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Some Kollár–Enoki Type Injectivity Theorems on Compact Kähler Manifolds

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Abstract
In this paper we prove some Kollár–Enoki type injectivity theorems on compact Kähler manifolds using the Hodge theory, the Bochner–Kodaira–Nakano identity and the analytic method provided by Fujino (Osaka J Math 49(3):833–852, 2012), Fujino and Matsumur (Trans Am Math Soc Ser B 8(27):849–884, 2021), Matsumura (J Algebraic Geom 27(2):305–337, 2018), Matsumura (Complex analysis and geometry, Springer, Tokyo, 2015). We have some straightforward corollaries. In particular, we will show that our injectivity theorem implies several Nadel type vanishing theorems on smooth projective manifolds.

Keywords Injectivity theorems · Compact Kahler manifolds · L^2 analytic methods · Nadel type vanishing theorems

Mathematics Subject Classification Primary 32L10 · Secondary 32Q15

1 Introduction

The subject of cohomology vanishing theorems for holomorphic vector bundles on complex manifolds occupies a role of central importance in several complex variables and algebraic geometry (cf. [9, 10, 18, 20, 22, 27]). Among various vanishing theorems the Kodaira vanishing theorem [26] is one of the most celebrated results in complex geometry and his original proof is based on his theory of harmonic integrals on compact Kähler manifolds. The injectivity theorem as one of the most important generalizations of the Kodaira vanishing theorem plays an important role when we study fundamental problems in higher dimensional algebraic geometry (cf. [11, 16, 19, 29, 30, 34]). Kollár obtained in [28] his famous injectivity theorem, which is one of the most important generalizations of the Kodaira vanishing theorem for smooth complex

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projective varieties. After Kollár's important work, Enoki recovered and generalized Kollár's injectivity theorem in [8] as an easy application of the theory of harmonic integrals on compact Kähler manifolds.

Recently, Fujino and Matsumura in [13, 14, 23, 31–33, 35] have obtained a series of important injectivity theorems on compact Kähler manifolds formulated by singular hermitian metrics and multiplier ideal sheaves by using the transcendental method based on the theory of harmonic integrals on complete noncompact Kähler manifolds. As is well known, the transcendental method often provides us some very powerful tools not only in complex geometry but also in algebraic geometry (cf. [6, 24, 37–42]). Thus it is natural and of interest to study various vanishing theorems, injectivity theorems and other related topics by using the transcendental method. For a comprehensive and further description about this method, we recommend the reader to see the papers [4, 5, 11, 12, 15–17, 21] and also the references therein.

In this paper, we consider some Kollár–Enoki type injectivity theorems on compact Kähler manifolds by using the Hodge theory, the Bochner–Kodaira–Nakano identity on compact Kähler manifolds and the analytic method provided by Fujino and Matsumura in [13, 23, 32, 35]. Our first main result is the following Theorem 1.1 which contains the famous Enoki injectivity theorem as a special case.

Theorem 1.1 *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F (resp. E) is a holomorphic line (resp. vector) bundle over X with a smooth hermitian metric h_F (resp. h_E) such that*

(1) $\sqrt{-1}\Theta_{h_F}(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$
(2) $\sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E \otimes \sqrt{-1}\Theta_{h_L}(L) \geq_{Nak} 0$ *in the sense of Nakano*

for some positive constants $a, b > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F) \to H^q(X, K_X \otimes E \otimes F \otimes L),$$

is injective for every $q \geq 0$, where K_X is the canonical line bundle of X.

Although the assumptions in Theorem 1.1 may look a little bit artificial it is very useful and has some interesting applications. For instance, by applying Theorem 1.1 we obtain the following Corollaries 1.2 and 1.3. Corollary 1.2 is just the original Enoki injectivity theorem. Corollary 1.3 generalizes the Enoki injectivity theorem to the case twisted by Nakano semi-positive vector bundles.

Corollary 1.2 (Enoki injectivity theorem cf. [8, 19, 23, 32]) *Let L be a semi-positive line bundle over a compact Kähler manifold X. Then for a nonzero section $s \in H^0(X, L^l)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes L^k) \to H^q(X, K_X \otimes L^{l+k}),$$

is injective for any $k, l \geq 1$ and $q \geq 0$.

Corollary 1.3 *Let L (resp. E) be a semi-positive line bundle (resp. a Nakano semi-positive vector bundle) over a compact Kähler manifold X. Then for a nonzero section $s \in H^0(X, L^l)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes E \otimes L^k) \to H^q(X, K_X \otimes E \otimes L^{l+k}),$$

is injective for any $k, l \geq 1$ and $q \geq 0$.

Motivated by the profound work obtained by Fujino and Matsumura in a series of papers (cf. [13, 23, 32, 35]) we can generalize Theorem 1.1 to the case formulated by singular hermitian metrics and multiplier ideal sheaves as follows.

Theorem 1.4 *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F (resp. E) is a holomorphic line (resp. vector) bundle over X with a singular hermitian metric h (resp. a smooth hermitian metric h_E) such that*

(1) $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ in the sense of currents
(2) $\sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E \otimes \sqrt{-1}\Theta_{h_L}(L) \geq_{Nak} 0$ in the sense of Nakano

for some positive constants $a, b > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf of h.

Here we remark that Theorem 1.4 has many straightforward applications. For instance, by applying Theorem 1.4 we have the following Corollaries 1.5 and 1.6. Corollary 1.5 is the main injectivity theorem in [23] and Corollary 1.6 is the Theorem 6.6 in [23].

Corollary 1.5 (Theorem A in [23]) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F is a holomorphic line bundle over X with a singular hermitian metric h such that $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ in the sense of currents for some positive constants $a > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$.

