 18, 36–38] and the monograph [1].

2.1 Sasakian Structure (φ, ξ, η, g) of the Unit Sphere $\mathbb{S}^{2n+1}(1)$

As a real hypersurface of the complex Euclidean space $\mathbb{C}^{n+1}$ with canonical complex structure J, the $(2n + 1)$-dimensional unit sphere $\mathbb{S}^{2n+1}(1)$ admits a natural Sasakian structure (φ, ξ, η, g): $\xi = J\bar{N}$ is the structure vector field with $\bar{N}$ being the unit normal vector field of the inclusion $\mathbb{S}^{2n+1}(1) \hookrightarrow \mathbb{C}^{n+1}$; g is the induced metric on $\mathbb{S}^{2n+1}(1)$; $\eta(X) = g(X, \xi)$ and $\varphi X = JX - \langle JX, \bar{N}\rangle\bar{N}$ for any tangent vector field X on $\mathbb{S}^{2n+1}(1)$, where $\langle\cdot, \cdot\rangle$ is the standard Hermitian metric on $\mathbb{C}^{n+1}$.

For any tangent vector fields X, Y on $\mathbb{S}^{2n+1}(1)$, the Sasakian structure (φ, ξ, η, g) of $\mathbb{S}^{2n+1}(1)$ satisfies the properties:

$$\begin{cases} g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \\ \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \mathrm{rank}\,(\varphi) = 2n, \\ \varphi^2 X = -X + \eta(X)\xi, \quad d\eta(X, Y) = g(X, \varphi Y), \\ \bar{\nabla}_X\xi = -\varphi X, \quad (\bar{\nabla}_X\varphi)Y = g(X, Y)\xi - \eta(Y)X, \end{cases} \tag{2.1}$$

where $\bar{\nabla}$ is the Levi–Civita connection with respect to the induced metric g on $\mathbb{S}^{2n+1}(1)$.

2.2 Anti-invariant Submanifolds of the Unit Sphere $\mathbb{S}^{2n+1}(1)$

Let M^{n+1} denote an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ which means that $T_xM^{n+1} \perp \varphi(T_xM^{n+1})$ for each point $x \in M^{n+1}$, and so the structure vector field ξ of $\mathbb{S}^{2n+1}(1)$ is tangent to M^{n+1} (cf. Lemma 2.1 of [36]). Denote by N a unit normal vector field along M^{n+1} and by X, Y, Z the tangent vector fields on M^{n+1} in the subsequent paragraphs. Then, we have the Gauss and Weingarten formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

where ∇ is the Levi–Civita connection of the induced metric on M^{n+1}, still denoted by g, h (resp. A_N) is the corresponding second fundamental form (resp. shape operator with respect to N), and $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M^{n+1}$. In particular, it can be checked from (2.2) that

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.3}$$

Notice from (2.1), (2.2), and the fact $\xi \in TM^{n+1}$ that

$$\nabla_X \xi = 0, \quad h(X, \xi) = -\varphi X, \quad \nabla^{\perp}_X \varphi Y = \varphi \nabla_X Y, \tag{2.4}$$

$$A_{\varphi X} Y = -\varphi h(X, Y) + \eta(X) Y - g(X, Y)\xi. \tag{2.5}$$

Now, in order to utilize the moving frame method, we shall take the following range convention of indices:

$$i, j, k, \ell, m, r, q = 1, \dots, n; \quad a, b, c, d, p = 0, 1, \dots, n;$$
$$i^*, j^*, k^*, \ell^*, m^*, r^*, q^* = i+n, j+n, k+n, \ell+n, m+n, r+n, q+n.$$

As usual, a local orthonormal frame $\{e_0 = \xi, e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}\}$ of $\mathbb{S}^{2n+1}(1)$ can be chosen such that, restricted to M^{n+1}, $\{e_0, \dots, e_n\}$ is an orthonormal frame of M^{n+1}, and $e_{1^*} = \varphi e_1, \dots, e_{n^*} = \varphi e_n$ are the orthonormal normal vector fields on M^{n+1} of $\mathbb{S}^{2n+1}(1)$. Denote by $\{\theta_0 = \eta, \theta_1, \dots, \theta_n\}$ the dual frame of $\{e_0, e_1, \dots, e_n\}$. Let θ_{ab} and $\theta_{i^*j^*}$ denote the connection 1-forms of TM^{n+1} and $T^{\perp}M^{n+1}$, respectively, defined by

$$\nabla e_a = \sum_b \theta_{ab} e_b, \quad \nabla^{\perp} e_{i^*} = \sum_j \theta_{i^*j^*} e_{j^*},$$

where $\theta_{ab} + \theta_{ba} = \theta_{i^*j^*} + \theta_{j^*i^*} = 0$. By (2.2) and (2.4), we have $\theta_{i^*j^*} = \theta_{ij}$ and $\theta_{i0} = 0$.

Put $h_{ab}^{k^*} = g(h(e_a, e_b), \varphi e_k)$. It follows from (2.3)–(2.5) that

$$h_{ab}^{k^*} = h_{ba}^{k^*},\ \ h_{ij}^{k^*} = h_{ik}^{j^*},\ \ h_{0b}^{k^*} = -\delta_{bk},\ \ \forall\, a, b, i, j, k. \tag{2.6}$$

Let $R_{abcd} = g(R(e_a, e_b)e_d, e_c)$ and $R_{abk^*\ell^*} = g(R(e_a, e_b)e_{\ell^*}, e_{k^*})$ be the components of the curvature tensors of ∇ and $\nabla^\perp$, respectively, corresponding to the above frame. Then, the equations of Gauss, Ricci, and Codazzi are, respectively, given by

$$R_{abcd} = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \sum_m (h_{ac}^{m^*} h_{bd}^{m^*} - h_{ad}^{m^*} h_{bc}^{m^*}), \tag{2.7}$$

$$R_{abk^*\ell^*} = (\delta_{ak}\delta_{b\ell} - \delta_{a\ell}\delta_{bk}) + \sum_m (h_{ak}^{m^*} h_{b\ell}^{m^*} - h_{a\ell}^{m^*} h_{bk}^{m^*}), \tag{2.8}$$

$$h_{ab,c}^{k^*} = h_{ac,b}^{k^*}, \tag{2.9}$$

where $h_{ab,c}^{k^*}$ are the components of the covariant differentiation of h, defined by

$$\sum_c h_{ab,c}^{k^*}\theta_c = dh_{ab}^{k^*} + \sum_c h_{ac}^{k^*}\theta_{cb} + \sum_c h_{bc}^{k^*}\theta_{ca} + \sum_j h_{ab}^{j^*}\theta_{j^*k^*}.$$

Furthermore, direct calculations with (2.4) and (2.9) show that

$$h_{ij,\ell}^{k^*} = h_{j\ell,k}^{i^*} = h_{\ell k,i}^{j^*} = h_{ki,j}^{\ell^*},\ \ h_{0b,c}^{k^*} = 0,\ \ \forall\, b, c, i, j, k, \ell. \tag{2.10}$$

In particular, the Ricci identity can be written as

$$h_{ab,cd}^{k^*} - h_{ab,dc}^{k^*} = \sum_p h_{pb}^{k^*} R_{pacd} + \sum_p h_{ap}^{k^*} R_{pbcd} + \sum_\ell h_{ab}^{\ell^*} R_{dck^*\ell^*}, \tag{2.11}$$

where the second covariant derivative $h_{ab,cd}^{k^*}$ is defined by

$$\sum_d h_{ab,cd}^{k^*}\theta_d = dh_{ab,c}^{k^*} + \sum_d h_{db,c}^{k^*}\theta_{da} + \sum_d h_{ad,c}^{k^*}\theta_{db} + \sum_d h_{ab,d}^{k^*}\theta_{dc} + \sum_j h_{ab,c}^{j^*}\theta_{j^*k^*}.$$

