The Journal of Geometric Analysis (2023) 33:376
https://doi.org/10.1007/s12220-023-01432-5

A New Conformal Heat Flow of Harmonic Maps

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Received: 5 December 2022 / Accepted: 6 September 2023 / Published online: 29 September 2023

Abstract
We introduce and study a conformal heat flow of harmonic maps defined by an evolution equation for a pair consisting of a map and a conformal factor of metric on the two-dimensional domain. This flow is designed to postpone finite time singularity but does not get rid of possibility of bubble forming. We show that Struwe type global weak solution exists, which is smooth except at most finitely many points.

Keywords Harmonic maps · Conformal heat flow · Existence · Bubbling

Mathematics Subject Classification 58E20 · 53E99 · 53C43 · 35K58

1 Introduction

Consider a map $f_0 : \Sigma \times [0, T) \to N$ from a compact Riemann surface (Σ, g_0) with metric g_0 to a Riemannian manifold (N, h). Under the usual harmonic map heat flow, f_0 evolves to a map $f(t)$ according to the evolution equation $f_t = \tau_{g_0}(f)$, where $\tau_g(f) = \mathrm{tr}_g(\nabla^g \mathrm{d}f)$ is the tension field with respect to the metric g. In this paper we consider the generalization in which both the map and the metric evolve with $(f(t), g(t))$ satisfying the equations

$$\begin{cases} f_t &= \tau_g(f), \qquad (1a) \\ g_t &= (2b|\mathrm{d}f|_g^2 - 2a)g, \qquad (1b) \end{cases}$$

where $a, b > 0$ are constants and $|\mathrm{d}f|_g^2 = g^{ij}h_{\alpha\beta}f_i^\alpha f_j^\beta$ is the energy density. We assume that the initial map $f(0) = f_0$ and metric $g(0) = g_0$ are smooth.

The first of these equations is the harmonic map heat flow, with varying metric g. The second equation is designed to attenuate energy concentration. If the energy

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density become large in some region $\Omega \subset \Sigma$, then under the flow (1b), the metric is conformally enlarged; this increases the area of Ω and decreases the energy density. This suggests that the system (1) may be better behaved than the harmonic map heat flow, where energy concentration at points is an impediment to convergence.

Writing the metric $g(t) = \mathrm{e}^{2u} g_0$ for a real-valued function $u(t)$, equations (1) are equivalent to the following equations for the pair $(f(t), u(t))$:

$$\begin{cases} f_t &= \mathrm{e}^{-2u}\tau(f), \qquad (2a)\\ u_t &= b\mathrm{e}^{-2u}|\mathrm{d}f|^2 - a, \qquad (2b)\end{cases}$$

where τ and $|\mathrm{d}f|^2$ are with respect to the fixed metric g_0, and where the initial conditions are $f(0) = f_0$, $u(0) = 0$. In this form, the flow is more easily analyzed.

The main Theorem of this paper is the following.

Theorem 1 (Existence of global weak solution) *For any $f_0 \in W^{3,2}(\Sigma, N)$, a global weak solution (f, u) of (2) exists on $\Sigma \times [0, \infty)$ which is smooth on $\Sigma \times (0, \infty)$ except at most finitely many points.*

There is a long history of harmonic maps and related fields. We could not list all such literatures but only few, including [1–12] and therein. In terms of heat flow of harmonic maps, see for example [13–25] and therein. Note that usual heat flow can have finite time singularity, see Chang–Ding–Ye [26], Raphael–Schweyer [27], or more recently Dávila–Del Pino–Wei [28].

There are several directions to allow metric change along harmonic map heat flow. The most well-known direction is Teichmüller flow, where metric lies in Teichmüller space of constant curvature. Teichmüller flow is the L^2 gradient flow of the energy and hence reduce the energy in the fastest sense. A pioneering work in this direction was the result of Ding–Li–Liu [29] in the torus case, and later in higher genus case by Rupflin [30], Rupflin–Topping [31], and Rupflin–Topping–Zhu [32]. For further references, see for example Rupflin–Topping [33], Huxol–Rupflin–Topping [34] or Rupflin–Topping [35] and therein. Another direction is Ricci-harmonic map flow. This is a combination of harmonic map heat flow and Ricci flow of the metric. Surprisingly, this flow is more regular than both harmonic map heat flow and Ricci flow. See for example, Muller [36], Williams [37] or Buzano–Rupflin [38] among others. Recently in Huang–Tam [39], harmonic map heat flow together with evolution equation of metric is considered under time-dependent curvature restriction and smooth short time existence is obtained. Because we do not assume a priori curvature bounds of the domain, the result cannot be applied into our case.

The paper is organized as follows. In Sect. 2 we look at some preliminaries, including volume formula and its asymptotic limit if the map f is steady solution, that is, harmonic. Next, in Sect. 3 we define Hilbert spaces X, Y, Z and their closed subsets B, B'. So, from Sect. 3 we consider $f \in B$ and $u \in B'$. Then Sect. 4 defines the operator S_1, S_2 and shows their properties. Briefly, we can show that $S_1 : B \times B' \to B$ and $S_2 : B \times B' \to B'$ and they satisfy twisted partial contraction properties, see Lemmas 6, 7, 10, and 11. In Section 5 we define the operator $\mathcal{S}$ on $B \times B'$ mapping into itself defined by $\mathcal{S} = (S_1, S_2)$. For T small enough, $\mathcal{S}$ is a contraction and hence we can prove short time existence.

Next we are working on types of singularity. Ultimately we will show that the solution is singular only when energy concentrates, similar with Struwe's solution for harmonic map heat flow. In Sect. 6 we show local estimate and obtain bounds for $\iint e^{2u}|f_t|^4$. This is used in Sect. 7 to show $W^{2,2}$ and higher estimate, which implies boundedness of $|\mathrm{d}f|$. Finally in Sect. 8 we prove the main theorem 1 and in Sect. 9 some remarks about finite time singularity are provided.

1.1 Notation

Even though our equation is heat-type equation for varying metric, we use initial metric g_0 as default. So, all computations use the metric g_0 unless we specify the metric. For example, $|\mathrm{d}f|^2$ is calculated in terms of g_0 and $|\mathrm{d}f|_g^2$ is calculated in terms of g. If the volume form is calculated in terms of metric g, we denote it as dvol_g. We also omit dvol_{g_0} and $\mathrm{d}t$ if there is no confusion. We also use the simplifications $\|\cdot\|_{W^{k,p}} = \|\cdot\|_{W^{k,p}(\Sigma\times[0,T])}$, $\|\cdot\|_{C^0} = \|\cdot\|_{C^0(\Sigma\times[0,T])}$ and $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Sigma\times[0,T])}$. Also, the constant c is universal and changed line by line.

2 Preliminaries

Before we show the main result, we record a few facts about solutions to the flow equations (2).

2.1 Energy and Volume

First note that the 2-form $|\mathrm{d}f|^2\,\mathrm{dvol}_g$ is conformally invariant, and that the energy

$$E(t) = \tfrac{1}{2}\int |\mathrm{d}f|^2\,\mathrm{dvol}_g \tag{3}$$

satisfies

$$E'(t) = \int \langle \mathrm{d}f, \mathrm{d}f_t\rangle \;=\; -\int \langle \nabla \mathrm{d}f, e^{-2u}\tau(f)\rangle \;=\; -\int e^{-2u}|\tau(f)|^2 \;\le\; 0. \tag{4}$$

Thus $E(t) \le E_0$ for all t.

Lemma 2 *The volume satisfies $V(t) \le e^{-2at}V(0) + \frac{2b}{a}E_0$, and hence is finite for all t.*

Proof The second Eq. (2b) can be explicitly solved, yielding

$$e^{2u} = e^{-2at}\left(1 + 2b\int_0^t e^{2as}|\mathrm{d}f|^2(s)\mathrm{d}s\right). \tag{5}$$

Consequently, the volume

$$V(t) = \int_{\Sigma} \mathrm{dvol}_{g(t)} = \int_{\Sigma} \mathrm{e}^{2u} \mathrm{dvol}_{g_0} \tag{6}$$

can be written as

$$V(t) = \mathrm{e}^{-2at} \left(V(0) + 4b \int_0^t \mathrm{e}^{2as} E(s) \mathrm{d}s \right). \tag{7}$$

The lemma follows by noting that $E(s) \leq E_0$ and integrating. □

2.2 Asymptotic Behavior of Steady Solution

In this subsection we consider the steady solution.

