



Global Existence and Full Convergence of the Möbius-Invariant Willmore Flow in the 3-Sphere

Ruben Jakob¹

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Abstract

In this article, we prove two global existence and full convergence theorems for flow lines of the Möbius-invariant Willmore flow, and we use the latter result in order to prove that fully and smoothly convergent flow lines of the Möbius-invariant Willmore flow are stable with respect to small perturbations of their initial immersions in any $C^{4,\gamma}$ -norm, provided they converge to a umbilic-free C^4 -local minimizer of the Willmore functional among C^4 -immersions of a smooth compact torus into either \mathbf{R}^3 or \mathbf{S}^3 . The proofs of our two main theorems rely on the author's recent achievements about the Möbius-invariant Willmore flow, on Weiner's investigation of the stability of the Clifford torus with respect to the Willmore functional, and on Escher's, Mayer's, and Simonett's work from the 1990s on invariant center manifolds for uniformly parabolic quasilinear evolution equations and their special applications to the Willmore- and surface diffusion flow near round 2-spheres in \mathbf{R}^3 .

Keywords Willmore flow · Conformal invariance · Clifford torus · Center manifolds

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1 Introduction and Main Results

The central mathematical object of this article is the *Willmore functional*

$$\mathcal{W}(f) := \int_{\Sigma} K_f^M + \frac{1}{4} |\mathbf{H}_f|^2 d\mu_f, \quad (1)$$

which can be defined for C^2 -immersions $f : \Sigma \rightarrow M$, mapping any closed smooth Riemannian orientable surface Σ into an arbitrary smooth Riemannian manifold M ,

✉ Ruben Jakob
rubenj@technion.ac.il

¹ Mathematics Department, Israel Institute of Technology, 3200003Haifa, Israel

where the function $x \mapsto K_f^M(x)$ in (1) denotes the sectional curvature of M with respect to the “immersed tangent plane” $Df_x(T_x \Sigma)$ in $T_{f(x)}M$ and \mathbf{H}_f the mean curvature vector along the immersion f ; see here Definition (6) below. Regarding the aims of this article, we will only have to consider the simple cases $M = \mathbf{R}^n$ or $M = \mathbf{S}^n$ —especially for $n = 3$ —in which there simply holds $K_f^M \equiv 0$ or $K_f^M \equiv 1$, respectively. In this paper, the author aims to continue his investigation of the Möbius-invariant Willmore flow (MIWF) in [17] and [19], which is the evolution equation

$$\partial_t f_t = -\frac{1}{2} \frac{1}{|A_{f_t}^0|^4} \left(\Delta_{f_t}^\perp \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t}) \right) \equiv -\frac{1}{|A_{f_t}^0|^4} \nabla_{L^2} \mathcal{W}(f_t), \tag{2}$$

being well defined for differentiable families of sufficiently smooth and umbilic-free immersions f_t of some arbitrarily fixed smooth compact torus Σ into \mathbf{R}^n or \mathbf{S}^n . We recall here from [17] and [19] that the MIWF (2) exists uniquely at least for a very short time, if it starts moving in a sufficiently smooth and umbilic-free surface of genus 1, and that the MIWF can only decrease the Willmore energy, although it is not the L^2 -gradient flow (3) of \mathcal{W} . Here, one should also recall from [17], that the existence of a smooth immersion f of a compact surface Σ into \mathbf{R}^n satisfying $\min_{x \in \Sigma} |A_f^0|^2(x) > 0$ yields a non-vanishing section of the bundle $T^*\Sigma \otimes T^*\Sigma$, implying that the Euler-characteristic of Σ vanishes and thus that Σ can only be a compact torus. This might appear to be a tough restriction, regarding differential-geometric applications of the MIWF. On the other hand, the MIWF has the remarkable property, to be conformally invariant, which means that any family of smooth immersions $\{f_t\}_{t \in [0, T]}$ of a fixed smooth torus Σ into \mathbf{R}^n solves equation (2) classically on $\Sigma \times [0, T]$, if and only if for any Möbius-transformation $M \in \text{Möb}(\mathbf{R}^n)$, for which the transformed family $\{M(f_t)\}_{t \in [0, T]}$ is well defined on $\Sigma \times [0, T]$, $\{M(f_t)\}_{t \in [0, T]}$ solves equation (2) on $\Sigma \times [0, T]$ again; see here also Corollary 1 in [17]. This property of the MIWF is particularly desirable because of the well-known invariance of the Willmore functional $\mathcal{W}(f)$ in (1) with respect to composition $f \mapsto M \circ f$ with any applicable $M \in \text{Möb}(\mathbf{R}^n)$ and it starkly contrasts the behavior of the classical Willmore flow

$$\partial_t f_t = -\frac{1}{2} \left(\Delta_{f_t}^\perp \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t}) \right) \equiv -\nabla_{L^2} \mathcal{W}(f_t) \tag{3}$$

with respect to conformal transformations of the ambient space \mathbf{R}^n . The classical Willmore flow is well defined for smooth immersions f_t of any compact, orientable surface Σ into \mathbf{R}^n , but it has only been particularly well studied in the simplest, e.g., *spherical* case: $\Sigma \cong \mathbf{S}^2$. In Simonett’s famous Theorem 1.2 of his paper [44], he proved exponential attractivity of round 2-spheres within sufficiently small $h^{2+\beta}$ -neighborhoods for the classical Willmore flow (3), and nearly at the same time Kuwert and Schätzle started to publish their series of seminal papers [22]– [24], in which they estimated the life span of general flow lines of (3), characterized singular flow lines in terms of *curvature concentration*, and finally proved global existence and smooth convergence of any flow line $\{f_t\}$ of (3), which starts moving in a spherical

immersion $f_0 : \mathbf{S}^2 \rightarrow \mathbf{R}^3$ with energy $\mathcal{W}(f_0) \leq 8\pi$.¹ We actually know on account of [32], Theorem 1.4, and [30], Theorem A, that there is only a finite number of critical values of the Willmore functional—considered as a map from the set of smooth, closed and orientable surfaces in \mathbf{R}^3 into \mathbf{R}_+ —within the interval $(2\pi^2, 8\pi - \delta)$, for any fixed small $\delta > 0$. But still any sort of global existence and smooth convergence result that *only requires an energy condition on the initial immersion* $f_0 : \Sigma \rightarrow \mathbf{R}^3$ seems to be out of reach in the general case of $\text{genus}(\Sigma) > 0$, both for the classical Willmore flow and also for the MIWF in \mathbf{R}^3 . The deeper reason for this lack of concrete knowledge lies in the fact that a sequence of immersions $f_j : \Sigma \rightarrow \mathbf{R}^n$ of fixed positive genus might in general *degenerate in moduli space*,² which means that the conformal classes of the pullback metrics $f_j^*(g_{\text{euc}})$ might not stay in any compact subset of the moduli space $\mathcal{M}(\text{genus}(\Sigma))$. And even if such a bad behavior can be somehow ruled out—for example by means of an appropriate Willmore energy bound³—then still the conformal factors u_j , appearing in Poincaré’s identity $f_j^*(g_{\text{euc}}) = e^{2u_j} g_{\text{Poin},j}$, might be unbounded in $L^\infty(\Sigma)$, at least if we cannot correct the immersions f_j in terms of appropriate Möbius-transformations, which is exactly the statement of Theorem 4.2 in [25]. Obviously a flow line $\{f_t\}_{t \in [0, T_{\text{Max}}]}$ of the classical Willmore flow cannot be corrected by general Möbius-transformations, and therefore, the conformal factors u_t of the immersions $\{f_t\}$ indeed might be unbounded in $L^\infty(\Sigma)$, and in this case, the genera of the images $f_t(\Sigma)$ would have to finally drop in the limit, as $t \nearrow T_{\text{Max}}$, on account of Proposition 2.4 in [37], even if the initial energy was supposed to be smaller than 8π or any other reasonable threshold.

Now, in spite of the conformal invariance of the MIWF, we cannot simply apply Theorem 4.2 in [25] to sequences $\{f_{t_j}\}$ along a general flow line $\{f_t\}$ of the MIWF neither. However, the flow lines of the MIWF can be conformally mapped—by means of stereographic projection—from \mathbf{R}^n into \mathbf{S}^n . Of special interest is here the 3-sphere \mathbf{S}^3 , because it contains the Clifford torus $\mathcal{C} := \frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1)$, an embedded closed minimal surface in \mathbf{S}^3 with constant intrinsic Gaussian curvature -1 —thus in particular an embedded closed Willmore surface—which can be isometrically mapped onto a very simple trapezoid in the complex plane. Hence, especially the spectrum and the dimension of each eigenspace of the Beltrami-Laplacian $\Delta_{\mathcal{C}}$ can be computed precisely, and in addition its most basic eigenspaces $\text{Eig}_{-2}(\Delta_{\mathcal{C}})$ and $\text{Eig}_{-4}(\Delta_{\mathcal{C}})$ can be explicitly described in terms of conformal vector fields on \mathbf{S}^3 restricted to \mathcal{C} ⁴—a striking insight going back to Simons’ [45] seminal investigation of closed minimal submanifolds of n -spheres—which will play a very important role in the proof of the first main theorem of this paper, Theorem 1.

At this point, we should also mention that there are actually concrete examples of singular flow lines of the classical Willmore flow moving *rotationally symmetric immersions* into \mathbf{R}^3 whose initial Willmore energies are only slightly bigger than 8π ; see [7] for the important genus-0-case and [9] for the genus-1-case, respectively.

¹ See Definition 2 (b), (d) and (e) in [19], where the terms “flow line”, “life span” and “singular time” have been introduced.

² See here Sect. 5.1 in [36] for a very nice exposition.

³ See here Theorem 5.1 in [25].

⁴ Compare here with our Lemma 3 and also with the proof of our Lemma 4 below.

Unfortunately, such a *concrete divergence result* is still missing for the MIWF, although the existence of singular flow lines of the MIWF should be expected in view of its singular evolution equation (2).

Moreover, we mention here, that the author has proved in the third part of Theorem 1.2 in [18], that—up to smooth reparametrizations—the unique flow lines of the classical Willmore flow (3) in \mathbf{S}^3 converge smoothly and fully to immersions F^* , which parametrize conformally transformed Clifford tori in \mathbf{S}^3 , provided those flow lines start moving in a smooth parametrization $F_0 : \Sigma \rightarrow \mathbf{S}^3$ of a Hopf-torus with initial Willmore energy $\mathcal{W}(F_0) \leq \frac{8\pi^2}{\sqrt{2}}$.⁵ Interestingly, such a result does not seem to hold for flow lines of the MIWF (2), meeting the same start conditions, since even this narrow class of flow lines of the MIWF might develop curvature singularities, even under the condition that their initial Willmore energies are smaller than the prominent threshold 8π . One should see here Theorem 1.3 in [20] for a precise criterion for *full convergence* of global flow lines of the MIWF which start moving in smooth Hopf-tori in \mathbf{S}^3 with Willmore energies smaller than 8π . Actually, the proof of Theorem 1.3 in [20] strongly relies on the result of our Theorem 1 below.

Now in this article, we will show the first two *global existence and convergence results* for the MIWF and derive a *stability result* for fully convergent flow lines of the MIWF to local minimizers of the Willmore functional in \mathbf{S}^3 , respectively \mathbf{R}^3 . Firstly, in Theorem 1 below, we will prove an analog—here for the MIWF near the Clifford torus—of Simonett’s Theorem 1.2 in [44], which we had already mentioned above. Our proof of Theorem 1 relies on a combination of particular computations due to Weiner [48] and Simons [45] with Escher’s, Mayer’s, and Simonett’s technique in [13], [14], [42], [43], and [44] of *invariant center manifolds* for uniformly parabolic quasilinear evolution equations and their special application to the Willmore flow near round 2-spheres in \mathbf{R}^3 .

Theorem 1 (Full Convergence Theorem I) *Let Σ be a smooth compact torus, and let $F^* : \Sigma \xrightarrow{\cong} M(\frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1))$ be a smooth diffeomorphic parametrization of a compact torus in \mathbf{S}^3 , which is conformally equivalent to the standard Clifford torus $\frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1)$ via some conformal transformation $M \in \text{Möb}(\mathbf{S}^3)$, and let some $\beta \in (0, 1)$ and $k \in \mathbf{N}$ be fixed. Then, there is some small neighborhood $W = W(\Sigma, F^*, k)$ about F^* in $h^{2+\beta}(\Sigma, \mathbf{R}^4)$, such that for every C^∞ -smooth initial immersion $F_1 : \Sigma \rightarrow \mathbf{S}^3$, which is contained in W , the unique flow line $\{\mathcal{P}(t, 0, F_1)\}_{t \geq 0}$ of the MIWF exists globally and converges—up to smooth reparametrization—fully to a smooth and diffeomorphic parametrization of a torus in \mathbf{S}^3 , which is again conformally equivalent to the Clifford torus $\frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1)$. This full convergence takes place with respect to the $C^k(\Sigma, \mathbf{R}^4)$ -norm and at an exponential rate. \square*

In Theorem 1, the symbol “ $h^{2+\beta}(\Sigma, \mathbf{R})$ ” denotes the “little Hölder space,” modeled on Σ , of differentiation order $2 + \beta$, see here e.g., [13], p. 1419, or [40], p. 219,

⁵ Here and in the sequel, “full convergence of a flow line” $\{f_t\}_{t \in [0, \infty)}$ of either evolution equation (2) or equation (3) in a Banach space $(X, \|\cdot\|_X)$ means that $\|f_t - f^*\|_X \rightarrow 0$ for a unique limit $f^* \in X$, as $t \rightarrow \infty$, in contrast to “subconvergence” of $\{f_t\}_{t \in [0, \infty)}$ with respect to $\|\cdot\|_X$, which means that only for certain sequences $t_j \nearrow \infty$ the immersions f_{t_j} converge to certain limits in X , that might in general depend on those special sequences $\{t_j\}$.

for a precise definition. It should also be noted here, that the proof of Theorem 1 does not require any type of *Lojasiewicz-Simon gradient inequality* for the Willmore functional.

However, we are going to prove the following *full convergence theorem* by means of the well-known trick using the *Lojasiewicz-Simon gradient inequality* for a certain real-analytic functional \mathcal{F} —see e.g., [8] or [41]—in order to obtain simultaneously global existence of a flow line of the corresponding L^2 -gradient flow and also its full C^k -convergence.

Theorem 2 (Full Convergence Theorem II) *Let Σ be a smooth compact torus, $k \in \mathbf{N}$, with $k \geq 4$, and $\alpha \in (0, 1)$ be given, and let $F^* : \Sigma \rightarrow \mathbf{S}^3$ be a umbilic-free and C^∞ -smooth Willmore immersion, which locally minimizes the Willmore functional in the $C^k(\Sigma, \mathbf{R}^4)$ -norm, in the sense that there exists some $\delta > 0$, such that for any immersion $f : \Sigma \rightarrow \mathbf{S}^3$ with $\|f - F^*\|_{C^k(\Sigma, \mathbf{R}^4)} < \delta$ there holds*

$$\mathcal{W}(f) \geq \mathcal{W}(F^*). \quad (4)$$

Then there exists some $\varepsilon = \varepsilon(\Sigma, F^, k, \alpha) \in (0, \delta)$, such that for any C^∞ -smooth immersion $f_0 : \Sigma \rightarrow \mathbf{S}^3$ with $\|f_0 - F^*\|_{C^{k,\alpha}(\Sigma, \mathbf{R}^4)} < \varepsilon$ the corresponding flow line $\{\mathcal{P}(t, 0, f_0)\}_{t \geq 0}$ of the MIWF exists globally and converges fully with respect to the $C^k(\Sigma, \mathbf{R}^4)$ -norm—up to smooth reparametrization—to a umbilic-free Willmore immersion F_∞ , as $t \rightarrow \infty$, and this limit immersion is a C^k -local minimizer of the Willmore energy as well, satisfying $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$. \square*

Hence, the MIWF can be used in order to detect C^k -local minimizers of the Willmore energy in both \mathbf{S}^3 and \mathbf{R}^3 . One can quickly check that Theorem 2 is in exact analogy with Theorem 1.2 in [8] for the classical Willmore flow (3).

Remark 1 (1) Employing the conformal invariance of the Willmore functional, of its L^2 -gradient and of the MIWF-equation (2) it does not matter, if we state Theorems 1 and 2 for the MIWF in \mathbf{S}^3 or in \mathbf{R}^3 , and this flexibility will turn out crucial, at least for the proof of Theorem 2 in its present form above. As we will see below, Theorem 1 cannot be proven directly for the MIWF in \mathbf{R}^3 , whereas vice versa Theorem 2 should not be proven directly for the MIWF in \mathbf{S}^3 , as it is actually stated above. We will therefore prove Theorem 2 for the MIWF in \mathbf{R}^3 and then project the entire result back into \mathbf{S}^3 by means of inverse stereographic projection S^{-1} . See here also the comments at the beginning of the proof of Theorem 2.

- (2) We should also mention here, that the limit Willmore immersion F_∞ in Theorem 2—and also in Theorem 4 below—is actually C^∞ -smooth, and not only of class $C^k(\Sigma, \mathbf{R}^4)$, respectively $C^4(\Sigma, \mathbf{R}^4)$ —according to the types of convergence obtained in these two theorems. In order to see this, we replace F_∞ by its composition $S \circ F_\infty$ with stereographic projection into \mathbf{R}^3 , and we use Lemma 4.1, respectively, Theorem 4.3 in [36], in order to construct a conformal structure c on Σ by means of only finitely many *isothermal* charts $\{\psi_j : B_1^2(0) \xrightarrow{\cong} \Omega_j\}_{j=1, \dots, N}$ with respect to F_∞ . Hence, there is actually a conformal atlas $\{\psi_j\}_{j=1, \dots, N}$ of Σ with the additional property that each composition $S \circ F_\infty \circ \psi_j : B_1^2(0) \rightarrow \mathbf{R}^3$ is

conformal, which means that the constructed conformal structure respects the conformal class of $(S \circ F_\infty)^* g_{\text{euc}}$. Now only using the fact that the immersion F_∞ is *classically Willmore* and at least of class C^k with $k \geq 4$, we can write down locally in the charts ψ_j both the Euler–Lagrange equation $\Delta_{F_\infty}^\perp \mathbf{H}_{F_\infty} + Q(A_{F_\infty}^0)(\mathbf{F}_\infty) = 0$ of the Willmore functional for F_∞ and also the general identity $\mathbf{H}_{F_\infty} = \Delta_{F_\infty}(F_\infty)$ for the mean curvature vector of F_∞ , in order to infer from ordinary Schauder regularity theory that F_∞ is actually of class $C^{l+\gamma}$, for each $l \geq k$ and any fixed $\gamma \in (0, 1)$. This result already proves our claim above, because every complex structure on Σ automatically determines a smooth structure on Σ , and moreover, there is only one smooth structure on Σ on account of the well-known classification of compact, orientable, smooth 2-manifolds. Lemma 4.1 and Theorem 4.3 of Rivière’s lecture notes [36] originate from Hélein’s, Müller’s, and Sverák’s seminal work in [16] and [31] combining *the method of moving frames* and ideas from *Gauge Theory* with the phenomenon of *integrability by compensation*; see here Sects. 3.1–3.4, Theorem 5.1.1 and also Sect. 5.4 in [16] and Sects. 3 and 4 in [31], and for further technical improvements also Sect. 6 of [25] and Appendix A of [37]. □

2 Preparations for the Proofs of Theorems 1 and 2

Definition 1 Let Σ be a smooth compact torus, and let $M = \mathbf{R}^3$ or $M = \mathbf{S}^3$. We denote by $C_{\text{imm}}^\infty(\Sigma, M)$ the set of C^∞ -smooth immersions $F : \Sigma \rightarrow M$ of the torus Σ into M . □

Now, given any such immersion f of Σ into M , we endow the torus Σ with the pullback f^*g_{euc} of the Euclidean metric of either \mathbf{R}^3 or \mathbf{R}^4 , i.e., with coefficients $g_{ij} := \langle \partial_i f, \partial_j f \rangle$, and we let $(A_f)_{\mathbf{R}^3}$ and $(A_f)_{\mathbf{S}^3}$ denote the second fundamental form of the immersion f , either mapping into \mathbf{R}^3 or into \mathbf{S}^3 , defined on pairs of tangent vector fields X, Y on Σ by:

$$\begin{aligned} (A_f)_{\mathbf{R}^3}(X, Y) &:= D_X(D_Y(f)) - P^{\text{Tan}(f), \mathbf{R}^3}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp_{f, \mathbf{R}^3}} \\ (A_f)_{\mathbf{S}^3}(X, Y) &:= D_X(D_Y(f)) - P^{\text{Tan}(f), \mathbf{S}^3}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp_{f, \mathbf{S}^3}} \end{aligned}$$

where $D_X(V)|_x$ denotes the projection of the classical derivative of a vector field $V : \Sigma \rightarrow \mathbf{R}^3$, respectively, $V : \Sigma \rightarrow \mathbf{R}^4$ in direction of the tangent vector field $X \in \Gamma(T\Sigma)$ onto the respective fiber $T_{f(x)}\mathbf{R}^3 = \mathbf{R}^3$ of $T\mathbf{R}^3$, respectively, $T_{f(x)}\mathbf{S}^3$ of $T\mathbf{S}^3$, and where

$$\begin{aligned} P^{\text{Tan}(f), \mathbf{R}^3} &: \bigcup_{x \in \Sigma} \{x\} \times \mathbf{R}^3 \longrightarrow \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}(f(\Sigma)) =: \text{Tan}(f) \\ P^{\text{Tan}(f), \mathbf{S}^3} &: \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}\mathbf{S}^3 \longrightarrow \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}(f(\Sigma)) = \text{Tan}(f) \end{aligned} \tag{5}$$

denote the bundle morphisms which project the entire tangent spaces \mathbf{R}^3 , respectively, $T_{f(x)}\mathbf{S}^3$ orthogonally into their subspaces $T_{f(x)}(f(\Sigma))$ —the tangent spaces of the

immersion f in the points $f(x)$ for every $x \in \Sigma$. Furthermore, $(A_f^0)_{\mathbf{R}^3}$ and $(A_f^0)_{\mathbf{S}^3}$ denote the tracefree parts of $(A_f)_{\mathbf{R}^3}$ and $(A_f)_{\mathbf{S}^3}$, respectively, i.e.,

$$(A_f^0)_{\mathbf{S}^3}(X, Y) := (A_f)_{\mathbf{S}^3}(X, Y) - \frac{1}{2} g_f(X, Y) \mathbf{H}_{f, \mathbf{S}^3}$$

with

$$\mathbf{H}_{f, \mathbf{S}^3} := \text{trace}((A_f)_{\mathbf{S}^3}) \equiv (A_f)_{\mathbf{S}^3}(e_i, e_i) \tag{6}$$

(“Einstein’s summation convention”) denotes the mean curvature vector of the immersion $f : \Sigma \rightarrow \mathbf{S}^3$, where $\{e_i\}$ denotes a local orthonormal frame along the tangent bundle $T\Sigma$. Finally, in both settings $Q(A_f)$, respectively, $Q(A_f^0)$ operate on vector fields ϕ which are sections of the normal bundle of f , i.e., which are normal along f , by assigning

$$Q(A_f)(\phi) := A_f(e_i, e_j)\langle A_f(e_i, e_j), \phi \rangle,$$

which is by definition again a section of the normal bundle of f . Moreover, in equation (2), we consider the *normal Beltrami–Laplace operator* Δ_f^\perp for an arbitrary C^2 -immersion $f : \Sigma \rightarrow \mathbf{S}^3$. As introduced in Sect. 1 of [45] or also in Sect. 1 of [48], this is a differential operator of 2nd order acting on those sections of the pullback-bundle $f^*(T\mathbf{S}^3)$, which are normal along f within $T\mathbf{S}^3$, and again outputting such sections, i.e., sections of the normal subbundle Nf of $f^*(T\mathbf{S}^3)$. The operator Δ_f^\perp is constructed by means of the composition of the unique Riemannian connection $\nabla^{\mathbf{S}^3}$ on $f^*(T\mathbf{S}^3)$ with pointwise orthogonal projection of each fiber of the pullback bundle $f^*(T\mathbf{S}^3)$ into the corresponding fiber of its normal subbundle Nf . Alternatively, this notion can be defined via coordinate patches on Σ , as for example in Definition 3.1 of [18].