Finally, the mean curvature vector field H along M^{n+1} of $\mathbb{S}^{2n+1}(1)$ satisfies

$$H = \tfrac{1}{n+1}\sum_a h(e_a, e_a) = \sum_k H^{k^*} e_{k^*},\ \ H^{k^*} = \tfrac{1}{n+1}\sum_a h_{aa}^{k^*}.$$

This combining with (2.9), (2.10) and $\nabla_{e_p}^\perp H = \sum_k H_{,p}^{k^*} e_{k^*}$ implies that

$$H_{,\ell}^{k^*} = H_{,k}^{\ell^*}\ \ H_{,0}^{k^*} = 0,\ \ \forall\, k, \ell. \tag{2.12}$$

By means of (2.7), the components R_{ab} of Ricci tensor Ric and the scalar curvature R of M^{n+1} become

$$R_{ac} = n\delta_{ac} + (n+1)\sum_m h_{ac}^{m^*} H^{m^*} - \sum_{b,m} h_{ab}^{m^*} h_{bc}^{m^*}, \tag{2.13}$$

$$R = n(n+1) + (n+1)^2\|H\|^2 - S, \quad S = \|h\|^2, \tag{2.14}$$

where $\|H\|^2 = \sum_m (H^{m^*})^2$ and $\|h\|^2 = \sum_{a,b,m}(h_{ab}^{m^*})^2$.

2.3 Some Results on Anti-invariant Submanifolds of the Unit Sphere $\mathbb{S}^{2n+1}(1)$

In this subsection, as the preparation for proving Theorems 1.1–1.3, we are going to present the following results on anti-invariant submanifolds of the unit sphere $\mathbb{S}^{2n+1}(1)$.

Lemma 2.1 (cf. [36]) *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. Then, M^{n+1} is locally isometric to a Riemannian product $\mathbb{R} \times N^n$ and $\mathbb{R}$ is the 1-dimensional subspace generated by ξ.*

Let $U_xN^n = \{u \in T_xN^n \mid g(u,u) = 1\}$ and $e_0 = \xi(x)$ at the point $x \in M^{n+1}$. We can define a function f on U_xN^n by $f(u) = g(h(u,u), \varphi u)$ and it then follows from (2.6) that all indices of $g(h(u,v), \varphi w)$ are totally symmetric.

Lemma 2.2 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. There exists an orthonormal basis $\{e_0 = \xi(x), e_1, \dots, e_n\}$ of T_xM^{n+1} such that e_i are tangent to N^n, $i = 1, \dots, n$, and satisfy the following:*

(1) *$h(e_1, e_i) = \lambda_i \varphi e_i$, $i = 1, \dots, n$, where λ_1 is the maximum of f on U_xN^n;*
(2) *$\lambda_1 \geq 2\lambda_j$, $j = 2, \dots, n$. Moreover, if $\lambda_1 = 2\lambda_j$ for some $j \geq 2$, then $f(e_j) = 0$.*

Proof Since U_xN^n is compact, there exists a unit vector $e_1 \in U_xN^n$ at which the above function $f(u)$ attains an absolute maximum, denoted by $\lambda_1 := g(h(e_1,e_1), \varphi e_1)$, and so $g(h(e_1,e_1), \varphi w) = 0$ for any $w \in U_xN^n$ orthogonal to e_1. In this situation, by definition, we can pointwisely define a self-adjoint operator $\mathcal{A}_x : T_xN^n \to T_xN^n$ by

$$\mathcal{A}_x(v) := A_{\varphi e_1} v - g(A_{\varphi e_1} v, \xi(x))\xi(x). \tag{2.15}$$

It is easily seen from (2.5) that $\mathcal{A}_x(e_1) = \lambda_1 e_1$. Therefore, we can obtain an orthonormal basis $\{e_i\}_{i=1}^n$ of T_xN^n, which consists of eigenvectors of $\mathcal{A}_x$ with associated eigenvalues $\{\lambda_i\}_{i=1}^n$, satisfying the following relations:

$$A_{\varphi e_1} e_1 = \lambda_1 e_1 - \xi(x), \quad A_{\varphi e_1} e_j = \lambda_j e_j, \quad 2 \leq j \leq n. \tag{2.16}$$

Finally, according to Lemma 5.1 of [17], Lemma 2.2 has been proved. □

Let $\mathbb{C}P^n(4)$ denote the complex projective space with constant holomorphic sectional curvature 4, complex structure J, and Kähler metric G. Then, it is known from [29] that, for such a canonical projection π as in Example 1.2 from the unit sphere $\mathbb{S}^{2n+1}(1)$ onto the complex projective space $\mathbb{C}P^n(4)$, we have the lemma below:

Lemma 2.3 (cf. [29, 31]) *The canonical projection* $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{C}P^n(4)$ *as stated above is a Riemannian submersion. Moreover, the following properties hold:*

(1) *The vertical subspace of* π *at* $x \in \mathbb{S}^{2n+1}(1)$ *is equal to the span of* ξ_x*;*
(2) G *on* $\mathbb{C}P^n(4)$ *and the contact metric* g *on* $\mathbb{S}^{2n+1}(1)$ *are related by* $g = \pi^*G + \eta\otimes\eta$*;*
(3) $\varphi X = (J\pi_* X)^*$*, for any vector field* X *on* $\mathbb{S}^{2n+1}(1)$*, where* $\cdot^*$ *is the horizontal lift of* $\cdot$ *with respect to* η*.*

3 Properties of Anti-invariant Submanifolds of the Unit Sphere $\mathbb{S}^{2n+1}(1)$

In this section, before proving the main results of this paper, we will give the following two lemmas related to the properties of anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$.

Lemma 3.1 *Let* M^{n+1} *be an* $(n+1)$*-dimensional anti-invariant submanifold of the unit sphere* $\mathbb{S}^{2n+1}(1)$*. Then*

$$\|\bar{\nabla} h\|^2 \geq \tfrac{3(n+1)^2}{n+2}\|\nabla^\perp H\|^2, \tag{3.1}$$

where, with respect to the local orthonormal frame $\{e_A\}_{A=0}^{2n}$ *as described above,*

$$\|\bar{\nabla} h\|^2 = \sum_{a,b,c,k} (h^{k^*}_{ab,c})^2, \quad \|\nabla^\perp H\|^2 = \sum_{a,k} (H^{k^*}_{,a})^2.$$

Moreover, the equality in (3.1) holds identically if and only if

$$h^{k^*}_{ij,\ell} = \tfrac{n+1}{n+2}(H^{k^*}_{,i}\delta_{j\ell} + H^{k^*}_{,j}\delta_{i\ell} + H^{k^*}_{,\ell}\delta_{ij}), \quad 1 \leq i, j, k, \ell \leq n, \tag{3.2}$$

or equivalently,

$$h^{k^*}_{ij,\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} + \delta_{ij}\delta_{k\ell}), \quad 1 \leq i, j, k, \ell \leq n, \tag{3.3}$$

where $\mu = \frac{n+1}{n(n+2)}\sum_\ell H^{\ell^*}_{,\ell}$.