Lemma 3 *Let (f, u) be a solution of (2) and $f(0)$ a harmonic with energy E. Then $f(t)$ is harmonic for all t and as $t \to \infty$,*

$$\mathrm{e}^{2u} \to \tfrac{b}{a} |\mathrm{d}f|^2$$

and hence by (6) the volume $V(t)$ converges to

$$V(\infty) = \tfrac{2b}{a} E.$$

Proof If $f(0)$ is harmonic, then $f_t = 0$ and hence f and $|\mathrm{d}f|^2$ are independent of t. Integrating (5) then shows that, as $t \to \infty$,

$$\begin{aligned} \mathrm{e}^{2u} &= \mathrm{e}^{-2at} \left(1 + 2b|\mathrm{d}f|^2 \frac{\mathrm{e}^{2at} - 1}{2a} \right) \\ &= \mathrm{e}^{-2at} + \tfrac{b}{a} |\mathrm{d}f|^2 (1 - \mathrm{e}^{-2at}) \to \tfrac{b}{a} |\mathrm{d}f|^2. \end{aligned}$$

□

This means that, for solutions as in Lemma 3, the energy density $|\mathrm{d}f|_g^2 = |\mathrm{d}f|^2 \mathrm{e}^{-2u}$ converges as $t \to \infty$ to the constant $\frac{a}{b}$. Hence the conformal heat flow forces the conformal factor and the energy density be distributed evenly. Remark that, because the image $f(\Sigma)$ does not change, this flow modifies the domain toward the space which is similar to the image with the similarity ratio $\frac{a}{b}$.

3 Construction of Hilbert Spaces

In this section we build Hilbert spaces X_T, Y_T, Z_T and their closed subsets B, B'. For parabolic theory used here, see Mantegazza–Martinazzi [40], Evans [41] or Lieberman [42]. From now on, we consider the target manifold being isometrically embedded, $N \hookrightarrow \mathbb{R}^L$.

3.1 Spaces *X*, *Y* and *Z*

The set

$$Y_T = L^2([0,T], W^{4,2}(\Sigma,\mathbb{R}^L)) \cap W^{1,2}([0,T], W^{2,2}(\Sigma,\mathbb{R}^L)) \cap W^{2,2}([0,T], L^2(\Sigma,\mathbb{R}^L))$$

is a Hilbert space with norm

$$\|f\|_Y^2 = \int_0^T \int_\Sigma |\nabla^4 f|^2 + |f|^2 + |\nabla^2 f_t|^2 + |f_t|^2 + |f_{tt}|^2 \,\mathrm{dvol}_{g_0}\,\mathrm{d}t.$$

As in Proposition 4.1 in [40],

$$Y_T \hookrightarrow C^0([0,T], C^1(\Sigma,\mathbb{R}^L)) \cap L^4([0,T], W^{3,4}(\Sigma,\mathbb{R}^L)) \cap W^{1,4}([0,T], W^{1,4}(\Sigma,\mathbb{R}^L))$$

and there is a constant c such that

$$\|f\|_{C^0} + \|\nabla f\|_{C^0} + \|\nabla^3 f\|_{L^4} + \|\nabla f_t\|_{L^4} \le c\|f\|_Y. \tag{8}$$

Also, by standard parabolic theory (see, for example, [41]), $f \in Y_T$ implies $f \in C^0([0,T], W^{3,2}(\Sigma,\mathbb{R}^L))$, $f_t \in C^0([0,T], W^{1,2}(\Sigma,\mathbb{R}^L))$ and

$$\max_{0\le t\le T}\|f(t)\|_{W^{3,2}(\Sigma)}, \max_{0\le t\le T}\|f_t(t)\|_{W^{1,2}(\Sigma)} \le c\|f\|_Y. \tag{9}$$

This also implies that

$$\max_{0\le t\le T}\|f(t)\|_{W^{2,8}(\Sigma)} \le c\|f\|_Y. \tag{10}$$

Next, denote

$$X_T = L^2([0,T], W^{2,2}(\Sigma,\mathbb{R}^L)) \cap W^{1,2}([0,T], L^2(\Sigma,\mathbb{R}^L))$$

be another Hilbert space with norm

$$\|f\|_X^2 = \int_0^T \int_\Sigma |f|^2 + |\nabla^2 f|^2 + |f_t|^2 \,\mathrm{dvol}_{g_0}\,\mathrm{d}t.$$

Note that in the notation of [40], $Y = P^2$ and $X = P^1$.

Now we define spaces for u. The set

$$Z_T = L^2([0,T], W^{3,2}(\Sigma)) \cap W^{1,2}([0,T], W^{1,2}(\Sigma))$$

is a Hilbert space with norm

$$\|u\|_Z^2 = \int_0^T \int_\Sigma |\nabla^3 u|^2 + |u|^2 + |\nabla u_t|^2 + |u_t|^2 \,\mathrm{dvol}_{g_0}\,\mathrm{d}t.$$

Similar to above, there is a constant c such that

$$\|\nabla^2 u\|_{L^4} + \|u_t\|_{L^4} \le c\|u\|_Z \tag{11}$$

and

$$\max_{0\le t\le T} \|u(t)\|_{W^{2,2}(\Sigma)} + \max_{0\le t\le T} \|u_t(t)\|_{L^2(\Sigma)} \le c\|u\|_Z. \tag{12}$$

Also, by Sobolev embedding, we have

$$\max_{0\le t\le T} \|u(t)\|_{W^{1,8}(\Sigma)} \le c\|u\|_Z. \tag{13}$$

Moreover, u is continuous and there is a constant C_2 such that for all $u \in Z_T$,

$$\|u\|_{C^0} \le C_2\|u\|_Z. \tag{14}$$

3.2 The Ball *B* and *B′*

Now we fix $f_0 \in W^{3,2}(\Sigma)$ throughout the section and thereafter. Consider the operator $\partial_t - \mathrm{e}^{-2u}\Delta$. If $\|u\|_{C^0} \le 1$, this operator is uniformly elliptic. So, Proposition 2.3 of [40] then says that the map $f \mapsto \big(f_0, (\partial_t - \mathrm{e}^{-2u}\Delta)f\big)$ is a linear isomorphism

$$Y_T \to W^{3,2}(\Sigma) \times X_T.$$

Hence there is a constant C_1 such that for each $f_0 \in W^{3,2}(\Sigma)$ and $g \in X_T$, there is a unique solution $h(t,x) \in Y_T$ of the initial value problem

$$(\partial_t - \mathrm{e}^{-2u}\Delta)h = g \quad h(0) = f_0 \tag{15}$$

with

$$\|h\|_Y \le C_1\big(\|f_0\|_{3,2} + \|g\|_X\big). \tag{16}$$

Let $h_0(t,x)$ be the unique solution of

$$(\partial_t - \Delta)h = 0 \quad h(0) = f_0. \tag{17}$$

By (16) there is a constant C_0, depending on C_1 and $\|f_0\|_{3,2}$ such that

$$\|h_0\|_Y \le C_0. \tag{18}$$

Because of (8), $\{f \in Y_T \mid f(0) = f_0\}$ is a closed affine subspace of Y_T. Hence the ball

$$B = B_\delta \;=\; \left\{ f \in Y_T \;\middle|\; f(0) = f_0 \text{ and } \|f - h_0\|_Y \le \delta \right\} \tag{19}$$

is a closed subset of Y_T. Note that each $f \in B_\delta$ satisfies

$$\|f\|_Y \;\le\; \|f - h_0\|_Y + \|h_0\|_Y \;\le\; \delta + C_0. \tag{20}$$

Also let the ball

$$B' = B'_{\delta'} = \{u \in Z_T \mid u(0) = 0 \text{ and } \|u\|_Z \le \delta'\}$$

be a closed subset of Z_T. Obviously $h_0 \in B_\delta$ and $0 \in B'_{\delta'}$. For simplicity, we denote $B = B_\delta$ and $B' = B'_{\delta'}$.

Now fix $\delta > 0$ and define

$$C_3 := 1600 C_0 C_1 C_2. \tag{21}$$

Choose δ' small enough so that $C_2\delta' < 1$ which implies $\|u\|_{C^0} \le 1$. Also we assume $\delta' \le \frac{\delta}{C_3}$.

4 Construction of Operators

In this section we will construct operators $S_1 : Y_T \times Z_T \to Y_T$ and $S_2 : Y_T \times Z_T \to Z_T$. First fix $f \in Y_T$ and $u \in Z_T$. f and u are considered to be fixed throughout this section and after unless we mention any choice of them.

First we show a lemma that is needed in several places.