In order to transfer the method in [13], [14], [43], and [44] to the MIWF (2) for families of immersions $f_t : \Sigma \rightarrow \mathbf{S}^3$, we have to establish “Fermi coordinates” in a sufficiently small open neighborhood U of the standard Clifford torus $\mathcal{C} := \frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1)$ in \mathbf{S}^3 . Intentionally, we will work in \mathbf{S}^3 and not in \mathbf{R}^3 throughout the entire proof of Theorem 1. This is due to the circumstance that the stereographic projection of the Clifford torus into \mathbf{R}^3 is not a minimal surface anymore and neither satisfies $|A^0|^2 \equiv \text{const}$. But these are key properties of the standard Clifford torus in \mathbf{S}^3 which will turn out crucial in the proof of our central Lemma 2, yielding a fairly simple linearization of our basic differential operator G from (25), respectively, (28) at $\rho = 0$ and thus paving the path to Lemma 4; see here also Remark 2 below Lemma 2. Firstly, we recall from [26], p. 108, that the tangent bundle of \mathbf{S}^3 splits along \mathcal{C} into a direct sum of vector bundles

$$T\mathbf{S}^3|_{\mathcal{C}} = T\mathcal{C} \oplus N\mathcal{C} \tag{7}$$

namely into the tangent bundle and the normal bundle of \mathcal{C} within $T\mathbf{S}^3$. Now we can construct Fermi coordinates in a canonical way by means of the restriction of the

exponential map $\exp \equiv \exp^{\mathbf{S}^3} : \mathcal{D}(\exp^{\mathbf{S}^3}) \subset T\mathbf{S}^3 \longrightarrow \mathbf{S}^3$ to the normal bundle NC - a smooth subbundle of $T\mathbf{S}^3$:

$$\exp|_{NC} : \text{dom}(\exp) \cap NC \longrightarrow \mathbf{S}^3,$$

because the proof of Theorem 5.1 in [26] guarantees, that there is a small open neighborhood Z of the zero-section in the total space of NC and an open neighborhood U of the torus \mathcal{C} in \mathbf{S}^3 , such that

$$\exp|_{NC} : Z \xrightarrow{\cong} U \tag{8}$$

is a smooth diffeomorphism. Hence, the restriction $\exp|_{NC}$ to a sufficiently small open neighborhood Z of the zero-section of the bundle $NC \longrightarrow \mathcal{C}$ is a *tubular map*, and the corresponding open neighborhood U of \mathcal{C} thus turns out to be a *tube* about \mathcal{C} , in the language of differential topology, see e.g., [26], p. 108. In other words, having chosen a global unit normal field $\nu_{\mathcal{C}}$ along the orientable surface \mathcal{C} , i.e., a smooth section of the bundle NC of constant length 1, any point $p \in U$ can be written as $p = \exp_x(r \nu_{\mathcal{C}}(x))$ for a unique point $x = x(p) \in \mathcal{C}$ and a unique real number $r = r(p)$, thus yielding globally defined *Fermi coordinates* $(x, r) \in \mathcal{C} \times \mathbf{R}$ for points $p \in U$. Hence, having chosen a unit normal field $\nu_{\mathcal{C}}$ along \mathcal{C} , statement (8) yields a smooth diffeomorphism

$$X : \mathcal{C} \times (-a, a) \xrightarrow{\cong} \text{image}(X) =: U_a, \quad X(x, r) := \exp_x(r \nu_{\mathcal{C}}(x)), \tag{9}$$

onto an open neighborhood U_a of the torus \mathcal{C} in \mathbf{S}^3 , provided $a > 0$ is chosen sufficiently small, more precisely smaller than the width of the tube Z about the zero-section in NC .⁶ Taking the inverse of the smooth diffeomorphism X in (9), we obtain a well-defined and unique pair of smooth coordinate functions

$$S : U_a \longrightarrow \mathcal{C} \quad \text{and} \quad \Lambda : U_a \longrightarrow (-a, a). \tag{10}$$

Now, suppose there is some smooth manifold \mathcal{E} in U_a , which has the property that the coordinate function S maps \mathcal{E} bijectively onto \mathcal{C} . We thus obtain a unique smooth function

$$\rho \equiv \rho_{\mathcal{E}} : \mathcal{C} \longrightarrow (-a, a) \quad \text{by setting} \quad \rho(x) := \Lambda \circ (S|_{\mathcal{E}})^{-1}(x), \quad x \in \mathcal{C}, \tag{11}$$

where Λ and S are the smooth coordinate functions from line (10). Obviously, the function $\rho_{\mathcal{E}}$ measures the pointwise “signed geodesic distance” between the set \mathcal{E} and any chosen point $x \in \mathcal{C}$, and we therefore recover \mathcal{E} as a graph over \mathcal{C} :

$$\mathcal{E} = \text{image}([\mathcal{C} \ni x \mapsto X(x, \rho(x))])$$

⁶ See here also Sect. 4.1 in [35] for the construction of the map X in Euclidean space, and especially formulae (27) and (34) in [35] for a simple upper bound on a in terms of the maximal principle curvature of the base surface, which is here the Clifford torus $\mathcal{C} \hookrightarrow \mathbf{S}^3$.

just by construction of the diffeomorphism X in (9), of S and Λ in (10), and of ρ in (11). Conversely, suppose that there is some function $\rho : \mathcal{C} \rightarrow (-a, a)$ of class $h^{2+\alpha}(\mathcal{C})$, for some arbitrarily fixed $\alpha \in (0, \beta)$ —where $\beta \in (0, 1)$ is already given by the asserted statement of Theorem 1—then the set

$$\mathcal{E}(\rho) := \text{image}([\mathcal{C} \ni x \mapsto X(x, \rho(x))]) \subset U_a \tag{12}$$

is a 2-dimensional manifold of class $h^{2+\alpha}$ —provided $a > 0$ is sufficiently small—and moreover, $\mathcal{E}(\rho)$ is the level set $\{p \in U_a \mid \Phi_\rho(p) = 0\}$ of the function

$$\Phi_\rho : U_a \rightarrow \mathbf{R} \text{ defined by } \Phi_\rho(p) := \Lambda(p) - \rho(S(p)). \tag{13}$$

Obviously, the function Φ_ρ is just as smooth as the function ρ is, thus here it is of class $h^{2+\alpha}(U_a)$. Suppose now that we have a time-dependent function $\rho : \mathcal{C} \times [0, T) \rightarrow (-a, a)$ of class $C^0([0, T); h^{2+\alpha}(\mathcal{C}))$ for the above fixed $\alpha \in (0, \beta)$. We thus consider the time-dependent function

$$\Phi_{\rho_t}(p, t) := \Lambda(p) - \rho(S(p), t),$$

and we obtain closed and compact $h^{2+\alpha}$ -manifolds $\mathcal{E}(\rho_t)$, for $t \in [0, T)$, in the neighborhood U_a of \mathcal{C} in \mathbf{S}^3 as level sets:

$$\mathcal{E}(\rho_t) = \{p \in U_a \mid \Phi_{\rho_t}(p, t) = 0\}. \tag{14}$$

In combination with equation (12), we have the equation

$$\Phi_{\rho_t}(X(x, \rho(x, t)), t) = 0 \quad \forall (x, t) \in \mathcal{C} \times [0, T). \tag{15}$$

Now, provided the distance function ρ is sufficiently smooth in $t \in (0, T)$, then differentiating (15) with respect to t and the chain rule yield:

$$\begin{aligned} 0 &= \left\langle \nabla_p^{\mathbf{S}^3} \Phi_{\rho_t}(X(x, \rho(x, t)), t), \frac{\partial}{\partial t}(X(x, \rho(x, t))) \right\rangle + \frac{\partial}{\partial t} \Phi_{\rho_t}(X(x, \rho(x, t)), t) \\ &= \pm \left| \nabla_p^{\mathbf{S}^3} \Phi_{\rho_t}(X(x, \rho(x, t)), t) \right| \left| (\partial_t)^{\perp x}(X(x, \rho(x, t))) \right| - \frac{\partial \rho}{\partial t}(x, t) \end{aligned} \tag{16}$$

for $(x, t) \in \mathcal{C} \times (0, T)$, where “ $\frac{\partial}{\partial t} \Phi_{\rho_t}(X(x, \rho(x, t)), t)$ ” means $\frac{\partial}{\partial t} \Phi_{\rho_t}(p, t)|_{p=X(x, \rho(x, t))}$. Here, “ $(\partial_t)^{\perp x}(X(x, \rho(x, t)))$ ” denotes the normal component of the velocity vector $\partial_t(X(\cdot, \rho(\cdot, t)))$ of the family $\{\mathcal{E}(\rho_t)\}_{t \in [0, T)}$ of parametrized moving manifolds from (14), evaluated in their points $X(x, \rho(x, t))$, and this vector is actually parallel to the gradient $\nabla_p^{\mathbf{S}^3} \Phi_{\rho_t}(X(x, \rho(x, t)), t)$ on account of formulae (12)–(15). Furthermore, the sign in (16) depends on the direction of the normal velocity $(\partial_t)^{\perp x}(X(x, \rho(x, t)))$ of the surfaces $\mathcal{E}(\rho_t)$ compared to the direction of the chosen unit normal $\nu_{\mathcal{C}}$ along the Clifford torus because of (9), (10) and (13). Now, flipping $\nu_{\mathcal{C}}$ to $-\nu_{\mathcal{C}}$ would force us to also change ρ_t to $-\rho_t$ —according to the introduction of

the signed distance function $\rho_{\mathcal{E}}$ in formula (11)—and the “signed speed” $V(x, t)$ of $(\partial_t)^\perp X(X(x, \rho(x, t)))$ should be positive if and only if $\frac{\partial \rho}{\partial t}(x, t)$ is. We therefore infer from (16) as on p. 1423 in [13] or as on p. 272 in [14] the fundamental description

$$V(x, t) = \frac{\frac{\partial \rho}{\partial t}(x, t)}{|\nabla_{\mathbb{S}^3} \Phi_{\rho_t}(X(x, \rho(x, t)), t)|}, \quad \text{for } (x, t) \in \mathcal{C} \times (0, T), \quad (17)$$

of the speed of the surfaces $\mathcal{E}(\rho_t)$ in normal direction of motion—with respect to the chosen orientation of \mathcal{C} - provided our distance function ρ is sufficiently smooth in t . In view of Theorem 3 below, we can actually assume until the formulation of Lemma 1, that our distance function ρ will additionally be of class $C^\infty((0, T); C^\infty(\mathcal{C}))$, thus giving rise to the smooth family

$$\theta_{\rho_t}(x, t) := \exp_x(\rho(x, t) \nu_{\mathcal{C}}(x)), \quad \text{for } (x, t) \in \mathcal{C} \times (0, T), \quad (18)$$

of C^∞ -smooth and diffeomorphic parametrizations of smooth surfaces $\mathcal{E}(\rho_t)$, moving according to (17). Moreover, since for any fixed $t \in [0, T)$, the normal bundle $N\mathcal{E}(\rho_t)$ of the submanifold $\mathcal{E}(\rho_t) \hookrightarrow \mathbb{S}^3$ is only one-dimensional and possesses the non-vanishing section $\nu_{\mathcal{E}(\rho_t)}$ of constant length 1 with respect to $g_{\mathbb{S}^3}$, any smooth section $V \in \Gamma(N\mathcal{E}(\rho_t))$ can be written in the form

$$V = f_V \nu_{\mathcal{E}(\rho_t)} \quad \text{on } \mathcal{E}(\rho_t) \quad (19)$$

for a uniquely determined smooth function $f_V : \mathcal{E}(\rho_t) \rightarrow \mathbf{R}$. Hence, we obtain a linear bijection

$$\Gamma(N\mathcal{E}(\rho_t)) \ni V \longleftrightarrow f_V \in C^\infty(\mathcal{E}(\rho_t)) \quad (20)$$

between the set of smooth sections of $N\mathcal{E}(\rho_t)$ and functions of class $C^\infty(\mathcal{E}(\rho_t))$. Now, the connection ∇^\perp in the normal bundle $N\mathcal{E}(\rho_t)$ —see Sect. 2.1 in [45]—maps sections of $N\mathcal{E}(\rho_t)$ into $N\mathcal{E}(\rho_t)$ again and maps the unit normal $\nu_{\mathcal{E}(\rho_t)}$ to 0. Hence, defining covariant differentiation “ $\nabla_{\partial_i}^{\mathcal{E}(\rho_t)}$ ” of smooth functions $f \in C^\infty(\mathcal{E}(\rho_t))$ in the direction of some locally defined partial derivative ∂_i by $\nabla_{\partial_i}^{\mathcal{E}(\rho_t)}(f) := \partial_i(f)$ and $(\nabla^{\mathcal{E}(\rho_t)})_{\partial_i, \partial_j}^2(f) := \partial_{ij}(f) - (\Gamma_{\mathcal{E}(\rho_t)})_{ij}^k \partial_k(f)$, where $(\Gamma_{\mathcal{E}(\rho_t)})_{ij}^k$ denote the Christoffel symbols of the Euclidean metric induced by the injection $\mathcal{E}(\rho_t) \hookrightarrow \mathbb{S}^3$, we infer from the Leibniz-rule for linear connections:

$$\nabla_{\partial_i}^\perp(f \nu_{\mathcal{E}(\rho_t)}) = \nabla_{\partial_i}^{\mathcal{E}(\rho_t)}(f) \nu_{\mathcal{E}(\rho_t)} \quad \text{on } \mathcal{E}(\rho_t),$$

and thus by definition of the Beltrami-Laplace operator, associated both to the linear connection ∇^\perp in the normal bundle $N\mathcal{E}(\rho_t)$ and to the covariant derivative $\nabla^{\mathcal{E}(\rho_t)}$ on $\mathcal{E}(\rho_t)$:

$$\begin{aligned} \Delta_{\mathcal{E}(\rho_t)}^\perp(f \nu_{\mathcal{E}(\rho_t)}) &= g_{\mathcal{E}(\rho_t)}^{ij} (\nabla_{\partial_i, \partial_j}^\perp)^2(f \nu_{\mathcal{E}(\rho_t)}) \\ &= g_{\mathcal{E}(\rho_t)}^{ij} (\nabla^{\mathcal{E}(\rho_t)})_{\partial_i, \partial_j}^2(f) \nu_{\mathcal{E}(\rho_t)} = \Delta_{\mathcal{E}(\rho_t)}(f) \nu_{\mathcal{E}(\rho_t)} \end{aligned} \quad (21)$$

for any function $f \in C^\infty(\mathcal{E}(\rho_t))$. Now we infer from (21) that

$$\Delta_{\mathcal{E}(\rho_t)}^\perp \mathbf{H}_{\mathcal{E}(\rho_t)} = \Delta_{\mathcal{E}(\rho_t)}^\perp (H_{\mathcal{E}(\rho_t)} \nu_{\mathcal{E}(\rho_t)}) = \Delta_{\mathcal{E}(\rho_t)} (H_{\mathcal{E}(\rho_t)}) \nu_{\mathcal{E}(\rho_t)}. \tag{22}$$

Moreover, we have

$$|A_{\mathcal{E}(\rho_t)}^0|^2 \mathbf{H}_{\mathcal{E}(\rho_t)} = \left(\frac{1}{2} |H_{\mathcal{E}(\rho_t)}|^2 - 2 K_{\mathcal{E}(\rho_t)}\right) H_{\mathcal{E}(\rho_t)} \nu_{\mathcal{E}(\rho_t)} \tag{23}$$

on $\mathcal{E}(\rho_t)$, for every $t \in [0, T)$, where the symbols $H_{\mathcal{E}(\rho_t)}$ and $K_{\mathcal{E}(\rho_t)}$ denote the trace and the determinant, respectively, of the scalar second fundamental form $(A_{\mathcal{E}(\rho_t)})_{\mathbb{S}^3}$ of the submanifold $\mathcal{E}(\rho_t) \hookrightarrow \mathbb{S}^3$ with respect to a fixed unit normal $\nu_{\mathcal{E}(\rho_t)}$; see here formula (5) on p. 22 in [48].

Now we fix an arbitrary family of compact surfaces $\mathcal{E}(\rho_t)$ which are contained in the open neighborhood U_a of \mathcal{C} in \mathbb{S}^3 and implicitly given by equation (14) in terms of a time-dependent distance function $\rho : \mathcal{C} \times [0, T) \rightarrow (-a, a)$ of class

$$\rho \in C^0([0, T); h^{2+\beta}(\mathcal{C})) \cap C^\infty((0, T); C^\infty(\mathcal{C})),$$

where $\beta \in (0, 1)$ had been prescribed in the statement of Theorem 1. Exactly as in formulae (1.1)–(2.2) of [13], we infer now from a combination of formulae (2), (17), (21), (22), and (23), that a family of immersions $f_t : \Sigma \rightarrow \mathbb{S}^3$, $t \in [0, T)$, parametrizing the compact $h^{2+\beta}$ -manifolds $\mathcal{E}(\rho_t)$, moves according to the “relaxed variant”

$$(\partial_t)^\perp f_t = -\frac{1}{2} \frac{1}{|A_{f_t}^0|^4} \left(\Delta_{f_t}^\perp \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t}) \right) \equiv -\frac{1}{|A_{f_t}^0|^4} \nabla_{L^2} \mathcal{W}(f_t) \tag{24}$$

of the MIWF (2) on $\Sigma \times (0, T)$, if and only if the prescribed distance function $\rho = \{\rho_t\}$ satisfies the evolution equation

$$\begin{aligned} \frac{\partial \rho}{\partial t}(x, t) &= -\frac{|\nabla_p^{\mathbb{S}^3} \Phi_{\rho_t}(\theta_{\rho_t}(x, t), t)|}{2 |A_{\mathcal{E}(\rho_t)}^0(\theta_{\rho_t}(x, t))|^4} \left(\theta_{\rho_t}^* (\Delta_{\mathcal{E}(\rho_t)} H_{\mathcal{E}(\rho_t)})(x, t) \right. \\ &\quad \left. + \left(\frac{1}{2} |H_{\mathcal{E}(\rho_t)}(\theta_{\rho_t}(x, t))|^2 - 2 K_{\mathcal{E}(\rho_t)}(\theta_{\rho_t}(x, t))\right) H_{\mathcal{E}(\rho_t)}(\theta_{\rho_t}(x, t)) \right) \\ &=: G(\rho_t)(x) \end{aligned} \tag{25}$$

for $(x, t) \in \mathcal{C} \times (0, T)$, whose initial value ρ_0 is determined by the initial $h^{2+\beta}$ -manifold $\mathcal{E}_0 \subset U_a$ on account of formulae (9)–(12),⁷ In view of formula (25) and of

⁷ We can easily infer from formula (5) in [48] that—on the one hand—the values on both sides of (22) and (23) do not change if we flip the unit normal from $\nu_{\mathcal{E}(\rho_t)}$ to $-\nu_{\mathcal{E}(\rho_t)}$. On the other hand, in equation (25) we have simply dropped the unit normal $\nu_{\mathcal{E}(\rho_t)}$ appearing in equations (22) and (23). Therefore the \pm -ambiguity in (16), leading to $V(x, t) = \pm |(\partial_t)^\perp X(X(x, \rho(x, t)))|$ on account of (17), disappears in (25), if we choose a continuous field of unit normals $\nu_{\mathcal{E}(\rho_t)}$ along the moving surfaces $\mathcal{E}(\rho_t)$ in such a way that $\nu_{\mathcal{E}(\rho_t)} = \nu_{\mathcal{C}}$ holds for $\rho_t \equiv 0$, just as asserted in [38], formula (1.1), or in [39], formula (5.1).

Lemma 1 below, we recall here the transformation law

$$\theta_{\rho_t}^* (\Delta_{\mathcal{E}_{\rho_t}}(h)) \equiv (\Delta_{\mathcal{E}_{\rho_t}}(h)) \circ \theta_{\rho_t} = \Delta_{\theta_{\rho_t}^*(g_{\mathbb{S}^3 \lfloor \mathcal{E}(\rho_t)})}(h \circ \theta_{\rho_t}), \tag{26}$$

for $h \in C^\infty(\mathcal{E}_{\rho_t})$, between the Beltrami–Laplacian $\Delta_{\mathcal{E}_{\rho_t}}$ on the submanifold $\mathcal{E}_{\rho_t} \hookrightarrow \mathbb{S}^3$, endowed with the Euclidean metric, and the Laplacian $\Delta_{\theta_{\rho_t}^*(g_{\mathbb{S}^3 \lfloor \mathcal{E}(\rho_t)})}$ on \mathcal{C} with respect to the metric $\sigma(\rho_t) := \theta_{\rho_t}^*(g_{\mathbb{S}^3 \lfloor \mathcal{E}(\rho_t)})$ pulled back by the diffeomorphism $\theta_{\rho_t}(\cdot, t) : \mathcal{C} \xrightarrow{\cong} \mathcal{E}(\rho_t)$ from line (18), for any fixed time $t \in [0, T)$. We shall adopt the notation in [13, 38, 39, 44] and abbreviate in the sequel

$$\Delta_{\rho_t}(f) := \Delta_{\theta_{\rho_t}^*(g_{\mathbb{S}^3 \lfloor \mathcal{E}(\rho_t)})}(f) = \sigma^{jk}(\rho_t) \left(\partial_{jk}(f) - \gamma_{jk}^i(\rho_t) \partial_i(f) \right), \tag{27}$$

for $f \in C^\infty(\mathcal{C})$ and for every fixed $t \in [0, T)$, using the coefficients $\sigma^{jk}(\rho_t)$ of the dual metric tensor $\sigma^*(\rho_t)$ and the Christoffel-symbols $\gamma_{jk}^i(\rho_t)$, $i, j, k = 1, 2$, with respect to $\sigma(\rho_t)$. Now we choose some $\beta_0 \in (\alpha, \beta)$, where $\alpha \in (0, \beta)$ was arbitrarily fixed above line (12), and we define the open subset

$$\mathcal{U}_{\beta_0}^a := \{ \rho \in h^{2+\beta_0}(\mathcal{C}) \mid \| \rho \|_{L^\infty(\mathcal{C})} < a \}$$

of the Banach space $h^{2+\beta_0}(\mathcal{C})$, for some sufficiently small $a > 0$ as in (9). Closely following the proofs of Lemma 2.1 in [13] and of Lemma 3.1 in [14], we will prove here the following fundamental result.

Lemma 1 *The differential operator*

$$G(\rho) \equiv - \frac{|\nabla_p^{\mathbb{S}^3} \Phi_\rho \circ \theta_\rho|}{2|A_\rho^0|^4} \left(\Delta_\rho H_\rho + H_\rho \left(\frac{1}{2} H_\rho^2 - 2K_\rho \right) \right) \tag{28}$$

from line (25), having abbreviated here $A_\rho^0 := A_{\mathcal{E}(\rho)}^0 \circ \theta_\rho$, $H_\rho := H_{\mathcal{E}(\rho)} \circ \theta_\rho$ and $K_\rho := K_{\mathcal{E}(\rho)} \circ \theta_\rho$, is a uniformly elliptic quasilinear operator. More precisely, G can be decomposed in the following way:

$$G(\rho) = -P(\rho) \cdot \rho + F(\rho) \tag{29}$$

for every $\rho \in \mathcal{V}_\alpha^a := h^{4+\alpha}(\mathcal{C}) \cap \mathcal{U}_{\beta_0}^a$, where

$$P : \mathcal{U}_{\beta_0}^a \longrightarrow \mathcal{L}(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C}))$$

is a uniformly elliptic quasilinear operator of class $C^\infty(\mathcal{U}_{\beta_0}^a, \mathcal{L}(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C})))$, and $F \in C^\infty(\mathcal{U}_{\beta_0}^a, h^{\beta_0}(\mathcal{C}))$ is a non-linear operator of only second order, satisfying $F(0) = 0$ on \mathcal{C} . In particular, $-P(\rho)$ generates a strongly continuous analytic semigroup on $h^\alpha(\mathcal{C})$, i.e., $P(\rho) \in \mathcal{H}(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C}))$, for every $\rho \in \mathcal{U}_{\beta_0}^a$.