Proof Let us define a tensor $T : TM^{n+1} \times TM^{n+1} \times TM^{n+1} \to T^\perp M^{n+1}$, satisfying

$$T^{k^*}_{abc} = h^{k^*}_{ab,c} - \tfrac{n+1}{n+2}\Big((\delta_{ab} - \eta_a\eta_b)H^{k^*}_{,c} + \sum_\ell \delta_{a\ell}\delta_{bk}H^{\ell^*}_{,c} + \sum_\ell \delta_{b\ell}\delta_{ak}H^{\ell^*}_{,c}\Big). \tag{3.4}$$

Here, $\eta_a = \eta(e_a) = g(e_a, \xi)$. Note from (2.10) and (2.12) that $T^{k^*}_{abc} = 0$ if any of a, b, c is equal to 0. Consequently, (3.4) reduces to

$$T^{k^*}_{ij\ell} = h^{k^*}_{ij,\ell} - \tfrac{n+1}{n+2}(\delta_{ij}H^{k^*}_{,\ell} + \delta_{jk}H^{i^*}_{,\ell} + \delta_{ik}H^{j^*}_{,\ell}), \tag{3.5}$$

and therefore,

$$0 \leq \|T\|^2 = \|\bar{\nabla} h\|^2 - \tfrac{3(n+1)^2}{n+2}\|\nabla^\perp H\|^2, \tag{3.6}$$

where, to derive the above equation, we made use of the facts $h^{k^*}_{0b,c} = 0$ and $H^{k^*}_{,0} = 0$ for $0 \leq b, c \leq n$ and $1 \leq k \leq n$. As the result, we obtain from (3.6) the inequality in (3.1), and it becomes an equality if and only if (3.2) holds identically.

Assume that (3.2) holds on M^{n+1}. Then we get the relation by exchanging k and ℓ:

$$h^{\ell^*}_{ij,k} = \tfrac{n+1}{n+2}(H^{\ell^*}_{,i}\delta_{jk} + H^{\ell^*}_{,j}\delta_{ik} + H^{\ell^*}_{,k}\delta_{ij}), \quad 1 \leq i, j, k, \ell \leq n, \tag{3.7}$$

which combining with (3.2) implies that

$$H^{k^*}_{,i}\delta_{j\ell} + H^{k^*}_{,j}\delta_{i\ell} + H^{k^*}_{,\ell}\delta_{ij} = H^{\ell^*}_{,i}\delta_{jk} + H^{\ell^*}_{,j}\delta_{ik} + H^{\ell^*}_{,k}\delta_{ij}. \tag{3.8}$$

By contracting the indices i and ℓ in (3.8), we deduce from (2.12) that

$$H^{k^*}_{,j} = \tfrac{1}{n}\sum_\ell H^{\ell^*}_{,\ell}\delta_{jk}, \quad 1 \leq j, k, \ell \leq n. \tag{3.9}$$

From the above equation, we obtain (3.3), and in this case, $\nabla^\perp H = \lambda\varphi$ for $\lambda = \frac{1}{n}\sum_\ell H^{\ell^*}_{,\ell}$. On the other hand, if (3.3) holds, summing over i and j, by (2.6) we conclude that

$$(n+1)H^{k^*}_{,\ell} = (n+2)\mu\delta_{k\ell}. \tag{3.10}$$

This implies that (3.2) holds. Hence, Lemma 3.1 has been proved. □

Lemma 3.2 *Let M^{n+1} be an $(n+1)$-dimensional minimal anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. Then it holds the identity:*

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla} h\|^2 - \|\mathrm{Rie}\|^2 - \|\mathrm{Ric}\|^2 + (n+1)R, \tag{3.11}$$

where, with respect to the local orthonormal frame $\{e_A\}_{A=0}^{2n}$ as described above,

$$\|\mathrm{Rie}\|^2 = \sum_{a,b,c,d}(R_{abcd})^2, \quad \|\mathrm{Ric}\|^2 = \sum_{a,b}(R_{ab})^2.$$

Proof Choose the orthonormal frame $\{e_A\}_{A=0}^{2n}$ as in Section 2. By definition, we have

$$\tfrac{1}{2}\Delta S = \tfrac{1}{2}\Delta\Big(\sum_{a,b,k}(h^{k^*}_{ab})^2\Big) = \sum_{a,b,c,k}(h^{k^*}_{ab,c})^2 + \sum_{a,b,k}h^{k^*}_{ab}\Delta h^{k^*}_{ab}. \tag{3.12}$$

Since M^{n+1} is minimal, we deduce from (2.9) and (2.11) that

$$\begin{aligned}\Delta h_{ab}^{k^*} &= \sum_c h_{ab,cc}^{k^*} = \sum_c h_{ac,bc}^{k^*}\\ &= \sum_c h_{ac,cb}^{k^*} + \sum_{c,p} h_{pc}^{k^*} R_{pabc} + \sum_{c,p} h_{ap}^{k^*} R_{pcbc} + \sum_{c,m} h_{ac}^{m^*} R_{cbk^*m^*}\\ &= \sum_{c,p} h_{pc}^{k^*} R_{pabc} + \sum_p h_{ap}^{k^*} R_{pb} + \sum_{c,m} h_{ac}^{m^*} R_{cbk^*m^*},\end{aligned} \tag{3.13}$$

which together with (2.7) and (2.8) shows that

$$\begin{aligned}\sum_{a,b,k} h_{ab}^{k^*}\Delta h_{ab}^{k^*} = &\sum_{a,b,c,p,k} h_{ab}^{k^*} h_{pc}^{k^*} R_{pabc} + \sum_{a,b,p,k} h_{ab}^{k^*} h_{ap}^{k^*} R_{pb}\\ &+ \sum_{a,b,c,k,m} h_{ab}^{k^*} h_{ac}^{m^*} R_{cbkm}.\end{aligned} \tag{3.14}$$

On the one hand, with the help of (2.6) and (2.7), it is easy to see that

$$\begin{aligned}&\sum_{a,b,c,p,k} h_{ab}^{k^*} h_{pc}^{k^*} R_{pabc} + \sum_{a,b,c,k,m} h_{ab}^{k^*} h_{ac}^{m^*} R_{cbkm}\\ &= \sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{m\ell}^{k^*} R_{mij\ell} + \sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{\ell jkm} - R.\end{aligned} \tag{3.15}$$

This combining with the relations

$$\begin{aligned}\sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{m\ell}^{k^*} R_{mij\ell} &= \sum_{i,j,k,\ell,m} h_{ij}^{m^*} h_{k\ell}^{m^*} R_{kij\ell},\\ \sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{\ell jkm} &= -\sum_{i,j,k,\ell,m} h_{jk}^{m^*} h_{i\ell}^{m^*} R_{kij\ell},\end{aligned} \tag{3.16}$$

and (2.7) yields that

$$\sum_{a,b,c,p,k} h_{ab}^{k^*} h_{pc}^{k^*} R_{pabc} + \sum_{a,b,c,k,m} h_{ab}^{k^*} h_{ac}^{m^*} R_{cbkm} = -\|\mathrm{Rie}\|^2 + R, \tag{3.17}$$

where, according to (2.7) and (2.13), we used the following facts:

$$\begin{aligned}\|\mathrm{Rie}\|^2 &= \sum_{a,b,c,d} (R_{abcd})^2 = \sum_{i,j,k,\ell} (R_{ijk\ell})^2,\\ R &= \sum_{a,b} \delta_{ab} R_{ab} = \sum_{i,j} \delta_{ij} R_{ij}.\end{aligned} \tag{3.18}$$