Lemma 4 *Fix $f_0 \in W^{3,2}(\Sigma)$. Then there is an $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \le T_0$, for each $h \in B$ and $u_1, u_2 \in B'$,*

$$\|(\mathrm{e}^{2u_2 - 2u_1} - 1)\partial_t h\|_X \le \frac{C_3}{2C_1}\|u_1 - u_2\|_Z. \tag{22}$$

Proof Denote

$$g := (\mathrm{e}^{2u_2 - 2u_1} - 1)\partial_t h.$$

Recall that

$$\left|\mathrm{e}^{2u_1 - 2u_2} - 1\right| \le \mathrm{e}^{2|u_1 - u_2|}\left|1 - \mathrm{e}^{-2|u_1 - u_2|}\right| \le 2\mathrm{e}^4|u_1 - u_2|$$

if $\|u_1 - u_2\|_{C^0} \leq 2$, which comes from $u_1, u_2 \leq B'$. Using $2\mathrm{e}^4 \leq 200$ and by (9) and (20),

$$\begin{aligned}\|g\|_{L^2}^2 &\leq 200^2 \|u_1 - u_2\|_{C^0}^2 \max_{0 \leq t \leq T} \|\partial_t h_2(t)\|_{L^2}^2 \, T \\ &\leq \frac{C_3^2}{16C_1^2} \|u_1 - u_2\|_Z^2\end{aligned}$$

if we choose T small enough.

Next, consider $\left|\nabla^2 g\right|^2$.

$$\begin{aligned}\left|\nabla^2 g\right| &= \left|\nabla^2 \left((\mathrm{e}^{2u_2 - 2u_1} - 1)\partial_t h_2\right)\right| \\ &\leq 800|u_1 - u_2||\nabla(u_1 - u_2)|^2|\partial_t h_2| + 400|u_1 - u_2||\nabla^2(u_1 - u_2)||\partial_t h_2| \\ &\quad + 400|u_1 - u_2||\nabla(u_1 - u_2)||\nabla\partial_t h_2| + 200|u_1 - u_2||\nabla^2\partial_t h_2|.\end{aligned}$$

Hence, by integrating, we have

$$\begin{aligned}\|\nabla^2 g\|_{L^2}^2 \leq\ & 1600^2 \|u_1 - u_2\|_{C^0}^2 \|\nabla(u_1 - u_2)\|_{L^8}^4 \max_{0 \leq t \leq T} \|\partial_t h_2(t)\|_{L^4(\Sigma)}^2 \, T^{1/2} \\ & + 800^2 \|u_1 - u_2\|_{C^0}^2 \|\nabla^2(u_1 - u_2)\|_{L^4}^2 \max_{0 \leq t \leq T} \|\partial_t h_2(t)\|_{L^8(\Sigma)}^4 \, T^{1/2} \\ & + 800^2 \|u_1 - u_2\|_{C^0}^2 \max_{0 \leq t \leq T} \|\nabla(u_1(t) - u_2(t))\|_{L^4(\Sigma)}^2 \|\nabla\partial_t h_2\|_{L^4}^2 \, T^{1/2} \\ & + 400^2 \|u_1 - u_2\|_{C^0}^2 \|\nabla^2\partial_t h_2\|_{L^2}^2 \\ \leq\ & 400^2 C_2^2 \|u_1 - u_2\|_Z^2 2C_0^2 \\ =\ & \frac{C_3^2}{8C_1^2} \|u_1 - u_2\|_Z^2\end{aligned}$$

if we choose T small enough.

Finally, we will compute $\|g_t\|_{L^2}^2$.

$$|g_t| \leq 400|u_1 - u_2||\partial_t h_2||(u_1 - u_2)_t| + 200|u_1 - u_2||\partial_{tt} h_2|.$$

Hence,

$$\begin{aligned}\|g_t\|_{L^2}^2 &\leq 2(400)^2 \|u_1 - u_2\|_{C^0}^2 \|(u_1 - u_2)_t\|_{L^4}^2 \max_{0 \leq t \leq T} \|\partial_t h_2\|_{L^4(\Sigma)}^2 \, T^{1/2} \\ &\quad + 2(200)^2 \|u_1 - u_2\|_{C^0}^2 \|\partial_{tt} h_2\|_{L^2}^2 \\ &\leq 2(200)^2 C_2^2 \|u_1 - u_2\|_Z^2 2C_0^2 \\ &= \frac{C_3^2}{16C_1^2} \|u_1 - u_2\|_Z^2\end{aligned}$$

if we choose T small enough.

Combining all the estimates above, we get

$$\|(\mathrm{e}^{2u_2-2u_1}-1)\partial_t h\|_X \leq \frac{C_3}{2C_1}\|u_1-u_2\|_Z$$

which proves the lemma. □

4.1 The Construction S_1

Define an operator

$$S_1 : Y_T \times Z_T \to Y_T$$

by $S_1(f, u) = h$ where $h \in Y_T$ is the unique solution of

$$(\partial_t - \mathrm{e}^{-2u}\Delta)h = \mathrm{e}^{-2u}A_f(\mathrm{d}f, \mathrm{d}f) \quad h(0) = f_0. \tag{23}$$

Lemma 5 *Fix $f_0 \in W^{3,2}(\Sigma)$. Then there is $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \leq T_0$, S_1 restricts to an operator $S_1 : B \times B' \to B$.*

Proof We also can assume $\|A\|, \|DA\|, \|D^2A\|, \|D^3A\| \leq c$ where c depends only on the geometry of N. Then the vector-valued function $A_f(\mathrm{d}f, \mathrm{d}f)$ satisfies the pointwise bound $|A_f(\mathrm{d}f, \mathrm{d}f)|^2 \leq c|\mathrm{d}f|^4$. Fix $f \in B$ and $u \in B'$.

Now we estimate X norm of

$$g = \mathrm{e}^{-2u(t)}A_f(\mathrm{d}f, \mathrm{d}f).$$

First, $|g|^2 \leq c|\mathrm{d}f|^4$, so $\|g\|^2_{L^2} \leq c\|f\|^4_Y|\Sigma|T$. Hence if we choose T small enough, we have $\|g\|^2_{L^2} \leq \frac{\delta^2}{6C_1^2}$. Next, compute $|\nabla^2 g|^2$.

$$\begin{aligned}|\nabla^2 g| \leq{}& c|\mathrm{d}f|^2|\nabla^2 u| + c|\mathrm{d}f|^2|\nabla u|^2 + c|\mathrm{d}f|^3|\nabla u| + c|\nabla\mathrm{d}f||\mathrm{d}f||\nabla u|\\ &+ c|\mathrm{d}f|^4 + c|\mathrm{d}f|^2|\nabla\mathrm{d}f| + c|\nabla^3 f||\mathrm{d}f| + c|\nabla\mathrm{d}f|^2.\end{aligned}$$

So, using Young's inequality

$$c\|\mathrm{d}f\|^2_{C^0}\iint|\nabla\mathrm{d}f|^2|\nabla u|^2 \leq c\|\mathrm{d}f\|^4_{C^0}\iint|\nabla u|^4 + c\iint|\nabla\mathrm{d}f|^4,$$

we get, by (8), (9), (10), (12) and (20),

$$\begin{aligned}\|\nabla^2 g\|^2_{L^2} \leq{}& c\|\mathrm{d}f\|^4_{C^0}\iint(|\nabla^2 u|^2 + |\nabla u|^4) + c\|\mathrm{d}f\|^6_{C^0}\iint|\nabla u|^2\\ &+ c\|\mathrm{d}f\|^2_{C^0}\iint|\nabla\mathrm{d}f|^2|\nabla u|^2 + c\|\mathrm{d}f\|^8_{C^0}|\Sigma|T + c\|\mathrm{d}f\|^4_{C^0}\iint|\nabla\mathrm{d}f|^2\end{aligned}$$

$$
\begin{aligned}
&+ c\|\mathrm{d}f\|_{C^0}^2 \iint |\nabla^3 f|^2 + c \iint |\nabla \mathrm{d} f|^4 \\
&\leq c\|f\|_Y^4 \left(\max_{0\leq t\leq T} \|\nabla^2 u(t)\|_{L^2(\Sigma)}^2 + \max_{0\leq t\leq T} \|\nabla u(t)\|_{L^4(\Sigma)}^4 \right) T \\
&+ c\|f\|_Y^6 \max_{0\leq t\leq T} \|\nabla u(t)\|_{L^2(\Sigma)}^2 T + c\|f\|_Y^8 |\Sigma| T \\
&+ c\|f\|_Y^4 \max_{0\leq t\leq T} \|\nabla \mathrm{d} f(t)\|_{L^2(\Sigma)}^2 T + c\|f\|_Y^2 \max_{0\leq t\leq T} \|\nabla^3 f(t)\|_{L^2(\Sigma)}^2 T \\
&+ c \max_{0\leq t\leq T} \|\nabla \mathrm{d} f\|_{L^4(\Sigma)}^4 T \\
&\leq \frac{\delta^2}{6C_1^2}
\end{aligned}
$$

if we choose T small enough. Finally,

$$
|g_t| \leq c|\mathrm{d}f|^2|u_t| + c|\mathrm{d}f|^2|f_t| + c|\mathrm{d}f_t||\mathrm{d}f|
$$

and

$$
\begin{aligned}
\|g_t\|_{L^2}^2 &\leq c\|\mathrm{d}f\|_{C^0}^4 \iint |u_t|^2 + |f_t|^2 + c\|\mathrm{d}f\|_{C^0}^2 \iint |\mathrm{d}f_t|^2 \\
&\leq c\|f\|_Y^4 \left(\max_{0\leq t\leq T} \|u_t(t)\|_{L^2(\Sigma)}^2 + \max_{0\leq t\leq T} \|f_t(t)\|_{L^2(\Sigma)}^2 \right) T \\
&\quad + c\|f\|_Y^2 \max_{0\leq t\leq T} \|\nabla f_t\|_{L^2(\Sigma)}^2 T \\
&\leq \frac{\delta^2}{6C_1^2}
\end{aligned}
$$

if we choose T small enough.