Proof : First of all, as in the beginning of the proof of Lemma 2.1 in [13], we choose an atlas $\{\mathcal{O}_l \mid l = 1, \dots, m\}$ of open coordinate neighborhoods \mathcal{O}_l on \mathcal{C} , yielding partial derivatives $\partial_j, j = 1, 2$, on \mathcal{O}_l , and we pull back the Euclidean metric $g_{\mathbb{S}^3}$ via the diffeomorphism

$$X_l := X|_{\mathcal{O}_l \times (-a, a)}: \mathcal{O}_l \times (-a, a) \xrightarrow{\cong} \text{image}(X|_{\mathcal{O}_l \times (-a, a)}) =: R_l(a) \subset \mathbb{S}^3$$

which is a restriction of the diffeomorphism X in line (9) to $\mathcal{O}_l \times (-a, a)$. Now, on account of one of Gauss’s famous results in Riemannian Geometry, the exponential map $\exp_x : T_x \mathbb{S}^3 \rightarrow \mathbb{S}^3$ yields a radial isometry, at least locally about 0 in $T_x \mathbb{S}^3$ for any fixed $x \in \mathcal{C}$; see here Lemma 3.5 in [10]. We can therefore verify by means of formula (9) and the usual chain rule that we actually obtain a smooth product metric

$$g_l := X_l^*(g_{\mathbb{S}^3}|_{R_l(a)}) = w_l(r) + dr \otimes dr \quad \text{on } T(\mathcal{O}_l \times (-a, a)), \tag{30}$$

where $w_l(r)$ is the metric on $T(\mathcal{O}_l \times \{r\}) \cong T\mathcal{O}_l$ whose coefficients can be explicitly given by

$$(w_l(r))_{jk}(x) = g_{\mathbb{S}^3}(\partial_j X_l(x, r), \partial_k X_l(x, r)), \quad \text{for } (x, r) \in \mathcal{O}_l \times (-a, a)$$

and for $l = 1, \dots, m$, where ∂_j, ∂_k are the partial derivatives on \mathcal{O}_l introduced above. Moreover, for $\rho \in \mathcal{U}_{\beta_0}^a$, we use the notion of the metrics $w_l(r)$ in order to globally define the metric $w(\rho)$ on TC by

$$(w(\rho))_{jk}(x) := (w_l(\rho(x)))_{jk}(x) \equiv g_{\mathbb{S}^3}(\partial_j X(x, \rho(x)), \partial_k X(x, \rho(x))), \quad \text{for } x \in \mathcal{O}_l,$$

and for $l = 1, \dots, m$. Compare here also with [13], p. 1424, and [14], p. 275. In other words, $w(\rho)$ is the unique metric η on TC such that there holds:

$$\eta(x) + dr \otimes dr = (g_l)_{(x, \rho(x))} \quad \text{on } T_{(x, \rho(x))}(\mathcal{O}_l \times (-a, a)) \quad \forall x \in \mathcal{O}_l,$$

and for each $l = 1, \dots, m$. From $w(\rho)$, we also obtain the metric $w^*(\rho)$ on the cotangent bundles $T^*(\mathcal{O}_l)$. Throughout this proof, we will simultaneously use the metric $w(\rho)$ and the other metric $\sigma(\rho)$ on TC which we had obtained already above between formulae (26) and (27) and which enters here via the pulled back Laplacian Δ_ρ on $(\mathcal{C}, \theta_\rho^*(g_{\mathbb{S}^3}|_{\mathcal{E}(\rho)}))$ from lines (27) and (28). Now, exactly as in the proof of Lemma 2.1 in [13], we can conclude from $\Phi_\rho(p) = \Lambda(p) - \rho(S(p))$ and from the definition of the metric $w(\rho)$, that there holds

$$L_\rho^2(x) := |\nabla_\rho^{\mathbb{S}^3} \Phi_\rho(X_l(x, \rho(x)))|^2 = 1 + w^*(\rho)_x(d\rho(x), d\rho(x)) \quad \forall x \in \mathcal{O}_l, \tag{31}$$

and for each $l = 1, \dots, m$. Moreover, as in Lemma 3.1 in [14] or as in Lemma 2.1 in [13], we pull back the scalar mean curvature $H_{\mathcal{E}(\rho)}$ to $H_\rho := \theta_\rho^* H_{\mathcal{E}(\rho)} \equiv H_{\mathcal{E}(\rho)} \circ \theta_\rho$ and we notice that our introduction of the mean curvature vector in (6)—following

exactly the conventional, general definition via formula (5) in [48]—coincides with the expression in formula (3.1) in [14], p. 274, at least up to a minus sign. We can therefore follow the lines of the proof of Part B of Lemma 3.1 in [14] and express the function H_ρ in terms of L_ρ , the metric $w(\rho)$, and in terms of the Christoffel symbols Γ_{jk}^i of the product metric g_l from formula (30) on each product $\mathcal{O}_l \times (-a, a)$, evaluated in the points $(x, \rho(x))$:

$$H_\rho = P_1(\rho) \cdot \rho + F_1(\rho) \quad \text{on } \mathcal{C} \tag{32}$$

for any $\rho \in \mathcal{U}_{\beta_0}^a$, where we have exactly

$$\begin{aligned} -P_1(\rho) &= \frac{1}{L_\rho^3} \left((-L_\rho^2 w^{jk}(\rho) + w^{jl}(\rho) w^{km}(\rho) \partial_l \rho \partial_m \rho) \partial_j \partial_k \right. \\ &\quad + (L_\rho^2 w^{jk}(\rho) \Gamma_{jk}^i(\rho) + w^{jl}(\rho) w^{ki}(\rho) \Gamma_{jk}^3(\rho) \partial_l \rho + 2 w^{km}(\rho) \Gamma_{3k}^i(\rho) \partial_m \rho \\ &\quad \left. - w^{jl}(\rho) w^{km}(\rho) \Gamma_{jk}^i(\rho) \partial_l \rho \partial_m \rho) \partial_i \right), \end{aligned} \tag{33}$$

and

$$F_1(\rho) = \frac{1}{L_\rho} w^{jk}(\rho) \Gamma_{jk}^3(\rho), \tag{34}$$

on account of formula (3.3) in [14]. In formulae (33) and (34), summation only runs from 1 to 2 for repeated indices, and we have explicitly

$$\Gamma_{jk}^3(\rho)(x) \equiv \Gamma_{jk}^3(x, \rho(x)) = g_{\mathbb{S}^3}(\partial_j \partial_k X_l, \partial_3 X_l)(x, \rho(x)) \quad \text{for } x \in \mathcal{O}_l \tag{35}$$

and for each $l = 1, \dots, m$, as on p. 275 in [14]. As in the proof of Lemma 2.1 in [13], one can derive from formulae (31) and (33) and Cauchy–Schwarz inequality that the symbol $P_1^\pi(\rho)$ of the leading 2nd order term of the operator $P_1(\rho)$ in (33) satisfies

$$-P_1^\pi(\rho)(\xi) \geq \frac{1}{L_\rho^3} w^*(\rho)(\xi, \xi) \quad \forall \xi \in T^*(\mathcal{C}). \tag{36}$$

Combining now formulae (28) and (32)–(35), one can start proving equation (29) for any fixed $\rho \in \mathcal{V}_\alpha^a \equiv h^{4+\alpha}(\mathcal{C}) \cap \mathcal{U}_{\beta_0}^a$ as in the proof of Lemma 2.1 in [13], p. 1425. First of all, one can compute by means of formulae (28), (32), and (33) that the leading 4th order term of the operator $P(\rho)$ on the right hand side of equation (29) is explicitly given by

$$P^\pi(\rho) := \frac{1}{2 L_\rho^2 |A_\rho^0|^4} \sigma^{rs}(\rho) \left(L_\rho^2 w^{jk}(\rho) - w^{jl}(\rho) w^{km}(\rho) \partial_l \rho \partial_m \rho \right) \partial_r \partial_s \partial_j \partial_k, \tag{37}$$

for any fixed $\rho \in \mathcal{U}_{\beta_0}^a$. Moreover, one can argue as on p. 1425 in [13] that applying the Laplacian Δ_ρ to the right hand side in (33) leads to a quasilinear differential operator

of fourth order, which can be decomposed by means of $P^\pi(\rho)$ in (37) in the following way:

$$\frac{1}{2|A_\rho^0|^4} L_\rho \Delta_\rho(P_1(\rho) \cdot \rho) = P^\pi(\rho) \cdot \rho + Q(\rho) \cdot \rho \quad \text{for } \rho \in \mathcal{U}_{\beta_0}^a \cap h^{4+\alpha}(\mathcal{C}), \quad (38)$$

where $[\rho \mapsto Q(\rho) \cdot \rho]$ is a quasilinear differential operator of third order, which acts on third-order partial derivatives only linearly. Hence, for any fixed $\rho \in \mathcal{U}_{\beta_0}^a$, we have that $Q(\rho) \in \mathcal{L}(h^{3+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C}))$. Now, comparing the explicit formula (37) with formula (33), and using estimate (36) for the symbol $p_1^\pi(\rho)$ of the leading 2nd order term of the operator $P_1(\rho)$ in (33), the symbol $p^\pi(\rho)$ of the operator $P^\pi(\rho)$ from line (37) turns out to satisfy

$$p^\pi(\rho)(\xi) \geq \frac{1}{2L_\rho^2|A_\rho^0|^4} \sigma^*(\rho)(\xi, \xi) w^*(\rho)(\xi, \xi) \quad \forall \xi \in T^*(\mathcal{C}),$$

proving that the operator $P^\pi(\rho)$ is uniformly elliptic of fourth order. Moreover, as in the proof of Lemma 2.1 in [13], we write the operator $P(\rho)$ on the right hand side of equation (29) as a sum of the principal quasilinear operator $P^\pi(\rho)$ of fourth order and two further quasilinear operators $[\rho \mapsto Q(\rho) \cdot \rho]$ and $[\rho \mapsto R(\rho) \cdot \rho]$ of third order, which contain all partial derivatives of third order of the operator G in formula (28), i.e.,

$$P(\rho) := P^\pi(\rho) + Q(\rho) + R(\rho) \quad \text{for } \rho \in \mathcal{U}_{\beta_0}^a \cap h^{4+\alpha}(\mathcal{C}), \quad (39)$$

where the quasilinear operator $R(\rho)$ is concretely given by

$$R(\rho) \cdot \rho := \frac{L_\rho}{2|A_\rho^0|^4} \left(\Delta_\rho \left(\frac{1}{L_\rho} \right) \right) L_\rho F_1(\rho) \quad \text{for } \rho \in \mathcal{U}_{\beta_0}^a \cap h^{3+\alpha}(\mathcal{C}).$$

Hence, combining formulae (28), (32), (38), and (39), we see as in the proof of Lemma 2.1 in [13] that the remaining term in formula (29) has to be the non-linear operator

$$\begin{aligned} F(\rho) := & -\frac{L_\rho}{2|A_\rho^0|^4} \left(\Delta_\rho F_1(\rho) + H_\rho \left(\frac{1}{2}|H_\rho|^2 - 2K_\rho \right) \right) \\ & + R(\rho) \cdot \rho \quad \text{for } \rho \in \mathcal{U}_{\beta_0}^a \cap h^{3+\alpha}(\mathcal{C}). \end{aligned} \quad (40)$$

Moreover, it follows as in the proofs of Lemma 2.1 in [13] and of Lemma 2.1 in [44], that the non-linear operator F in (40) is—just by its construction—of only second order and smooth, more precisely F is of second order and of class $C^\infty(\mathcal{U}_{\beta_0}^a, h^{\beta_0}(\mathcal{C}))$. Similarly one can infer from formulae (37)–(39), that P is a non-linear operator of class $C^\infty(\mathcal{U}_{\beta_0}^a, \mathcal{L}(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C})))$. This has completed the proof of formula (29). Furthermore, we can verify that the uniform ellipticity of $P(\rho)$ implies that $-P(\rho)$ is sectorial in $h^\alpha(\mathcal{C})$ by Theorem 3.3 in [40], for any fixed $\rho \in \mathcal{U}_{\beta_0}^a$. Since $h^{4+\alpha}(\mathcal{C})$ embeds densely into $h^\alpha(\mathcal{C})$, $-P(\rho)$ therefore generates a strongly continuous analytic

semigroup in $h^\alpha(\mathcal{C})$, in classical notation “ $P(\rho) \in \mathcal{H}(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C}))$ ”, for every $\rho \in \mathcal{U}_{\beta_0}^a$; see here also Theorem 3.5 in [40]. Finally, combining the quasilinear structure in (29) of the non-linear operator G from lines (25) and (28), which we have just verified above, with the correspondence between equations (24) and (25) and with the well-known fact that the Clifford torus \mathcal{C} is “Willmore” in \mathbf{S}^3 , we infer indeed that $F(0) = G(0) = 0$ on \mathcal{C} from simply evaluating formula (29) at $\rho = 0$, just as asserted below formula (29). \square

Relying on the proof of Theorem 2.2 in [13], we infer the following fundamental existence, uniqueness, and regularity result for the quasilinear parabolic equation (25) from Lemma 1 and Sect. 12 in [3]. See also Theorem 3.1 in [43]. We recall here, that we have chosen $\beta_0 \in (\alpha, \beta)$ before the statement of Lemma 1, implying that $h^{2+\beta}(\mathcal{C})$ embeds compactly into $h^{2+\beta_0}(\mathcal{C})$.

Theorem 3 *For any $\rho_0 \in \mathcal{U}_\beta^a := \{\rho \in h^{2+\beta}(\mathcal{C}) \mid \|\rho\|_{L^\infty(\mathcal{C})} < a\}$ there is a unique, non-extendable solution*

$$[t \mapsto \rho(t, \rho_0)] \in C^0([0, t^+), \mathcal{U}_\beta^a) \cap C^\infty((0, t^+), C^\infty(\mathcal{C}))$$

of the initial value problem

$$\frac{\partial \rho}{\partial t}(x, t) = G(\rho_t)(x) \text{ for } t \geq 0, \text{ with } \rho(x, 0) = \rho_0(x) \text{ for every } x \in \mathcal{C}, \quad (41)$$

where $t^+ = t^+(\rho_0) > 0$ denotes the “time of maximal existence” of the smooth solution of problem (41) and where G denotes the quasilinear differential operator from lines (25) and (28). Moreover, the map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a smooth local semiflow on \mathcal{U}_β^a in the sense of Sect. 12 in [3]. \square

Now, using some computations from Weiner’s article [48] about the Willmore functional, we obtain the following counterpart to Lemma 3.1 in [13].

Lemma 2 *The operator $G : \mathcal{V}_\alpha^a \equiv h^{4+\alpha}(\mathcal{C}) \cap \mathcal{U}_{\beta_0}^a \longrightarrow h^\alpha(\mathcal{C})$ from line (28) is C^∞ -smooth, and its Fréchet derivative in $\rho = 0 \in \mathcal{V}_\alpha^a$ is precisely the uniformly elliptic linear operator*

$$D_\rho G(\rho)|_{\rho=0} \equiv D_\rho F(0) - P(0) = -\frac{1}{8} (\Delta_{\mathcal{C}} + 4) \circ (\Delta_{\mathcal{C}} + 2) : h^{4+\alpha}(\mathcal{C}) \longrightarrow h^\alpha(\mathcal{C}),$$

where $\Delta_{\mathcal{C}}$ denotes the standard Beltrami–Laplace operator on \mathcal{C} with respect to the Euclidean metric induced by the injection $\mathcal{C} \hookrightarrow \mathbf{S}^3$, i.e., $\Delta_{\mathcal{C}} = \Delta_\rho|_{\rho=0}$ in the terminology of equation (27).

Proof We can immediately infer from Lemma 1 that the operator $G : \mathcal{V}_\alpha^a \longrightarrow h^\alpha(\mathcal{C})$ is C^∞ -smooth and thus continuously Fréchet-differentiable in \mathcal{V}_α^a . Moreover, the manifold $\mathcal{E}(0)$ from (12) is simply the Clifford torus \mathcal{C} in \mathbf{S}^3 by the above construction. We may therefore use Weiner’s computation in [48], pp. 24–25 and p. 34, and formula

(21), in order to infer that the Fréchet derivative of the non-linear mean curvature operator $[\rho \mapsto H_\rho]$ of the manifolds \mathcal{E}_ρ evaluated in $\rho = 0$ is concretely given by:⁸

$$D_\rho H_\rho|_{\rho=0} = \Delta_C + 4 : h^{2+\alpha}(C) \longrightarrow h^\alpha(C). \tag{42}$$

Now we compute the Fréchet derivative of the first term in (28). To this end, we notice that in our situation, there holds simply $L_0 \equiv 1$, $H_\rho|_{\rho=0} \equiv H_C \equiv 0$, and $|A_C^0|^2 \equiv 2$. We can therefore proceed exactly as in the proof of Lemma 3.1 in [13], deducing from the chain rule for bilinear composition of non-linear operators combined with equation (42) that

$$D_\rho \left(\frac{L_\rho}{2|A_\rho^0|^4} \Delta_\rho H_\rho \right) |_{\rho=0} = \frac{1}{8} \Delta_C \circ (\Delta_C + 4). \tag{43}$$

Moreover, in order to compute the Fréchet derivative of the second term in (28), we can employ again equation (42) and the equations $L_0 \equiv 1$, $H_0 = H_C \equiv 0$, and $K_0 = K_C \equiv -1$ and obtain here via the chain rule for bilinear composition:

$$\begin{aligned} D_\rho \left(\frac{L_\rho}{2|A_\rho^0|^4} H_\rho \left(\frac{1}{2} H_\rho^2 - 2 K_\rho \right) \right) |_{\rho=0} &= \frac{1}{8} D_\rho H_\rho|_{\rho=0} \left(\frac{1}{2} H_0^2 - 2 K_0 \right) \\ &= \frac{1}{4} (\Delta_C + 4). \end{aligned} \tag{44}$$

Hence, adding formulae (43) and (44), we finally obtain

$$D_\rho G(\rho)|_{\rho=0} = -\frac{1}{8} \left(\Delta_C^2 + 4 \Delta_C + 2 \Delta_C + 8 \right) = -\frac{1}{8} (\Delta_C + 4) \circ (\Delta_C + 2),$$

on account of equation (28). Compare here also with p. 29 in [48]. □

Remark 2 We should remark here that Lemma 2 would become tremendously complicated—and additionally useless—if we would start working between formulae (8) and (25) in a tubular neighborhood of the stereographically projected Clifford torus “Cliff” in \mathbf{R}^3 , being conformally parametrized by the standard embedding $F(x, y) := \frac{1}{\sqrt{2-\sin(y)}} \left(\cos(x), \sin(x), \cos(y) \right)$, for $x, y \in [0, 2\pi]$, because its mean and Gaussian curvatures are non-constant functions, concretely given by $H_F(x, y) = \sqrt{2} \sin(y)$ and $K_F(x, y) = \sqrt{2} \sin(y) - 1$, and already our formula (42) would become instead: $D_\rho H_\rho|_{\rho=0} = \Delta_{\text{Cliff}} + (\kappa_1^2 + \kappa_2^2)$ on account of Appendix A in [2], where the principal curvatures along Cliff are concretely given by $\kappa_1(x, y) = 1$ and $\kappa_2(x, y) = \sqrt{2} \sin(y) - 1$ via F . □

In view of the proof of Lemma 4 and of Theorem 1 below, we recall here some differential-geometric key-insights from [45] and [48]. First of all, replacing the smooth submanifolds $\mathcal{E}(\rho_t) \hookrightarrow \mathbf{S}^3$ in (19)–(21) simply by the Clifford torus \mathcal{C} , we

⁸ Here our procedure clearly differs from the one in [13] and [44], where the authors relied on the computations in Lemma 3.1 of [14], which we only used in the proof of Lemma 1.

have $\nabla_{\partial_i}^\perp(f v_C) = \nabla_{\partial_i}^C(f) v_C$ on C and therefore also $\Delta_C^\perp(f v_C) = \Delta_C(f) v_C$, for any function $f \in C^\infty(C)$, where g_C and Δ_C denote the Euclidean metric restricted to TC and the standard Beltrami–Laplace operator on C , which are both induced by the injection $C \hookrightarrow \mathbf{S}^3$. Moreover, following Weiner [48], Sect. 3, we define two different elementary types of smooth sections of the normal bundle NC of the Clifford torus C within TS^3 , recalling here the direct bundle decomposition $TS^3|_C = TC \oplus NC$ from formula (7), splitting each fiber $T_x\mathbf{S}^3$ of the entire tangent bundle TS^3 , for $x \in C$, into the 2-dimensional tangent space T_xC of C and its 1-dimensional orthogonal complement N_xC , yielding the bundle morphism

$$P^{\text{Tan}(C),\mathbf{S}^3} : \bigcup_{x \in C} \{x\} \times T_x\mathbf{S}^3 \longrightarrow \bigcup_{x \in C} \{x\} \times T_x(C) = TC \tag{45}$$

from formula (5) for $\Sigma = C$ and $f = \text{inclusion} : C \hookrightarrow \mathbf{S}^3$.

Definition 2 (1) We term elements W of the 6-dimensional Lie-algebra Ω of the isometry group $\text{Iso}(\mathbf{S}^3) \equiv O(4)$ Killing fields on \mathbf{S}^3 . We restrict every Killing field W to the Clifford torus, and we consider its orthogonal projection $W^N(x) := W(x) - P^{\text{Tan}(C),\mathbf{S}^3}(W)(x)$, for $x \in C$, into the fibers N_xC of the normal bundle NC . We denote the linear space of all these sections $W^N \in \Gamma(NC)$ by Ω^N .

(2) We denote by $\bar{\nabla}$ the standard Euclidean connection on \mathbf{R}^4 , and we call a vector field V on \mathbf{R}^4 parallel, if it satisfies $\bar{\nabla}(V) \equiv 0$ on \mathbf{R}^4 . We restrict any such vector field V to \mathbf{S}^3 , and then we project each vector $V(x)$, for $x \in \mathbf{S}^3$, orthogonally into $T_x\mathbf{S}^3$, i.e., we consider the tangent vector field $Z(x) := V(x) - \langle V(x), x \rangle_{\mathbf{R}^4} x$, for $x \in \mathbf{S}^3$. We denote the 4-dimensional vector space of all such tangential projections Z of parallel vector fields into TS^3 by ξ .

(3) We restrict every $Z \in \xi$ to the Clifford torus and we consider its orthogonal projection $Z^N(x) := Z(x) - P^{\text{Tan}(C),\mathbf{S}^3}(Z)(x)$, for $x \in C$, into the fibers N_xC of the normal bundle NC . We denote the linear space of all these sections $Z^N \in \Gamma(NC)$ by ξ^N . □

Definition 3 (1) We define the subspace of $\Gamma(TS^3)$ consisting of all conformal vector fields on \mathbf{S}^3 by

$$\mathbf{Moeb}(\mathbf{S}^3) := \{V \in \Gamma(TS^3) \mid \exists \{\Phi_t\}_{t \in \mathbf{R}} \subset \text{Möb}(\mathbf{S}^3) \text{ with } V = \partial_t \Phi_t|_{t=0}\}.$$

(2) We define the space of normal conformal directions⁹ along C by

$$\mathbf{Moeb}^\perp(C) := \{V|_C - P^{\text{Tan}(C),\mathbf{S}^3}(V|_C) \mid V \in \mathbf{Moeb}(\mathbf{S}^3)\}.$$

Remark 3 The linear spaces Ω and ξ from Definition 2 are vector subspaces of $\Gamma(TS^3)$ with trivial intersection, and one can easily check that each element of either of these vector spaces generates a one-parameter family of Möbius-transformations of \mathbf{S}^3 . Hence according to Definition 3, Ω and ξ are both contained in the Lie algebra

⁹ Compare here also with p.31 in [48], with Appendix B in [33] or with Sect. 6 in [46].

$\mathbf{Moeb}(\mathbf{S}^3)$ of $\mathbf{Möb}(\mathbf{S}^3)$, which is a 10-dimensional Lie-group because of the isomorphism $\mathbf{Möb}(\mathbf{S}^{n-1}) \cong \mathbf{SO}^+(1, n)$, whose dimension is known to be $\frac{(n+1)n}{2}$.¹⁰ Adding the dimensions of Ω and ξ —see here Definition 2 above—we therefore obtain that $\Omega \oplus \xi = \mathbf{Moeb}(\mathbf{S}^3)$ exactly holds; see here also pp. 30–33 in [48]. \square

Now, by Lemmata 5.1.3 and 5.1.7 and Theorem 3.3.1 in [45] and by Lemmata 3.3, 3.4, and 3.5 in [48], we have the following fundamental results:

Lemma 3 (1) *Any vector field $W^N \in \Omega^N$ satisfies the partial differential equation*

$$\Delta_{\mathcal{C}}^{\perp}(W^N) = -4 W^N.$$

(2) *Any vector field $Z^N \in \xi^N$ satisfies the partial differential equation*

$$\Delta_{\mathcal{C}}^{\perp}(Z^N) = -2 Z^N.$$

Moreover, both Ω^N and ξ^N are 4-dimensional \mathbf{R} -vector spaces and their direct sum exactly constitutes the linear space $\mathbf{Moeb}^{\perp}(\mathcal{C})$ of all normal conformal directions along \mathcal{C} from Definition 3, implying that $\dim_{\mathbf{R}}(\mathbf{Moeb}^{\perp}(\mathcal{C})) = 8$.

\square

Remark 4 The operator $-(\Delta_{\mathcal{C}}^{\perp} + 4)$ appearing in Lemma 3 is the *Jacobi operator* along the Clifford torus, i.e., corresponds to the second variation of the area functional evaluated in the Clifford torus \mathcal{C} with respect to sections of the normal bundle $N\mathcal{C}$ along the Clifford torus, and a smooth vector field $V \in \Gamma(N\mathcal{C})$ is termed a *Jacobi field* along \mathcal{C} , iff it satisfies $(\Delta_{\mathcal{C}}^{\perp} + 4)(V) = 0$, i.e., iff V is contained in the eigenspace $\text{Eig}_{-4}(\Delta_{\mathcal{C}}^{\perp})$. The first part of Lemma 3 therefore shows us that orthogonal projections $W^N \in \Omega^N$ of Killing fields along the Clifford torus \mathcal{C} into its normal bundle are Jacobi fields along \mathcal{C} . \square

Combining Lemmata 2 and 3 with ideas from Lemma 3.2 and Corollary 1 in [48], we finally arrive at the following result, which will substitute Lemmata 3.2 and 3.3 in [13], [44] or Proposition 5.4 of [14] in the proof of Theorem 1.

Lemma 4 *The spectrum of the Fréchet derivative*

$$D_{\rho}G(\rho)|_{\rho=0} \equiv D_{\rho}F(0) - P(0) = -\frac{1}{8} (\Delta_{\mathcal{C}} + 4) \circ (\Delta_{\mathcal{C}} + 2)$$

is discrete and non-positive, and its kernel is an 8-dimensional \mathbf{R} -vector subspace of $C^{\infty}(\mathcal{C})$, which corresponds to the vector subspace $\mathbf{Moeb}^{\perp}(\mathcal{C})$ of $\Gamma(N\mathcal{C})$ of all normal conformal directions along the Clifford torus \mathcal{C} from Lemma 3 via the explicit linear bijection (20):

$$\text{Ker}(D_{\rho}G(\rho)|_{\rho=0}) \cong \Omega^N \oplus \xi^N = \mathbf{Moeb}^{\perp}(\mathcal{C}). \tag{46}$$

¹⁰ Compare here also with Proposition B.1 in [33].