On the other hand, using (2.13) and $\|\mathrm{Ric}\|^2 = \sum_{a,b}(R_{ab})^2 = \sum_{i,j}(R_{ij})^2$, we have

$$\sum_{a,b,p,k} h^{k^*}_{ab} h^{k^*}_{ap} R_{pb} = \sum_{i,j} (n\delta_{ij} - R_{ij}) R_{ij} = nR - \|\mathrm{Ric}\|^2. \tag{3.19}$$

Finally, by substituting (3.17) and (3.19) into (3.14), we conclude that

$$\sum_{a,b,k} h^{k^*}_{ab} \Delta h^{k^*}_{ab} = -\|\mathrm{Rie}\|^2 - \|\mathrm{Ric}\|^2 + (n+1)R. \tag{3.20}$$

Obviously, the assertion follows from (3.12) and (3.20) immediately. □

4 The Lemma and Propositions Involving the Function μ

In this section, we always assume that M^{n+1} is an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. When working at the point $x \in M^{n+1}$, we also assume that an orthonormal basis is chosen such that Lemma 2.2 is satisfied. While if we work at a neighborhood of $x \in M^{n+1}$ and if not stated otherwise, we will choose an orthonormal frame $\{E_0 = \xi, E_1, \ldots, E_n\}$ with $E_0(x) = e_0, E_1(x) = e_1, \ldots, E_n(x) = e_n$, where $e_0, e_1, \ldots, e_n$ are given as in Lemma 2.2.

Lemma 4.1 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. Assume that (3.3) holds. Then it holds that*

$$e_t(\mu) = 0, \quad t = 2, \ldots, n, \tag{4.1}$$

$$e_1(\mu) = (2\lambda_t - \lambda_1)(1 + \lambda_1\lambda_t - \lambda_t^2), \; t = 2, \ldots, n, \tag{4.2}$$

$$(\lambda_s - \lambda_t)(2\lambda_k - \lambda_1) h^{s^*}_{kt} = 0, \;\; 2 \le s \ne t \le n, \;\; 1 \le k \le n, \tag{4.3}$$

$$(1 + \lambda_1\lambda_t - \lambda_t^2) h^{t^*}_{tt} = 0, \;\; t = 2, \ldots, n. \tag{4.4}$$

Proof We first take the covariant derivative of (3.3) to obtain that

$$h^{k^*}_{ij,\ell m} = e_m(\mu)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} + \delta_{ij}\delta_{k\ell}). \tag{4.5}$$

Exchanging the indices ℓ and m in (4.5) gives

$$h^{k^*}_{ij,m\ell} = e_\ell(\mu)(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} + \delta_{ij}\delta_{km}). \tag{4.6}$$

Moreover, by means of the Ricci identity (2.11), we deduce from (2.7) and (2.8) that

$$\begin{aligned} h^{k^*}_{ij,\ell m} - h^{k^*}_{ij,m\ell} &= \sum_p h^{k^*}_{pj} R_{pi\ell m} + \sum_p h^{k^*}_{ip} R_{pj\ell m} + \sum_r h^{r^*}_{ij} R_{m\ell k^* r^*} \\ &= \sum_r h^{k^*}_{rj} R_{ri\ell m} + \sum_r h^{k^*}_{ir} R_{rj\ell m} + \sum_r h^{r^*}_{ij} R_{m\ell k^* r^*} \\ &= \sum_r h^{k^*}_{rj} R_{ri\ell m} + \sum_r h^{k^*}_{ir} R_{rj\ell m} + \sum_r h^{r^*}_{ij} R_{m\ell kr}, \end{aligned} \tag{4.7}$$

which together with (4.5) and (4.6) yields that

$$\begin{aligned} &e_m(\mu)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} + \delta_{ij}\delta_{k\ell}) - e_\ell(\mu)(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} + \delta_{ij}\delta_{km}) \\ &= \sum_r h^{k^*}_{rj}(\delta_{r\ell}\delta_{im} - \delta_{rm}\delta_{i\ell} + \sum_q h^{q^*}_{r\ell} h^{q^*}_{im} - \sum_q h^{q^*}_{rm} h^{q^*}_{i\ell}) \\ &\quad + \sum_r h^{k^*}_{ir}(\delta_{r\ell}\delta_{jm} - \delta_{rm}\delta_{j\ell} + \sum_q h^{q^*}_{r\ell} h^{q^*}_{jm} - \sum_q h^{q^*}_{rm} h^{q^*}_{j\ell}) \\ &\quad + \sum_r h^{r^*}_{ij}(\delta_{r\ell}\delta_{km} - \delta_{rm}\delta_{k\ell} + \sum_q h^{q^*}_{r\ell} h^{q^*}_{km} - \sum_q h^{q^*}_{rm} h^{q^*}_{k\ell}). \end{aligned} \tag{4.8}$$

Next, letting $i = j = m = 1$ and $\ell = t \geq 2$ in (4.8), we see from Lemma 2.2 that

$$e_1(\mu)\delta_{kt} - 3e_t(\mu)\delta_{1k} = (2\lambda_k - \lambda_1)\delta_{kt} + (3\lambda_1\lambda_k^2 - 2\lambda_t\lambda_k^2 - \lambda_1^2\lambda_k)\delta_{kt}. \tag{4.9}$$

In this case, we immediately obtain (4.1) by choosing $k = 1$ in (4.9), and further obtain (4.2) by choosing $k = t \geq 2$ in (4.9). Finally, letting $i = j = 1$ and $2 \leq m = s \neq \ell = t \leq n$ in (4.8), we apply (4.1) to get (4.3), whereas letting $i = \ell = 1$ and $j = k = m = t \geq 2$ in (4.8), we apply (4.1) again to get (4.4). In conclusion, Lemma 4.1 has been proved. □

In the following, noticing from (2.7), (2.8), (2.10), and the Ricci identity (2.11) that $\xi(\mu) = 0$ if (3.3) holds identically, we shall consider the following two cases, depending on the function μ.

4.1 The Relation (3.3) Holds with μ = constant

In this situation, we can prove the proposition as follows:

Proposition 4.1 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ such that (3.3) holds everywhere with $\mu =$ constant. Then, $\mu = 0$ and the second fundamental form h is parallel.*

Proof First of all, we fix a point $x \in M^{n+1}$ and choose an orthonormal basis $\{e_a\}_{a=0}^n$ of $T_x M^{n+1}$ as in Lemma 2.2 such that

$$h(e_1, e_1) = \lambda_1 \varphi e_1, \ h(e_1, e_i) = \lambda_i \varphi e_i, \ \ 2 \leq i \leq n. \tag{4.10}$$

Then, we take a geodesic $\gamma(t)$ passing through x in the direction of e_1. Let $\{E_1, \ldots, E_n\}$ be parallel orthonormal vector fields along γ, satisfying $E_i(x) = e_i$ for $1 \le i \le n$ and $E_1 = \gamma'(t)$. Then, using (2.4) and (3.3), by definition, we have

$$\frac{\partial}{\partial t} g(h(E_1, E_1), \varphi E_i) = g((\bar{\nabla}_{E_1} h)(E_1, E_1), \varphi E_i) = 0, \quad 2 \le i \le n, \tag{4.11}$$

$$\frac{\partial}{\partial t} g(h(E_1, E_i), \varphi E_j) = g((\bar{\nabla}_{E_1} h)(E_1, E_i), \varphi E_j) = 0, \quad 2 \le i \ne j \le n, \tag{4.12}$$

and therefore, it holds that

$$g(h(E_1, E_1), \varphi E_i) = g(h(e_1, e_1), \varphi e_i) = 0, \quad 2 \le i \le n, \tag{4.13}$$

$$g(h(E_1, E_i), \varphi E_j) = g(h(e_1, e_i), \varphi e_j) = 0, \quad 2 \le i \ne j \le n. \tag{4.14}$$