Therefore, if we choose T small enough, we have $\|g\|_X^2 \leq \frac{\delta^2}{2C_1^2}$. Noting that $S(f) - h_0 = h - h_0$ satisfies

$$
(\partial_t - \mathrm{e}^{-2u}\Delta)(h - h_0) = g + (\mathrm{e}^{-2u} - 1)\Delta h_0 \quad (h - h_0)(0) = 0.
$$

The bounds (16) give

$$
\begin{aligned}
\|S(f) - h_0\|_Y^2 \; &\leq \; C_1^2 \left(\|g\|_X^2 + \|(\mathrm{e}^{-2u} - 1)\Delta h_0\|_X^2 \right) \\
&= C_1^2 \left(\|g\|_X^2 + \|(\mathrm{e}^{-2u} - 1)\partial_t h_0\|_X^2 \right)
\end{aligned}
$$

because h_0 satisfies (17).

Now by Lemma 4 with $h = h_0$, $u_1 = u$, $u_2 = 0$,

$$
\|(\mathrm{e}^{-2u} - 1)\partial_t h_0\|_X \leq \frac{C_3}{2C_1}\|u\|_Z \leq \frac{\delta}{2C_1}.
$$

This implies

$$\|S(f) - h_0\|_Y^2 \le C_1^2 \left(\frac{\delta^2}{2C_1^2} + \frac{\delta^2}{4C_1^2} \right) \le \delta^2.$$

Therefore $S(f) \in B$. □

Lemma 6 *Fix $f_0 \in W^{3,2}(\Sigma)$ and $u \in B'$. Then there is an $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \le T_0$ and for each $f_1, f_2 \in B$,*

$$\|S_1(f_1, u) - S_1(f_2, u)\|_Y \le \frac{1}{3}\|f_1 - f_2\|_Y. \tag{24}$$

Proof Set $h_i = S_1(f_i, u)$ and $g_i = \mathrm{e}^{-2u} A_{f_i}(\mathrm{d}f_i, \mathrm{d}f_i)$ for $i = 1, 2$ and subtracting, the function $h_1 - h_2$ satisfies

$$(\partial_t - \mathrm{e}^{-2u}\Delta)(h_1 - h_2) = g_1 - g_2 \quad (h_1 - h_2)(0) = 0.$$

Hence (15) gives a bound

$$\|h_1 - h_2\|_Y \le C_1\|g_1 - g_2\|_X. \tag{25}$$

Next, we have

$$\begin{aligned} g_1 - g_2 &= \mathrm{e}^{-2u}(A_{f_1} - A_{f_2})(\mathrm{d}f_1, \mathrm{d}f_1) + \mathrm{e}^{-2u} A_{f_2}(\mathrm{d}f_1 + \mathrm{d}f_2, \mathrm{d}f_1 - \mathrm{d}f_2) \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

So, there is a constant c with

$$|g_1 - g_2|^2 \le c|f_1 - f_2|^2|\mathrm{d}f_1|^4 + c|\mathrm{d}f_1 - \mathrm{d}f_2|^2 \left(|\mathrm{d}f_1|^2 + |\mathrm{d}f_2|^2 \right).$$

Integrating and applying Holder's inequality, (8), and (20) gives

$$\begin{aligned} \|g_1 - g_2\|_{L^2}^2 &\le c\|f_1 - f_2\|_{C^0}^2 \iint |\mathrm{d}f_1|^4 + c\|\mathrm{d}f_1 - \mathrm{d}f_2\|_{L^4}^2 \left(\|\mathrm{d}f_1\|_{L^4}^2 + \|\mathrm{d}f_2\|_{L^4}^2 \right) \\ &\le c\|f_1 - f_2\|_Y^2 \|f_1\|_Y^4 |\Sigma| T + c\|f_1 - f_2\|_Y^2 (\|f_1\|_Y^2 + \|f_2\|_Y^2)|\Sigma|^{1/2} T^{1/2} \\ &\le \frac{1}{27C_1^2} \|f_1 - f_2\|_Y^2 \end{aligned}$$

if we choose T small enough.

For $\nabla^2(g_1 - g_2)$, first note that

$$\begin{aligned} \nabla(A_{f_1} - A_{f_2}) &= DA_{f_1}\mathrm{d}f_1 - DA_{f_2}\mathrm{d}f_2 = (DA_{f_1} - DA_{f_2})\mathrm{d}f_1 + DA_{f_2}(\mathrm{d}f_1 - \mathrm{d}f_2) \\ \nabla(DA_{f_1} - DA_{f_2}) &= (D^2A_{f_1} - D^2A_{f_2})\mathrm{d}f_1 + D^2A_{f_2}(\mathrm{d}f_1 - \mathrm{d}f_2). \end{aligned}$$

So, we get

$$
\begin{aligned}
|\nabla^2 I| \le\ & c|f_1 - f_2||\mathrm{d}f_1|^2|\nabla^2 u| + c|f_1 - f_2||\mathrm{d}f_1|^2|\nabla u|^2 + c|f_1 - f_2||\mathrm{d}f_1|^3|\nabla u| \\
& + c|\mathrm{d}f_1 - \mathrm{d}f_2||\mathrm{d}f_1|^2|\nabla u| + c|f_1 - f_2||\nabla \mathrm{d}f_1||\mathrm{d}f_1||\nabla u| \\
& + c|f_1 - f_2||\mathrm{d}f_1|^4 + c|f_1 - f_2||\nabla \mathrm{d}f_1||\mathrm{d}f_1|^2 \\
& + c|\mathrm{d}f_1 - \mathrm{d}f_2||\mathrm{d}f_2||\mathrm{d}f_1|^2 + c|\nabla \mathrm{d}f_1 - \nabla \mathrm{d}f_2||\mathrm{d}f_1|^2 \\
& + c|\mathrm{d}f_1 - \mathrm{d}f_2||\nabla \mathrm{d}f_1||\mathrm{d}f_1| \\
& + c|f_1 - f_2||\nabla^3 f_1||\mathrm{d}f_1| + c|f_1 - f_2||\nabla \mathrm{d}f_1|^2.
\end{aligned}
$$

Using (8), (9), (10), (12), (13), and using Young's inequality, we can estimate it term by term.

$$
\begin{aligned}
\iint |f_1 - f_2|^2|\mathrm{d}f_1|^4|\nabla^2 u|^2 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^4 \max_{0\le t\le T}\|\nabla^2 u(t)\|_{L^2(\Sigma)}^2 T \\
\iint |f_1 - f_2|^2|\mathrm{d}f_1|^4|\nabla u|^4 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^4 \max_{0\le t\le T}\|\nabla u(t)\|_{L^4(\Sigma)}^4 T \\
\iint |f_1 - f_2|^2|\mathrm{d}f_1|^6|\nabla u|^2 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^6 \max_{0\le t\le T}\|\nabla u(t)\|_{L^2(\Sigma)}^2 T \\
\iint |f_1 - f_2|^2|\nabla \mathrm{d}f_1|^2|\mathrm{d}f_1|^2|\nabla u|^2 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^4 \max_{0\le t\le T}\|\nabla u(t)\|_{L^4(\Sigma)}^4 T \\
&\quad + \|f_1 - f_2\|_Y^2 \max_{0\le t\le T}\|\nabla \mathrm{d}f_1(t)\|_{L^4(\Sigma)}^4 T \\
\iint |f_1 - f_2|^2|\mathrm{d}f_1|^8 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^8|\Sigma|T \\
\iint |f_1 - f_2|^2|\nabla \mathrm{d}f_1|^2|\mathrm{d}f_1|^4 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^4 \max_{0\le t\le T}\|\nabla \mathrm{d}f_1(t)\|_{L^2(\Sigma)}^2 T \\
\iint |\mathrm{d}f_1 - \mathrm{d}f_2|^2|\mathrm{d}f_2|^2|\mathrm{d}f_1|^4 &\le \|f_1 - f_2\|_Y^2\|f_2\|_Y^2\|f_1\|_Y^4|\Sigma|T \\
\iint |\nabla \mathrm{d}f_1 - \nabla \mathrm{d}f_2|^2|\mathrm{d}f_1|^4 &\le \|f_1\|_Y^4 \max_{0\le t\le T}\|\nabla \mathrm{d}f_1(t) - \nabla \mathrm{d}f_2(t)\|_{L^2(\Sigma)}^2 T \\
\iint |\mathrm{d}f_1 - \mathrm{d}f_2|^2|\nabla \mathrm{d}f_1|^2|\mathrm{d}f_1|^2 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^2 \max_{0\le t\le T}\|\nabla \mathrm{d}f_1(t)\|_{L^2(\Sigma)}^2 T \\
\iint |f_1 - f_2|^2|\nabla^3 f_1|^2|\mathrm{d}f_1|^2 &\le \|f_1 - f_2\|_Y^2\|f_1\|_Y^2 \max_{0\le t\le T}\|\nabla^3 f_1(t)\|_{L^2(\Sigma)}^2 T \\
\iint |f_1 - f_2|^2|\nabla \mathrm{d}f_1|^4 &\le \|f_1 - f_2\|_Y^2 \max_{0\le t\le T}\|\nabla \mathrm{d}f_1(t)\|_{L^4(\Sigma)}^4 T.
\end{aligned}
$$