Proof Using the particularly simple form of the uniformly elliptic operator

$$T_C := -D_\rho G(\rho)|_{\rho=0} = \frac{1}{8} (\Delta_C + 4) \circ (\Delta_C + 2) : W^{4,2}(C) \longrightarrow L^2(C), \quad (47)$$

which we had proven in Lemma 2, one can prove as in Sect. 3 of [46] that T_C is a compact perturbation of an isomorphism between $W^{4,2}(C)$ and $L^2(C)$ and thus a Fredholm operator of index 0. Moreover, integration by parts and Cauchy-Schwarz inequality show that there is some constant $c > 0$ such that

$$T_C + c \text{Id}_{W^{4,2}(C)} : W^{4,2}(C) \longrightarrow L^2(C)$$

is injective and thus a topological isomorphism. Since the composition

$$\iota \circ (T_C + c \text{Id}_{W^{4,2}(C)})^{-1} : L^2(C) \longrightarrow L^2(C) \quad (48)$$

of the inverse $(T_C + c \text{Id}_{W^{4,2}(C)})^{-1}$ with the compact embedding $\iota : W^{4,2}(C) \hookrightarrow L^2(C)$ is a compact and selfadjoint operator, classical spectral theory, e.g., Theorem 12.12 in [1], guarantees that the spectrum of T_C consists of countably many real and isolated eigenvalues $-c < \nu_1 < \nu_2 < \nu_3 < \dots \in \mathbf{R}$ and that $L^2(C)$ is the closure of the direct and L^2 -orthogonal sum of the eigenspaces $\text{Eig}_{\nu_j}(T_C)$ of T_C . Obviously, the same method applies to the Laplacian $\Delta_C : W^{2,2}(C) \longrightarrow L^2(C)$ as well, and we therefore also obtain a direct and L^2 -orthogonal decomposition of $L^2(C)$ into eigenspaces $\text{Eig}_{\lambda_k}(\Delta_C)$ of Δ_C , yielding

$$\overline{\bigoplus_{j \in \mathbf{N}} \text{Eig}_{\nu_j}(T_C)}^{L^2} = L^2(C) = \overline{\bigoplus_{k \in \mathbf{N}} \text{Eig}_{\lambda_k}(\Delta_C)}^{L^2}. \quad (49)$$

Now, one can easily verify that for each $k \in \mathbf{N}$ there is a unique $j = j(k)$, such that $\text{Eig}_{\lambda_k}(\Delta_C) \subseteq \text{Eig}_{\nu_{j(k)}}(T_C)$, and that there holds

$$\nu_{j(k)} = \frac{1}{8} (\lambda_k + 4)(\lambda_k + 2) =: p(\lambda_k)$$

for the corresponding eigenvalues. Using the additional fact that all eigenspaces in (49) are only finite-dimensional, one infers from (49) the stronger statement that every eigenspace $\text{Eig}_{\nu_j}(T_C)$ is a finite direct sum of certain eigenspaces of Δ_C , i.e., that there holds

$$\text{Eig}_{\nu_j}(T_C) = \text{Eig}_{\lambda_{k_1(j)}}(\Delta_C) \oplus \dots \oplus \text{Eig}_{\lambda_{k_r(j)}}(\Delta_C), \quad (50)$$

for $r \geq 1$ pairwise different eigenvalues $\lambda_{k_i(j)}$ of Δ_C , which have to satisfy the polynomial equation $\nu_j = \frac{1}{8} (\lambda_{k_i(j)} + 4)(\lambda_{k_i(j)} + 2) = p(\lambda_{k_i(j)})$. This insight has two important consequences. Firstly, for any fixed eigenvalue $\nu \in \text{Spec}(T_C)$, there has to be at least one eigenvalue $\lambda \in \text{Spec}(\Delta_C)$ such that $\nu = p(\lambda)$, proving that exactly

$$\text{Spec}(T_C) = \left\{ \frac{1}{8}(\lambda^2 + 6\lambda + 8) \mid \lambda \in \text{Spec}(\Delta_C) \right\}. \tag{51}$$

Secondly, since for any fixed $v \in \text{Spec}(T_C)$, the polynomial equation $p(\lambda) = v$ has at least one and at most two different real solutions, the decomposition in (50) actually reduces to the two simple cases in which either $r = 1$ or $r = 2$. In particular, there holds

$$\text{Eig}_v(T_C) = \text{Eig}_{\lambda_{k_1}}(\Delta_C) \oplus \text{Eig}_{\lambda_{k_2}}(\Delta_C), \tag{52}$$

if there is more than only one eigenvalue λ of Δ_C solving the polynomial equation $p(\lambda) = v$.¹¹ Since we particularly aim to exclude negative eigenvalues of T_C , we should note here that the polynomial $p(\lambda) = \frac{1}{8}(\lambda^2 + 6\lambda + 8)$ arising in (51) is only negative on the open interval $(-4, -2)$ and that its roots are the two endpoints -4 and -2 of this interval. Now, again motivated by (51), we should try to locate all eigenvalues of Δ_C on the Clifford torus. First of all, by Proposition 1 in [34], the Clifford torus can be *isometrically* mapped onto the flat torus \mathbf{C}/Γ^* , where the lattice Γ^* is spanned by the two vectors $v_1 := (2\pi, 0)$ and $v_2 := (\pi, \pi)$ in the complex plane, which reduces our eigenvalue problem to the simpler one of the Euclidean Laplacian on \mathbf{C}/Γ^* . Now, motivated by the solution of the eigenvalue problem of the Euclidean Laplacian on cuboids $\prod_{i=1}^N [0, R_i]$ in \mathbf{R}^N —see E.12.5 in [1]—we follow the classical “Ansatz” to use complex functions of the form $\exp(i(c_1x_1 + c_2x_2)) = \exp(ic_1x_1) \exp(ic_2x_2)$ or equivalently linear combinations of the functions $\sin(c_1x_1 + c_2x_2)$ and $\cos(c_1x_1 + c_2x_2)$, for appropriate $c_1, c_2 \in \mathbf{Z}$ and $(x_1, x_2) \in \mathbf{C}/\Gamma^*$, in order to determine all eigenvalues and eigenfunctions of the Euclidean Laplacian on \mathbf{C}/Γ^* . One can quickly verify that our a-priori condition on any considered function

$$u(x_1, x_2) = A \sin(c_1x_1 + c_2x_2) + B \cos(c_1x_1 + c_2x_2), \text{ for } A, B \in \mathbf{R}, \tag{53}$$

to be actually well defined on \mathbf{C}/Γ^* , implies that $u(x_1, x_2)$ in (53) can only be an eigenfunction of the Euclidean Laplacian on \mathbf{C}/Γ^* —and then necessarily with eigenvalue $-(c_1^2 + c_2^2)$ —if the vector $c := (c_1, c_2) \in \mathbf{Z}^2$ satisfies the two compatibility conditions

$$\langle c, v_1 \rangle = 2\pi k_1 \text{ and } \langle c, v_2 \rangle = 2\pi k_2, \tag{54}$$

for appropriate $k_1, k_2 \in \mathbf{Z}$. Now, for any pair of natural numbers $m, n \in \mathbf{N}_0$, we can choose $c_1 = m - n$ and $c_2 = m + n$ or vice versa $c_1 = m + n$ and $c_2 = m - n$, yielding in each case an admissible vector $c = (c_1, c_2)$ in (54) which additionally satisfies $|c|^2 = 2(m^2 + n^2)$. Hence, at least the inclusion

$$\text{Spec}(\Delta_C) \supseteq \{-2(m^2 + n^2) \mid m, n \in \mathbf{N}_0\} \tag{55}$$

has to hold for the Beltrami–Laplacian on the Clifford torus. Now noting that the area of the trapezoid spanned by $v_1 = (2\pi, 0)$ and $v_2 = (\pi, \pi)$ is $2\pi^2$, one might guess that

¹¹ Compare here with the statements of Lemma 3.2 in [48].

the choices $A = \frac{1}{\sqrt{2}\pi}$ and $B = \pm \frac{1}{\sqrt{2}\pi}$ in (53) yield an $L^2(\mathbf{C}/\Gamma^*, \mathbf{R})$ -orthonormal system

$$\left\{ \frac{1}{\sqrt{2}\pi} \sin((m-n)x_1 + (m+n)x_2) \pm \frac{1}{\sqrt{2}\pi} \cos((m-n)x_1 + (m+n)x_2), \right. \\ \left. \frac{1}{\sqrt{2}\pi} \sin((m+n)x_1 + (m-n)x_2) \pm \frac{1}{\sqrt{2}\pi} \cos((m+n)x_1 + (m-n)x_2) \right\}, \tag{56}$$

for $m, n \in \mathbf{N}_0$, of doubly periodic eigenfunctions of the Euclidean Laplacian on \mathbf{C}/Γ^* according to condition (54). Our assertion that these functions are additionally of length 1 and mutually orthogonal in $L^2(\mathbf{C}/\Gamma^*, \mathbf{R})$ can be verified by means of concrete computation, using the variable transformation $(x_1, x_2) = (z_1 + \frac{1}{2}z_2, \frac{1}{2}z_2)$, for $(z_1, z_2) \in [0, 2\pi]^2$, Fubini’s Theorem and the entire discussion of the eigenvalue problem for the Laplacian on the interval $[-\pi, \pi]$; see Examples 9.9 and E12.4 in [1]. Moreover, the set (56) constitutes a Schauder basis of $L^2(\mathbf{C}/\Gamma^*, \mathbf{R})$, because the set of functions

$$\left\{ (e^{ix_1})^{(m-n)} \cdot (e^{ix_2})^{(m+n)}, (e^{ix_1})^{(m+n)} \cdot (e^{ix_2})^{(m-n)} \mid m, n \in \mathbf{N}_0 \right\}$$

spans a dense subset of $L^2(\mathbf{C}/\Gamma^*, \mathbf{C})$ on account of the complex version of Stone–Weierstrass’ Theorem. Therefore, the set (56) actually spans each eigenspace of the Euclidean Laplacian on \mathbf{C}/Γ^* on account of the Spectral Theorem for compact, self-adjoint linear operators; see [1], Theorems 12.12 and 12.17. Hence, our first guess in (55) about the eigenvalues of $\Delta_{\mathcal{C}}$ turns out to have been very effective, and our entire reasoning proves equality in (55), i.e.,¹²

$$\text{Spec}(\Delta_{\mathcal{C}}) = \{-2(m^2 + n^2) \mid m, n \in \mathbf{N}_0\}. \tag{57}$$

Now, since the intersection of the set $\{-2(m^2 + n^2) \mid m, n \in \mathbf{N}_0\}$ with the interval $(-4, -2)$ is empty, there indeed cannot exist any negative eigenvalues of $T_{\mathcal{C}} = -D_{\rho}G(\rho)|_{\rho=0}$ on account of statements (51) and (57). Moreover, the set $\{-2(m^2 + n^2) \mid m, n \in \mathbf{N}_0\}$ actually contains the two numbers -2 and -4 —the two roots of our polynomial $p(\lambda) = \frac{1}{8}(\lambda^2 + 6\lambda + 8)$ from (51)—and we can therefore infer from (52) and (57) that there holds exactly

$$\text{Eig}_0(T_{\mathcal{C}}) = \text{Eig}_{-2}(\Delta_{\mathcal{C}}) \oplus \text{Eig}_{-4}(\Delta_{\mathcal{C}}). \tag{58}$$

Moreover, by formulae (19)–(21), our eigenvalue problem $\Delta_{\mathcal{C}}(f) = \lambda f$, for $f \in C^{\infty}(\mathcal{C})$, is equivalent to the eigenvalue problem $\Delta_{\mathcal{C}}^{\perp}(V) = \lambda V$, for $V \in \mathcal{N}\mathcal{C}$. We therefore recall, that by Lemma 3 there holds

$$\text{Eig}_{-2}(\Delta_{\mathcal{C}}^{\perp}) \supseteq \xi^N \quad \text{and} \quad \text{Eig}_{-4}(\Delta_{\mathcal{C}}^{\perp}) \supseteq \Omega^N, \tag{59}$$

¹² This result is not new, and it was stated without proof on page 34 in [48].

and that ξ^N and Ω^N are two 4-dimensional \mathbf{R} -vector spaces, respectively. Moreover, using our considerations between (54) and (56), we can easily check that exactly the three vectors $c = (1, 1), (-1, 1), (1, -1)$ have squared distance $|c|^2 = 2$ and are additionally admissible in (54), and similarly that exactly the two vectors $c = (2, 0), (0, 2)$ have squared distance $|c|^2 = 4$ and are additionally admissible in (54). Then one can quickly infer from formula (56) itself by elementary inspection that both the triplet of vectors $c = (1, 1), c = (-1, 1)$, and $c = (1, -1)$ and also the pair of vectors $c = (2, 0), c = (0, 2)$ give rise to only two different pairs of linearly independent eigenfunctions of Δ on \mathbf{C}/Γ^* , respectively, implying the non-trivial fact that both $\text{Eig}_{-2}(\Delta_{\mathcal{C}})$ and $\text{Eig}_{-4}(\Delta_{\mathcal{C}})$ are 4-dimensional; compare here again with [48], p. 34. Hence, on account of statement (59), the eigenspaces $\text{Eig}_{-2}(\Delta_{\mathcal{C}}^{\perp})$ and $\text{Eig}_{-4}(\Delta_{\mathcal{C}}^{\perp})$ exactly coincide with the 4-dimensional vector spaces ξ^N and Ω^N , respectively, i.e., there hold two equalities in (59). Hence, combining this again with formulae (21) and (58), the assertion of the lemma follows. \square

3 Proofs of Theorems 1 and 2

Proof of Theorem 1 Part (i) Without loss of generality, we can assume that $F^* : \Sigma \xrightarrow{\cong} \frac{1}{\sqrt{2}}(\mathbf{S}^1 \times \mathbf{S}^1) \equiv \mathcal{C}$ is a diffeomorphic parametrization of exactly the Clifford torus in \mathbf{S}^3 . Following the proof of Theorem 1.2 in [13], respectively, the lines of Sect. 6 in [14], we are going to use Lemmata 1, 2, and 4 and Theorem 3 and adopt the procedure in Sect. 4 of [43], in order to construct an invariant “center manifold” \mathcal{M}^c for the flow equation (25) as a graph over the 8-dimensional space $\text{Ker}(T_{\mathcal{C}})$. To this end, we use again the fact that the composition $\iota \circ (T_{\mathcal{C}} + c \text{Id}_{W^{4,2}(\mathcal{C})})^{-1}$ from line (48) is compact and selfadjoint, implying that the Hilbert space $L^2(\mathcal{C})$ can be decomposed orthogonally with respect to $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$ into the finite-dimensional eigenspaces $\text{Ker}(T_{\mathcal{C}}) \subset C^{\infty}(\mathcal{C})$ and $\text{Eig}_{\mu_j}(T_{\mathcal{C}}) \subset C^{\infty}(\mathcal{C})$ of $T_{\mathcal{C}}$, for the positive eigenvalues μ_j of the linear operator $T_{\mathcal{C}}$ in $L^2(\mathcal{C})$, which means precisely

$$L^2(\mathcal{C}) = \text{Ker}(T_{\mathcal{C}}) \oplus \overline{\bigoplus_{\mu > 0} \text{Eig}_{\mu}(T_{\mathcal{C}})}^{L^2}, \tag{60}$$

where we have also used Lemma 4. Now, by the second statement of Lemma 4, we can choose an $L^2(\mathcal{C})$ -orthonormal system of 8 smooth eigenfunctions $\{Y_k\}_{k=1,\dots,8}$ of $T_{\mathcal{C}}$ in $\text{Ker}(T_{\mathcal{C}})$, and we define the continuous linear projection

$$\pi^c := \sum_{k=1}^8 \langle \cdot, Y_k \rangle_{L^2(\mathcal{C})} Y_k : L^2(\mathcal{C}) \longrightarrow \text{Ker}(T_{\mathcal{C}}) \tag{61}$$

of $L^2(\mathcal{C})$ onto the *center subspace* $\text{Ker}(T_{\mathcal{C}})$ of the linear operator $T_{\mathcal{C}}$, which is orthogonal with respect to $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$ on account of (60). The restrictions of π^c in (61) to the Banach spaces $h^r(\mathcal{C})$, for $r > 0$, are still continuous, linear projections onto the finite-dimensional subspace $\text{Ker}(T_{\mathcal{C}})$, and we therefore obtain as in Sect. 4 of [43] a

unique decomposition of $h^r(\mathcal{C})$ into two closed linear subspaces:

$$h^r(\mathcal{C}) = \text{range}(\pi^c \lfloor_{h^r(\mathcal{C})}) \oplus \text{ker}(\pi^c \lfloor_{h^r(\mathcal{C})}) \equiv \text{Ker}(T_{\mathcal{C}}) \oplus h^r_s(\mathcal{C}), \tag{62}$$

for any fixed $r > 0$, especially for $r = \alpha$ or $r = 4 + \alpha$. Obviously, $T_{\mathcal{C}}$ is a symmetric operator in $L^2(\mathcal{C})$ on account of the concrete formula (47), which implies together with (61) that

$$\pi^c(T_{\mathcal{C}}(f)) = \sum_{k=1}^8 \langle T_{\mathcal{C}}(f), Y_k \rangle_{L^2(\mathcal{C})} Y_k = \sum_{k=1}^8 \langle f, T_{\mathcal{C}}(Y_k) \rangle_{L^2(\mathcal{C})} Y_k = 0$$

for every function $f \in W^{4,2}(\mathcal{C})$. In particular, there holds therefore:

$$T_{\mathcal{C}} \circ \pi^c \lfloor_{h^{4+\alpha}(\mathcal{C})} = 0 = \pi^c \lfloor_{h^{\alpha}(\mathcal{C})} \circ T_{\mathcal{C}} \quad \text{on } h^{4+\alpha}(\mathcal{C}),$$

i.e., that $T_{\mathcal{C}}$ descends to a direct sum $T_{\mathcal{C}}^c \oplus T_{\mathcal{C}}^s$ of linear operators, which respects the direct sum decomposition $\text{Ker}(T_{\mathcal{C}}) \oplus h^{4+\alpha}_s(\mathcal{C})$ of $h^{4+\alpha}(\mathcal{C})$ in (62). Obviously, by (60) and (62), the restriction $T_{\mathcal{C}}^s$ of $T_{\mathcal{C}}$ to $h^{4+\alpha}_s(\mathcal{C})$ has only positive eigenvalues $0 < \mu_1 < \mu_2 < \dots$. As in [13], [14], and [43], we shall consider besides the projection π^c in (61) the L^2 -orthogonal projection $\pi^s := \text{id}_{L^2(\mathcal{C})} - \pi^c$ of $L^2(\mathcal{C})$ onto $\text{ker}(\pi^c)$, whose restriction to $h^r(\mathcal{C})$ maps $h^r(\mathcal{C})$ onto the ‘‘stable subspace’’ $h^r_s(\mathcal{C}) \subset h^r(\mathcal{C})$ in (62) with respect to $T_{\mathcal{C}}$. Now, following Sect. 2 in [42], Sect. 4 of [43], or Sect. 9.2.1 in [27], we rewrite equation (25) in the equivalent form

$$\partial_t \rho_t + T_{\mathcal{C}}(\rho_t) \equiv \partial_t \rho_t + (P(0) - D_{\rho}F(0)).(\rho_t) = g(\rho_t), \tag{63}$$

with $g(\rho) := (P(0) - P(\rho)).(\rho) + F(\rho) - D_{\rho}F(0).(\rho)$, satisfying $g(0) = 0$ and $Dg(0) = 0$, because of $F(0) = 0$ by Lemma 1. Moreover, as in formula (4.21) in [43] or as in Sect. 9.2.1 in [27], decomposition (62) yields the equivalent formulation

$$\begin{aligned} \partial_t x_t + T_{\mathcal{C}}^c(x_t) &= \pi^c g(x_t, y_t) \\ \partial_t y_t + T_{\mathcal{C}}^s(y_t) &= \pi^s g(x_t, y_t) \end{aligned} \tag{64}$$

of equation (63), respectively, of equation (25) as a coupled system, for two separate functions $x : [0, T) \rightarrow \text{Ker}(T_{\mathcal{C}})$ and $y : [0, T) \rightarrow h^{4+\alpha}_s(\mathcal{C})$. Now we shall quickly check the assumptions (4.1)–(4.8) of Theorem 4.1 in [43], in order to obtain a *locally invariant center manifold* for our evolution equation (63), respectively (64). To this end, we should firstly follow the proof of Theorem 2.2 in [13] and translate our framework of Lemma 1 correctly into the specific language of Sects. 2–4 in [43]. In view of our Lemma 1, we should firstly choose the basic pair of Banach spaces $X_1 \hookrightarrow X_0$ in (4.2) of [43] as $X_1 := h^{4+\alpha}(\mathcal{C})$ and $X_0 := h^{\alpha}(\mathcal{C})$, then we choose the pair of interpolation parameters $0 < \beta < \alpha < 1$ in [43] as $\beta := \frac{2-\alpha+\beta_0}{4}$, whereas we only rename $\alpha := \gamma \in (\frac{2-\alpha+\beta_0}{4}, 1)$ first of all. Finally, we see that here $U_{\beta} := \mathcal{U}_{\beta_0}^{\alpha}$ is an open subset of the interpolation space $X_{\beta} = h^{2+\beta_0}(\mathcal{C})$ between X_0

and X_1 . On account of Lemma 1 and line (25), we can now immediately verify that the quasilinear operator $A := P$ and our non-linear operator F satisfy exactly conditions (4.1) and (4.2) in [43]. Moreover, choosing slightly larger spaces $E_0 := h^{\alpha'}(\mathcal{C})$ and $E_1 := h^{4+\alpha'}(\mathcal{C})$, with $\alpha' \in (0, \alpha)$ and fixing now the interpolation parameter γ as $\gamma := \frac{2-\alpha+\beta'_0}{4}$ for some $\beta'_0 \in (\beta_0, \beta)$, we obtain here $X_\gamma = h^{2+\beta'_0}(\mathcal{C})$ and its open subset $U_\gamma := \mathcal{U}_{\beta'_0}^a \cap X_\gamma = \{\rho \in h^{2+\beta'_0}(\mathcal{C}) \mid \|\rho\|_{L^\infty(\mathcal{C})} < a\}$, and we can infer from our Lemma 1, that indeed $P(\rho) \in \mathcal{H}(h^{4+\alpha'}(\mathcal{C}), h^{\alpha'}(\mathcal{C}))$, for any $\rho \in U_\gamma$, and also that $E_1 = h^{4+\alpha'}(\mathcal{C}) \hookrightarrow X_\beta \equiv h^{2+\beta_0}(\mathcal{C}) \hookrightarrow h^{\alpha'}(\mathcal{C}) = E_0$ is an interpolating triple. Now condition (4.3) in [43], which expresses *maximal regularity* in the sense that $P(\rho) \in \mathcal{M}_\gamma(h^{4+\alpha}(\mathcal{C}), h^\alpha(\mathcal{C}))$ for each $\rho \in U_\gamma$, follows right away from Theorem 2.2 in [43]. Condition (4.5) is true by our Lemma 1, and the linear operator L in (4.6) is simply our operator $T_C = P(0) - D_\rho F(0)$ from line (47), which indeed satisfies the spectral conditions (4.7) and (4.8) in [43] on account of our Lemma 4. Hence, Theorem 4.1 in [43] guarantees us, that for some fixed $m \in \mathbf{N}$ there exists a neighborhood $U = U(m)$ of 0 in $\text{Ker}(T_C)$ and a function

$$\gamma \in C^m(U, h_s^{4+\alpha}(\mathcal{C})), \text{ with } \gamma(0) = 0 \text{ and } D\gamma(0) = 0, \tag{65}$$

such that $\mathcal{M}^c := \text{graph}(\gamma)$ is a “locally invariant center manifold” for the semiflow generated by the unique maximal solutions of equation (25), respectively of the coupled system (64), provided by Theorem 3 above. Obviously, by construction and statement (65) \mathcal{M}^c is a submanifold of $h^{4+\alpha}(\mathcal{C})$ with tangent space $T_0(\mathcal{M}^c) = \text{Ker}(T_C)$, which is 8-dimensional on account of Lemma 4. In addition, the invariant manifold \mathcal{M}^c is “exponentially attractive” by Theorem 5.8 in [43]. This means here precisely the following: Due to Lemmata 1, 2, and 4 and Theorem 3, we may apply Theorem 5.8 in [43], and this theorem guarantees us, that there is some appropriate $\omega \in (0, \mu_1)$ —where μ_1 is the smallest positive eigenvalue of T_C respectively T_C^S by decomposition (62)—a positive constant $c = c(\omega, \beta, \alpha)$ and a neighborhood W of 0 in $h^{2+\beta}(\mathcal{C})$, such that

$$\begin{aligned} & \|\pi^s(\rho(t, \rho_0)) - \gamma(\pi^c(\rho(t, \rho_0)))\|_{h^{4+\alpha}(\mathcal{C})} \\ & \leq c \frac{e^{-\omega t}}{t^{1-\theta}} \|\pi^s(\rho_0) - \gamma(\pi^c(\rho_0))\|_{h^{2+\beta}(\mathcal{C})} \end{aligned} \tag{66}$$

for each $\rho_0 \in W$ and for $t \in (0, t^+(\rho_0))$, as long as there holds $\pi^c \rho(t, \rho_0) \in U$, and where we set $\theta := \frac{(2+\beta-\alpha)}{4}$. Here, $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$, $t \in [0, t^+(\rho_0))$ denotes the unique classical and maximal solution of initial value problem (41) from Theorem 3 above.