Corollary 1.6 (Theorem 6.6 in [23]) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold (X, ω) equipped with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$ and E a Nakano semi-positive vector bundle over X. If F is a holomorphic line bundle over X with a singular hermitian metric h such that $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ for some positive constants $a > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$.

Moreover, by applying Theorem 1.4 we can also prove some vanishing theorems of Nadel type on smooth projective manifolds.

Corollary 1.7 *Let* X *be a smooth projective manifold with a Kähler form* ω *and* E *a Nakano semi-positive vector bundle on* X. *Let* F *be a holomorphic line bundle on* X *with a singular hermitian metric* h *such that* $\sqrt{-1}\Theta_h(F) \geq \varepsilon\omega$ *in the sense of currents for some* $\varepsilon > 0$. *Then for every* $q > 0$ *we have*

$$H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) = 0.$$

In particular we have

Corollary 1.8 (Nadel vanishing theorem due to Demailly: [3, Theorem 4.5]) *Let* X *be a smooth projective manifold with a Kähler form* ω *and* F *be a holomorphic line bundle on* X *with a singular hermitian metric* h *such that* $\sqrt{-1}\Theta_h(F) \geq \varepsilon\omega$ *in the sense of currents for some* $\varepsilon > 0$. *Then for every* $q > 0$ *we have*

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0.$$

This paper is organizied as follows. In Sect. 2, we recall some basic definitions and collect several preliminary lemmas. Section 3 is devoted to the proof of the main Kollár–Enoki type injectivity theorems on compact Kähler manifolds. We will give the proof of Theorem 1.1 at first and then generalize Theorems 1.1 to 1.4 by applying the deep method provided by Fujino and Matsumura in [13, 23, 32, 35].

2 Preliminaries

In this section, we collect some basic definitions and results from complex analytic and differential geometry. For details, see, for example, [4, 5].

2.1 Singular Hermitian Metrics and Multiplier Ideal Sheaves

Next let us recall the definition of singular hermitian metrics and its multiplier ideal sheaves. For the details, we recommend the reader to see [4]. Let F be a holomorphic line bundle on a complex manifold X.

Definition 2.1 A singular hermitian metric on F is a metric h_F which is given in every trivialization $\theta : F|_\Omega \simeq \Omega \times \mathbb{C}$ by $|\xi|_{h_F} = |\theta(\xi)|e^{-\varphi}$ on Ω, where ξ is a section of F on Ω and $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function. Here $L^1_{\mathrm{loc}}(\Omega)$ is the space of locally integrable functions on Ω. We usually call φ the weight function of the metric with respect to the trivialization θ. The curvature current of a singular hermitian metric h_F is defined by $\sqrt{-1}\Theta_{h_F}(F) := 2\sqrt{-1}\partial\overline{\partial}\varphi$, where φ is a weight function and $\partial\overline{\partial}\varphi$ is taken in the sense of distributions. It is easy to see that the right hand side does not depend on the choice of trivializations (cf. [4]).

Definition 2.2 A holomorphic line bundle F is said to be pseudo-effective if F admits a singular hermitian metric h_F with semi-positive curvature current.

The notion of multiplier ideal sheaves introduced by Nadel in [36] is very important in the recent developments of complex geometry and algebraic geometry.

Definition 2.3 A quasi-plurisubharmonic function by definition is a function φ which is locally equal to the sum of a plurisubharmonic function and of a smooth function. If φ is a quasi-plurisubharmonic function on a complex manifold X, then the multiplier ideal sheaf $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{J}(\varphi)) := \{f \in \mathcal{O}_X(U) \,|\, |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\},$$

for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf of $\mathcal{O}_X$ (see [4, (5.7) Lemma] for example).

Definition 2.4 Let F be a holomorphic line bundle over a complex manifold X and let h_F be a singular hermitian metric on F. We assume $\sqrt{-1}\Theta_{h_F}(F) \geq \gamma$ for some smooth $(1, 1)$-form γ on X. We fix a smooth hermitian metric h_∞ on F. Then we can write $h_F = h_\infty e^{-2\psi}$ for some $\psi \in L^1_{\mathrm{loc}}(X)$ and ψ coincides with a quasi-plurisubharmonic function φ on X almost everywhere. In this situation, we put $\mathcal{J}(h_F) := \mathcal{J}(\varphi)$. We note that $\mathcal{J}(h_F)$ is independent of h_∞ and is thus well-defined.

2.2 Equisingular Approximations

The following Lemma 2.5 is the well-known Demailly–Peternell–Schneider equisingular approximation theorem, which is frequently used in this paper. For details, see [7, Theorem 2.3] and [32, Theorem 2.3].

Lemma 2.5 *Let F be a holomorphic line bundle on a compact Kähler manifold (X, ω) with a singular hermitian metric h with semi-positive curvature current. Then there exists a countable family $\{h_\varepsilon\}_{1\gg\varepsilon>0}$ of singular hermitian metrics on F with the following properties:*

(a) h_ε is smooth on $Y_\varepsilon := X \setminus Z_\varepsilon$, where Z_ε is a proper closed subvariety on X.
(b) $h_{\varepsilon'} \leq h_{\varepsilon''} \leq h$ holds on X when $\varepsilon' > \varepsilon'' > 0$.
(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$ on X.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq a\sqrt{-1}\Theta_{h_F}(F) - \varepsilon\omega$ on X.