Hence, using (20), if we choose T small enough, we get

$$
\|\nabla^2 I\|_{L^2}^2 \le \frac{1}{54C_1^2}\|f_1 - f_2\|_Y^2.
$$

We obtain similar result for II if we choose T small enough:

$$\|\nabla^2 II\|_{L^2}^2 \leq \frac{1}{54C_1^2}\|f_1 - f_2\|_Y^2.$$

Hence, we obtain that $\|\nabla^2(g_1 - g_2)\|_{L^2}^2 \leq \frac{1}{27C_1^2}\|f_1 - f_2\|_Y^2$.

Finally, compute $\partial_t(g_1 - g_2)$. As above, note that

$$\partial_t(A_{f_1} - A_{f_2}) = DA_{f_1}\partial_t f_1 - DA_{f_2}\partial_t f_2 = (DA_{f_1} - DA_{f_2})\partial_t f_1 + DA_{f_2}(\partial_t(f_1 - f_2)).$$

So,

$$\begin{aligned}|\partial_t(g_1 - g_2)| \leq{}& c|f_1 - f_2||\mathrm{d}f_1|^2|u_t| + c|f_1 - f_2||\partial_t f_1||\mathrm{d}f_1|^2 + c|\partial_t(f_1 - f_2)||\mathrm{d}f_1|^2\\ &+ c|f_1 - f_2||\partial_t \mathrm{d}f_1||\mathrm{d}f_1|\\ &+ c|\mathrm{d}f_1 + \mathrm{d}f_2||\mathrm{d}f_1 - \mathrm{d}f_2||u_t| + c|\partial_t f_2||\mathrm{d}f_1 + \mathrm{d}f_2||\mathrm{d}f_1 - \mathrm{d}f_2|\\ &+ c|\partial_t(\mathrm{d}f_1 + \mathrm{d}f_2)||\mathrm{d}f_1 - \mathrm{d}f_2| + c|\mathrm{d}f_1 + \mathrm{d}f_2||\partial_t(\mathrm{d}f_1 - \mathrm{d}f_2)|.\end{aligned}$$

Similar with above, by (8), (9), (10), (12), (13) and (20),

$$\begin{aligned}\|\partial_t(g_1 - g_2)\|_{L^2}^2 \leq{}& c\|f_1\|_Y^4\|f_1 - f_2\|_Y^2\left(\max_{0\leq t\leq T}\|u_t(t)\|_{L^2(\Sigma)}^2 + \max_{0\leq t\leq T}\|\partial_t f_1(t)\|_{L^2(\Sigma)}^2\right)T\\ &+ c\|f_1\|_Y^4\max_{0\leq t\leq T}\|\partial_t(f_1(t) - f_2(t))\|_{L^2(\Sigma)}^2 T\\ &+ c\|f_1\|_Y^2\|f_1 - f_2\|_Y^2\max_{0\leq t\leq T}\|\partial_t \mathrm{d}f_1(t)\|_{L^2(\Sigma)}^2 T\\ &+ c(\|f_1\|_Y^2 + \|f_2\|_Y^2)\|f_1 - f_2\|_Y^2\left(\max_{0\leq t\leq T}\|u_t(t)\|_{L^2(\Sigma)}^2 + \max_{0\leq t\leq T}\|\partial_t f_2(t)\|_{L^2(\Sigma)}^2\right)T\\ &+ c\|f_1 - f_2\|_Y^2\left(\max_{0\leq t\leq T}\|\partial_t \mathrm{d}f_1(t)\|_{L^2(\Sigma)}^2 + \max_{0\leq t\leq T}\|\partial_t \mathrm{d}f_2(t)\|_{L^2(\Sigma)}^2\right)T\\ &+ c(\|f_1\|_Y^2 + \|f_2\|_Y^2)\max_{0\leq t\leq T}\|\partial_t(\mathrm{d}f_1(t) - \mathrm{d}f_2(t))\|_{L^2(\Sigma)}^2 T\\ \leq{}& \frac{1}{27C_1^2}\|f_1 - f_2\|_Y^2\end{aligned}$$

if we choose T small enough.

Combine all of them,

$$\|h_1 - h_2\|_Y \leq C_1\|g_1 - g_2\|_X \leq \frac{1}{3}\|f_1 - f_2\|_Y$$

which proves the lemma. □

Lemma 7 *Fix $f_0 \in W^{3,2}(\Sigma)$ and $f \in B$. Then there is an $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \leq T_0$ and for each $u_1, u_2 \in B'$,*

$$\|S_1(f, u_1) - S_1(f, u_2)\|_Y \leq \frac{C_3}{2}\|u_1 - u_2\|_Z. \tag{26}$$

Proof Set $h_i = S_1(f, u_i)$. Multiplying e^{2u_i} to the equation for h_i respectively and subtracting them gives

$$\begin{aligned} \mathrm{e}^{2u_1}\partial_t(h_1 - h_2) - \Delta(h_1 - h_2) &= -(\mathrm{e}^{2u_1} - \mathrm{e}^{2u_2})\partial_t h_2, \\ (\partial_t - \mathrm{e}^{-2u_1}\Delta)(h_1 - h_2) &= (\mathrm{e}^{2u_2 - 2u_1} - 1)\partial_t h_2. \end{aligned}$$

So, $h_1 - h_2$ satisfies the estimate from (16), and by Lemma 4,

$$\|h_1 - h_2\|_Y \leq C_1\|(\mathrm{e}^{2u_2 - 2u_1} - 1)\partial_t h_2\|_X \leq \frac{C_3}{2}\|u_1 - u_2\|_Z$$

if we choose T small enough.

□

4.2 The Construction S_2

Define an operator

$$S_2 : Y_T \times Z_T \to Z_T$$

by $S_2(f, u) = v$ where $v \in Z_T$ is the unique solution of

$$\partial_t v = b|\mathrm{d}f|^2\mathrm{e}^{-2u} - a \quad v(0) = 0. \tag{27}$$

Lemma 8 *In the above definition, $v \in Z_T$.*

Proof From (27), we directly get

$$v(t) = \int_0^t (b|\mathrm{d}f|^2\mathrm{e}^{-2u} - a). \tag{28}$$

So, $\|v\|_{L^2}$ and $\|v_t\|_{L^2}$ is trivially bounded if $f \in Y_T$ and $u \in Z_T$. (Because $u \in Z_T$, we have e^{-2u} is pointwise uniformly bounded by $\mathrm{e}^{C\|u\|_Z}$.) Applying Cauchy–Schwarz, we obtain the pointwise bound

$$\begin{aligned} |\nabla^3 v|^2 &= \left| b\int_0^T \nabla^2\left(\langle\nabla\mathrm{d}f, \mathrm{d}f\rangle\mathrm{e}^{-2u} - 2|\mathrm{d}f|^2\mathrm{e}^{-2u}\nabla u\right)\right|^2 \\ &\leq cT\int_0^T \Big(|\nabla^4 f|^2|\mathrm{d}f|^2 + |\nabla^3 f|^2|\nabla\mathrm{d}f|^2 + |\nabla^3 f|^2|\mathrm{d}f|^2|\nabla u|^2 + |\nabla^2 f|^4|\nabla u|^2 \end{aligned}$$

$$+ |\nabla^2 f|^2 |\mathrm{d}f|^2 (|\nabla u|^4 + |\nabla^2 u|^2) + |\mathrm{d}f|^4 (|\nabla u|^6 + |\nabla^2 u|^2 |\nabla u|^2 + |\nabla^3 u|^2) \Big)$$

so

$$\begin{aligned}\|\nabla^3 v\|_{L^2}^2 \le\ & cT^2 \|\mathrm{d}f\|_{C^0}^2 \|\nabla^4 f\|_{L^2}^2 + cT^2 \|\nabla^3 f\|_{L^4}^2 \|\nabla^2 f\|_{L^4}^2 \\ & + cT^2 \|\mathrm{d}f\|_{C^0}^2 \|\nabla^3 f\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + cT^2 \|\nabla^2 f\|_{L^8}^4 \|\nabla u\|_{L^4}^2 \\ & + cT^2 \|\mathrm{d}f\|_{C^0}^2 \|\nabla^2 f\|_{L^4}^2 \left(\|\nabla u\|_{L^8}^4 + \|\nabla^2 u\|_{L^4}^2 \right) \\ & + cT^2 \|\mathrm{d}f\|_{C^0}^4 \left(\|\nabla u\|_{L^6}^6 + \|\nabla^2 u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + \|\nabla^3 u\|_{L^2}^2 \right)\end{aligned}$$

which is bounded if $f \in Y_T$ and $u \in Z_T$.