Thus, there exist functions $\tilde{\lambda}_j$ for $1 \le j \le n$, defined along γ, such that

$$h(E_1, E_1) = \tilde{\lambda}_1 \varphi E_1, \quad h(E_1, E_i) = \tilde{\lambda}_i \varphi E_i, \quad 2 \le i \le n, \tag{4.15}$$

where $\tilde{\lambda}_j(x) = \lambda_j$ for $1 \le j \le n$. Consequently, we conclude that $\mathcal{A}E_j = \tilde{\lambda}_j E_j$ holds for the operator $\mathcal{A}(X) = A_{\varphi E_1}X - g(A_{\varphi E_1}X, \xi)\xi$ for $X \in TN^n$, defined in (2.15) pointwisely. Applying the fact $\mu =$ constant and following the proof of (4.2), along γ, we further have

$$0 = E_1(\mu) = (2\tilde{\lambda}_i - \tilde{\lambda}_1)(1 + \tilde{\lambda}_1\tilde{\lambda}_i - \tilde{\lambda}_i^2), \quad 2 \le i \le n. \tag{4.16}$$

With the help of (3.3) once more, we derive along γ that

$$\frac{\partial}{\partial t}\tilde{\lambda}_1(t) = \frac{\partial}{\partial t} g(h(E_1, E_1), \varphi E_1) = g((\bar{\nabla}_{E_1} h)(E_1, E_1), \varphi E_1) = 3\mu, \tag{4.17}$$

$$\frac{\partial}{\partial t}\tilde{\lambda}_1(t) = \frac{\partial}{\partial t} g(h(E_1, E_1), \varphi E_1) = g((\bar{\nabla}_{E_1} h)(E_1, E_1), \varphi E_1) = 3\mu,$$

$$\frac{\partial}{\partial t}\tilde{\lambda}_i(t) = \frac{\partial}{\partial t} g(h(E_1, E_i), \varphi E_i) = g((\bar{\nabla}_{E_1} h)(E_1, E_i), \varphi E_i) = \mu, \quad i \ge 2. \tag{4.18}$$

In this situation, we can take the derivative of (4.16) three times along $\gamma(t)$ to get

$$12\mu^3 = 0.$$

This combining with (3.3) immediately says that $\mu = 0$ and $h_{ij,\ell}^{k^*} = 0$ for $1 \le i, j, k, \ell \le n$. Therefore, by means of (2.10), we conclude that $\bar{\nabla} h = 0$ since $h_{ab,c}^{k^*} = 0$ for $0 \le a, b, c \le n$ and $1 \le k \le n$, that is, the second fundamental form h is parallel. □

4.2 The Relation (3.3) Holds with $\mu \ne$ constant

Now, we consider the $(n+1)$-dimensional anti-invariant submanifold M^{n+1} of $\mathbb{S}^{2n+1}(1)$ such that (3.3) holds with $\mu \ne$ constant. Since our result is local in nature, the condition $\mu \ne$ constant allows us to assume that $\{x \in M^{n+1} \mid X(\mu) = 0, \ \forall X \in$

$T_xM^{n+1}\}$ is not an open subset. For this reason, with the fact $\xi(\mu)=0$, we shall carry our discussion in the open dense subset:

$$\Omega=\{x\in M^{n+1}\mid \text{there exists } X\in T_xM^{n+1} \text{ such that } X(\mu)\neq 0\}. \tag{4.19}$$

Proposition 4.2 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ such that (3.3) holds everywhere with $\mu\neq$ constant. Then, there exists a smooth non-vanishing function κ such that the second fundamental form h satisfies the following property:*

$$\begin{aligned} h(E_1,E_1)=3\kappa\varphi E_1,\ h(E_1,E_i)=\kappa\varphi E_i,\\ h(E_i,E_j)=\kappa\delta_{ij}\varphi E_1,\ \ 2\le i,j\le n, \end{aligned} \tag{4.20}$$

where $\{\xi,E_1,\dots,E_n\}$ is an orthonormal frame of M^{n+1} with $\{E_1,\dots,E_n\}$ tangent to N^n and φE_1 is parallel to mean curvature vector field H. Moreover, (4.20) implies that

$$\begin{aligned} h(X,Y)=\tfrac{n+1}{n+2}\Big((g(X,Y)-\eta(X)\eta(Y))H+(g(\varphi X,H)-\tfrac{n+2}{n+1}\eta(X))\varphi Y\\ +(g(\varphi Y,H)-\tfrac{n+2}{n+1}\eta(Y))\varphi X\Big) \end{aligned} \tag{4.21}$$

for any tangent vector fields X,Y on M^{n+1}.

Proof As the key of proof, it should be necessary to figure out how the assumptions of Proposition 4.2 constrain the eigenvalues of $\mathcal{A}_x: T_xN^n\to T_xN^n$ defined in (2.15) for any point $x\in\Omega$. For this purpose, we first derive from Lemma 4.1 that

$$e_1(\mu)=(2\lambda_t-\lambda_1)(1+\lambda_1\lambda_t-\lambda_t^2),\ \ t=2,\dots,n, \tag{4.22}$$

which, with the fact $\mu\neq$ constant, says that

$$\lambda_1>0,\ \ \lambda_1-2\lambda_t>0,\ \ 1+\lambda_1\lambda_t-\lambda_t^2\neq 0,\ \ t=2,\dots,n. \tag{4.23}$$

Then, putting $y_t=\lambda_1-2\lambda_t>0$, we can rewrite (4.22) as

$$4e_1(\mu)-y_t(y_t^2-\lambda_1^2-4)=0,\ \ y_t>0,\ \ t=2,\dots,n. \tag{4.24}$$

Related to the solution y of (4.24), it is sufficient to consider the following three cases:

(1) If $-\lambda_1^2-4\ge 0$, then (4.24) implies that $e_1(\mu)>0$, and as an equation of y_t, (4.24) has only one positive solution, i.e., $y_2=\cdots=y_n$. This shows that $\lambda_2=\cdots=\lambda_n$;
(2) If $-\lambda_1^2-4<0$ and $e_1(\mu)>0$, similarly (4.24) has only one positive solution and thus $\lambda_2=\cdots=\lambda_n$;
(3) If $-\lambda_1^2-4<0$ and $e_1(\mu)<0$, then (4.24) has at most two positive solutions. In this case, at most two of $\{\lambda_2,\dots,\lambda_n\}$ are distinct.

It is known from (4.23) that $\lambda_1 > \lambda_t$ for all $t \geq 2$. Based on these above, we can conclude that the number of distinct eigenvalues of $\mathcal{A}_x$ can be at most 3. In particular, it is equal to 2 or 3. Consequently, the study of anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$ such that (3.3) holds everywhere with $\mu \neq$ constant can be divided into the following two cases:

Case I: $\lambda_2 = \cdots = \lambda_t \neq \lambda_{t+1} = \cdots = \lambda_n, \quad 2 \leq t \leq n-1$;

Case II: $\lambda_2 = \lambda_3 = \cdots = \lambda_n$.

Now, according to the cases above, we separate the remaining proof into three steps.