Part (ii) Estimate (66) tells us immediately, that the invariant manifold \mathcal{M}^c contains all smooth equilibria ρ of equation (25) which are contained in a sufficiently small neighborhood of 0 in $h^{2+\beta}(\mathcal{C})$, because the restriction of the linear projection π^c from (61) to $h^{2+\beta}(\mathcal{C})$ is particularly a continuous map from $h^{2+\beta}(\mathcal{C})$ onto $\text{ker}(T_C)$ by (62). Now, again following closely the proof of Theorem 1.2 in [13], respectively, of Proposition 6.4 in [14], we will show that— at least locally about 0—the manifold

\mathcal{M}^c consists only of equilibria of equation (25), and even more precisely: locally about 0 the 8-dimensional manifold \mathcal{M}^c consists only of smooth distance functions ρ , whose induced maps $\theta_\rho(x) := \exp_x(\rho(x) \nu_{\mathcal{C}}(x))$, $x \in \mathcal{C}$, from line (18) yield C^∞ -diffeomorphisms between \mathcal{C} and embedded tori in \mathbf{S}^3 , which are congruent to the Clifford torus \mathcal{C} modulo the action of $\text{M\"ob}(\mathbf{S}^3)$. Following the notation of the proof of Theorem 1.2 in [13], we call the set of these special equilibria of equation (25) “ \mathcal{M} ”. Now, in order to prove the above assertion, we firstly recall from Remark 3, that $\text{M\"ob}(\mathbf{S}^3) \cong \text{SO}^+(1, 4)$ is a 10-dimensional Lie-group whose Lie algebra $\text{Moeb}(\mathbf{S}^3)$ is the direct sum of the particular vector spaces ξ and Ω , introduced in Definition 2. Now we choose a system of 10 linearly independent conformal vector fields $\{\mathbf{v}_k\}_{k=1, \dots, 10} \subset \xi \oplus \Omega \subset \Gamma(T\mathbf{S}^3)$. For any tuple $z = (z_1, \dots, z_{10}) \in B_1^{10}(0) \subset \mathbf{R}^{10}$, the linear combination $V_z := \sum_{k=1}^{10} z_k \mathbf{v}_k \in \xi \oplus \Omega$ is smooth and generates—according to the first part of Definition 3—a smooth 1-parameter family of conformal transformations $U_z(t) \in \text{M\"ob}(\mathbf{S}^3)$, $t \in \mathbf{R}$, namely in terms of the flow $\Psi_z : \mathbf{S}^3 \times \mathbf{R} \rightarrow \mathbf{S}^3$, which is generated by the flow lines of the initial value problem

$$\partial_t y(t) = V_z(y(t)), \quad y(0) = y_0 \in \mathbf{S}^3, \tag{67}$$

for an unknown smooth function $y : \mathbf{R} \rightarrow \mathbf{S}^3$, and then setting: $U_z(t) := \Psi_z(\cdot, t)$, for every fixed $z \in B_1^{10}(0)$. Now, for technical reasons, we extend the above conformal vector fields $\{\mathbf{v}_k\}_{k=1, \dots, 10} \subset \Gamma(T\mathbf{S}^3)$ —and thus automatically any linear combination V_z of them—smoothly and with compact support $\{y \in \mathbf{R}^4 \mid \text{dist}(y, \mathbf{S}^3) \leq \frac{1}{4}\}$ from \mathbf{S}^3 onto entire \mathbf{R}^4 , and we shall not distinguish in our notation between these extensions and the original vector fields $\{\mathbf{v}_k\}_{k=1, \dots, 10}$. Hence, we can interpret problem (67) as an initial value problem on both the compact, closed manifold \mathbf{S}^3 and also on entire \mathbf{R}^4 with smooth right hand side, which in both cases additionally depends on the 10 real parameters (z_1, \dots, z_{10}) . We can therefore infer from Theorem 1.5.3 in [21], i.e., from classical theory about ordinary differential equations in Euclidean spaces, that the map

$$U_{(\cdot)}(1) : B_1^{10}(0) \rightarrow \text{M\"ob}(\mathbf{S}^3) \tag{68}$$

is C^∞ -smooth. Now, the images $U_z(t)(\mathcal{C})$ are embedded, compact tori in \mathbf{S}^3 being conformally equivalent to \mathcal{C} , for any $z \in B_1^{10}(0)$ and for any $t \in \mathbf{R}$. In particular, the tori $U_z(t)(\mathcal{C})$ are Willmore tori, i.e., any immersion $f_{z,t} : \Sigma \rightarrow \mathbf{S}^3$ parametrizing $U_z(t)(\mathcal{C})$ is a critical point of \mathcal{W} . Now we choose some small $\varepsilon > 0$, consider the tori $\mathcal{C}_z := U_z(1)(\mathcal{C})$ for any $z \in B_\varepsilon^{10}(0)$ and obtain via Fermi coordinates (10), respectively, via formula (11) a unique smooth function $\rho_z \equiv \rho_{\mathcal{C}_z}$, which measures the pointwise, signed geodesic distance between points $x \in \mathcal{C}$ and the torus \mathcal{C}_z . Hence, the function $[x \mapsto X(x, \rho_z(x))]$, appearing already in formula (12), parametrizes \mathcal{C}_z diffeomorphically, i.e.,

$$X(\cdot, \rho_z(\cdot)) : \mathcal{C} \xrightarrow{\cong} \mathcal{C}_z, \tag{69}$$

as a graph over \mathcal{C} via the exponential map, for any fixed $z \in B_\varepsilon^{10}(0)$, provided $\varepsilon > 0$ is sufficiently small. On account of the smoothness of the map $U_{(\cdot)}(1)$ in (68), and on account of formula (11) combined with the smoothness of the Fermi coordinate functions S and Λ in (10), we easily infer also the smoothness of the non-linear operator

$$\rho_{(\cdot)} : B_\varepsilon^{10}(0) \longrightarrow h^{4+\alpha}(\mathcal{C}), \tag{70}$$

assigning to $z \in B_\varepsilon^{10}(0)$ the unique smooth distance function ρ_z , that we have just obtained via formula (11). Since the tori \mathcal{C}_z are Willmore tori, every distance function ρ_z is an equilibrium of the corresponding evolution equation (25), for $z \in B_\varepsilon^{10}(0)$, which implies that

$$\rho_z \in \mathcal{M}^c \quad \forall z \in B_\varepsilon^{10}(0) \tag{71}$$

according to estimate (66), provided $\varepsilon > 0$ is sufficiently small, where we have used that $U_0(1) = \text{id}_{\mathbb{S}^3}$ implies that $\rho_0 = 0$ in $h^{4+\alpha}(\mathcal{C})$. As in the proof of Theorem 1.2 in [13], we consider now the composition

$$F := \pi^c \circ \rho_{(\cdot)} : B_\varepsilon^{10}(0) \longrightarrow \ker(T_{\mathcal{C}}), \tag{72}$$

for $\varepsilon > 0$ as small as in (71). We note here for later use that statement (70) implies, that the map F is a smooth map between finite-dimensional flat manifolds, and moreover, we note that $\rho_0 = 0$ implies that $F(0) = 0$ in $\ker(T_{\mathcal{C}})$. Now, the vector space $\Omega \oplus \xi \subset T(\mathbb{S}^3)$ of all conformal vector fields on \mathbb{S}^3 is 10-dimensional, whereas the vector space $\mathbf{Moeb}^\perp(\mathcal{C}) \equiv \xi^N \oplus \Omega^N \subset \Gamma(N\mathcal{C})$ of normal conformal directions along \mathcal{C} is only 8-dimensional by our Lemma 4, and the proofs of Lemmata 3.4 and 3.5 in [48] show that the kernel of the homomorphism $\Omega \longrightarrow \Omega^N$, mapping $V \mapsto V^N$, is two-dimensional, whereas the projection of ξ onto ξ^N is isomorphic. Hence, we may assume without loss of generality that the 10 basis vectors \mathbf{v}_k of $\Omega \oplus \xi$ are chosen in such a way that $(\mathbf{v}_k)^N \equiv 0$, for $k = 9, 10$, i.e., that $\{(\mathbf{v}_k)^N\}_{k=1,\dots,8}$ constitutes a basis of the vector space $\mathbf{Moeb}^\perp(\mathcal{C}) = \Omega^N \oplus \xi^N$ of normal conformal directions along \mathcal{C} . Moreover, we should notice here, that the unique distance function $\rho_z \equiv \rho_{\mathcal{C}_z}$ satisfying (69) can be written down more precisely by means of formula (11) and our definition $\mathcal{C}_z := \Psi_z(\mathcal{C}, 1)$:

$$\rho_z(x) = \Lambda \circ (S|_{\Psi_z(\mathcal{C},1)})^{-1}(x), \quad \forall x \in \mathcal{C}, \tag{73}$$

and for any fixed $z \in B_\varepsilon^{10}(0)$. Taking now also $V_0 \equiv 0$ on \mathbb{S}^3 and thus $\Psi_0(\cdot, t) \equiv \text{Id}_{\mathbb{S}^3}$, $\forall t \in \mathbf{R}$, and the definition of the map S in (10) into account, we can compute the partial derivative with respect to z_k of the smooth composition (73) in $z = 0$ by means of the chain rule:

$$\begin{aligned} D_{z_k} \rho_z(x)|_{z=0} &= \langle (\nabla^{\mathbb{S}^3} \Lambda \circ \text{Id}_{\mathcal{C}})(x), D_{z_k} \Psi_z(x, 1)|_{z=0} \rangle_{\mathbb{S}^3} \\ &= \langle \nu_{\mathcal{C}}(x), D_{z_k} \Psi_z(x, 1)|_{z=0} \rangle_{\mathbb{S}^3} \quad \forall x \in \mathcal{C}, \end{aligned} \tag{74}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{S}^3}$ denotes the Euclidean scalar product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ restricted to $T\mathbb{S}^3$. In (74), we have also used the general fact that the gradient of the signed distance function, being defined in a narrow tube about some smooth, orientable, and compact surface $M \hookrightarrow \mathbb{S}^3$ yields exactly one of the two globally defined unit normals in $N(M)$, when restricted to M itself.¹³ Again using the fact that our equation (67) can be extended to an ordinary differential equation in \mathbb{R}^4 with 10 additional real parameters, we can apply here formula (1.5.3) in Theorem 1.5.3 in [21] and compute exactly

$$\begin{aligned} D_{z_k} \Psi_z(x, 1)|_{z=0} &= D_{z_k} \Psi_z(x, 1)|_{z=0} - D_{z_k} \Psi_z(x, 0)|_{z=0} \\ &= \int_0^1 \frac{d}{dt} (D_{z_k} \Psi_z(x, t))|_{z=0} dt = \int_0^1 D_{z_k} V_z(\Psi_z(x, t))|_{z=0} dt \\ &= \int_0^1 \mathbf{v}_k(x) dt = \mathbf{v}_k(x), \quad \forall x \in \mathcal{C}, \end{aligned}$$

for $k = 1, \dots, 10$, where we have used the fact that also the extended vector fields V_z satisfy $V_0 \equiv 0$ on entire \mathbb{R}^4 and that therefore $\Psi_0(\cdot, t)$ reduces to $\text{Id}_{\mathbb{R}^4}$, for every $t \in \mathbb{R}$. Together with (74), we arrive at the following formula:

$$D_{z_k} \rho_z(x)|_{z=0} = \langle \nu_{\mathcal{C}}(x), \mathbf{v}_k(x) \rangle_{\mathbb{S}^3} \equiv \langle \nu_{\mathcal{C}}(x), (\mathbf{v}_k)^N(x) \rangle_{\mathbb{S}^3}, \quad \forall x \in \mathcal{C}, \quad (75)$$

for $k = 1, \dots, 10$. Using the isomorphism (46) between the vector space $\ker(T_{\mathcal{C}})$ and the 8-dimensional vector space $\mathbf{Moeb}^\perp(\mathcal{C}) \equiv \xi^N \oplus \Omega^N$, we obtain from the chosen basis vectors $\{(\mathbf{v}_k)^N\}_{k=1, \dots, 8}$ of $\mathbf{Moeb}^\perp(\mathcal{C})$ unique coordinate functions $\{v_k\}_{k=1, \dots, 8}$ which form a basis of the vector space $\ker(T_{\mathcal{C}})$ and which turn out to satisfy by equation (75):

$$D_{z_k} \rho_z|_{z=0} = \langle (\mathbf{v}_k)^N, \nu_{\mathcal{C}} \rangle_{\mathbb{S}^3} = v_k \in \ker(T_{\mathcal{C}}), \quad \text{for } k = 1, \dots, 8.$$

Hence, on account of the definition of F in (72) and the chain rule, the partial derivative of F in $z = 0$ in direction of the coordinate z_k reads

$$D_{z_k} F(0) = \pi^c(D_{z_k} \rho_z|_{z=0}) = \pi^c(v_k) = v_k, \quad \text{for } k = 1, \dots, 8,$$

showing that the entire differential

$$DF(0) : \mathbb{R}^{10} \longrightarrow \ker(T_{\mathcal{C}})$$

is an epimorphism. Since we also know already that there holds $F(0) = 0$, we can now infer from the classical *Open Mapping Theorem* for C^1 -maps between finite-dimensional vector spaces, that there is some small open ball $B_\delta(0) \subset U$ about 0 in $\ker(T_{\mathcal{C}})$ —where U is as in (65) and $\delta > 0$ depends on the size of ε in (71) and (72) depending in turn on the size of U on account of statement (66)—which satisfies $B_\delta(0) \subset F(B_\varepsilon^{10}(0))$. By definition of the map F in (72), this means that the projection

¹³ See here our construction of Fermi coordinates about \mathcal{C} in (8)–(10) and formula (35) in [35] for the similar case in which the ambient space is \mathbb{R}^3 .

π^c of $h^{4+\alpha}(\mathcal{C})$ onto $\ker(T_{\mathcal{C}})$ restricted to the set of distance functions $\{\rho_z \mid z \in B_\varepsilon^{10}(0)\}$ covers $B_\delta(0)$ in $\ker(T_{\mathcal{C}})$, i.e.,

$$\pi^c(\{\rho_z \mid z \in B_\varepsilon^{10}(0)\}) \supseteq B_\delta(0), \tag{76}$$

if $\delta > 0$ is sufficiently small. Furthermore, on account of statement (71), we know that $\{\rho_z \mid z \in B_\varepsilon^{10}(0)\}$ is contained in the 8-dimensional center manifold \mathcal{M}^c , i.e., in the graph of the function $\gamma \in C^m(U, h_s^{4+\alpha}(\mathcal{C}))$ in (65) over the neighborhood U of 0 in $\ker(T_{\mathcal{C}})$, provided $\varepsilon > 0$ has been chosen sufficiently small. We can therefore infer from (76) the sharper statement

$$B_\delta(0) \subseteq \pi^c(\{\rho_z \mid z \in B_\varepsilon^{10}(0)\}) = \pi^c(\{\rho_z \mid z \in B_\varepsilon^{10}(0)\} \cap \text{graph}(\gamma)). \tag{77}$$

Now, recalling the direct decomposition (62) of $h^{4+\alpha}(\mathcal{C})$ into the closed subspaces $\ker(T_{\mathcal{C}})$ and $h_s^{4+\alpha}(\mathcal{C})$, we trivially have $\pi^c(v, \gamma(v)) = v$ for every $v \in U$ on account of the definition of γ in (65). We can therefore refine statements (76) and (77) even further:

$$B_\delta(0) \subseteq \pi^c(\{\rho_z \mid z \in B_\varepsilon^{10}(0)\} \cap \text{graph}(\gamma|_{B_\delta(0)})) \subseteq B_\delta(0),$$

implying that the sets $\{\rho_z \mid z \in B_\varepsilon^{10}(0)\} \cap \text{graph}(\gamma|_{B_\delta(0)})$ and $\text{graph}(\gamma|_{B_\delta(0)})$ exactly coincide, provided $\delta > 0$ is sufficiently small. But this is equivalent to the statement that $\text{graph}(\gamma|_{B_\delta(0)})$ is contained in $\{\rho_z \mid z \in B_\varepsilon^{10}(0)\}$. Recalling now the definition of the set \mathcal{M} of particular equilibria of equation (25) at the beginning of Part (ii) of this proof and the fact that all the tori \mathcal{C}_z in (69) are—just by construction—conformally equivalent to the Clifford torus, we therefore finally conclude that $\text{graph}(\gamma|_{B_\delta(0)})$ is contained in the set of smooth equilibria of equation (25) of type \mathcal{M} , which means—again by (65)—that at least locally about the zero-function the center manifold $\mathcal{M}^c = \text{graph}(\gamma)$ only consists of smooth equilibria of equation (25) of type \mathcal{M} .

Part (iii) As in the proof of Theorem 1.2 in [13], we can infer from the result of step (ii), that the locally “reduced flow” of equation (25) on $\mathcal{M}^c = \text{graph}(\gamma)$ —which is determined by flow lines $\{z_t\}$ of class $C^0([0, t^+), \ker(T_{\mathcal{C}})) \cap C^\infty((0, t^+), \ker(T_{\mathcal{C}}))$ of the “reduced evolution equation”

$$\partial_t z_t + T_{\mathcal{C}}^c(z_t) = \pi^c g(z_t, \gamma(z_t)), \quad z_0 \in B_\delta(0) \subset \ker(T_{\mathcal{C}}), \tag{78}$$

according to the decomposition in (64)—consists of equilibria only, i.e., the locally “reduced flow” of equation (25) does not move at all, if it starts moving in a sufficiently small neighborhood about 0 in $\ker(T_{\mathcal{C}})$. In particular, the zero-function is a stable equilibrium of the reduced equation (78). Hence, by Proposition 3.2, respectively, Theorem 3.3 in [42] also the point $(0, \gamma(0)) = (0, 0)$ is a stable equilibrium of the original evolution equation (25) in $h^{2+\beta}(\mathcal{C})$. This means precisely that there exists for every neighborhood W_1 of 0 in $h^{2+\beta}(\mathcal{C})$ another neighborhood W_2 of 0 in $h^{2+\beta}(\mathcal{C})$ such that any solution of evolution equation (25) exists globally and stays within W_1 , provided its initial value ρ_0 is contained in W_2 . Combining this with statement (66), we obtain even some more precise information: There is a neighborhood W of 0 in

$h^{2+\beta}(\mathcal{C})$, depending on the size of the neighborhood U of 0 in $\ker(T_C)$ from line (65) such that any flow line of evolution equation (25), which starts moving in some initial function $\rho_0 \in W$, exists globally and approaches the center manifold $\mathcal{M}^c = \text{graph}(\gamma)$ asymptotically in the $h^{4+\alpha}(\mathcal{C})$ -norm for all $t > 0$, according to estimate (66). See here also some technical explanations in the proof of Theorem 6.5 in [14].

Part (iv) Now we can follow exactly the last step of the proof of Theorem 1.2 in [13], i.e., we can apply the reasoning of the proof of Theorem 6.5(b) in [14]—in which the statements of Propositions 6.2 and 6.4 and Theorem 6.5(a) of [14] correspond to the results of our steps (i)–(iii) above—in combination with the bootstrap-technique of Proposition 6.6 in [14], in order to draw the following conclusions: For any fixed $k \in \mathbb{N}$ and for some appropriately chosen $\omega \in (0, \mu_1)$ —where μ_1 is the smallest positive eigenvalue of T_C respectively of T_C^s —there exists a neighborhood $W = W(k, \omega)$ of 0 in $h^{2+\beta}(\mathcal{C})$ with the following properties: Given an initial function $\rho_0 \in W$, the unique maximal and smooth solution $\{\rho(t, \rho_0)\}_{t \in [0, t^+(\rho_0))}$ of initial value problem (41) exists globally, and there exist a constant $c = c(k, \omega) > 0$ and a unique function $z_0 = z_0(\rho_0) \in B_\delta(0) \subset \ker(T_C)$, such that

$$\begin{aligned} & \| (\pi^c(\rho(t, \rho_0)), \pi^s(\rho(t, \rho_0))) - (z_0, \gamma(z_0)) \|_{C^k(\mathcal{C})} \\ & \leq c e^{-\omega t} \| \pi^s(\rho_0) - \gamma(\pi^c(\rho_0)) \|_{h^{2+\beta}(\mathcal{C})} \end{aligned} \tag{79}$$

holds for all $t \geq 1$. Now, again using the result of step (ii), we know that for $z_0 \in B_\delta(0)$ the pair $(z_0, \gamma(z_0)) \in \mathcal{M}^c$ actually has to be contained in the set of equilibria of equation (25) of type \mathcal{M} . Hence, $(z_0, \gamma(z_0))$ is a smooth distance function on \mathcal{C} , whose induced map $\theta_{(z_0, \gamma(z_0))}(x) = \exp_x((z_0, \gamma(z_0))(x) \nu_{\mathcal{C}}(x))$, $x \in \mathcal{C}$, from line (18) yields a C^∞ -smooth diffeomorphism between \mathcal{C} and some embedded torus in \mathbb{S}^3 , which is congruent to the Clifford torus \mathcal{C} in \mathbb{S}^3 . Thus statement (79) guarantees that having fixed some $k \in \mathbb{N}$ and some appropriate $\omega \in (0, \mu_1)$, for any initial distance function ρ_0 taken from a sufficiently small neighborhood $W = W(k, \omega)$ of 0 in $h^{2+\beta}(\mathcal{C})$ the unique maximal and smooth solution $\rho(\cdot, \rho_0)$ of equation (25) exists globally and converges fully to a smooth distance function $(z_0, \gamma(z_0))$, which yields—via formula (18)—a smooth diffeomorphic parametrization $\theta_{(z_0, \gamma(z_0))} : \mathcal{C} \xrightarrow{\cong} \mathbb{S}^3$ of an embedded torus in \mathbb{S}^3 , which is conformally equivalent to the Clifford torus, and this convergence is at an exponential rate with respect to the $C^k(\mathcal{C})$ -norm as $t \rightarrow \infty$. Finally, we remark that we require the initial immersion $F_1 : \Sigma \rightarrow \mathbb{S}^3$ to be C^∞ -smooth. On account of Theorem 1 in [17], this implies that there is a unique smooth and maximal flow line $\{\mathcal{P}(t, 0, F_1)\}_{t \in [0, T_{\max})}$ of the MIWF, starting in F_1 at time $t = 0$, and moreover the proof of Theorem 1 in [17] shows that any given C^∞ -smooth solution $\{f_t\}_{t \in [0, T)}$ of the “relaxed MIWF-equation” (24) on $\Sigma \times [0, T)$, with $f_0 = F_1$ and with arbitrarily fixed $T > 0$, yields the unique smooth flow line $\{\mathcal{P}(\cdot, 0, F_1)\}_{t \in [0, T)}$ of the original MIWF, starting in F_1 at time $t = 0$, by means of reparametrization with a C^∞ -smooth family of C^∞ -smooth diffeomorphisms from Σ onto itself.¹⁴ Hence, on account of the “correspondence” between C^∞ -smooth flow lines of the relaxed MIWF-equation (24) and C^∞ -smooth flow lines of evolution equation (25)—as explained in formulae (12)–(25) of Sect. 2—the above results prove the assertion of this theorem. \square

¹⁴ See here also the second part of Theorem 5 below.

Proof of Theorem 2 First of all, we fix an integer $k \geq 4$ and some $\alpha \in (0, 1)$ as in the statement of the theorem. On account of the compactness of the torus Σ and on account of the conformal invariance of the MIWF, we may assume that the image $f_0(\Sigma)$ of any immersion $f_0 : \Sigma \rightarrow \mathbf{S}^3$ satisfying $\|f_0 - F^*\|_{C^{k,\alpha}(\Sigma, \mathbf{R}^4)} < \varepsilon$ is still contained in $\mathbf{S}^3 \setminus \{(0, 0, 0, 1)\}$, for any sufficiently small $\varepsilon > 0$, to be further specified below. Again on account of the invariance of the MIWF and also of the Willmore functional itself, we may therefore compose the given C^k -local minimizer F^* with stereographic projection $\mathcal{S} : \mathbf{S}^3 \setminus \{(0, 0, 0, 1)\} \rightarrow \mathbf{R}^3$ and prove the entire statement of the theorem for the MIWF in \mathbf{R}^3 . We choose this detour in order to easily adopt the basic ideas of the proofs of Lemma 4.1 in [8] and of Theorem 1.2 in [10], especially the technique of Theorem 5.1 in [46], respectively, of Lemma 4.1 in [10] to represent immersions into \mathbf{R}^n as normal graphs over some fixed smooth reference immersion—together with corresponding estimates—and then to reduce the complexity of the non-linear MIWF-equation by means of exactly this geometrically motivated graph-ansatz, as elaborated in formulae (4.3)–(4.6) of [8]. Now, we require by assumption that our initial immersion $f_0 : \Sigma \rightarrow \mathbf{R}^3$ is smooth and satisfies

$$\|f_0 - F^*\|_{C^{k,\alpha}(\Sigma, \mathbf{R}^3)} < \varepsilon \tag{80}$$

for some sufficiently small $\varepsilon \in (0, \varepsilon_0)$, to be determined only below in (117). As in the proof of Lemma 4.1 in [8], we conclude from condition (80) that we can represent the immersion f_0 as a graph over the given Willmore immersion F^* , which means precisely on account of Theorem 5.1 in [46]: There is a smooth section N_0 of the normal bundle of F^* and a smooth diffeomorphism $\Phi_0 : \Sigma \xrightarrow{\cong} \Sigma$, such that there holds:

$$f_0 \circ \Phi_0 = F^* + N_0 \quad \text{on } \Sigma. \tag{81}$$

Furthermore, we infer from condition (80) and from the proof of Theorem 5.1 in [46] combined with Lemma 3.1 in [15] for $k = 0 = q$,¹⁵ that there is some continuous and monotonically increasing function $C^o : [0, \varepsilon_0] \rightarrow \mathbf{R}_+$ with $C^o(0) = 0$, depending on the immersion F^* and on the above $k \geq 4$, such that

$$\|N_0\|_{C^{k,\alpha}(\Sigma)} = \|f_0 \circ \Phi_0 - F^*\|_{C^{k,\alpha}(\Sigma)} < C^o(\varepsilon) \tag{82}$$

holds for the same positive, small ε as the one in (80). In order to smartly use (81) and (82), we consider now the modified Cauchy-problem

$$\partial_t^{\perp_{\tilde{f}_t}}(\tilde{f}_t) = -\frac{1}{|A_{\tilde{f}_t}^0|^4} \nabla_{L^2} \mathcal{W}(\tilde{f}_t), \quad \tilde{f}_0 = f_0 \circ \Phi_0 \quad \text{on } \Sigma \tag{83}$$

of the MIWF, which is solved by any smooth reparametrization $\{\mathcal{P}(t, 0, f_0) \circ \Phi_t\}_{t \geq 0}$ of the smooth flow line $\mathcal{P}(\cdot, 0, f_0)$ of the MIWF, starting in f_0 . Now, again Theorem

¹⁵ See here also Sect. 2 in [12], Sect. 3 in [28] and [11].