Finally, compute ∇v_t from (27).

$$\begin{aligned}|\nabla v_t|^2 &\le c|\nabla \mathrm{d}f||\mathrm{d}f| + |\mathrm{d}f|^2 |\nabla u|, \\ \|\nabla v_t\|_{L^2}^2 &\le c\|\mathrm{d}f\|_{C^0}^2 \|\nabla \mathrm{d}f\|_{L^2}^2 + c\|\mathrm{d}f\|_{C^0}^4 \|\nabla u\|_{L^2}^2\end{aligned}$$

which is bounded if $f \in Y_T$ and $u \in Z_T$. □

In fact, we can show further.

Lemma 9 *Fix $f_0 \in W^{3,2}(\Sigma)$. Then there is $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \le T_0$, S_2 restricts to an operator $S_2 : B \times B' \to B'$.*

Proof From previous calculation, we have

$$\|\nabla^3 v\|_{L^2}^2 \le cT^2 \|f\|_Y^4 (1 + \|u\|_Z^2 + \|u\|_Z^4 + \|u\|_Z^6).$$

So, if we choose T small enough, we get $\|\nabla^3 v\|_{L^2}^2 \le \frac{\delta'^2}{4}$. Also, because $|v_t| \le c(\|\mathrm{d}f\|_{C^0} + 1)$ and $|v| \le cT(\|\mathrm{d}f\|_{C^0} + 1)$, we can make $\|v_t\|_{L^2}^2, \|v\|_{L^2}^2 \le \frac{\delta'^2}{4}$ if we choose T small. Finally,

$$\|\nabla v_t\|_{L^2}^2 \le c\|\mathrm{d}f\|_{C^0}^2 \max_{0 \le t \le T} \|\nabla \mathrm{d}f(t)\|_{L^2(\Sigma)}^2 T + c\|\mathrm{d}f\|_{C^0}^4 \max_{0 \le t \le T} \|\nabla u(t)\|_{L^2(\Sigma)}^2 T$$

so if we choose T small enough, we get $\|\nabla v_t\|_{L^2}^2 \le \frac{\delta'^2}{4}$. This proves the lemma. □

Lemma 10 *Fix $f_0 \in W^{3,2}(\Sigma)$ and $u \in B'$. Then there is an $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \le T_0$ and for each $f_1, f_2 \in B$,*

$$\|S_2(f_1, u) - S_2(f_2, u)\|_Z \le T^{1/4} \|f_1 - f_2\|_Y. \tag{29}$$

Proof Set $v_i = S_2(f_i, u)$. Then from (27), subtracting them gives

$$(v_1 - v_2)_t = b(|\mathrm{d}f_1|^2 - |\mathrm{d}f_2|^2)\mathrm{e}^{-2u} = b\mathrm{e}^{-2u} \langle \mathrm{d}f_1 + \mathrm{d}f_2, \mathrm{d}f_1 - \mathrm{d}f_2 \rangle,$$

$$v_1 - v_2 = b \int_0^t \mathrm{e}^{-2u} \langle \mathrm{d} f_1 + \mathrm{d} f_2, \mathrm{d} f_1 - \mathrm{d} f_2 \rangle.$$

So,
$$\begin{aligned}\|v_1 - v_2\|_{L^2}^2 &\leq cT^2 (\|\mathrm{d} f_1\|_{C^0}^2 + \|\mathrm{d} f_2\|_{C^0}^2) \|\mathrm{d} f_1 - \mathrm{d} f_2\|_{L^2}^2 \\ &\leq \frac{\sqrt{T}}{4} \| f_1 - f_2\|_Y^2\end{aligned}$$

if we choose T small enough. Also,

$$\begin{aligned}\|(v_1 - v_2)_t\|_{L^2}^2 &\leq c(\| f_1\|_Y^2 + \| f_2\|_Y^2) \max_{0 \leq t \leq T} \|\mathrm{d} f_1(t) - \mathrm{d} f_2(t)\|_{L^2(\Sigma)}^2 \, T \\ &\leq \frac{\sqrt{T}}{4} \| f_1 - f_2\|_Y^2\end{aligned}$$

if we choose T small enough.

Next, compute $\nabla^3(v_1 - v_2)$.

$$\begin{aligned}\nabla(v_1 - v_2) = b \int_0^t \mathrm{e}^{-2u} \Big(&\langle \nabla(\mathrm{d} f_1 + \mathrm{d} f_2), \mathrm{d} f_1 - \mathrm{d} f_2 \rangle + \langle \mathrm{d} f_1 + \mathrm{d} f_2, \nabla(\mathrm{d} f_1 - \mathrm{d} f_2) \rangle \\ &- \langle \mathrm{d} f_1 + \mathrm{d} f_2, \mathrm{d} f_1 - \mathrm{d} f_2 \rangle 2\nabla u \Big).\end{aligned}$$

So,

$$\begin{aligned}|\nabla^3(v_1 - v_2)| \leq c \int_0^T \mathrm{e}^{-2u} \Big(&|\nabla^3(\mathrm{d} f_1 + \mathrm{d} f_2)||\mathrm{d} f_1 - \mathrm{d} f_2| + |\nabla^2(\mathrm{d} f_1 + \mathrm{d} f_2)||\nabla(\mathrm{d} f_1 - \mathrm{d} f_2)| \\ &+ |\nabla(\mathrm{d} f_1 + \mathrm{d} f_2)||\nabla^2(\mathrm{d} f_1 - \mathrm{d} f_2)| + |\mathrm{d} f_1 + \mathrm{d} f_2||\nabla^3(\mathrm{d} f_1 - \mathrm{d} f_2)| \\ &+ |\nabla^2(\mathrm{d} f_1 + \mathrm{d} f_2)||\mathrm{d} f_1 - \mathrm{d} f_2||\nabla u| + |\nabla(\mathrm{d} f_1 + \mathrm{d} f_2)||\nabla(\mathrm{d} f_1 - \mathrm{d} f_2)||\nabla u| \\ &+ |\mathrm{d} f_1 + \mathrm{d} f_2||\nabla^2(\mathrm{d} f_1 - \mathrm{d} f_2)||\nabla u| \\ &+ |\nabla(\mathrm{d} f_1 + \mathrm{d} f_2)||\mathrm{d} f_1 - \mathrm{d} f_2|(|\nabla u|^2 + |\nabla^2 u|) \\ &+ |\mathrm{d} f_1 + \mathrm{d} f_2||\nabla(\mathrm{d} f_1 - \mathrm{d} f_2)|(|\nabla u|^2 + |\nabla^2 u|) \\ &+ |\mathrm{d} f_1 + \mathrm{d} f_2||\mathrm{d} f_1 - \mathrm{d} f_2|(|\nabla u|^3 + |\nabla^2 u||\nabla u| + |\nabla^3 u|) \Big).\end{aligned}$$

Integrating over $\Sigma \times [0, T]$ gives

$$\begin{aligned}&\|\nabla^3(v_1 - v_2)\|_{L^2}^2 \\ &\leq cT^2 \|\mathrm{d} f_1 - \mathrm{d} f_2\|_{C^0}^2 \|\nabla^4(f_1 + f_2)\|_{L^2}^2 + cT^2 \|\nabla^3(f_1 + f_2)\|_{L^4}^2 \|\nabla^2(f_1 - f_2)\|_{L^4}^2 \\ &\quad + cT^2 \|\nabla^2(f_1 + f_2)\|_{L^4}^2 \|\nabla^3(f_1 - f_2)\|_{L^4}^2 + cT^2 \|\mathrm{d} f_1 + \mathrm{d} f_2\|_{C^0}^2 \|\nabla^4(f_1 - f_2)\|_{L^2}^2 \\ &\quad + cT^2 \|\mathrm{d} f_1 - \mathrm{d} f_2\|_{C^0}^2 \|\nabla^3(f_1 + f_2)\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \\ &\quad + cT^2 \|\nabla^2(f_1 - f_2)\|_{L^4}^2 \|\nabla^2(f_1 + f_2)\|_{L^8}^4 \|\nabla u\|_{L^8}^4 \\ &\quad + cT^2 \|\mathrm{d} f_1 + \mathrm{d} f_2\|_{C^0}^2 \|\nabla^3(f_1 - f_2)\|_{L^4}^2 \|\nabla u\|_{L^4}^2\end{aligned}$$

$$
\begin{aligned}
&+cT^2\|\mathrm{d}f_1-\mathrm{d}f_2\|_{C^0}^2\|\nabla^2(f_1+f_2)\|_{L^4}^2(\|\nabla u\|_{L^8}^4+\|\nabla^2 u\|_{L^4}^2)\\
&+cT^2\|\mathrm{d}f_1+\mathrm{d}f_2\|_{C^0}^2\|\nabla^2(f_1-f_2)\|_{L^4}^2(\|\nabla u\|_{L^8}^4+\|\nabla^2 u\|_{L^4}^2)\\
&+cT^2\|\mathrm{d}f_1-\mathrm{d}f_2\|_{C^0}^2\|\mathrm{d}f_1+\mathrm{d}f_2\|_{C^0}^2(\|\nabla u\|_{L^6}^6+\|\nabla^2 u\|_{L^4}^2\|\nabla u\|_{L^4}^2+\|\nabla^3 u\|_{L^2}^2)\\
\le{}& \frac{\sqrt{T}}{4}\|f_1-f_2\|_Y^2
\end{aligned}
$$

if we choose T small enough.