Step 1. *In both Case I and Case II, the second fundamental form h takes the form:*

$$\begin{aligned} h(e_1, e_1) = \lambda_1 \varphi e_1,\ h(e_1, e_i) = \lambda_i \varphi e_i, \\ h(e_i, e_j) = \lambda_i \delta_{ij} \varphi e_1, \quad 2 \leq i, j \leq n. \end{aligned} \tag{4.25}$$

If Case I occurs, with the fact $e_1(\mu) \neq 0$, it is easily seen from (4.2) and (4.4) that

$$g(h(e_k, e_k), \varphi e_k) = 0, \quad 2 \leq k \leq t. \tag{4.26}$$

By linearization, we have

$$g(h(e_i, e_j), \varphi e_k) = 0, \quad 2 \leq i, j, k \leq t. \tag{4.27}$$

Furthermore, applying (4.3) and (4.23), we deduce that

$$g(h(e_i, e_j), \varphi e_k) = 0, \quad 2 \leq i, j \leq t, \quad t+1 \leq k \leq n. \tag{4.28}$$

This combining with (4.27) and Lemma 2.2 implies that

$$h(e_i, e_j) = \lambda_i \delta_{ij} \varphi e_1, \quad 2 \leq i, j \leq t. \tag{4.29}$$

Similarly, it can be verified that

$$h(e_i, e_j) = \lambda_i \delta_{ij} \varphi e_1, \quad t+1 \leq i, j \leq n. \tag{4.30}$$

Direct calculations with (4.3) and (4.23) give

$$g(h(e_i, e_j), \varphi e_k) = 0, \quad 2 \leq i \leq t, \quad t+1 \leq j \leq n, \quad 1 \leq k \leq n. \tag{4.31}$$

Consequently, $h(e_i, e_j) = 0$ for $2 \leq i \neq j \leq n$. Hence, Step 1 has been proved for Case I.

If Case II occurs, using (4.2) and (4.4) again, we conclude by linearization that

$$g(h(e_i, e_j), \varphi e_k) = 0, \quad 2 \leq i, j, k \leq n. \tag{4.32}$$

Following the proof similar to that of Case I, we can prove Step 1 for Case II.

Step 2. *Case I does not occur.*

Suppose on the contrary that Case I does occur. By the fact $\xi(\mu) = 0$ and the relations

$$\begin{aligned} g(\mathrm{grad}\,\mu, e_1) &= e_1(\mu) \neq 0, \\ g(\mathrm{grad}\,\mu, e_k) &= e_k(\mu) = 0, \quad 2 \le k \le n, \end{aligned} \tag{4.33}$$

we get $e_1 = \pm\frac{\mathrm{grad}\,\mu}{\|\mathrm{grad}\,\mu\|}(x)$. Without loss of generality, we can assume that $e_1 = \frac{\mathrm{grad}\,\mu}{\|\mathrm{grad}\,\mu\|}(x)$. Thus, in a neighborhood Ω' around x, a unit vector field $E_1 = \frac{\mathrm{grad}\,\mu}{\|\mathrm{grad}\,\mu\|}$ can be defined, and according to the proof of (4.33), it then follows that for each $\tilde{x} \in \Omega'$, the function f should attain its absolute maximum over $U_{\tilde{x}}N^n$ exactly at $E_1(\tilde{x})$. By the continuity of eigenvalue functions of $\mathcal{A}(X) = A_{\varphi E_1}X - g(A_{\varphi E_1}X, \xi)\xi$ for $X \in TN^n$, defined pointwisely in (2.15), we further conclude that the multiplicity of each of its eigenvalue functions is constant. It follows from Lemma 1.2 of [32] that there is a smooth eigenvector extension of $\mathcal{A}$, from $\{e_1, e_2, \dots, e_n\}$ at x to $\{E_1(\tilde{x}), E_2(\tilde{x}), \dots, E_n(\tilde{x})\}$ at arbitrary point $\tilde{x}$ in the neighborhood Ω' of x, such that $\mathcal{A}E_i = \tilde{\lambda}_i E_i$ for $1 \le i \le n$, with the functions $\{\tilde{\lambda}_i\}_{i=1}^n$ satisfying $\tilde{\lambda}_1 > 2\tilde{\lambda}_j$ for $j \ge 2$ and

$$\tilde{\lambda}_2 = \cdots = \tilde{\lambda}_t < \tilde{\lambda}_{t+1} = \cdots = \tilde{\lambda}_n, \quad 2 \le t \le n-1. \tag{4.34}$$

It should be pointed out that, with respect to the local orthonormal frame $\{E_i\}_{i=1}^n$, the foregoing results involving the orthonormal basis $\{e_i = E_i(x)\}$ still remain valid, in view of which we will calculate by using them directly without further explanation.

Next, by definition, we have

$$\begin{aligned} (\bar{\nabla}_{E_i}h)(E_i, E_i) &= \nabla^{\perp}_{E_i}h(E_i, E_i) - 2h(\nabla_{E_i}E_i, E_i), \\ &= \nabla^{\perp}_{E_i}h(E_i, E_i) - 2g(\nabla_{E_i}E_i, E_1)h(E_1, E_i) \\ &\quad - 2\sum_{k=2}^{n} g(\nabla_{E_i}E_i, E_k)h(E_k, E_i) - 2g(\nabla_{E_i}E_i, \xi)h(\xi, E_i), \end{aligned} \tag{4.35}$$

which together with (2.4) and (4.25) yields that

$$\begin{aligned} (\bar{\nabla}_{E_i}h)(E_i, E_i) &= E_i(\tilde{\lambda}_i)\varphi E_1 + 3\tilde{\lambda}_i g(\nabla_{E_i}E_1, E_i)\varphi E_i \\ &\quad + \sum_{k\neq i} \tilde{\lambda}_i g(\nabla_{E_i}E_1, E_k)\varphi E_k, \end{aligned} \tag{4.36}$$

where $2 \le i \le n$. Similarly, we also obtain that

$$(\bar{\nabla}_{E_i}h)(E_1, E_1) = E_i(\tilde{\lambda}_1)\varphi E_1 + \sum_{k=2}^{n}(\tilde{\lambda}_1 - 2\tilde{\lambda}_k)g(\nabla_{E_i}E_1, E_k)\varphi E_k, \quad i \ge 2. \tag{4.37}$$

As the result of (3.3), (4.36), and (4.37), it holds that

$$3\mu = h^{i^*}_{ii,i} = 3\tilde\lambda_i g(\nabla_{E_i} E_1, E_i),\ \ 2 \le i \le n, \tag{4.38}$$

$$3\mu = 3h^{i^*}_{11,i} = 3(\tilde\lambda_1 - 2\tilde\lambda_i) g(\nabla_{E_i} E_1, E_i),\ \ 2 \le i \le n. \tag{4.39}$$

Since $\tilde\lambda_1 > 2\tilde\lambda_i$ for $2 \le i \le n$, comparing (4.38) and (4.39) immediately gives

$$\tilde\lambda_1 = 3\tilde\lambda_i,\ \ 2 \le i \le n. \tag{4.40}$$

This shows that $\tilde\lambda_2 = \cdots = \tilde\lambda_n$ which is a contradiction. Hence, Case I does not occur.
Step 3. *Completion of the proof of Proposition* 4.2.