5.1 in [46], condition (80), and equation (81) motivate us to momentarily assume the existence of a—sufficiently smooth—short-time solution $\{\tilde{f}_t\}$ of equation (83) of the particularly simple form $\tilde{f}_t = F^* + N_t$, for a family of normal sections N_t along F^* , starting in the smooth immersion $F^* + N_0 = f_0 \circ \Phi_0$ at time $t = 0$. In order to prove this intuitive idea rigorously, we follow the lines of the proof of Lemma 4.1 in [8], that is we reformulate the short-time existence problem at hand equivalently by means of the function $\phi_t := \langle N_t, \nu_{F^*} \rangle_{\mathbf{R}^3}$, the signed length of any fixed normal section N_t along F^* , and we argue as in formulae (4.4)–(4.6) of [8] that a sufficiently smooth family of immersions $\tilde{f}_t = F^* + N_t$ solves equation (83) classically on some short-time interval, if and only if the family of functions $\{\phi_t\} = \{\langle N_t, \nu_{F^*} \rangle_{\mathbf{R}^3}\}$ classically solves the uniformly parabolic quasilinear 4th order equation

$$\begin{aligned} \partial_t(\phi_t) + \frac{1}{2} \frac{1}{|A_{F^*+\phi_t\nu_{F^*}}^0|^4} g_{F^*+\phi_t\nu_{F^*}}^{ij} g_{F^*+\phi_t\nu_{F^*}}^{kl} \nabla_{ijkl}^{F^*}(\phi_t) \\ = B(\cdot, \phi_t, D_x\phi_t, D_x^2\phi_t, D_x^3\phi_t) \quad \text{on } \Sigma, \end{aligned} \tag{84}$$

on the same time interval, where $B : \Sigma \times \mathbf{R}^{1+2+4+8} \rightarrow \mathbf{R}$ denotes a globally defined function, which is rational in its 15 real arguments and has smooth coefficients—depending on the fixed smooth immersion F^* only—at least as long as there holds

$$\|N_t\|_{C^2(\Sigma)} \equiv \|\tilde{f}_t - F^*\|_{C^2(\Sigma)} < \tilde{\delta}(F^*), \tag{85}$$

for some sufficiently small chosen positive number $\tilde{\delta}(F^*) > 0$. Indeed, as in formula (4.5) of [8], one can compute that inequality (85) implies

$$|P^{\perp_{\tilde{f}_t}}(\hat{N}_t)| \geq \frac{1}{2} |\hat{N}_t| \quad \text{on } \Sigma \tag{86}$$

for the projection $P^{\perp_{\tilde{f}_t}}(\hat{N}_t)$ of any smooth normal field \hat{N}_t along F^* onto the normal bundle of \tilde{f}_t , and obviously inequality (85) also implies

$$\min_{\Sigma} |A_{F^*+\phi_t\nu_{F^*}}^0|^2 \geq \frac{1}{2} \min_{\Sigma} |A_{F^*}^0|^2 > 0, \tag{87}$$

which actually lets us adopt the decisive computation in formulae (4.4) and (4.5) of [8] without any significant changes. One can check elementarily by means of the computation in formula (4.4) of [8] that indeed the geometrically motivated idea, to solve the modified equation (83) by means of functions $\tilde{f}_t = F^* + \phi_t \nu_{F^*}$, reduces the fully non-linear MIWF-equation (2) to the quasilinear parabolic equation (84), thus yielding the equivalence of the existence of a short-time solution of equation (83) in graph representation $\tilde{f}_t = F^* + N_t$ to the existence of a short-time solution $\{\phi_t\}$ of equation (84). As in formula (4.7) of [8], we should note here that condition (82) implies

$$\|\phi_0\|_{C^{j,\alpha}(\Sigma)} < C(F^*, j) C^o(\varepsilon), \quad \text{for each } j = 1, \dots, k, \tag{88}$$

for the initial function of the desired short-time solution $\{\phi_t\}$ of equation (84), where $C^o(\varepsilon)$ denotes here the same function as in (82) and $C(F^*, j)$ some positive constant only depending on F^* and j . Now, in order to actually prove the existence of a unique maximal strict solution¹⁶ $\{\phi_t\}_{t \in [0, T_{\max})}$ of the scalar equation (84)—starting in $\langle N_0, \nu_{F^*} \rangle_{\mathbf{R}^3}$ —with values in the Banach space $C^{4,\alpha}(\Sigma, \mathbf{R})$, we shall introduce the parabolic Hölder space

$$Z_{T,\beta} := C^{4+\beta, 1+\frac{\beta}{4}}(\Sigma \times [0, T], \mathbf{R})$$

for any fixed $\beta \in (\alpha, 1)$ and $T > 0$, and its open subsets

$$U_{F^*,\beta,\varrho,T} := \left\{ \{\phi_t\} \in Z_{T,\beta} \mid \|\phi_t\|_{C^2(\Sigma)} < \varrho \quad \forall t \in [0, T] \right\} \tag{89}$$

with $0 < \varrho < \tilde{\delta}(F^*)$ that small, such that the normal field $N_t := \phi_t \nu_{F^*}$ satisfies inequality (85) $\forall t \in [0, T]$, for any fixed function $\varphi \in U_{F^*,\beta,\varrho,T}$. Now, by statements (85)–(87) and (89), the non-linear differential operator

$$\begin{aligned} \partial_t + \frac{1}{2} \frac{1}{|A_{F^*+(\cdot)\nu_{F^*}}^0|^4} g_{F^*+(\cdot)\nu_{F^*}}^{ij} g_{F^*+(\cdot)\nu_{F^*}}^{kl} \nabla_{ijkl}^{F^*} : \\ U_{F^*,\beta,\varrho,T} \subset Z_{T,\beta} \longrightarrow C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbf{R}) \end{aligned} \tag{90}$$

is well defined for any fixed $0 < \varrho < \tilde{\delta}(F^*)$ as in (89), and we can infer exactly as in Theorem 2 in [17], that it is a C^1 -map, that the highest order term of its Fréchet derivative in any chosen $\varphi \in U_{F^*,\beta,\varrho,T}$ is the uniformly parabolic linear operator

$$\begin{aligned} \partial_t + \mathcal{L}_{F^*,\varphi} := \partial_t + \frac{1}{2} \frac{1}{|A_{F^*+\varphi\nu_{F^*}}^0|^4} g_{F^*+\varphi\nu_{F^*}}^{ij} g_{F^*+\varphi\nu_{F^*}}^{kl} \nabla_{ijkl}^{F^*} : \\ Z_{T,\beta} \longrightarrow C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbf{R}) \end{aligned} \tag{91}$$

for any fixed $\beta \in (\alpha, 1)$ and $T > 0$, and that this linear operator satisfies all requirements of Proposition 2 in [19]. We can therefore argue as in the proof of that proposition, that for any fixed $\varphi \in U_{F^*,\beta,\varrho,T}$ the linear differential operators

$$\begin{aligned} \mathcal{L}_{F^*,\varphi_t} := \frac{1}{2} \frac{1}{|A_{F^*+\varphi_t\nu_{F^*}}^0|^4} g_{F^*+\varphi_t\nu_{F^*}}^{ij} g_{F^*+\varphi_t\nu_{F^*}}^{kl} \nabla_{ijkl}^{F^*} : \\ C^{4,\beta'}(\Sigma, \mathbf{R}) \longrightarrow C^{0,\beta'}(\Sigma, \mathbf{R}) \end{aligned} \tag{92}$$

are $(\mathcal{E}, \frac{3}{4}\pi, 4)$ -elliptic for some appropriate constant $\mathcal{E} = \mathcal{E}(F^*, \varrho) > 1$ in the terminology of [40], p. 228, for any fixed $\beta' \in (0, \beta]$ and for any fixed $t \in [0, T]$. We can therefore derive from Theorem 3.3 in [40], that the linear operators $\mathcal{L}_{F^*,\varphi_t}$ in (92)

¹⁶ See here Definition 4.1.1 in [27].

are sectorial in the Hölder space $C^{0,\beta'}(\Sigma, \mathbf{R})$ ¹⁷ with constants $\omega > 0$ and $\mathcal{N} > 1$ depending only on F^* , $\|\varphi\|_{C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T])}$ and β' , in the terminology of Theorem 3.3 in [40], for any fixed $\beta' \in (0, \beta)$ and uniformly for every fixed $t \in [0, T]$, where $T > 0$ and $\beta \in (\alpha, 1)$ have already been chosen above in (89). Taking now $\beta' = \alpha$ and recalling also the smoothness of the initial function $\phi_0 := \langle N_0, \nu_{F^*} \rangle_{\mathbf{R}^3}$, we may therefore apply Theorems 8.1.1 and 8.1.3 and Proposition 8.2.1 in [27] with Banach space pair $X := C^{0,\alpha}(\Sigma, \mathbf{R})$ and $D := C^{4,\alpha}(\Sigma, \mathbf{R})$, and we obtain the existence of a unique and maximal strict solution $\{\phi_t\}_{t \in [0, T_{\max})}$ of equation (84) with values in the Banach space $C^{4,\alpha}(\Sigma, \mathbf{R})$, meeting the additional condition

$$\|\phi_t\|_{C^2(\Sigma)} < \varrho \quad \forall t \in [0, T_{\max}) \tag{93}$$

from line (89) and starting in the initial function $\phi_0 := \langle N_0, \nu_{F^*} \rangle_{\mathbf{R}^3}$ at time $t = 0$, provided ε is sufficiently small in view of condition (88). Moreover, this solution is of class $C^\gamma([0, T], C^{4,\alpha}(\Sigma, \mathbf{R})) \cap C^{1,\gamma}([0, T], C^{0,\alpha}(\Sigma, \mathbf{R}))$, $\forall \gamma \in (0, 1)$ and for every $T \in (0, T_{\max})$, where $0 < \alpha < 1$ had already been chosen at the beginning of the proof. We may therefore apply Proposition 3 of [17]—successively for every $k \in \mathbf{N}_0$ —in order to conclude that our maximal solution $\{\phi_t\}$ of equation (84) satisfies the following Schauder a-priori estimates:

$$\begin{aligned} & \|\phi_t\|_{C^{4+l+\mu,1+\frac{l+\mu}{4}}(\Sigma \times [0,T])} \\ & \leq C \left(\|B(\cdot, \phi(\cdot, t), \dots, \partial_{x_{xx}}\phi(\cdot, t))\|_{C^{l+\mu, \frac{l+\mu}{4}}(\Sigma \times [0,T])} \right. \\ & \quad \left. + \|\phi_t\|_{L^\infty(\Sigma \times [0,T])} + \|\phi_0\|_{C^{4+l+\mu}(\Sigma)} \right), \end{aligned} \tag{94}$$

for every fixed $T \in (0, T_{\max})$, for every $l \in \mathbf{N}_0$ and every $\mu \in (0, \alpha)$, and for some large constant $C = C(\Sigma, F^*, T, \mu, l)$. From this result, we immediately infer the C^∞ -smoothness of our constructed, maximal solution $\{\phi_t\}$ of equation (84) and thus automatically the existence of a corresponding solution of equation (83) of the special, desired form $\tilde{f}_t = F^* + N_t$, with $N_t = \phi_t \nu_{F^*}$, being of class $C^\infty(\Sigma \times [0, T], \mathbf{R}^3)$ for every $T \in [0, T_{\max})$ and starting in the smooth immersion $\tilde{f}_0 = f_0 \circ \Phi_0$ at time $t = 0$. As pointed out in the proofs of Lemma 4.1 in [8] and Theorem 1.2 in [10], this smooth solution $\tilde{f}_t = F^* + N_t$ of equation (84) can be reparametrized by a smooth family of smooth diffeomorphisms $\Psi_t : \Sigma \xrightarrow{\cong} \Sigma$, with $\Psi_0 = \text{Id}_\Sigma$, such that

$$\tilde{f}_t \circ \Psi_t = \mathcal{P}(t, 0, f_0 \circ \Phi_0) \equiv \mathcal{P}(t, 0, f_0) \circ \Phi_0 \quad \text{for } t \in [0, T_{\max}), \tag{95}$$

which will be of significant importance later on. Now we fix some positive $\sigma < \min\{\delta, \varrho\}$ —where δ was determined in assumption (4) and ϱ in line (89)—such that the Lojasiewicz-Simon-gradient-inequality for the Willmore functional, Theorem 3.1 in [8], holds for every C^4 -immersion $f : \Sigma \rightarrow \mathbf{R}^3$ with $\|f - F^*\|_{C^4(\Sigma)} \leq \sigma$, and we choose the $\varepsilon > 0$ in (80) and (82) that small, such that we have: $C^o(\varepsilon) < \sigma$. As in [8],

¹⁷ See here also Definition 2.0.1 in [27].

p. 359, or as in [10], p. 2190, we shall now choose a possibly smaller “maximal” time $T(\sigma) \in (0, T_{\max}]$ —depending on σ , but not on ε —by means of imposing the following additional, quantitative smallness condition on the normal sections $N_t = \phi_t \nu_{F^*}$:

$$\| \tilde{f}_t - F^* \|_{C^k(\Sigma, \mathbf{R}^3)} \equiv \| N_t \|_{C^k(\Sigma, \mathbf{R}^3)} \leq \sigma \quad \forall t \in [0, T(\sigma)). \tag{96}$$

We should note here, that condition (96) implies the inequality

$$\| \phi_t \|_{C^j(\Sigma, \mathbf{R})} \leq C(F^*, j) \sigma \quad \forall t \in [0, T(\sigma)), \tag{97}$$

for each $j = 1, \dots, k$, similarly to the relation between estimates (82) and (88). Comparing initial condition (82) with the additional smallness condition (96) and recalling that we have $C^o(\varepsilon) < \sigma$, we can conclude that

$$0 < T(\sigma) \leq T_{\max} \leq \infty.$$

In order to prove that statement (96) actually holds for $T(\sigma) = \infty$, we shall follow the strategy of the proof of Theorem 1.2 in [9], pp. 2190–2191: We assume firstly, that the time $T(\sigma)$ was finite and that there would hold $T(\sigma) < T_{\max}$. Now, first of all our conditions (93) and (96) imply that estimates (85)–(87) do actually hold for the immersions $F^* + \phi_t \nu_{F^*}$ on $[0, T(\sigma)]$. Taking also estimate (97) into account, we can therefore roughly estimate the right hand side in (84) by

$$\| B(\cdot, \phi_t, D_x \phi_t, D_x^2 \phi_t, D_x^3 \phi_t) \|_{L^\infty(\Sigma)} \leq C(\Sigma, F^*, \sigma) \quad \forall t \in [0, T(\sigma)], \tag{98}$$

for some appropriate, large constant $C = C(\Sigma, F^*, \sigma)$, and we can easily verify by the same reasoning that the coefficients on the left hand side of (84), respectively, in (91) with $\{\varphi_t\} = \{\phi_t\}$ are continuous and uniformly bounded on $\Sigma \times [0, T(\sigma)]$ by another large constant $C = C(\Sigma, F^*, \sigma)$. We can therefore apply Proposition 2 in [19] to the linear operator $\mathcal{L}_{F^*, \phi}$ in (91)—here on $\Sigma \times [0, T(\sigma)]$ and with any fixed $p \in (1, \infty)$ —and we infer from that proposition, together with estimates (88) and (98) and with $C^o(\varepsilon) < \sigma$, that the smooth solution $\{\phi_t\}$ of equation (84) has bounded norm in the parabolic L^p -space¹⁸

$$X_{T,p} := W^{1,p}([0, T]; L^p(\Sigma, \mathbf{R})) \cap L^p([0, T]; W^{4,p}(\Sigma, \mathbf{R})), \tag{99}$$

for any fixed $p \in (1, \infty)$, precisely

$$\begin{aligned} \| \{\phi_t\} \|_{X_{T(\sigma),p}} &\leq C(\Sigma, F^*, \sigma, T(\sigma), p) \left(\| B(\cdot, \phi_t, \dots, D_x^3 \phi_t) \|_{L^p([0, T(\sigma)]; L^p(\Sigma, \mathbf{R}))} \right. \\ &\quad \left. + \| \phi_0 \|_{W^{4,p}(\Sigma, \mathbf{R})} \right) \leq C^*(\Sigma, F^*, \sigma, T(\sigma), p), \end{aligned} \tag{100}$$

for some large constant $C^*(\Sigma, F^*, \sigma, T(\sigma), p)$. Moreover, as explained in Theorem B.5 in [47], we can use interpolation results from [5], in order to obtain for any

¹⁸ See here pp. 88–89 in [4] for an exact definition of parabolic L^p -function spaces.

$p \in (1, \infty)$ and $\theta \in (0, 1)$ with $4(1 - \theta) \notin \mathbf{N}$ the continuous embedding

$$\begin{aligned} X_{T,p} &= W^{1,p}([0, T]; L^p(\Sigma, \mathbf{R})) \cap L^p([0, T]; W^{4,p}(\Sigma, \mathbf{R})) \\ &\hookrightarrow (W^{1,p}([0, T]; L^p(\Sigma, \mathbf{R})), L^p([0, T]; W^{4,p}(\Sigma, \mathbf{R})))_{\theta,p} \\ &= W^{\theta,p}([0, T]; W^{4(1-\theta),p}(\Sigma, \mathbf{R})), \end{aligned} \tag{101}$$

for any finite $T > 0$. Furthermore, the proof of Lemma 3.3 in [47] can be slightly adapted, in order to see that for $p \in (6, \infty)$, for any small $\epsilon > 0$ still satisfying $p > 6 + 4\epsilon$ and for $\theta := \frac{1+\epsilon}{p} \in (0, \frac{1}{4})$ the general Sobolev embedding from Theorem B.4 in [47] yields

$$W^{\theta,p}([0, T]; W^{4(1-\theta),p}(\Sigma, \mathbf{R})) \hookrightarrow C^{q_1}([0, T]; C^{3,q_2}(\Sigma, \mathbf{R})), \tag{102}$$

for any finite $T > 0$ and for sufficiently small exponents $q_1, q_2 \in (0, \frac{1}{8})$. Hence, combining embeddings (101) and (102) with estimate (100)—here with any fixed $p \in (6, \infty)$ —we obtain the existence of sufficiently small $q_1, q_2 \in (0, \frac{1}{8})$, such that the smooth solution $\phi_t = \langle N_t, \nu_{F^*} \rangle_{\mathbf{R}^3}$ of equation (84) satisfies

$$\| \{ \phi_t \} \|_{C^{q_1}([0, T(\sigma)], C^{3,q_2}(\Sigma, \mathbf{R}))} \leq C(\Sigma, F^*, q_1, q_2, p, T(\sigma), \sigma), \tag{103}$$

for some appropriate constant $C = C(\Sigma, F^*, q_1, q_2, p, T(\sigma), \sigma) > 0$. On account of the mean value theorem, estimate (103) particularly implies the estimate

$$\| B(\cdot, \phi_t, \dots, D_x^3 \phi_t) \|_{C^{\bar{\mu}, \frac{\bar{\mu}}{4}}(\Sigma \times [0, T(\sigma)], \mathbf{R})} \leq C(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}), \tag{104}$$

for any small $\bar{\mu} \in (0, \frac{1}{8})$ with $\bar{\mu} < \min\{4q_1, q_2\}$ and for another appropriate constant $C = C(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu})$, which does not depend on any more data of the solution $\{ \phi_t \}$, especially not on the size of ϵ from lines (80) and (82). Hence, we obtain from the parabolic Schauder a-priori estimates (94)—here with $l = 0$ and $\mu = \bar{\mu}$ —combined with estimates (88), (97), and (104), that the above smooth solution $\{ \phi_t \}$ of equation (84) satisfies

$$\| \{ \phi_t \} \|_{C^{4+\bar{\mu}, 1+\frac{\bar{\mu}}{4}}(\Sigma \times [0, T(\sigma)], \mathbf{R})} \leq C_0(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}), \tag{105}$$

for some sufficiently small Hölder-exponent $\bar{\mu} \in (0, \frac{1}{8})$, where the above constant $C_0 = C_0(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu})$ does not depend on the size of $\epsilon > 0$ from lines (80) and (82) neither. Now we recall, that “ k ” in conditions (80) and (82) was a fixed integer ≥ 4 . If $k = 4$, then estimate (105) does not have to be improved any more. But if $k > 4$, then estimate (105) should be used, in order to improve estimate (104) by means of another application of the mean value theorem:

$$\| B(\cdot, \phi_t, \dots, D_x^3 \phi_t) \|_{C^{1+\bar{\mu}, 1+\frac{1+\bar{\mu}}{4}}(\Sigma \times [0, T(\sigma)], \mathbf{R})} \leq \tilde{C}_1(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}), \tag{106}$$

for the same exponent $\bar{\mu}$ as in estimate (105) and for another appropriate constant $\tilde{C}_1 = \tilde{C}_1(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu})$. Since we have proved already that the solution $\{\phi_t\}$ of equation (84) is C^∞ -smooth, estimates (105) and (106) can be combined again with conditions (88) and (97), in order to infer from another application of Schauder estimates (94)—but now with $l = 1$ and $\mu = \bar{\mu}$:

$$\|\phi_t\|_{C^{5+\bar{\mu}, \frac{5+\bar{\mu}}{4}}(\Sigma \times [0, T(\sigma)], \mathbf{R})} \leq C_1(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}), \tag{107}$$

for the same exponent $\bar{\mu}$ as in estimate (105) and for another appropriate constant $C_1 = C_1(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu})$. Hence, by finite induction—stopping after exactly $k-4$ steps on account of condition (88)—we arrive in this way at the optimal Schauder estimate

$$\|\phi_t\|_{C^{k+\bar{\mu}, \frac{k+\bar{\mu}}{4}}(\Sigma \times [0, T(\sigma)], \mathbf{R})} \leq C_{k-4}(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}), \tag{108}$$

for the same exponent $\bar{\mu}$ as in estimate (105) and for another appropriate constant $C_{k-4} = C_{k-4}(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu})$, which does not depend on the size of $\varepsilon > 0$ from (82) and (88) neither. Estimate (108) immediately implies

$$\begin{aligned} \|\tilde{f}_t - F^*\|_{C^{k, \bar{\mu}}(\Sigma, \mathbf{R}^3)} &\equiv \|N_t\|_{C^{k, \bar{\mu}}(\Sigma, \mathbf{R}^3)} \\ &\leq \tilde{C}_{k-4}(\Sigma, F^*, T(\sigma), \sigma, \bar{\mu}) \quad \forall t \in [0, T(\sigma)], \end{aligned} \tag{109}$$

for the corresponding smooth solution $\tilde{f}_t = F^* + N_t \equiv F^* + \phi_t \nu_{F^*}$ of equation (83), for some small exponent $\bar{\mu} \in (0, \frac{1}{8})$, which is exactly the analog of formula (5.13) in [10]. Now, on account of condition (96), due to the choice $\sigma < \min\{\delta, \varrho\}$ and since F^* was supposed to be a C^k -local minimizer of the Willmore functional, we know that

$$\mathcal{W}(\tilde{f}_t) \geq \mathcal{W}(F^*) \tag{110}$$

for $t \in [0, T(\sigma)]$. Moreover, using the fact that the smooth family $\{\tilde{f}_t\} = \{F^* + \phi_t \nu_{F^*}\}$ solves equation (83), we can infer that

$$\begin{aligned} \frac{d}{dt} \mathcal{W}(\tilde{f}_t) &= \int_{\Sigma} \langle \partial_t^\perp \tilde{f}_t(\tilde{f}_t), \nabla_{L^2}(\tilde{f}_t) \rangle_{\mathbf{R}^3} d\mu_{\tilde{f}_t} \\ &= - \int_{\Sigma} \frac{1}{|A_{\tilde{f}_t}^0|^4} |\nabla_{L^2}(\tilde{f}_t)|^2 d\mu_{\tilde{f}_t} \leq 0 \end{aligned} \tag{111}$$

for $t \in [0, T(\sigma)]$, i.e., that $\mathcal{W}(\tilde{f}_t)$ does not increase for $t \in [0, T(\sigma)]$. Moreover, due to $T(\sigma) < T_{\max}$, there holds equation (95) on $[0, T(\sigma)]$, implying that

$$\mathcal{W}(\tilde{f}_t) = \mathcal{W}(\mathcal{P}(t, 0, f_0)) \quad \text{for } t \in [0, T(\sigma)]. \tag{112}$$