Finally consider $\nabla(v_1-v_2)_t$.

$$
\begin{aligned}
\nabla(v_1-v_2)_t = b\mathrm{e}^{-2u}\,&(\langle\nabla(\mathrm{d}f_1+\mathrm{d}f_2),\mathrm{d}f_1-\mathrm{d}f_2\rangle+\langle\mathrm{d}f_1+\mathrm{d}f_2,\nabla(\mathrm{d}f_1-\mathrm{d}f_2)\rangle\\
&-\langle\mathrm{d}f_1+\mathrm{d}f_2,\mathrm{d}f_1-\mathrm{d}f_2\rangle 2\nabla u)\,.
\end{aligned}
$$

So,

$$
\begin{aligned}
\|\nabla(v_1-v_2)_t\|_{L^2}^2 \le{}& c\|\mathrm{d}f_1-\mathrm{d}f_2\|_{C^0}^2\max_{0\le t\le T}\|\nabla^2(f_1(t)+f_2(t))\|_{L^2(\Sigma)}^2\,T\\
&+c\|\mathrm{d}f_1+\mathrm{d}f_2\|_{C^0}^2\max_{0\le t\le T}\|\nabla^2(f_1(t)-f_2(t))\|_{L^2(\Sigma)}^2\,T\\
&+c\|\mathrm{d}f_1+\mathrm{d}f_2\|_{C^0}^2\|\mathrm{d}f_1-\mathrm{d}f_2\|_{C^0}^2\max_{0\le t\le T}\|\nabla u(t)\|_{L^2(\Sigma)}^2\,T\\
\le{}& \frac{\sqrt{T}}{4}\|f_1-f_2\|_Y^2
\end{aligned}
$$

if we choose T small enough.

In summary, we get

$$
\|v_1-v_2\|_Z^2 \le \sqrt{T}\|f_1-f_2\|_Y^2
$$

which proves the lemma.

□

Lemma 11 *Fix $f_0\in W^{3,2}(\Sigma)$ and $f\in B$. Then there is an $T_0=T_0(C_0,\delta,\delta')>0$ such that for all $T\le T_0$ and for each $u_1,u_2\in B'$,*

$$
\|S_2(f,u_1)-S_2(f,u_2)\|_Z \le \frac{1}{3}\|u_1-u_2\|_Z. \tag{30}
$$

Proof Set $v_i=S_2(f,u_i)$. Subtracting them gives

$$
\begin{aligned}
(v_1-v_2)_t &= b|\mathrm{d}f|^2(\mathrm{e}^{-2u_1}-\mathrm{e}^{-2u_2}),\\
v_1-v_2 &= b\int_0^t|\mathrm{d}f|^2(\mathrm{e}^{-2u_1}-\mathrm{e}^{-2u_2}).
\end{aligned}
$$

Using $|\mathrm{e}^{-2u_1}-\mathrm{e}^{-2u_2}|\le c|u_1-u_2|$, we have

$$
\|v_1-v_2\|_{L^2}^2 \le cT^2\|\mathrm{d}f\|_{C^0}^4\|u_1-u_2\|_{L^2}^2,
$$

$$\|(v_1 - v_2)_t\|_{L^2}^2 \leq c\|\mathrm{d}f\|_{C^0}^4 \max_{0\leq t\leq T} \|u_1(t) - u_2(t)\|_{L^2(\Sigma)}^2 T,$$

so if we choose T small enough, we have that $\|v_1 - v_2\|_{L^2}^2, \|(v_1 - v_2)_t\|_{L^2}^2 \leq \frac{1}{36}\|u_1 - u_2\|_Z^2$.

Next, compute $\nabla^3(v_1 - v_2)$.

$$\nabla(v_1 - v_2) = b\int_0^t \Big(2\langle\nabla\mathrm{d}f, \mathrm{d}f\rangle(\mathrm{e}^{-2u_1} - \mathrm{e}^{-2u_2}) - |\mathrm{d}f|^2(\mathrm{e}^{-2u_1} - \mathrm{e}^{-2u_2})2(\nabla u_1 - \nabla u_2)\Big).$$

So,

$$\begin{aligned}
|\nabla^3(v_1 - v_2)| &\leq c\int_0^T \Big(|\nabla^4 f||\mathrm{d}f||u_1 - u_2| + |\nabla^3 f||\nabla^2 f||u_1 - u_2| \\
&\quad + (|\nabla^3 f||\mathrm{d}f| + |\nabla^2 f|^2)|u_1 - u_2||\nabla(u_1 - u_2)| \\
&\quad + |\nabla^2 f||\mathrm{d}f||u_1 - u_2|(|\nabla(u_1 - u_2)|^2 + |\nabla^2(u_1 - u_2)|) \\
&\quad + |\mathrm{d}f|^2|u_1 - u_2|(|\nabla(u_1 - u_2)|^3 + |\nabla^2(u_1 - u_2)||\nabla(u_1 - u_2)| + |\nabla^3(u_1 - u_2)|)\Big).
\end{aligned}$$

Now we integrate over $\Sigma \times [0, T]$.

$$\begin{aligned}
&\|\nabla^3(v_1 - v_2)\|_{L^2}^2 \\
&\quad \leq cT^2\|\mathrm{d}f\|_{C^0}^2\|u_1 - u_2\|_{C^0}^2\|\nabla^4 f\|_{L^2}^2 + cT^2\|u_1 - u_2\|_{C^0}^2\|\nabla^3 f\|_{L^4}^2\|\nabla^2 f\|_{L^4}^2 \\
&\qquad + cT^2(\|\mathrm{d}f\|_{C^0}^2\|\nabla^3 f\|_{L^4}^2 + \|\nabla^2 f\|_{L^8}^4)\|u_1 - u_2\|_{C^0}^2\|\nabla(u_1 - u_2)\|_{L^4}^2 \\
&\qquad + cT^2\|\mathrm{d}f\|_{C^0}^2\|u_1 - u_2\|_{C^0}^2\|\nabla^2 f\|_{L^4}^2(\|\nabla(u_1 - u_2)\|_{L^8}^4 + \|\nabla^2(u_1 - u_2)\|_{L^4}^2) \\
&\qquad + cT^2\|\mathrm{d}f\|_{C^0}^4\|u_1 - u_2\|_{C^0}^2\|\nabla(u_1 - u_2)\|_{L^6}^6 \\
&\qquad + cT^2\|\mathrm{d}f\|_{C^0}^4\|u_1 - u_2\|_{C^0}^2\|\nabla^2(u_1 - u_2)\|_{L^4}^2\|\nabla(u_1 - u_2)\|_{L^4}^2 \\
&\qquad + cT^2\|\mathrm{d}f\|_{C^0}^4\|u_1 - u_2\|_{C^0}^2\|\nabla^3(u_1 - u_2)\|_{L^2}^2 \\
&\quad \leq \frac{1}{36}\|u_1 - u_2\|_Z^2
\end{aligned}$$

if we choose T small enough.

Finally,

$$|\nabla(v_1 - v_2)_t| \leq c|\nabla\mathrm{d}f||\mathrm{d}f||u_1 - u_2| + |\mathrm{d}f|^2|u_1 - u_2||\nabla(u_1 - u_2)|$$

so

$$\begin{aligned}
\|\nabla(v_1 - v_2)_t\|_{L^2}^2 &\leq c\|\mathrm{d}f\|_{C^0}^2\|u_1 - u_2\|_{C^0}^2 \max_{0\leq t\leq T}\|\nabla^2 f(t)\|_{L^2(\Sigma)}^2 T \\
&\quad + c\|\mathrm{d}f\|_{C^0}^4\|u_1 - u_2\|_{C^0}^2 \max_{0\leq t\leq T}\|\nabla(u_1(t) - u_2(t))\|_{L^2(\Sigma)}^2 T \\
&\leq \frac{1}{36}\|u_1 - u_2\|_Z^2
\end{aligned}$$

if we choose T small enough.