2.3 L^2 Spaces and L^2 Estimates

Let X be a complex manifold with a positive $(1, 1)$-form ω and E be a holomorphic vector bundle over X with a smooth metric h. For E-valued (p, q)-forms u and v, the point-wise inner product $\langle u, v\rangle_{h,\omega}$ can be defined, and the global inner product $\langle\langle u, v\rangle\rangle_{h,\omega}$ can also be defined by

$$\langle\langle u, v\rangle\rangle_{h,\omega} := \int_X \langle u, v\rangle_{h,\omega} \, dV_\omega,$$

where $\mathrm{d}V_\omega := \omega^n/n!$ and n is the dimension of X. Recall that the Chern connection D_h on E determined by the holomorphic structure and the hermitian metric h can be written as $D_h = D'_h + \bar{\partial}$ with the $(1,0)$-connection D'_h and the $(0,1)$-connection $\bar{\partial}$ (the $\bar{\partial}$-operator). The connections D'_h and $\bar{\partial}$ can be regarded as a densely defined closed operator on the L^2-space $L^{p,q}_{(2)}(X, E)_{h,\omega}$ defined by

$$L^{p,q}_{(2)}(X, E)_{h,\omega} := \{u \mid u \text{ is an } E\text{-valued } (p,q)\text{-form such that } \|u\|_{h,\omega} < \infty\}.$$

The formal adjoints D'^*_h and $\bar{\partial}^*_h$ agree with the Hilbert space adjoints in the sense of Von Neumann if ω is a complete metric on X. For the L^2-space $L^{p,q}_{(2)}(X, E)_{h,\omega}$ of E-valued (p,q)-forms on X with respect to the inner product $\| \bullet \|_{h,\omega}$, we define the L^2 cohomology $H^{p,q}_{(2)}(X, E)_{h,\omega}$ by

$$H^{p,q}_{(2)}(X, E)_{h,\omega} := \frac{\operatorname{Ker} \bar{\partial} \cap L^{p,q}_{(2)}(X, F)_{h,\omega}}{\operatorname{Im} \bar{\partial} \cap L^{p,q}_{(2)}(X, F)_{h,\omega}}.$$

Finally, we require the following very famous Hörmander L^2 estimates, which will be used in the proof of our vanishing theorems.

Lemma 2.6 *([1, 4, 25]) Let (X, ω) be a complete Kähler manifold. Let (E, h) be an hermitian vector bundle over X. Assume that $A = [i\Theta_h(E), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^*X \otimes E$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^*X \otimes E)$ satisfying $\bar{\partial} g = 0$ and $\int_X (A^{-1}g, g) dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*X \otimes E)$ such that $\bar{\partial} f = g$ and*

$$\int_X |f|^2 \mathrm{d}V_\omega \leq \int_X (A^{-1}g, g) \mathrm{d}V_\omega.$$

3 Proof of Injectivity Theorems

Theorem 3.1 (=Theorem 1.1) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F (resp. E) is a holomorphic line (resp. vector) bundle over X with a smooth hermitian metric h_F (resp. h_E) such that*

(1) $\sqrt{-1}\Theta_{h_F}(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$
(2) $\sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E \otimes \sqrt{-1}\Theta_{h_L}(L) \geq_{Nak} 0$ *in the sense of Nakano*

for some positive constants $a, b > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F) \to H^q(X, K_X \otimes E \otimes F \otimes L),$$

is injective for every $q \geq 0$, where K_X is the canonical line bundle of X.

Proof Let ω be a fixed Kähler metric on X and $n = \dim X$. For simplicity, we denote $h_{E\otimes F} = h_E \otimes h_F$ and $h_{E\otimes F\otimes L} = h_E \otimes h_F \otimes h_L$. By the Hodge theory it is enough to show the map

$$\times s : \mathcal{H}^{n,q}(X, E \otimes F) \to \mathcal{H}^{n,q}(X, E \otimes F \otimes L), \tag{1}$$

is injective for every $q \geq 0$, where $\mathcal{H}^{n,q}(X, E \otimes F)$ is space of $E \otimes F$-valued forms u such that u is harmonic with respect to the metrics ω and $h_{E\otimes F}$, and the same for $\mathcal{H}^{n,q}(X, E \otimes F \otimes G)$.

We will show below the map (1) is well-defined, from which the injectivity is obvious to see. In fact, for any $u \in \mathcal{H}^{n,q}(X, E \otimes F)$ we have $\Delta''_{E\otimes F, h_{E\otimes F}} u = 0$, where

$$\Delta''_{E\otimes F, h_{E\otimes F}} = \nabla''_{E\otimes F} \nabla''^{*}_{E\otimes F, h_{E\otimes F}} + \nabla''^{*}_{E\otimes F, h_{E\otimes F}} \nabla''_{E\otimes F}$$

is the complex Laplace–Beltrami operator of $\nabla''_{E\otimes F}$. By the Nakano identity

$$\begin{aligned}
&\|\nabla''_{E\otimes F} u\|^2_{\omega, h_{E\otimes F}} + \|\nabla''^{*}_{E\otimes F, h_{E\otimes F}} u\|^2_{\omega, h_{E\otimes F}} \\
&= \|\nabla'^{*}_{E\otimes F} u\|^2_{\omega, h_{E\otimes F}} + \langle\langle \sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F)\Lambda_\omega u, u\rangle\rangle_{\omega, h_{E\otimes F}}.
\end{aligned}$$

We note that $\Delta''_{E\otimes F, h_{E\otimes F}} u = 0 \Leftrightarrow \nabla''_{E\otimes F} u = \nabla''^{*}_{E\otimes F, h_{E\otimes F}} u = 0$. Thus for any $u \in \mathcal{H}^{n,q}(X, E \otimes F)$ we have

$$0 = \|\nabla'^{*}_{E\otimes F} u\|^2_{\omega, h_{E\otimes F}} + \langle\langle \sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F)\Lambda_\omega u, u\rangle\rangle_{\omega, h_{E\otimes F}}. \tag{2}$$