Now, we can combine equations (110)–(112) with the strong regularity result in Theorem 3 (ii)¹⁹ of [19], in order to apply the same argument as on page 360 in [8], ruling out the special case, in which there might hold $\mathcal{W}(\tilde{f}_s) = \mathcal{W}(F^*)$ for some $s \in [0, T(\sigma))$. Hence, in the sequel, we can assume without loss of generality, that there holds $\mathcal{W}(\tilde{f}_t) > \mathcal{W}(F^*)$ for every $t \in [0, T(\sigma))$. Finally, we have to observe that condition (96) implies inequalities (85) and (87) to hold for $t \in [0, T(\sigma)]$, since we chose $\sigma < \varrho$ and $\varrho < \tilde{\delta}$. Hence, there is some small constant $c = c(F^*, \sigma) > 0$, such that

$$c(F^*, \sigma) \leq |A_{\tilde{f}_t}^0(x)|^2 \leq \frac{1}{c(F^*, \sigma)} \quad \text{for } (x, t) \in \Sigma \times [0, T(\sigma)].$$

We can therefore introduce the smooth, non-increasing, and positive function $[t \mapsto (\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^\theta]$, for $t \in [0, T(\sigma))$, where $\theta = \theta(F^*) \in (0, 1/2]$ denotes the exponent appearing in the Lojasiewicz-Simon-gradient-inequality for the Willmore functional, Theorem 3.1 in [8], in order to compute by means of Hölder’s inequality, again equation (83) and by the usual chain rule:

$$\begin{aligned} -\frac{d}{dt}(\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^\theta &= -\theta (\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^{\theta-1} \int_{\Sigma} \langle \nabla_{L^2} \mathcal{W}(\tilde{f}_t), \partial^{\perp_{\tilde{f}_t}}(\tilde{f}_t) \rangle d\mu_{\tilde{f}_t} \\ &= \theta (\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^{\theta-1} \int_{\Sigma} \frac{1}{|A_{\tilde{f}_t}^0|^4} |\nabla_{L^2} \mathcal{W}(\tilde{f}_t)|^2 d\mu_{\tilde{f}_t} \\ &\geq c(F^*, \sigma)^4 \theta (\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^{\theta-1} \left(\int_{\Sigma} |\nabla_{L^2} \mathcal{W}(\tilde{f}_t)|^2 d\mu_{\tilde{f}_t} \right)^{1/2} \cdot \\ &\quad \cdot \left(\int_{\Sigma} \frac{1}{|A_{\tilde{f}_t}^0|^8} |\nabla_{L^2} \mathcal{W}(\tilde{f}_t)|^2 d\mu_{\tilde{f}_t} \right)^{1/2} \\ &\geq \frac{c(F^*, \sigma)^4 \theta}{C_1^*(F^*)} \|\partial_t^{\perp_{\tilde{f}_t}}(\tilde{f}_t)\|_{L^2(\mu_{\tilde{f}_t})} \quad \text{for } t \in [0, T(\sigma)), \end{aligned} \tag{113}$$

where we have been able to apply the Lojasiewicz-Simon-gradient inequality in line (113) in a C^4 -ball of radius σ about the Willmore immersion F^* with appropriate constants $C_1^* = C_1^*(F^*) > 0$ and $\theta = \theta(F^*) \in (0, 1/2]$, taking estimate (96) for $t \in [0, T(\sigma)]$ into account. Now, estimate (96) also implies inequality (86) for $t \in [0, T(\sigma)]$, on account of $\sigma < \varrho < \tilde{\delta}$, and the time derivative $\partial_t \tilde{f}_t \equiv \partial_t N_t$ is actually a smooth section of the normal bundle of F^* , just as N_t is, for every $t \in [0, T(\sigma)]$. Hence, we infer from an integration of inequality (113) with respect to time and again from estimate (96) that

$$\begin{aligned} &\int_0^s \|\partial_t \tilde{f}_t\|_{L^2(\mu_{F^*})} dt \\ &\leq C(\sigma) \int_0^s \|\partial_t \tilde{f}_t\|_{L^2(\mu_{\tilde{f}_t})} dt \end{aligned}$$

¹⁹ It should be stressed here that this theorem indeed holds already for C^∞ -smooth, umbilic-free initial immersions.

$$\begin{aligned}
 &\leq 2 C(\sigma) \int_0^s \left\| \left(\partial_t \tilde{f}_t \right)^{\perp_{\tilde{f}_t}} \right\|_{L^2(\mu_{\tilde{f}_t})} dt \\
 &\leq -\frac{2 C(\sigma) C_1^*}{c(F^*, \sigma)^4 \theta} \int_0^s \frac{d}{dt} \left((\mathcal{W}(\tilde{f}_t) - \mathcal{W}(F^*))^\theta \right) dt \\
 &= \frac{2 C(\sigma) C_1^*}{c(F^*, \sigma)^4 \theta} \left((\mathcal{W}(\tilde{f}_0) - \mathcal{W}(F^*))^\theta - (\mathcal{W}(\tilde{f}_s) - \mathcal{W}(F^*))^\theta \right) \\
 &\leq \frac{2 C(\sigma) C_1^*}{c(F^*, \sigma)^4 \theta} \left(\mathcal{W}(F^* + N_0) - \mathcal{W}(F^*) \right)^\theta < \infty \quad \forall s \in [0, T(\sigma)]. \tag{114}
 \end{aligned}$$

Now, we can derive from a combination of condition (96) and estimate (114), together with the triangle inequality for the $L^2(\mu_{F^*})$ -norm:

$$\begin{aligned}
 \left\| \tilde{f}_s - F^* \right\|_{L^2(\mu_{F^*})} &\leq \left\| N_0 \right\|_{L^2(\mu_{F^*})} + \frac{2 C(\sigma) C_1^*}{c(F^*, \sigma)^4 \theta} \left(\mathcal{W}(F^* + N_0) - \mathcal{W}(F^*) \right)^\theta \\
 &\leq C \left\| N_0 \right\|_{C^2(\Sigma, \mathbf{R}^3)}^\theta \quad \forall s \in [0, T(\sigma)], \tag{115}
 \end{aligned}$$

for some appropriate constant $C = C(\Sigma, F^*, \theta, \sigma) > 0$. By Theorem 6.4.5 (iii) in [6], we can interpolate the Besov space $B_{p^*, p^*}^{\beta(k+\bar{\mu})}(\Sigma, \mathbf{R}^3)$, for $p^* = \frac{2}{1-\beta} \gg 1$ and $\beta \in (0, 1)$ close to 1, between the spaces $C^{k, \bar{\mu}}(\Sigma, \mathbf{R}^3) = B_{\infty, \infty}^{k+\bar{\mu}}(\Sigma, \mathbf{R}^3)$ and $L^2(\Sigma, \mathbf{R}^3) = B_{2, 2}^0(\Sigma, \mathbf{R}^3)$, and we can then use the fact that $B_{p^*, p^*}^{\beta(k+\bar{\mu})}(\Sigma, \mathbf{R}^3)$ embeds into $C^k(\Sigma, \mathbf{R}^3)$ by the fractional Sobolev embedding theorem, provided there holds $\beta(k + \bar{\mu}) - \frac{2}{p^*} \equiv \beta(k + \bar{\mu}) + \beta - 1 > k$. Consequently, we infer from estimates (82), (109), and (115) that

$$\begin{aligned}
 \left\| \tilde{f}_s - F^* \right\|_{C^k(\Sigma, \mathbf{R}^3)} &\leq C \left\| \tilde{f}_s - F^* \right\|_{C^{k, \bar{\mu}}(\Sigma, \mathbf{R}^3)}^\beta \left\| \tilde{f}_s - F^* \right\|_{L^2(\mu_{F^*})}^{1-\beta} \\
 &\leq C \left\| N_0 \right\|_{C^2(\Sigma, \mathbf{R}^3)}^{(1-\beta)\theta} \leq C^* (C^o(\varepsilon))^{(1-\beta)\theta} \quad \forall s \in [0, T(\sigma)], \tag{116}
 \end{aligned}$$

for some appropriately large constant $C^* = C^*(\Sigma, F^*, \bar{\mu}, k, \beta, \theta, T(\sigma), \sigma)$, which is independent of ε . It therefore turns out now, that we should choose $\varepsilon > 0$ above in estimate (80) that small, such that

$$C^* (C^o(\varepsilon))^{(1-\beta)\theta} \leq \frac{\sigma}{2} \tag{117}$$

holds, implying by estimate (116) that we thus would have

$$\left\| \tilde{f}_t - F^* \right\|_{C^k(\Sigma, \mathbf{R}^3)} \Big|_{t=T(\sigma)} \leq \frac{\sigma}{2}.$$

But this contradicts the fact that $\left\| \tilde{f}_t - F^* \right\|_{C^k(\Sigma, \mathbf{R}^3)} \Big|_{t=T(\sigma)} = \sigma$ would have to hold at time $t = T(\sigma)$ on account of condition (96), if the “maximal time” $T(\sigma)$ in (96) would have actually been finite and also smaller than T_{\max} . In the remaining special case “ $T(\sigma) = T_{\max} < \infty$ ” we could infer, that estimates (96), (97),f and (109)

would hold on every compact interval $[0, T]$ with $T < T_{\max} < \infty$. Since we also know that $\sigma < \varrho$, a comparison of conditions (93) and (97) shows us, that in this situation the smooth solution $\{\phi_t\}$ of the quasilinear parabolic equation (84) could be extended—using e.g., the methods of Theorems 2 and 3 in [17]—from $\Sigma \times [0, T_{\max})$ to $\Sigma \times [0, T']$, for some $T' > T_{\max}$, of class $C^{4+\mu, 1+\frac{\mu}{4}}(\Sigma \times [0, T'], \mathbf{R})$, for any fixed $\mu \in (0, \bar{\mu})$, and thus also of class $C^\infty(\Sigma \times [0, T'], \mathbf{R})$ on account of the above bootstrap argument employing estimates (94), but without violating condition (93) for $t \in [0, T']$. This contradicts the definition of the maximal time T_{\max} . Hence, we have proved that there actually has to hold “ $T(\sigma) = \infty$ ” in estimate (96), i.e., that the particular smooth solution $\tilde{f}_t = F^* + N_t$ of equation (83) exists globally and satisfies the smallness condition (96) at arbitrarily large times t :

$$\| \tilde{f}_t - F^* \|_{C^k(\Sigma, \mathbf{R}^3)} \leq \sigma \quad \forall t \in [0, \infty), \tag{118}$$

provided the initial immersion \tilde{f}_0 satisfies condition (82) with $\varepsilon > 0$ chosen that small in condition (80), such that inequality (117) finally holds. Combining now statement (118) with the entire reasoning which led us to estimate (109) on $\Sigma \times T(\sigma)$, one can exchange $T(\sigma)$ by any positive time T and then prove inductively—using (109) and (118) both in the induction basis and in the induction step—that there is a constant $K = K(\Sigma, F^*, \sigma, k, \bar{\mu})$ such that

$$\| \tilde{f}_t - F^* \|_{C^{k, \bar{\mu}}(\Sigma, \mathbf{R}^3)} \leq K \quad \forall t \in [0, \infty). \tag{119}$$

Moreover, since we especially know now that $T_{\max} = \infty$, we obtain equation (95) for every $t \geq 0$, i.e., there is a smooth family of smooth diffeomorphisms $\Psi_t : \Sigma \xrightarrow{\cong} \Sigma$, $\Psi_0 = \text{Id}_\Sigma$, such that

$$\tilde{f}_t = \mathcal{P}(t, 0, f_0) \circ \Phi_0 \circ \Psi_t^{-1} \quad \forall t \in [0, \infty). \tag{120}$$

Now, having chosen $\varepsilon > 0$ in (80) sufficiently small, we can let $s \rightarrow \infty$ in estimate (114) and obtain

$$\int_0^\infty \| \partial_t \tilde{f}_t \|_{L^2(\mu_{F^*})} dt \leq \frac{2C(\sigma)C_1^*}{c(F^*, \sigma)^4 \theta} \left(\mathcal{W}(F^* + N_0) - \mathcal{W}(F^*) \right)^\theta < \infty, \tag{121}$$

implying the existence of a unique function $F_\infty \in L^2((\Sigma, \mu_{F^*}), \mathbf{R}^3)$, such that

$$\tilde{f}_t \equiv F^* + N_t \longrightarrow F_\infty \quad \text{in } L^2(\Sigma, \mu_{F^*}) \tag{122}$$

as $t \rightarrow \infty$. Inserting now convergence (122) and estimate (119) into the first inequality in (116), we obtain together with equation (120):

$$\mathcal{P}(t, 0, f_0) \circ \Phi_0 \circ \Psi_t^{-1} = \tilde{f}_t \longrightarrow F_\infty \quad \text{in } C^k(\Sigma, \mathbf{R}^3) \tag{123}$$

as $t \rightarrow \infty$, where the limit function F_∞ satisfies additionally estimate (118) in the limit, i.e.,

$$\| F_\infty - F^* \|_{C^k(\Sigma, \mathbf{R}^3)} \leq \sigma, \tag{124}$$

and thus turns out to be a umbilic-free C^k -immersion, because estimate (118) implies inequalities (85) and (87) to hold for every $t \geq 0$. It remains to prove, that the limit immersion F_∞ of convergence (123) is “Willmore” and a C^k -local minimizer of the Willmore functional \mathcal{W} . Indeed, for any fixed $k \geq 4$, we infer from convergences (121) and (123) that

$$\begin{aligned} 0 \longleftarrow \| \partial_t^{\perp \tilde{f}_t}(\tilde{f}_t) \|_{L^2(\mu_{\tilde{f}_t})}^2 \Big|_{t=t_i} &= \int_{\Sigma} \frac{1}{|A_{\tilde{f}_t}^0|^8} |\nabla_{L^2} \mathcal{W}(\tilde{f}_t)|^2 d\mu_{\tilde{f}_t} \Big|_{t=t_i} \\ &\longrightarrow \int_{\Sigma} \frac{1}{|A_{F_\infty}^0|^8} |\nabla_{L^2} \mathcal{W}(F_\infty)|^2 d\mu_{F_\infty}, \end{aligned}$$

for some appropriate sequence $t_i \nearrow \infty$, showing that F_∞ is indeed a umbilic-free Willmore immersion, satisfying statement (124) for the considered $k \geq 4$.

Moreover, combining statement (124) again with the Lojasiewicz-Simon-gradient-inequality, Theorem 3.1 in [8], F_∞ turns out to satisfy $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$, proving that F_∞ is actually a C^k -local minimizer of \mathcal{W} as well, for any $k \geq 4$. Using now the conformal invariance of both the MIWF and the Willmore functional, we can project statements (123) and (124) back into \mathbf{S}^3 via inverse stereographic projection and thus obtain the entire assertion of Theorem 2. □

Remark 5 In contrast to the final steps of the proof of Lemma 4.1 in [8], we could not combine neither estimates (118), (119) nor the full C^k -convergence in (123) with localized L^∞ -estimates of covariant derivatives $\nabla^m A_{\tilde{f}_t}$ of the second fundamental forms of the converging immersions \tilde{f}_t in (122) and (123), in order to improve the quality of convergence (123) furthermore, e.g., from C^k - to smooth convergence, because such strong estimates have—so far—only been proven for flow lines of the classical and inverse Willmore flow in \mathbf{R}^n ; see here Sect. 4 in [23], respectively Sect. 3 in [22] and also Sects. 7–12 in [29]. The only available proof of such estimates relies strongly on the structural similarity between the leading term of the Willmore flow equation (3) and the simple heat equation of fourth order: $\partial_t \Phi_t = -\Delta_g^2(\Phi_t)$, for some fixed smooth metric g on Σ . It is actually this structure of the leading fourth order term of the Willmore flow equation, which leads to a fairly simple, inductive computation of the expressions $\partial_t^{\perp f_t}(\nabla^m A_{f_t}) + \Delta_{g_{f_t}}^2(\nabla^m A_{f_t})$ for any smooth solution $\{f_t\}_{t \in [0, T]}$ of equation (3), for each order $m \in \mathbf{N}_0$, yielding both $L^2(f_t^{-1}(B_r^n(x_0)))$ —and even $L^\infty(f_t^{-1}(B_r^n(x_0)))$ —estimates for the covariant derivatives $\nabla^m A_{f_t}$ of any order $m \in \mathbf{N}$, locally about some arbitrarily fixed $x_0 \in \mathbf{R}^n$ and uniformly in time $t \in [0, T]$, provided $\int_{f_t^{-1}(B_{2r}^n(x_0))} |A_{f_t}|^2 d\mu_{g_{f_t}}$ stays sufficiently small, for $t \in [0, T]$, i.e., provided there is no quantum of curvature that concentrates about some fixed point x_0 in \mathbf{R}^n , as $t \nearrow T$. Because of the fairly degenerate structure of the left hand

side of the MIWF-equation (2), one can neither adopt here the strategy of Sects. 2 and 4 in [23] nor its generalization and improvement in Sects. 7–12 of [29], treating the *inverse Willmore flow*. \square

4 Stability of Converging Flow Lines

Combining Theorem 2 with Theorem 5 below, we obtain a *stability result* for fully convergent flow lines of the MIWF into C^4 -local minimizers of the Willmore functional.

Theorem 4 (Stability Theorem) *Suppose that Σ is a smooth compact torus and that $F_0 : \Sigma \rightarrow \mathbf{S}^3$ is a C^∞ -smooth and umbilic-free immersion, whose corresponding flow line $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the MIWF exists globally and converges fully and smoothly—up to smooth reparametrization—to a C^∞ -smooth parametrization F^* of a umbilic-free C^4 -local minimizer of the Willmore functional \mathcal{W} , in the sense of formula (4) with $k = 4$. Then, for any fixed $\alpha \in (0, 1)$, there is an open ball $\mathcal{B}_r^{4,\alpha}(F_0)$ about F_0 in $C^{4,\alpha}(\Sigma, \mathbf{R}^4)$, with $r = r(\Sigma, F_0, F^*, \alpha) > 0$, such that For every C^∞ -smooth immersion $F : \Sigma \rightarrow \mathbf{S}^3$ being contained in the open ball $\mathcal{B}_r^{4,\alpha}(F_0)$, there is a smooth family of smooth diffeomorphisms $\Psi_t^F : \Sigma \xrightarrow{\cong} \Sigma$, for $t \geq 0$, such that the reparametrized flow line $\{\mathcal{P}(t, 0, F) \circ \Psi_t^F\}_{t \geq 0}$ of the MIWF converges fully in the $C^4(\Sigma, \mathbf{R}^4)$ -norm to a umbilic-free Willmore immersion F_∞ , as $t \rightarrow \infty$, and this limit immersion is a C^4 -local minimizer of the Willmore functional \mathcal{W} as well, satisfying $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$. \square*

In order to prepare the proof of Theorem 4, we shall adopt the methods of Theorems 1–4 in [19], but using here parabolic Hölder spaces and parabolic Schauder theory instead of optimal L^p - L^q -estimates, in order to apply the regularity bootstrap method of Theorem 3 (ii) in [17], via Schauder a-priori estimates. To this end, we consider evolution equations (2) and (24) for immersions $f_t : \Sigma \rightarrow \mathbf{R}^3$, and we recall from the author’s article [17] as in Sect. 2 of [19], that the differential operator

$$\begin{aligned} 2 |A_f^0|^{-4} \nabla_{L^2} \mathcal{W}(f) &\equiv |A_f^0|^{-4} \left(\Delta_f^\perp \mathbf{H}_f + Q(A_f^0)(\mathbf{H}_f) \right) \\ &= |A_f^0|^{-4} \left((\Delta_f \mathbf{H}_f)^\perp + 2 Q(A_f)(\mathbf{H}_f) - \frac{1}{2} |\mathbf{H}_f|^2 \mathbf{H}_f \right) \end{aligned} \tag{125}$$

arising on the right hand side of evolution equations (2) and (24) is not uniformly elliptic and that its leading term $(\Delta_f \mathbf{H}_f)^\perp$ can be written as

$$(\Delta_f \mathbf{H}_f)^\perp = g_f^{ij} g_f^{kl} \nabla_{ijkl}^f(f) - g_f^{ij} g_f^{kl} \langle \nabla_{ijkl}^f(f), \partial_m f \rangle g_f^{mr} \partial_r(f) \tag{126}$$

at least locally, in local coordinates on Σ , for any $W^{4,2}$ -immersion $f : \Sigma \rightarrow \mathbf{R}^3$, where $g_f := f^*(g_{\text{euc}})$ denotes the pullback-metric of the Euclidean metric of \mathbf{R}^3 .

Applying *DeTurck’s trick* as in Sect. 2 of [19], we add the globally well-defined differential operator²⁰ of fourth order

$$\mathcal{T}_{F_0}(f) := g_f^{ij} g_f^{kl} g_f^{mr} \langle \nabla_{ijkl}^f(f), \partial_m f \rangle \partial_r f - g_f^{ij} g_f^{kl} \nabla_{ij}^f((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m) \partial_m(f),$$

for some fixed smooth immersion $F_0 : \Sigma \rightarrow \mathbf{R}^3$, to the right hand side of equation (125) and obtain a quasilinear operator of fourth order whose leading term is $g_f^{ij} g_f^{kl} \nabla_{ijkl}^{F_0}(f)$, having a uniformly elliptic linearization in any umbilic-free $C^{4,\gamma}$ -immersion $f : \Sigma \rightarrow \mathbf{R}^3$. We are thus led to consider here the evolution equation

$$\partial_t(f_t) = -\frac{1}{2} |A_{f_t}^0|^{-4} (2 \nabla_{L^2} \mathcal{W}(f_t) + \mathcal{T}_{F_0}(f_t)) =: \mathcal{M}_{F_0}(f_t), \tag{127}$$

for some arbitrarily fixed C^∞ -smooth immersion $F_0 : \Sigma \rightarrow \mathbf{R}^3$, where the right-hand side $\mathcal{M}_{F_0}(f_t)$ of equation (127) can be expressed in local coordinates on Σ by

$$\begin{aligned} \mathcal{M}_{F_0}(f_t)(x) = & -\frac{1}{2} |A_{f_t}^0|^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_{ij}^{f_t} \nabla_{kl}^{F_0}(f_t)(x) \\ & + B(x, D_x f_t(x), D_x^2 f_t(x), D_x^3 f_t(x)), \end{aligned} \tag{128}$$

for $(x, t) \in \Sigma \times [0, T]$. Here, the symbols $D_x f_t, D_x^2 f_t, D_x^3 f_t$ abbreviate the matrix-valued functions $(\partial_1 f_t, \partial_2 f_t), (\nabla_{ij}^{F_0} f_t)_{i,j \in \{1,2\}}$ and $(\nabla_{ijk}^{F_0} f_t)_{i,j,k \in \{1,2\}}$, and $B : \Sigma \times \mathbf{R}^6 \times \mathbf{R}^{12} \times \mathbf{R}^{24} \rightarrow \mathbf{R}^3$ is a globally defined function, whose 3 components are rational functions in their 42 real variables. Following the lines of the author’s articles [17] and [19], we will collect some basic properties of the linearization of equation (127) or equivalently of equation

$$\partial_t(f_t) = -\frac{1}{2} |A_{f_t}^0|^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_{ij}^{f_t} \nabla_{kl}^{F_0}(f_t) + B(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t) \tag{129}$$

in any family of $C^{4,\gamma}$ -immersions $f_t : \Sigma \rightarrow \mathbf{R}^3$, which is sufficiently close to a prescribed flow line of the MIWF (2) in the parabolic Hölder space $C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$, $\gamma \in (0, 1)$, below in Proposition 1. To this end, we fix some umbilic-free immersion $F_0 \in C^\infty(\Sigma, \mathbf{R}^3)$ and denote by $T_{\max}(F_0)$ the maximal time such that the corresponding unique smooth solution $\mathcal{P}(\cdot, 0, F_0)$ of the MIWF exists on $\Sigma \times [0, T_{\max}(F_0)]$.²¹ We recall from the proof of Theorem 1 in [17] that there is a unique smooth family of smooth diffeomorphisms $\phi_t^{F_0} : \Sigma \rightarrow \Sigma$ with $\phi_0 = \text{Id}_\Sigma$, such that the reparametrization $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}$ solves evolution equations (127) and (129) on $\Sigma \times [0, T]$, for every final time $T \in (0, T_{\max}(F_0))$. Now we fix some $T \in$

²⁰ Compare here also with p. 1156 in [17] and with formula (12) in [19], where the tensorial character of the expressions $(\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m$ and $\mathcal{T}_{F_0}(f)$ had been explained.

²¹ Compare here with Definition 2 (d) in [19], introducing the life span of a flow line.