According to Step 1 and Step 2, if (3.3) holds everywhere with $\mu \neq$ constant, then Case II occurs everywhere, and corresponding to the orthonormal basis $\{e_a\}_{a=0}^n$ of M^{n+1} as in Lemma 2.2, the second fundamental form h takes the form in (4.25). Note that $\xi(\mu) = 0$ and (4.33) still holds. Without loss of generality, we shall assume that $e_1 = \frac{\mathrm{grad}\,\mu}{\|\mathrm{grad}\,\mu\|}(x)$ and $E_1 = \frac{\mathrm{grad}\,\mu}{\|\mathrm{grad}\,\mu\|}$. Similar argument as in the proof of Step 2 states that, for an arbitrary point $\tilde x$ in a neighborhood Ω' of x, the function f should attain its absolute maximum over $U_{\tilde x} N^n$ exactly at $E_1(\tilde x)$. For this reason, the operator $\mathcal{A}$ admits two distinct eigenvalues with multiplicities 1 and $n-1$ at $\tilde x$, respectively, and thus, we can apply Lemma 1.2 of [32] again to obtain local orthonormal eigenvector fields of $\mathcal{A}$, extending from $\{e_1, \ldots, e_n\}$ at x to $\{E_1, \ldots, E_n\}$ around x, such that $\mathcal{A}E_i = \tilde\lambda_i E_i$ for $1 \le i \le n$, with the eigenvalue functions $\{\tilde\lambda_i\}_{i=1}^n$ satisfying that $\tilde\lambda_2 = \cdots = \tilde\lambda_n$. In particular, with respect to $\{E_i\}_{i=1}^n$ and $\{\tilde\lambda_i\}_{i=1}^n$, the foregoing equations from (4.36) up to (4.40) are still valid. Therefore, we have $\tilde\lambda_1 = 3\tilde\lambda_i := 3\kappa$ for $i \ge 2$. This completes the proof of Proposition 4.2. □

5 The Proofs of Theorems 1.1 and 1.2

Let $\bar\nabla$ and $\tilde\nabla$ denote the Levi–Civita connections of $\mathbb{S}^{2n+1}(1)$ and $\mathbb{C}P^n(4)$, respectively. Then, for the Riemannian submersion $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{C}P^n(4)$ as stated in Lemma 2.3, it follows from the well-known O'Neill equations (cf. [30]) that

$$\bar\nabla_{X^*} Y^* = (\tilde\nabla_X Y)^* + \tfrac{1}{2}\eta([X^*, Y^*])\xi, \tag{5.1}$$

where X, Y are vector fields on $\mathbb{C}P^n(4)$ and $\cdot^*$ is the horizontal lift of $\cdot$ with respect to η. As the argument in [2], we see that there exists a Lagrangian submanifold N^n of $\mathbb{C}P^n(4)$ such that the following diagram commutes:

$$\begin{array}{ccc} M^{n+1} & \longrightarrow & \mathbb{S}^{2n+1}(1) \\ \big\downarrow & & \big\downarrow \pi \\ N^n & \longrightarrow & \mathbb{C}P^n(4) \end{array} \tag{5.2}$$

where M^{n+1} is the set of fibers over N^n. Then, we shall prove the theorems as follows:

Theorem 5.1 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. Then M^{n+1} is of parallel second fundamental form if and only if M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Lagrangian submanifolds (a)-(e) of $\mathbb{C}P^n(4)$, as described in Example 1.1.*

Proof Consider that the vector fields X, Y in (5.1) are tangent to N^n. Then, it holds from Lemma 2.1 and the relation (5.2) that $\eta([X^*, Y^*]) = 0$, and denoting by $\hat{h}$ and $\hat{H}$ the second fundamental form and the mean curvature vector field along N^n of $\mathbb{C}P^n(4)$, we can make use of the Gauss formula to obtain that

$$h(X^*, Y^*) = (\hat{h}(X, Y))^*, \quad (n+1)H = n\hat{H}^*, \tag{5.3}$$

where h and H are, respectively, the second fundamental form and the mean curvature vector field along M^{n+1} of $\mathbb{S}^{2n+1}(1)$. As M^{n+1} is anti-invariant, by definition, we deduce from (2.1) that

$$\eta(\bar{\nabla}_{X^*}Y^*) = -g(\bar{\nabla}_{X^*}\xi, Y^*) = g(\varphi X^*, Y^*) = 0, \tag{5.4}$$

which implies that M^{n+1} is locally isometric to a Riemannian product of $\mathbb{R}$ and the Lagrangian submanifold N^n of $\mathbb{C}P^n(4)$.

Now, using Lemma 2.3, (5.1), and (5.3), we easily get (cf. Corollary 2 of [31])

$$g((\bar{\nabla}h)(U^*, X^*, Y^*), \varphi Z^*) = G((\tilde{\nabla}\hat{h})(U, X, Y), JZ) \tag{5.5}$$

for tangent vector fields U, X, Y, Z on N^n. This together with the Codazzi equation and (2.10) shows that h is parallel if and only if $\hat{h}$ is parallel. By means of the classification theorem of Dillen-Li-Vrancken-Wang [11], we complete the proof of Theorem 5.1. □

Theorem 5.2 *Let M^{n+1} be an $(n+1)$-dimensional anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$. Then, (4.21) holds identically if and only if either M^{n+1} is a totally contact geodesic submanifold, or it is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Whitney spheres of $\mathbb{C}P^n(4)$, as described in Example 1.2.*

Proof Similar argument as in the proof of Theorem 5.1 says that (4.21) holds on M^{n+1} if and only if the second fundamental form $\hat{h}$ of the Lagrangian submanifold N^n of $\mathbb{C}P^n(4)$ satisfies the equation (4.8) of Theorem 3 in [3] (cf. also (2) of Theorem 1.1 in [15]), i.e.,

$$\hat{h}(X, Y) = \tfrac{n}{n+2}\big(G(X, Y)\hat{H} + G(JX, \hat{H})JY + G(JY, \hat{H})JX\big), \tag{5.6}$$

where X, Y are vector fields tangent to N^n of $\mathbb{C}P^n(4)$, and it then follows from Theorem A of [7] (cf. also Theorem 1.1 of [5] or Section 4 of [23]) that either N^n is totally geodesic, or it is a portion of one of the Whitney spheres in Example 1.2. Given that N^n is totally geodesic, $h(X^*, Y^*) = 0$ for any tangent vector fields X^*, Y^* on M^{n+1} orthogonal to ξ. Hence, M^{n+1} is totally contact geodesic. This completes the proof of Theorem 5.2. □

5.1 Completion of the Proof of Theorem 1.1

In order to determine those submanifolds satisfying (3.3) at every point, by means of the function μ, it is sufficient to just consider two cases: $\mu=$ constant, or $\mu\neq$ constant. In the former case, we obtain from Proposition 4.1 that the second fundamental form h of $M^{n+1}\to\mathbb{S}^{2n+1}(1)$ is parallel, and thus, M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of those Lagrangian submanifolds of $\mathbb{C}P^n(4)$ in Example 1.1, according to Theorem 5.1. As for the latter case, applying both Proposition 4.2 and Theorem 5.2, we conclude that M^{n+1} is locally isometric to the Riemannian product of $\mathbb{R}$ and a portion of one of the Whitney spheres of $\mathbb{C}P^n(4)$, as described in Example 1.2, provided that M^{n+1} is not a totally contact geodesic submanifold. Hence, we have completed the proof of Theorem 1.1. □