But

$$\begin{aligned}
&\sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F) \\
&= \sqrt{-1}\Theta_{h_E}(E) + Id_E \otimes \sqrt{-1}\Theta_{h_F}(F) \\
&\geq_{Nak} \sqrt{-1}\Theta_{h_E}(E) + Id_E \otimes a\sqrt{-1}\Theta_{h_L}(L) \\
&= \sqrt{-1}\Theta_{h_E}(E) + Id_E \otimes ((a-b)\sqrt{-1}\Theta_{h_L}(L) + b\sqrt{-1}\Theta_{h_L}(L)) \\
&= \sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E \otimes \sqrt{-1}\Theta_{h_L}(L) + bId_E \otimes \sqrt{-1}\Theta_{h_L}(L) \\
&\geq_{Nak} 0
\end{aligned}$$

which means that $E \otimes F$ is Nakano semi-positive. It follows that the curvature operator

$$[\sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F), \Lambda_\omega]u = \sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F)\Lambda_\omega u$$

is semi-positive for any (n, q)-forms u on X. Thus we obtain

$$\|\nabla'^{*}_{E\otimes F} u\|^2_{\omega, h_{E\otimes F}} = \langle\langle \sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F)\Lambda_\omega u, u\rangle\rangle_{\omega, h_{E\otimes F}} = 0$$

by equation (2). It follows that

$$\nabla'^{*}_{E\otimes F} u = \langle \sqrt{-1}\Theta_{h_{E\otimes F}}(E \otimes F)\Lambda_\omega u, u\rangle_{\omega, h_{E\otimes F}} = 0, \tag{3}$$

where $\langle\bullet\rangle_{\omega,h_{E\otimes F}}$ means the pointwise inner product on X with respect to ω and $h_{E\otimes F}$. Therefore

$$\nabla'^{*}{}_{E\otimes F\otimes L}(su) = -*\nabla''_{E\otimes F\otimes L}*(su) = s\nabla'^{*}_{E\otimes F}u = 0, \tag{4}$$

since s is a holomorphic L-valued (0,0)-form, where $*$ is the Hodge star operator with respect to the metric ω. By the Nakano identity again

$$\begin{aligned}&\|\nabla''_{E\otimes F\otimes L}su\|^2_{\omega,h_{E\otimes F\otimes L}} + \|\nabla''^{*}_{E\otimes F\otimes L,h_{E\otimes F\otimes L}}su\|^2_{\omega,h_{E\otimes F\otimes L}}\\ &= \|\nabla'^{*}_{E\otimes F\otimes L}su\|^2_{\omega,h_{E\otimes F\otimes L}} + \langle\langle\sqrt{-1}\Theta_{h_{E\otimes F\otimes L}}(E\otimes F\otimes L)\Lambda_\omega su, su\rangle\rangle_{\omega,h_{E\otimes F\otimes L}}.\end{aligned}$$

$\nabla''_{E\otimes F\otimes L}su = 0$ by the Leibnitz rule, since s is holomorphic and u is harmonic. It follows that

$$\|\nabla''^{*}_{E\otimes F\otimes L,h_{E\otimes F\otimes L}}su\|^2_{\omega,h_{E\otimes F\otimes L}} = \langle\langle\sqrt{-1}\Theta_{h_{E\otimes F\otimes L}}(E\otimes F\otimes L)\Lambda_\omega su, su\rangle\rangle_{\omega,h_{E\otimes F\otimes L}}.$$

On the other hand, we compute

$$\begin{aligned}&\sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F)\\ &= \sqrt{-1}\Theta_{h_E}(E) + Id_E\otimes\sqrt{-1}\Theta_{h_F}(F)\\ &\geq_{Nak} \sqrt{-1}\Theta_{h_E}(E) + Id_E\otimes a\sqrt{-1}\Theta_{h_L}(L)\\ &= \sqrt{-1}\Theta_{h_E}(E) + Id_E\otimes((a-b)\sqrt{-1}\Theta_{h_L}(L) + b\sqrt{-1}\Theta_{h_L}(L))\\ &= \sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E\otimes\sqrt{-1}\Theta_{h_L}(L) + bId_E\otimes\sqrt{-1}\Theta_{h_L}(L)\\ &\geq_{Nak} bId_E\otimes\sqrt{-1}\Theta_{h_L}(L)\end{aligned}$$

that is,

$$Id_E\otimes\sqrt{-1}\Theta_{h_L}(L) \leq_{Nak} \frac{1}{b}\sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F).$$

It follows that

$$\begin{aligned}&\sqrt{-1}\Theta_{h_{E\otimes F\otimes L}}(E\otimes F\otimes L)\\ &= \sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F) + Id_E\otimes\sqrt{-1}\Theta_{h_L}(L)\\ &\leq_{Nak} (1+\frac{1}{b})\sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F).\end{aligned}$$

Therefore, by equation (3) we have

$$\begin{aligned}&\langle\sqrt{-1}\Theta_{h_{E\otimes F\otimes L}}(E\otimes F\otimes L)\Lambda_\omega su, su\rangle_{\omega,h_{E\otimes F\otimes L}}\\ &\leq (1+\frac{1}{b})\langle\sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F)\Lambda_\omega su, su\rangle_{\omega,h_{E\otimes F\otimes L}}\\ &= (1+\frac{1}{b})|s|^2_{h_L}\langle\sqrt{-1}\Theta_{h_{E\otimes F}}(E\otimes F)\Lambda_\omega u, u\rangle_{\omega,h_{E\otimes F}}\\ &= 0\end{aligned}$$

and

$$\|\nabla''^{*}_{E\otimes F\otimes L,h_{E\otimes F\otimes L}} su\|^2_{\omega,h_{E\otimes F\otimes L}} = \langle\langle\sqrt{-1}\Theta_{h_{E\otimes F\otimes L}}(E\otimes F\otimes L)\Lambda_\omega su, su\rangle\rangle_{\omega,h_{E\otimes F\otimes L}} \leq 0.$$