($0, T_{\max}(F_0)$) and $\gamma \in (0, 1)$ arbitrarily, and we also choose some open neighborhood $W_{F_0, T, \gamma}$ of the above smooth solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}$ of (127) in the Banach space

$$X_T \equiv X_{T, \gamma} := C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3),$$

and we shall follow the strategy of [19] and [47]: using the fact that the restriction of elements of $X_{T, \gamma}$ at time $t = 0$:

$$r_0 : X_{T, \gamma} \longrightarrow C^{4, \gamma}(\Sigma, \mathbf{R}^3)$$

is a linear and continuous operator, i.e., that the *trace* of the Banach space X_T at time $t = 0$ is exactly

$$\text{Trace}(X_T) = C^{4, \gamma}(\Sigma, \mathbf{R}^3), \tag{130}$$

and considering the continuous, non-linear product operator

$$\Psi^{F_0, T} : W_{F_0, T, \gamma} \subset X_T \longrightarrow C^{4, \gamma}(\Sigma, \mathbf{R}^3) \times C^{\gamma, \frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3) =: Y_{T, \gamma} \equiv Y_T$$

defined by

$$\Psi^{F_0, T}(\{f_t\}_{t \in [0, T]}) := (f_0, \{\partial_t(f_t) - \mathcal{M}_{F_0}(f_t)\}_{t \in [0, T]}). \tag{131}$$

Now following the proofs of Theorem 1 in [19] and Theorem 2 in [17]—substituting here Proposition 2 in [19] by Corollary 3 in [17]—we can prove the following counterpart of Theorem 1 in [19] in the setting of *parabolic Schauder Theory*, aiming at basic properties of the Fréchet derivative of the operator $\Psi^{F_0, T}$ from (131) at the smooth solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ of equation (129) in view of the proof of Theorem 5 (1) below.

Proposition 1 *Let Σ be a smooth compact torus and $F_0 : \Sigma \longrightarrow \mathbf{R}^3$ a C^∞ -smooth and umbilic-free immersion, and let $0 < T < T_{\max}(F_0)$ and $\gamma \in (0, 1)$ be chosen arbitrarily, where $T_{\max}(F_0)$ denotes the time of maximal existence of the flow line $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the MIWF (2). There is a sufficiently small open neighborhood $W_{F_0, T, \gamma}$ about the smooth solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ of the modified MIWF-equation (129) in the Banach space X_T , such that the following three statements hold:*

- (1) *The map $\Psi^{F_0, T} : W_{F_0, T, \gamma} \longrightarrow Y_T$, defined in line (131), is of class C^1 on the open subset $W_{F_0, T, \gamma}$ of the Banach space X_T .*
- (2) *The Fréchet derivative of the second component of $\Psi^{F_0, T}$ at any fixed family $\{f_t\}_{t \in [0, T]} \in W_{F_0, T, \gamma}$ is a linear, uniformly parabolic operator of order 4 whose leading operator acts diagonally, i.e., on each component of $f = \{f_t\}_{t \in [0, T]}$ separately:*

$$\begin{aligned}
 (D(\Psi^{F_0,T})_2(f))_2(\eta) &\equiv \partial_t(\eta) - D(\mathcal{M}_{F_0})(f)_2(\eta) \\
 &= \partial_t(\eta) + \frac{1}{2} |A_{f_t}^0|^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_{ijkl}^{F_0}(\eta) + B_3^{ijk} \cdot \nabla_{ijk}^{F_0}(\eta) \\
 &\quad + B_2^{ij} \cdot \nabla_{ij}^{F_0}(\eta) + B_1^i \cdot \nabla_i^{F_0}(\eta)
 \end{aligned} \tag{132}$$

on $\Sigma \times [0, T]$, for any element $\eta = \{\eta_t\}$ of the tangent space $T_f W_{F_0,T,\gamma} = X_T$. Here, the coefficients $|A_{f_t}^0|^{-4} g_{f_t}^{ij} g_{f_t}^{kl}$ of the leading order term are of class $C^{2+\gamma, \frac{2+\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$, B_2^{ij} and B_1^i are the coefficients of $\text{Mat}_{3,3}(\mathbf{R})$ -valued, contravariant tensor fields of degrees 2 and 1, depending on $x, D_x f_t, D_x^2 f_t, D_x^3 f_t$ and on $D_x^4 f_t$ and therefore being of class $C^{\gamma, \frac{\gamma}{4}}(\Sigma \times [0, T], \text{Mat}_{3,3}(\mathbf{R}))$. Finally B_3^{ijk} are the coefficients of a $\text{Mat}_{3,3}(\mathbf{R})$ -valued, contravariant tensor field of degree 3, which depends on $x, D_x f_t$ and $D_x^2 f_t$ only and is therefore of class $C^{2+\gamma, \frac{2+\gamma}{4}}(\Sigma \times [0, T], \text{Mat}_{3,3}(\mathbf{R}))$.

(3) The Fréchet derivative of $\Psi^{F_0,T}$ yields a topological isomorphism

$$D\Psi^{F_0,T}(f) : T_f W_{F_0,T,\gamma} = X_T \xrightarrow{\cong} Y_T$$

in any fixed family of immersions $f \equiv \{f_t\}_{t \in [0,T]} \in W_{F_0,T,\gamma}$.

□

Proof The first two parts of the proposition are essentially repetitions of the first two parts of Theorem 2 in [17], up to exchanging the neighborhood $W_{F_0,T,\gamma}$ of the smooth solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0,T]}$ of evolution equation (127) by the open subset

$$X_{U_0,\gamma,\delta,T} := \{\{f_t\} \in X_{T,\gamma} \mid \|f_t - U_0\|_{C^4(\Sigma)} < \delta \text{ for } t \in [0, T], f_0 = U_0 \text{ on } \Sigma\}$$

- appearing on p. 1157 and in Theorem 2 of [17]—of the affine closed subspace

$$A_{\gamma,T,U_0} := \{\{f_t\}_{t \in [0,T]} \in X_{T,\gamma} \mid f_0 = U_0 \text{ on } \Sigma\} \tag{133}$$

of our basic Banach space $X_T \equiv X_{T,\gamma} = C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$, for any fixed umbilic-free initial immersion $U_0 \in C^{4,\gamma}(\Sigma, \mathbf{R}^3)$.²² The third statement of the proposition now follows in three fairly simple steps. First of all, we know from the first two parts of this proposition that the non-linear operator

$$\partial_t - \mathcal{M}_{F_0} : W_{F_0,T,\gamma} \longrightarrow C^{\gamma, \frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$$

is of class C^1 and that its Fréchet derivative at an arbitrary family $f = \{f_t\}_{t \in [0,T]} \in W_{F_0,T,\gamma}$ is a continuous linear operator from $T_f W_{F_0,T,\gamma} = X_T$ to $C^{\gamma, \frac{\gamma}{4}}(\Sigma \times$

²² The interested reader might also want to compare the first two parts of our Proposition 1 with the first three parts of Theorem 1 in [19], whose proofs are based on formulae (24)–(28) within the preparatory Lemma 1 in [19], where the quasilinear structure of the non-linear operator $[f \mapsto \mathcal{M}_{F_0}(f)]$ from lines (127), (128) and (131) above has been precisely analyzed.

$[0, T], \mathbf{R}^3$), being of the concrete, uniformly parabolic form (132) with Hölder-continuous coefficients. We can therefore apply Propositions 1 and 2 and Corollaries 2 and 3 of [17], in order to conclude that the restriction

$$D(\partial_t - \mathcal{M}_{F_0})(f) : A_{\gamma, T, 0} \longrightarrow C^{\gamma, \frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$$

of this Fréchet derivative to the linear Banach subspace $A_{\gamma, T, 0}$ of X_T from (133), with $U_0 = 0$, is an isomorphism, just as the biharmonic heat operator $\partial_t + \Delta_{F_0}^2$ is. Then it easily follows from this fact and from (130) that the Fréchet derivative of the entire product operator $\Psi^{F_0, T} : W_{F_0, T, \gamma} \longrightarrow Y_T$ is a bijective linear map from $T_f W_{F_0, T, \gamma} = X_T$ to Y_T , and the assertion follows from the open mapping theorem for continuous linear operators between Banach spaces.

Combining Proposition 1 with the proof of Theorem 3 (ii) in [17] via Proposition 3 in [17], we obtain the following theorem, similarly to Theorem 4 (i) in [19]. In Theorem 5 and also below in the proof of Theorem 4, we will abbreviate by “ $\mathcal{B}_\rho^{4, \gamma}(F_0)$ ” the open ball of radius $\rho > 0$ about any fixed immersion F_0 in the Banach space $C^{4, \gamma}(\Sigma, \mathbf{R}^3)$, for any fixed $\gamma \in (0, 1)$.

Theorem 5 *Let Σ be a smooth compact torus and $F_0 : \Sigma \longrightarrow \mathbf{R}^3$ a C^∞ -smooth and umbilic-free immersion, and let $0 < T < T_{\max}(F_0)$ and $\gamma \in (0, 1)$ be chosen arbitrarily, where $T_{\max}(F_0)$ denotes the time of maximal existence of the flow line $\mathcal{P}(\cdot, 0, F_0)$ of the MIWF (2).*

- (1) *There is some small $\rho = \rho(\Sigma, F_0, T, \gamma) > 0$, such that for every initial immersion $F \in C^{4, \gamma}(\Sigma, \mathbf{R}^3)$ with $\|F - F_0\|_{C^{4, \gamma}(\Sigma, \mathbf{R}^3)} < \rho$ there is a unique solution $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0, T]}$ of the “DeTurck modification” (127), respectively, (129) of the MIWF (2) in the Banach space $X_T = C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$, starting to move in the immersion F at time $t = 0$, and the resulting evolution operator*

$$\mathcal{P}^*(\cdot, 0, \cdot) : \mathcal{B}_\rho^{4, \gamma}(F_0) \subset C^{4, \gamma}(\Sigma, \mathbf{R}^3) \longrightarrow X_T, \tag{134}$$

mapping any element F of the open ball $\mathcal{B}_\rho^{4, \gamma}(F_0)$ about F_0 to the unique solution $\{\mathcal{P}^(t, 0, F)\}_{t \in [0, T]}$, is of class C^1 .*

- (2) *If the initial immersion $F \in \mathcal{B}_\rho^{4, \gamma}(F_0)$ from part (1) is additionally of class $C^\infty(\Sigma, \mathbf{R}^3)$, then the resulting solution $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0, T]}$ of evolution equation (129) from line (134) is of class $C^\infty(\Sigma \times [0, T], \mathbf{R}^3)$, and furthermore there is a smooth family of C^∞ -smooth diffeomorphisms $\psi_t^F : \Sigma \rightarrow \Sigma$, with $\psi_0^F = \text{Id}_\Sigma$, such that the composition $\mathcal{P}^*(t, 0, F) \circ \psi_t^F$ solves again evolution equation (2) on $\Sigma \times [0, T]$, i.e., such that there holds:*

$$\mathcal{P}^*(t, 0, F) \circ \psi_t^F = \mathcal{P}(t, 0, F) \text{ on } \Sigma, \quad \forall t \in [0, T].$$

Proof (1) Here, we can argue exactly as in the proof of Theorem 4 (i) in [19]. We assume that $F_0 : \Sigma \longrightarrow \mathbf{R}^3$ is a C^∞ -smooth and umbilic-free immersion which produces a maximal smooth flow line $\mathcal{P}(\cdot, 0, F_0)$ of the MIWF, starting in F_0 at

time $t = 0$. Hence, the proof of Theorem 1 in [17] shows that there is a unique smooth family of smooth diffeomorphisms $\phi_t^{F_0} : \Sigma \rightarrow \Sigma$ with $\phi_0^{F_0} = \text{Id}_\Sigma$, such that the reparametrization $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \geq 0}$ is the unique and maximal smooth solution of evolution equation (127) on Σ . Moreover, we know from Proposition 1 above, that there is some open neighborhood $W_{F_0, T, \gamma}$ about the solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ of equation (127), respectively, (129) in the space X_T , such that the operator $\Psi^{F_0, T}$ from line (131) is a C^1 -map from $W_{F_0, T, \gamma}$ to Y_T , whose Fréchet derivative in the particular element $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]} \in W_{F_0, T, \gamma}$ is a topological isomorphism between X_T and Y_T . Noting also that there holds

$$\Psi^{F_0, T}(\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}) = (F_0, 0) \in Y_T,$$

by definition of the operator $\Psi^{F_0, T}$ in (131) and since $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ solves equation (127), we can infer from the inverse mapping theorem for non-linear C^1 -operators, that there is some small open ball $B_\rho((F_0, 0)) \subset Y_T$, with $\rho = \rho(F_0, T, \gamma) > 0$, and an appropriate further open neighborhood $W_{F_0, T, \gamma}^* \subset W_{F_0, T, \gamma}$ about the smooth solution $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ of equation (129) in X_T , such that

$$\Psi^{F_0, T} : W_{F_0, T, \gamma}^* \xrightarrow{\cong} B_\rho((F_0, 0)) \tag{135}$$

is a C^1 -diffeomorphism. Hence, by definition of the map $\Psi^{F_0, T}$ and by Theorem 2 (ii) of [19], the restriction of the inverse mapping $(\Psi^{F_0, T})^{-1}$ from line (135) to the product $\mathcal{B}_\rho^{4, \gamma}(F_0) \times \{0\} \subset B_\rho((F_0, 0))$ yields exactly the evolution operator of the parabolic evolution equation (127):

$$\begin{aligned} X_T \supset W_{F_0, T, \gamma}^* \ni \{\mathcal{P}^*(t, 0, F)\}_{t \in [0, T]} &= (\Psi^{F_0, T})^{-1}((F, 0)) \\ \forall F \in \mathcal{B}_\rho^{4, \gamma}(F_0) &\subset C^{4, \gamma}(\Sigma, \mathbf{R}^3), \end{aligned} \tag{136}$$

and it consequently has to be of class C^1 as an operator from $C^{4, \gamma}(\Sigma, \mathbf{R}^3)$ to X_T . Here, $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0, T]}$ denotes the restriction of the unique maximal solution $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0, t+(F)]}$ of equation (127) from Theorem 2 (ii) in [19] to the interval $[0, T]$, noting that $C^{4, \gamma}(\Sigma, \mathbf{R}^3) \subset W^{4-\frac{4}{p}, p}(\Sigma, \mathbf{R}^3)$ and also $C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3) \subset W^{1, p}([0, T]; L^p(\Sigma, \mathbf{R}^3)) \cap L^p([0, T]; W^{4, p}(\Sigma, \mathbf{R}^3))$, for any fixed $p \in (3, \infty)$.

- (2) The second statement of the theorem now follows immediately from the first statement of the theorem, combined with Theorem 3 (i) in [19] and with another application of DeTurck’s trick relating smooth solutions of evolution equations (2) and (127); see also the proof of Theorem 1 in [17]. □

Proof of Theorem 4 On account of the assumptions of the theorem there is some smooth family of smooth diffeomorphisms $\Psi_t : \Sigma \xrightarrow{\cong} \Sigma, t > 0$, such that the reparametrized

flow line $\{\mathcal{P}(t, 0, F_0) \circ \Psi_t\}_{t>0}$ of the MIWF in \mathbf{S}^3 converges smoothly and fully to a smooth parametrization F^* of a umbilic-free C^4 -local minimizer of the Willmore functional. Now we choose some $\alpha \in (0, 1)$ and we obtain from Theorem 2 of this article some small number $\varepsilon = \varepsilon(\Sigma, F^*, \alpha) > 0$, such that for any smooth immersion $f_0 : \Sigma \rightarrow \mathbf{S}^3$ with $\|f_0 - F^*\|_{C^{4,\alpha}(\Sigma, \mathbf{R}^4)} < \varepsilon$ the unique smooth flow line $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the MIWF exists globally and converges—up to smooth reparametrization—fully in $C^4(\Sigma, \mathbf{R}^4)$ to a parametrization $F_\infty = F_\infty(f_0)$ of a umbilic-free Willmore immersion in \mathbf{S}^3 which is a C^4 -local minimizer of the Willmore functional as well, satisfying $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$. Now using the fact that the flow line $\{\mathcal{P}(t, 0, F_0) \circ \Psi_t\}$ of the MIWF in \mathbf{S}^3 converges smoothly and fully to the C^4 -local minimizer F^* of \mathcal{W} , we can choose some large but finite time $T = T(F_0, F^*, \alpha) \gg 1$, such that the immersion $\mathcal{P}(T, 0, F_0) \circ \Psi_T$ is contained in the specified ε -ball $\mathcal{B}_\varepsilon^{4,\alpha}(F^*)$ about the local minimizer F^* in $C^{4,\alpha}(\Sigma, \mathbf{R}^4)$. Now we recall that the stereographic projection \mathcal{S} from $\mathbf{S}^3 \setminus \{(0, 0, 0, 1)\}$ into \mathbf{R}^3 is a conformal diffeomorphism. Moreover, on account of the compactness of Σ and on account of the conformal invariance of the MIWF, we may assume that the image of the initial immersion $F_0 : \Sigma \rightarrow \mathbf{S}^3$ does not contain the north pole $(0, 0, 0, 1)$ of \mathbf{S}^3 . Now, again using the conformal invariance of the MIWF, the entire technique of Theorem 5 can be transported from \mathbf{R}^3 to \mathbf{S}^3 by means of stereographic projection $\mathcal{S} : \mathbf{S}^3 \setminus \{(0, 0, 0, 1)\} \rightarrow \mathbf{R}^3$ and its conformal inverse \mathcal{S}^{-1} . To this end, we firstly see that the requirements of Theorem 5 are trivially satisfied here for the initial immersion $\tilde{F}_0 := \mathcal{S} \circ F_0 : \Sigma \rightarrow \mathbf{R}^3$ and for any final time $T > 0$. Hence, the first part of Theorem 5 guarantees, that there is for \tilde{F}_0 and for any fixed $\gamma \in (0, 1)$ and $T > 0$ some small $\rho = \rho(\Sigma, \tilde{F}_0, T, \gamma) > 0$, such that for every immersion $\tilde{F} \in C^{4,\gamma}(\Sigma, \mathbf{R}^3)$ with $\|\tilde{F} - \tilde{F}_0\|_{C^{4,\gamma}(\Sigma, \mathbf{R}^3)} < \rho$ there is a unique solution $\{\mathcal{P}^*(t, 0, \tilde{F})\}_{t \in [0, T]}$ of the “DeTurck modification” (129) of the MIWF-equation (2) in the parabolic Hölder space $X_{T,\gamma} \equiv C^{4+\gamma, 1+\frac{\gamma}{4}}(\Sigma \times [0, T], \mathbf{R}^3)$, starting to move in the immersion \tilde{F} at time $t = 0$, and such that this unique solution of equation (129) in $X_{T,\gamma}$ depends in a C^0 -fashion on its initial immersion \tilde{F} , in the sense of statement (134) in Theorem 5. Now, combining this information with the second part of Theorem 5 and then applying again inverse stereographic projection \mathcal{S}^{-1} , we can therefore infer that in our situation, there is for any $\epsilon > 0$ some sufficiently small $r = r(\Sigma, F_0, T, \epsilon, \gamma) > 0$, such that for every C^∞ -smooth immersion $F : \Sigma \rightarrow \mathbf{S}^3$ with $\|F - F_0\|_{C^{4,\gamma}(\Sigma, \mathbf{R}^4)} < r$ the unique, maximal flow line $\mathcal{P}(\cdot, 0, F)$ of the MIWF in \mathbf{S}^3 exists at least on $\Sigma \times [0, T]$, and such that the flow lines $\{\mathcal{P}(t, 0, F_0)\}_{t \in [0, T]}$ and $\{\mathcal{P}(t, 0, F)\}_{t \in [0, T]}$ of the MIWF in \mathbf{S}^3 can be reparametrized by families of smooth diffeomorphisms $\phi_t^{F_0} : \Sigma \rightarrow \Sigma$ and $\phi_t^F : \Sigma \rightarrow \Sigma$ in such a way, that their reparametrizations $\{\mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\}_{t \in [0, T]}$ and $\{\mathcal{P}(t, 0, F) \circ \phi_t^F\}_{t \in [0, T]}$ satisfy

$$\|\mathcal{P}(t, 0, F) \circ \phi_t^F - \mathcal{P}(t, 0, F_0) \circ \phi_t^{F_0}\|_{C^{4,\gamma}(\Sigma, \mathbf{R}^4)} < \epsilon, \tag{137}$$

for every $t \in [0, T]$. Now, we had chosen $T = T(F_0, F^*, \alpha)$ that large such that the immersion $\mathcal{P}(T, 0, F_0) \circ \Psi_T$ was contained in the ε -ball $\mathcal{B}_\varepsilon^{4,\alpha}(F^*)$ about the limit immersion F^* in $C^{4,\alpha}(\Sigma, \mathbf{R}^4)$. Hence, by estimate (137) the diffeomorphism $\Theta_T^{F_0} := (\phi_T^{F_0})^{-1} \circ \Psi_T : \Sigma \xrightarrow{\cong} \Sigma$ has the property that both immersions

$\mathcal{P}(T, 0, F_0) \circ \phi_T^{F_0} \circ \Theta_T^{F_0} = \mathcal{P}(T, 0, F_0) \circ \Psi_T$ and $\mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0}$ are contained in $\mathcal{B}_\varepsilon^{4,\alpha}(F^*)$, at time $t = T$, provided $\varepsilon > 0$ was chosen sufficiently small in estimate (137), the initial smooth immersion $F : \Sigma \rightarrow \mathbf{S}^3$ was contained in the open ball $\mathcal{B}_r^{4,\gamma}(F_0)$ about F_0 in $C^{4,\gamma}(\Sigma, \mathbf{R}^4)$, and provided $r = r(\Sigma, F_0, T, \varepsilon, \gamma) = r(\Sigma, F_0, F^*, \varepsilon, \gamma, \alpha) > 0$ had also been chosen sufficiently small in estimate (137). Here, we have also used the obvious embedding $C^{4,\gamma}(\Sigma, \mathbf{R}^4) \hookrightarrow C^{4,\alpha}(\Sigma, \mathbf{R}^4)$, provided we have chosen $\gamma \geq \alpha$ above. Hence, we should simply take $\gamma = \alpha$ in (137) and in the sequel. Recalling from the statement of Theorem 2 that the size of the ε -ball $\mathcal{B}_\varepsilon^{4,\alpha}(F^*)$ only depends here on Σ, F^* , and α , we finally see that the above radius $r = r(\Sigma, F_0, F^*, \varepsilon, \gamma, \alpha)$ actually only depends here on Σ, F_0, F^* and on the parameter α , i.e., $r = r(\Sigma, F_0, F^*, \alpha)$, and we conclude if this number r is chosen sufficiently small, then for any smooth immersion $F : \Sigma \rightarrow \mathbf{S}^3$ being contained in the open ball $\mathcal{B}_r^{4,\alpha}(F_0)$, the reparametrized immersion $\mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0}$ is an element of the prescribed ε -ball $\mathcal{B}_\varepsilon^{4,\alpha}(F^*)$ about the limit immersion F^* . Now since we know already from above that the entire reparametrized flow line $\{\mathcal{P}(t, 0, F) \circ \phi_t^F\}_{t \in [0, T]}$ of the MIWF in \mathbf{S}^3 is of class $C^\infty(\Sigma \times [0, T], \mathbf{R}^4)$, we especially conclude that the immersion $\mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0}$ is C^∞ -smooth and certainly also umbilic-free on Σ . We can therefore choose the above initial immersion f_0 from the statement of Theorem 2 of this article as $f_0 := \mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0}$ and infer from Theorem 2 that the unique flow line $\{\mathcal{P}(t, 0, \mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0})\}_{t \geq 0}$ of the MIWF in \mathbf{S}^3 , starting in the smooth immersion $\mathcal{P}(T, 0, F) \circ \phi_T^F \circ \Theta_T^{F_0}$ at time $t = 0$, converges—up to smooth reparametrization—fully in $C^4(\Sigma, \mathbf{R}^4)$ to a umbilic-free parametrization $F_\infty = F_\infty(f_0)$ of a C^4 -local minimizer of the Willmore functional \mathcal{W} in \mathbf{S}^3 with $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$, provided $F : \Sigma \rightarrow \mathbf{S}^3$ is a smooth immersion being contained in the open ball $\mathcal{B}_r^{4,\alpha}(F_0)$ about F_0 in $C^{4,\alpha}(\Sigma, \mathbf{R}^4)$ and $r = r(\Sigma, F_0, F^*, \alpha)$ is sufficiently small. On account of the invariance of both the MIWF and the Willmore functional with respect to time-independent smooth reparametrizations this means that any flow line $\{\mathcal{P}(t, 0, F)\}_{t \geq 0}$ of the MIWF in \mathbf{S}^3 , which starts moving in some arbitrarily chosen smooth immersion $F : \Sigma \rightarrow \mathbf{S}^3$ belonging to the open ball $\mathcal{B}_r^{4,\alpha}(F_0)$, converges—up to smooth reparametrization—fully in $C^4(\Sigma, \mathbf{R}^4)$ to a umbilic-free parametrization F_∞ of a C^4 -local minimizer of the Willmore functional \mathcal{W} in \mathbf{S}^3 , satisfying $\mathcal{W}(F_\infty) = \mathcal{W}(F^*)$, provided $r = r(\Sigma, F_0, F^*, \alpha)$ was chosen sufficiently small. \square

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Data Availability Moreover, “Data sharing” is not applicable to this article, because datasets were neither generated nor analyzed during the current study.

Declarations

Conflict of interest The author declares that there are neither financial nor non-financial interests related to this article.

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