5.2 Completion of the Proof of Theorem 1.2

Choosing the local orthonormal frame $\{e_0=\xi, e_1,\dots,e_n\}$ on M^{n+1} as in Sect. 2, by definition we can apply (2.6) to obtain that

$$(n+1)^2\|H\|^2=\sum_k\Big(\sum_i(h^{k^*}_{ii})^2+2\sum_{i<j}h^{k^*}_{ii}h^{k^*}_{jj}\Big), \tag{5.7}$$

$$S=\|h\|^2=2n+\sum_{i,j,k}(h^k_{ij})^2,\ \ 1\le i,j,k\le n, \tag{5.8}$$

and it then follows from (2.14) that

$$R=n(n-1)+\sum_k\Big(\sum_i(h^{k^*}_{ii})^2+2\sum_{i<j}h^{k^*}_{ii}h^{k^*}_{jj}\Big)-\sum_{i,j,k}(h^k_{ij})^2. \tag{5.9}$$

As the same argument of (5.7) in [5], using (5.7) and (5.9), we immediately have

$$\begin{aligned}&(n+1)^2\|H\|^2-m(R-n(n-1))\\&=6m\sum_{k<i<j}(h^{k^*}_{ij})^2+(m-1)\sum_{k\neq i,j}\sum_{i<j}(h^{k^*}_{ii}-h^{k^*}_{jj})^2\\&\quad+\tfrac{1}{n-1}\sum_{i\neq j}\big(h^{i^*}_{ii}-(m-1)(n-1)h^{i^*}_{jj}\big)^2\ge 0,\end{aligned} \tag{5.10}$$

where $m=(n+2)/(n-1)$. From this, we can establish the inequality in (1.3), and the equality holds if and only if $h^{i^*}_{ii}=3\,h^{i^*}_{jj}$ and $h^{k^*}_{ij}=0$ for distinct i,j,k. Furthermore, e_1 can be chosen to satisfy that φe_1 is parallel to the mean curvature vector field H, and thus, $h^{k^*}_{ii}=0$ for all $1\le i\le n$ and $k\ge 2$. In this situation, if M^{n+1} is not totally contact geodesic, the second fundamental form h takes the form as in (4.20). Applying Proposition 4.2 and Theorem 5.2, we finally complete the proof of Theorem 1.2. □

6 The Proof of Theorem 1.3

Throughout this section, we shall assume that M^{n+1} is an $(n+1)$-dimensional minimal anti-invariant submanifold of the unit sphere $\mathbb{S}^{2n+1}(1)$ with η-Einstein induced metric.

First of all, from the components of the Weyl curvature tensor W of M^{n+1} for $n \geq 2$

$$\begin{aligned} W_{abcd} = R_{abcd} &- \tfrac{1}{n-1}(\delta_{ac}R_{bd} + \delta_{bd}R_{ac} - \delta_{ad}R_{bc} - \delta_{bc}R_{ad}) \\ &+ \tfrac{R}{n(n-1)}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \end{aligned} \tag{6.1}$$

we derive the expression as follows (cf. [14]):

$$\|\text{Rie}\|^2 = \|W\|^2 + \tfrac{4}{n-1}\|\text{Ric}\|^2 - \tfrac{2R^2}{n(n-1)}, \tag{6.2}$$

where $\|W\|^2 = \sum_{a,b,c,d}(W_{abcd})^2$. Then, with the help of (6.2) and (3.11), we easily get

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla}h\|^2 - \|W\|^2 - \tfrac{n+3}{n-1}\|\text{Ric}\|^2 + \tfrac{2}{n(n-1)}R^2 + (n+1)R. \tag{6.3}$$

Furthermore, a trace-free tensor $\tilde{\text{Ric}}$ of (0, 2)-type can be defined to satisfy

$$\tilde{R}_{ab} = R_{ab} - \tfrac{1}{n}R(\delta_{ab} - \eta_a\eta_b). \tag{6.4}$$

It follows that

$$\|\text{Ric}\|^2 = \|\tilde{\text{Ric}}\|^2 + \tfrac{1}{n}R^2, \tag{6.5}$$

where $\|\tilde{\text{Ric}}\|^2 = \sum_{a,b}(\tilde{R}_{ab})^2$. Substituting (6.5) into (6.3) yields that

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla}h\|^2 - \|W\|^2 - \tfrac{n+3}{n-1}\|\tilde{\text{Ric}}\|^2 - \tfrac{n+1}{n(n-1)}R^2 + (n+1)R, \tag{6.6}$$

which together with (2.14) yields that

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla}h\|^2 - \|W\|^2 - \tfrac{n+3}{n-1}\|\tilde{\text{Ric}}\|^2 + \tfrac{n+1}{n(n-1)}SR - \tfrac{2(n+1)}{n-1}R. \tag{6.7}$$

Secondly, based on the proof of Theorem 5.1, it is known that M^{n+1} is locally isometric to a Riemannian product of $\mathbb{R}$ and one Lagrangian submanifold N^n of $\mathbb{C}P^n(4)$. In this situation, we shall prove the following property relative to the anti-invariant submanifold M^{n+1} of $\mathbb{S}^{2n+1}(1)$ with η-Einstein induced metric.

Claim. M^{n+1} is *η-Einstein if and only if N^n is Einstein.*

Let $\hat{\nabla}$ be the Levi–Civita connection of N^n and then it holds from (5.1) and (5.3) that

$$\nabla_{X^*}Y^* = (\hat{\nabla}_X Y)^*, \tag{6.8}$$

where we used the fact $\eta([X^*, Y^*]) = 0$ for any tangent vector fields X, Y on N^n. As the result of Lemma 2.3 and (6.8), we see that

$$g(R(U^*, X^*)Y^*, Z^*) = G(\hat{R}(U, X)Y, Z) \tag{6.9}$$

for tangent vector fields U, X, Y, Z on N^n and the curvature tensor $\hat{R}$ of N^n given by

$$\hat{R}(U, X)Y = \hat{\nabla}_U \hat{\nabla}_X Y - \hat{\nabla}_X \hat{\nabla}_U Y - \hat{\nabla}_{[U,X]} Y \tag{6.10}$$

Thus, direct calculations by contraction of (6.9) imply that

$$\mathrm{Ric}(X^*, Y^*) = \hat{\mathrm{Ric}}(X, Y), \tag{6.11}$$

where $\hat{\mathrm{Ric}}$ denotes the Ricci tensor of N^n. Indeed, this states that M^{n+1} is η-Einstein if and only if N^n is Einstein. Hence, the Claim has been proved.

Next, noting from Lemma 2.1 that M^{n+1} is η-Einstein if and only if $\tilde{\mathrm{Ric}} = 0$ identically, we can rewrite (6.7) as

$$\frac{1}{2}\Delta S = \|\bar{\nabla} h\|^2 - \|W\|^2 + \frac{n+1}{n(n-1)} SR - \frac{2(n+1)}{n-1} R. \tag{6.12}$$

As the Claim says that the scalar curvature R of M^{n+1} is constant, it can be verified by combining with the minimality of M^{n+1} of $\mathbb{S}^{2n+1}(1)$, (2.14), and (6.12) that

$$0 \geq -\|W\|^2 + \frac{n+1}{n(n-1)} SR - \frac{2(n+1)}{n-1} R. \tag{6.13}$$

Consequently, we have the inequality in (1.4). Furthermore, according to (5.3), Theorem 5.1 and the Claim above, the inequality in (1.4) becomes an equality on M^{n+1} if and only if N^n is one of the minimal Lagrangian submanifolds of $\mathbb{C}P^n(4)$ with Einstein induced metrics and parallel second fundamental form. Finally, we complete the proof by applying the classification theorem of Dillen et al. [11] (cf. also Theorem 1.2 of Cheng et al. [8]). □

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