This means that

$$\|\nabla''^{*}_{E\otimes F\otimes L,h_{E\otimes F\otimes L}} su\|^2_{\omega,h_{E\otimes F\otimes L}} = 0 \text{ and } \nabla''^{*}_{E\otimes F\otimes L,h_{E\otimes F\otimes L}} su = 0.$$

Recall that $\nabla''_{E\otimes F\otimes G} su = 0$. Thus we conclude that the $E\otimes F\otimes L$-valued form su is harmonic with respect to the metrics ω and $h_{E\otimes F\otimes L}$, that is, $su \in \mathcal{H}^{n,q}(X, E\otimes F\otimes L)$. This means that the map (1) is well-defined. The proof is finished. □

Corollary 3.2 (=Corollary 1.2) *Let L be a semi-positive line bundle over a compact Kähler manifold X. Then for a nonzero section $s \in H^0(X, L^l)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes L^k) \to H^q(X, K_X \otimes L^{l+k}),$$

is injective for any $k, l \geq 1$ and $q \geq 0$.

Proof Let E be the trivial line bundle on X. For the semi-positive line bundle L we set $F = L^k$ and $L' = L^l$. Then the conditions (1) and (2) in Theorem 3.1 are easy to check for small positive constants $a > 0, b > 0$ with $a = b$. By Theorem 3.1 we know that for a nonzero section $s \in H^0(X, L')$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F) \to H^q(X, K_X \otimes E \otimes F \otimes L'),$$

that is,

$$\times s : H^q(X, K_X \otimes L^k) \to H^q(X, K_X \otimes L^{l+k}),$$

is injective for every $q \geq 0$. □

Corollary 3.3 (=Corollary 1.3) *Let L (resp. E) be a semi-positive line bundle (resp. a Nakano semi-positive vector bundle) over a compact Kähler manifold X. Then for a nonzero section $s \in H^0(X, L^l)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes E \otimes L^k) \to H^q(X, K_X \otimes E \otimes L^{l+k}),$$

is injective for any $k, l \geq 1$ and $q \geq 0$.

Proof For a semi-positive line bundle L and a Nakano semi-positive vector bundle E on X we let $F = L^k$ and $L' = L^l$. Then the conditions (1) and (2) in Theorem 3.1 are easy to check for small positive constants $a > 0, b > 0$ with $a = b$. By Theorem 3.1 we know that for a nonzero section $s \in H^0(X, L')$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F) \to H^q(X, K_X \otimes E \otimes F \otimes L'),$$

that is,

$$\times s : H^q(X, K_X \otimes E \otimes L^k) \to H^q(X, K_X \otimes E \otimes L^{l+k}),$$

is injective for any $k, l \geq 1$ and $q \geq 0$. □

Theorem 3.4 (=Theorem 1.4) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F (resp. E) is a holomorphic line (resp. vector) bundle over X with a singular hermitian metric h (resp. a smooth hermitian metric h_E) such that*

(1) $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ in the sense of currents
(2) $\sqrt{-1}\Theta_{h_E}(E) + (a-b)Id_E \otimes \sqrt{-1}\Theta_{h_L}(L) \geq_{Nak} 0$ in the sense of Nakano

for some positive constants $a, b > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf of h.

Proof We may assume $q > 0$ since the case $q = 0$ is obvious. For the proof, it is sufficient to show that an arbitrary cohomology class $\eta \in H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h))$ satisfying $s\eta = 0 \in H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L)$ is actually zero. We fix a Kähler form ω on X throughout the proof and represent the cohomology class $\eta \in H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h))$ by a $\overline{\partial}$-closed $E \otimes F$-valued (n, q)-form u with $\|u\|_{h_E h,\omega} < \infty$ by using the standard De Rham–Weil isomorphism

$$H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \cong \frac{\operatorname{Ker} \overline{\partial} : L^{n,q}_{(2)}(E \otimes F)_{h_E h,\omega} \to L^{n,q+1}_{(2)}(E \otimes F)_{h_E h,\omega}}{\operatorname{Im} \overline{\partial} : L^{n,q-1}_{(2)}(E \otimes F)_{h_E h,\omega} \to L^{n,q}_{(2)}(E \otimes F)_{h_E h,\omega}}.$$

For the given singular hermitian metric h on F, by the Demailly-Peternell-Schneider equisingular approximation theorem (Lemma 2.5), there is a countable family $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ of singular hermitian metrics on F with the following properties:

(a) h_ε is smooth on $Y_\varepsilon := X \setminus Z_\varepsilon$, where Z_ε is a proper closed subvariety on X.
(b) $h_{\varepsilon'} \leq h_{\varepsilon''} \leq h$ holds on X when $\varepsilon' > \varepsilon'' > 0$.
(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$ on X.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq a\sqrt{-1}\Theta_{h_F}(F) - \varepsilon\omega$ on X.

By [13, Section 3] we can take a complete Kähler form ω_ε on Y_ε such that: ω_ε is a complete Kähler form on Y_ε, $\omega_\varepsilon \geq \omega$ on Y_ε and $\omega_\varepsilon = \sqrt{-1}\partial\overline{\partial}\Psi_\varepsilon$ for some bounded function Ψ_ε on a neighborhood of every $p \in X$. We define a Kähler form $\omega_{\varepsilon,\delta}$ on Y_ε by

$$\omega_{\varepsilon,\delta} := \omega + \delta\omega_\varepsilon$$

for ε and δ with $0 < \delta \ll \varepsilon$. The following properties are easy to check

(A) $\omega_{\varepsilon,\delta}$ is a complete Kähler form on $Y_\varepsilon = X \setminus Z_\varepsilon$ for every $\delta > 0$.
(B) $\omega_{\varepsilon,\delta} \geq \omega$ on Y_ε for every $\delta > 0$.
(C) $\Psi + \delta\Psi_\varepsilon$ is a bounded local potential function of $\omega_{\varepsilon,\delta}$ and converges to Ψ as $\delta \to 0$ where Ψ is a local potential function of ω.

In the proof of Theorem 1.4, we actually consider only a countable sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ (resp. $\{\delta_\ell\}_{\ell=1}^{\infty}$) converging to zero since we need to apply Cantor's diagonal argument, but we often use the notation ε (resp. δ) for simplicity. In the following, we mainly consider the L^2-space $L^{n,q}_{(2)}(Y_\varepsilon, E\otimes F)_{h_E h_\varepsilon, \omega_{\varepsilon,\delta}}$ of $E\otimes F$-valued (n,q)-forms on Y_ε. We denote $L^{n,q}_{(2)}(E\otimes F)_{\varepsilon,\delta} := L^{n,q}_{(2)}(Y_\varepsilon, E\otimes F)_{h_E h_\varepsilon, \omega_{\varepsilon,\delta}}$ and $\| \bullet \|_{\varepsilon,\delta} := \| \bullet \|_{h_E h_\varepsilon, \omega_{\varepsilon,\delta}}$ for simplicity. The following inequality is easy to check

$$\|u\|_{\varepsilon,\delta} \le \|u\|_{h_E h, \omega_{\varepsilon,\delta}} \le \|u\|_{h_E h, \omega} < \infty. \tag{5}$$

In particular, the norm $\|u\|_{\varepsilon,\delta}$ is uniformly bounded with respect to ε, δ. □

There are various formulations for L^2-estimates for $\overline{\partial}$-equations, which originated from Hörmander's paper [25]. The following one is suitable for our purpose.

Lemma 3.5 (cf. [2, 4.1 Théorème]) *Assume that B is a Stein open set in X such that $\omega_{\varepsilon,\delta} = \sqrt{-1}\partial\overline{\partial}(\Psi + \delta\Psi_\varepsilon)$ on a neighborhood of $\overline{B}$. Then, for an arbitrary $\alpha \in \operatorname{Ker}\overline{\partial} \subset L^{n,q}_{(2)}(B\backslash Z_\varepsilon, E\otimes F)_{\varepsilon,\delta}$, there exist $\beta \in L^{n,q-1}_{(2)}(B\backslash Z_\varepsilon, E\otimes F)_{\varepsilon,\delta}$ and a positive constant $C_{\varepsilon,\delta}$ (independent of α) such that:* (1) $\overline{\partial}\beta = \alpha$ *and* $\|\beta\|^2_{\varepsilon,\delta} \le C_{\varepsilon,\delta}\|\alpha\|^2_{\varepsilon,\delta}$; (2) $\overline{\lim}_{\delta\to 0} C_{\varepsilon,\delta}$ *is finite and is independent of* ε.

Proof of Lemma 3.5 We may assume $\varepsilon < 1/2$. For the smooth hermitian metric $H_{\varepsilon,\delta}$ on $E\otimes F$ over $B \setminus Z_\varepsilon$ defined by $H_{\varepsilon,\delta} := h_E h_\varepsilon e^{-(\Psi+\delta\Psi_\varepsilon)}$, the curvature satisfies

$$\sqrt{-1}\Theta_{H_{\varepsilon,\delta}}(E\otimes F) \ge_{Nak} 1/2 \cdot Id_E \otimes \omega_{\varepsilon,\delta}$$

by property (B) and $\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F) \ge -\varepsilon Id_E \otimes \omega$. The L^2-norm $\|\alpha\|_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}$ with respect to $H_{\varepsilon,\delta}$ and $\omega_{\varepsilon,\delta}$ is finite since the function $\Psi + \delta\Psi_\varepsilon$ is bounded and $\|\alpha\|_{\varepsilon,\delta}$ is finite. Therefore, from the standard L^2-method for the $\overline{\partial}$-equation (cf. [2, 4.1 Théorème]), we obtain a solution β of the $\overline{\partial}$-equation $\overline{\partial}\beta = \alpha$ with

$$\|\beta\|^2_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}} \le \frac{2}{q}\|\alpha\|^2_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}.$$

It follows that

$$\|\beta\|^2_{\varepsilon,\delta} \le C_{\varepsilon,\delta}\|\alpha\|^2_{\varepsilon,\delta},$$

where $C_{\varepsilon,\delta} = \frac{2}{q}\frac{\sup_B e^{-(\Psi+\delta\Psi_\varepsilon)}}{\inf_B e^{-(\Psi+\delta\Psi_\varepsilon)}}$. It is easy to check $C_{\varepsilon,\delta}$ satisfies the above properties. □

By essentially using the property (C) and Lemma 3.5 we have the following De Rham–Weil isomorphism from the $\overline{\partial}$-L^2 cohomology on Y_ε to the Čech cohomology on X (cf. [13, Claim 1] and [32, Proposition 5.5])

$$H^q(X, K_X\otimes E\otimes F\otimes \mathcal{I}(h_\varepsilon)) \cong \frac{\operatorname{Ker}\overline{\partial} : L^{n,q}_{(2)}(E\otimes F)_{\varepsilon,\delta} \to L^{n,q+1}_{(2)}(E\otimes F)_{\varepsilon,\delta}}{\operatorname{Im}\overline{\partial} : L^{n,q-1}_{(2)}(E\otimes F)_{\varepsilon,\delta} \to L^{n,q}_{(2)}(E\otimes F)_{\varepsilon,\delta}}.$$

The following orthogonal decomposition then follows

$$L^{n,q}_{(2)}(E\otimes F)_{\varepsilon,\delta}=\operatorname{Im}\overline{\partial}\oplus\mathcal{H}^{n,q}_{\varepsilon,\delta}(E\otimes F)\oplus\operatorname{Im}\overline{\partial}^*_{\varepsilon,\delta}.$$

Now that the $E\otimes F$-valued (n,q)-form u belongs to $L^{n,q}_{(2)}(E\otimes F)_{\varepsilon,\delta}$ by (5), it can be decomposed into $u=\overline{\partial}w_{\varepsilon,\delta}+u_{\varepsilon,\delta}$ for some $w_{\varepsilon,\delta}\in\operatorname{Dom}\overline{\partial}\subset L^{n,q-1}_{(2)}(E\otimes F)_{\varepsilon,\delta}$ and $u_{\varepsilon,\delta}\in\mathcal{H}^{n,q}_{\varepsilon,\delta}(E\otimes F)$. The orthogonal projection of u to $\operatorname{Im}\overline{\partial}^*_{\varepsilon,\delta}$ is zero since u is $\overline{\partial}$-closed. We need the following Lemma 3.6 which can be proved by the same analytic method provided by Fujino and Matsumura in [23, 32]. Here we omit the proof for simplicity. For the details, we refer the reader to the proof of Proposition 5.7 in [23] in which the inequality (5) plays an important role. By Lemma 3.6 it is sufficient for the proof to study the asymptotic behavior of the norm of $su_{\varepsilon,\delta}$.

Lemma 3.6 *(cf. Proposition 5.7 in [23]) The cohomology class η is zero if*

$$\varliminf_{\varepsilon\to0}\varliminf_{\delta\to0}\|su_{\varepsilon,\delta}\|_{\varepsilon,\delta}=0$$

where $\|\bullet\|_{\varepsilon,\delta}:=\|\bullet\|_{h_Eh_\varepsilon h_L,\omega_{\varepsilon,\delta}}$ for an $E\otimes F\otimes L$-valued form $\bullet$.

Moreover, following the proof of Proposition 5.8 in [23] and Proposition 2.8 in [29] we have

Lemma 3.7

$$\lim_{\varepsilon\to0}\varlimsup_{\delta\to0}\|\overline{\partial}^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|_{\varepsilon,\delta}=0.$$

Proof of Lemma 3.7 By applying the Bochner–Kodaira–Nakano identity and the density lemma to $u_{\varepsilon,\delta}$ and $su_{\varepsilon,\delta}$ (see [23, Proposition 5.8] and [29, Proposition 2.8]), we have

$$0=\langle\!\langle\sqrt{-1}\Theta_{h_Eh_\varepsilon}(E\otimes F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta},u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}+\|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\tag{6}$$

$$\|\overline{\partial}^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}=\langle\!\langle\sqrt{-1}\Theta_{h_Eh_\varepsilon h_L}(E\otimes F\otimes L)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta},su_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}+\|D'^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\tag{7}$$

where we used the fact that $u_{\varepsilon,\delta}$ is harmonic and $\overline{\partial}(su_{\varepsilon,\delta})=s\overline{\partial}u_{\varepsilon,\delta}=0$. We have

$$\begin{aligned}&\sqrt{-1}\Theta_{h_Eh_\varepsilon}(E\otimes F)=\sqrt{-1}\Theta_{h_E}(E)+Id_E\otimes\sqrt{-1}\Theta_{h_\varepsilon}(F)\\&\geq_{Nak}\sqrt{-1}\Theta_{h_E}(E)+Id_E\otimes(a\sqrt{-1}\Theta_{h_L}(L)-\varepsilon\omega)\\&=\sqrt{-1}\Theta_{h_E}(E)+Id_E\otimes((a-b)\sqrt{-1}\Theta_{h_L}(L)+b\sqrt{-1}\Theta_{h_L}(L)-\varepsilon\omega)\\&\geq_{Nak}Id_E\otimes(b\sqrt{-1}\Theta_{h_L}(L)-\varepsilon\omega),\end{aligned}\tag{8}$$

by properties (d), (B) and the assumption (2) in Theorem 1.4. It follows that

$$\sqrt{-1}\Theta_{h_Eh_\varepsilon}(E\otimes F)\geq_{Nak}-\varepsilon Id_E\otimes\omega\geq_{Nak}-\varepsilon Id_E\otimes\omega_{\varepsilon,\delta}.$$

So the integrand $g_{\varepsilon,\delta}$ of the first term of (6) satisfies

$$-\varepsilon q|u_{\varepsilon,\delta}|^2_{\varepsilon,\delta} \le g_{\varepsilon,\delta} := \langle\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle_{\varepsilon,\delta}. \tag{9}$$

For the precise argument, see [32, Step 2 in the proof of Theorem 3.1]. By (6) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\left(\int_{\{g_{\varepsilon,\delta}\ge 0\}} g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}} + \|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\right)\\
&= \lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\left(-\int_{\{g_{\varepsilon,\delta}\le 0\}} g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}\right)\\
&\le \lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\left(\varepsilon q\int_{\{g_{\varepsilon,\delta}\le 0\}} |u_{\varepsilon,\delta}|^2_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}\right)\\
&\le \lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\left(\varepsilon q\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\right) = 0.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\int_{\{g_{\varepsilon,\delta}\ge 0\}} g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}} = 0 \text{ and } \lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} = 0. \tag{10}$$

Therefore, by Eq. (6) we have

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\langle\!\langle\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} = 0. \tag{11}$$

Thus, we obtain

$$0 \le \lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\langle\!\langle\sqrt{-1}\Theta_{h_E h_\varepsilon h_L}(E\otimes F\otimes L)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}, \tag{12}$$

thanks to

$$\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F) \le_{Nak} \sqrt{-1}\Theta_{h_E h_\varepsilon h_L}(E\otimes F\otimes L),$$

and

$$\langle\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle_{\varepsilon,\delta} \le \langle\sqrt{-1}\Theta_{h_E h_\varepsilon h_L}(E\otimes F\otimes L)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle_{\varepsilon,\delta}.$$

On the other hand, by formula (8) we have

$$\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F) \ge_{Nak} Id_E\otimes(b\sqrt{-1}\Theta_{h_L}(L)-\varepsilon\omega) \ge_{Nak} Id_E\otimes (b\sqrt{-1}\Theta_{h_L}(L)-\varepsilon\omega_{\varepsilon,\delta}).$$

It follows that

$$\langle\!\langle\sqrt{-1}\Theta_{h_E h_\varepsilon h_L}(E\otimes F\otimes L)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}$$

$$\begin{aligned}
\leq&\Big(1+\frac{1}{b}\Big)\int_{Y_\varepsilon}|s|^2_{h_L}g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}+\frac{\varepsilon q}{b}\int_{Y_\varepsilon}|s|^2_{h_L}|u_{\varepsilon,\delta}|^2_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}\\
\leq&\Big(1+\frac{1}{b}\Big)\sup_X|s|^2_{h_L}\int_{\{g_{\varepsilon,\delta}\geq 0\}}g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}+\frac{\varepsilon q}{b}\sup_X|s|^2_{h_L}\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\\
\leq&\Big(1+\frac{1}{b}\Big)\sup_X|s|^2_{h_L}\int_{\{g_{\varepsilon,\delta}\geq 0\}}g_{\varepsilon,\delta}\,\mathrm{d}V_{\omega_{\varepsilon,\delta}}+\frac{\varepsilon q}{b}\sup_X|s|^2_{h_L}\|u\|^2_{h_Eh_\varepsilon,\omega}
\end{aligned}$$

which leads to

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\langle\!\langle\sqrt{-1}\Theta_{h_Eh_\varepsilon h_L}(E\otimes F\otimes L)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta},su_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}=0, \tag{13}$$

by formulas (10) and (12). Moreover, we have

$$\|D'^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}=\|sD'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\leq\sup_X|s|^2_{h_L}\|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}$$

thanks to $D'^*_{\varepsilon,\delta}=-*\bar\partial*$ where $*$ is the Hodge star operator with respect to $\omega_{\varepsilon,\delta}$. By formula (10) it follows that

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\|D'^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}=0. \tag{14}$$

Therefore, we obtain the conclusion by the Eqs. (7), (13) and (14). □

Lemma 3.8 *There exist $E\otimes F\otimes L$-valued $(n,q-1)$-forms $v_{\varepsilon,\delta}$ on Y_ε such that $\bar\partial v_{\varepsilon,\delta}=su_{\varepsilon,\delta}$ and $\overline{\lim}_{\delta\to 0}\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta}$ can be bounded by a constant independent of ε.*

The proof of Lemma 3.8 is completely the same as that in Proposition 5.10 in [23] in which Lemma 3.5 is used to establish the De Rham–Weil isomorphism from the $\bar\partial$-L^2 cohomology on Y_ε to the Čech cohomology on X (cf. [13, Claim 1] and [32, Proposition 5.5]) and the inequality (5) is essentially used to control the bound. For the details, we refer the reader to the proof of Proposition 5.10 in [23] and here we omit it for simplicity.

Lemma 3.9

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\|su_{\varepsilon,\delta}\|_{\varepsilon,\delta}=0.$$

Proof of Lemma 3.9 For the solution $v_{\varepsilon,\delta}$ in Lemma 3.8, it is easy to check that

$$\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\|su_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}=\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\langle\!\langle\bar\partial^*_{\varepsilon,\delta}su_{\varepsilon,\delta},v_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta}\leq\lim_{\varepsilon\to 0}\overline{\lim_{\delta\to 0}}\,\|\bar\partial^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|_{\varepsilon,\delta}\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta}.$$

By Lemma 3.7 and Lemma 3.8 we conclude that the right hand side is zero.

Now we finish the proof of Theorem 3.4 by Lemma 3.6. □

Corollary 3.10 (=Corollary 1.5) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold X with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. If F is a holomorphic line bundle over X with a singular hermitian metric h such that $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ in the sense of currents for some positive constants $a > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$.

Proof We let E be the trivial line bundle on X. Then the conditions (1) and (2) in Theorem 3.4 are easy to check if we take $b = a$. By Theorem 3.4 we know that for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

that is,

$$\times s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf of h. □

Corollary 3.11 (=Corollary 1.6) *Let L be a semi-positive holomorphic line bundle over a compact Kähler manifold (X, ω) equipped with a smooth hermitian metric h_L satisfying $\sqrt{-1}\Theta_{h_L}(L) \geq 0$ and E a Nakano semi-positive vector bundle over X. If F is a holomorphic line bundle over X with a singular hermitian metric h such that $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_L}(L) \geq 0$ for some positive constants $a > 0$, then for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$*

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$.

Proof For a Nakano semi-positive vector bundle E on X the conditions (1) and (2) in Theorem 3.4 are easy to check if we take $b = a$. By Theorem 3.4 we know that for a nonzero section $s \in H^0(X, L)$ the multiplication map induced by $\otimes s$

$$\times s : H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(h) \otimes L),$$

is injective for every $q \geq 0$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf of h. □

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