In summary, we get

$$\|v_1 - v_2\|_Z^2 \leq \frac{1}{9}\|u_1 - u_2\|_Z^2$$

which proves the lemma. □

5 Existence of Fixed Point

Because Y_T and Z_T are Hilbert space, $Y_T \times Z_T$ is also a Hilbert space and we can equip the norm

$$\|(f, u)\|_{Y\times Z} = (C_3)^{-1}\|f\|_Y + \|u\|_Z. \tag{31}$$

Define an operator $\mathcal{S} : Y_T \times Z_T \to Y_T \times Z_T$ by

$$\mathcal{S}(f, u) = (S_1(f, u), S_2(f, u)). \tag{32}$$

Proposition 12 *Fix $f_0 \in W^{3,2}(\Sigma)$. Then there is an $T_0 = T_0(C_0, \delta, \delta') > 0$ such that for all $T \leq T_0$,*

(a) *$\mathcal{S}$ restricts to an operator $\mathcal{S} : B \times B' \to B \times B'$.*
(b) *For each $f_1, f_2 \in B$ and $u_1, u_2 \in B'$,*

$$\|\mathcal{S}(f_1, u_1) - \mathcal{S}(f_2, u_2)\|_{Y\times Z} \leq \frac{5}{6}\|(f_1, u_1) - (f_2 - u_2)\|_{Y\times Z}. \tag{33}$$

Proof By Lemmas 5 and 9, (a) is proved. For (b), using Lemmas 6, 7, 10, 11, there is $T_0 = T_0(\delta, \delta') > 0$ such that for all $T \leq T_0$,

$$\begin{aligned}
&\|\mathcal{S}(f_1, u_1) - \mathcal{S}(f_2, u_2)\|_{Y\times Z}\\
&\quad= (C_3)^{-1}\|S_1(f_1, u_1) - S_1(f_2, u_2)\|_Y + \|S_2(f_1, u_1) - S_2(f_2, u_2)\|_Z\\
&\quad\leq (C_3)^{-1}\|S_1(f_1, u_1) - S_1(f_2, u_1)\|_Y + (C_3)^{-1}\|S_1(f_2, u_1) - S_1(f_2, u_2)\|_Y\\
&\qquad+ \|S_2(f_1, u_1) - S_2(f_2, u_1)\|_Z + \|S_2(f_2, u_1) - S_2(f_2, u_2)\|_Z\\
&\quad\leq \frac{1}{3}(C_3)^{-1}\|f_1 - f_2\|_Y + \frac{1}{2}\|u_1 - u_2\|_Z\\
&\qquad+ T^{1/4}\|f_1 - f_2\|_Y + \frac{1}{3}\|u_1 - u_2\|_Z\\
&\quad\leq \frac{5}{6}\Big((C_3)^{-1}\|f_1 - f_2\|_Y + \|u_1 - u_2\|_Z\Big)\\
&\quad= \frac{5}{6}\|(f_1, u_1) - (f_2, u_2)\|_{Y\times Z}
\end{aligned}$$

if $T^{1/4} \leq \frac{1}{2}(C_3)^{-1}$. □

Theorem 13 (Short time existence for strong solution) *There is $T_0 > 0$ such that there exists a smooth solution $(f, u) \in B \times B'$ of* (2) *on $\Sigma \times [0, T_0]$.*

Proof The existence of solution f, u comes from Proposition 12. The fact $f(\Sigma \times [0, T_0]) \subset N$ can be easily shown using nearest point projection, see for example [24]. Moreover, the operator $\partial_t - \mathrm{e}^{-2u}\Delta$ is uniformly parabolic, so $|(\partial_t - \mathrm{e}^{-2u}\Delta)f| \in L^p(\Sigma \times [0, T_0])$ for any $1 \leq p < \infty$, by standard parabolic theory. This implies

$$\nabla^2 f, \partial_t f \in L^p(\Sigma \times [0, T_0])$$

for any $1 \leq p < \infty$.

Next, by direct computation from (2b), we have

$$\mathrm{e}^{2u} = \mathrm{e}^{-2at}\left(1 + 2b\int_0^t \mathrm{e}^{2as}|\mathrm{d}f|^2\right)$$

hence

$$\begin{aligned}\nabla u &= \mathrm{e}^{-2u-2at} 2b \int_0^t \mathrm{e}^{2as}\langle \nabla \mathrm{d}f, \mathrm{d}f\rangle \\ \int |\nabla u|^p &\leq (4b)^p \int \left(\int_0^t |\nabla \mathrm{d}f||\mathrm{d}f|\right)^p \\ &\leq (4b)^p t^{p-1} \int\int_0^t |\nabla \mathrm{d}f|^p |\mathrm{d}f|^p\end{aligned}$$

which implies $\nabla u \in L^p(\Sigma \times [0, T_0])$ for any $1 \leq p < \infty$. Now taking ∇ in the Eq. (2a) to get

$$\begin{aligned}|(\partial_t - \mathrm{e}^{-2u}\Delta)\nabla f| &\leq C\left(|\nabla u||\Delta f| + |\nabla u||\mathrm{d}f|^2 + |\nabla \mathrm{d}f||\mathrm{d}f| + |\mathrm{d}f|^3\right) \\ &\in L^p(\Sigma \times [0, T_0])\end{aligned}$$

which implies

$$\nabla^3 f, \partial_t(\nabla f) \in L^p(\Sigma \times [0, T_0])$$

for any $1 \leq p < \infty$.

Finally, from Sobolev embedding, we have $f, \mathrm{d}f \in C^\alpha(\Sigma \times [0, T_0])$ for some $\alpha > 0$. This implies $(\partial_t - \mathrm{e}^{-2u}\Delta)f \in C^{\alpha,\alpha/2}(\Sigma \times [0, T_0])$ where $C^{\alpha,\alpha/2}$ is parabolic Hölder space of exponent α. Now by Schauder estimate and standard bootstrapping argument, we conclude that f is smooth, so u is. □

6 Local Estimate

To get global weak solution, we will follow Struwe's idea: run the flow until singularity occurs. Then take weak limit as new initial condition, run the flow again. Keep going this process and we will have only finitely many singularities due to finiteness of the energy. Because our flow is coupled, we need to re-establish the whole process with f and u. And this requires some condition on b, which can be interpreted as the sensitiveness of the conformal evolution of the metric with respect to high energy density. Let $C_N > 0$ be a constant only depending on the embedding $N \hookrightarrow \mathbb{R}^L$ such that $\|R^N\|, \|A\|, \|DA\| \le C_N$ where R^N is the Riemannian curvature tensor of N. And from now on, assume $b \ge C_N^2$.

6.1 Energy Estimate

Now we establish local energy estimate. Fix B_{2r} and let φ be a cut-off function supported on B_{2r} such that $\varphi \equiv 1$ on B_r, $0 \le \varphi \le 1$ and $|\nabla\varphi| \le \frac{4}{r}$.

Proposition 14 *For solutions* (f, u) *of* (2), *we have*

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2 + \int_{B_{2r}} |\mathrm{d}f|^2\varphi^2(t_2) - \int_{B_{2r}} |\mathrm{d}f|^2\varphi^2(t_1)\\ &\le \frac{4^2}{ar^2}(e^{2at_2} - e^{2at_1})E_0.\end{aligned} \tag{34}$$

Especially, we have

$$E(B_r, t_2) - E(B_{2r}, t_1) \le \frac{4^2}{2ar^2}(e^{2at_2} - e^{2at_1})E_0. \tag{35}$$

Proof From the Eq. (2a), multiplying $e^{2u} f_t\varphi^2$ gives

$$\begin{aligned}\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2 &= \int_{B_{2r}} \langle f_t, \tau(f)\varphi^2\rangle\\ &= -\int_{B_{2r}} \langle \mathrm{d}f_t, \mathrm{d}f\varphi^2\rangle - 2\int_{B_{2r}} \langle f_t, f_i\rangle\varphi\nabla_i\varphi\\ &\le -\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{B_{2r}} |\mathrm{d}f|^2\varphi^2 + \frac{1}{2}\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2 + 2\int_{B_{2r}} e^{-2u}|\mathrm{d}f|^2|\nabla\varphi|^2.\end{aligned}$$

So, we have

$$\begin{aligned}\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2 + \frac{d}{\mathrm{d}t}\int_{B_{2r}} |\mathrm{d}f|^2\varphi^2 &\le 4\int_{B_{2r}} e^{-2u}|\mathrm{d}f|^2|\nabla\varphi|^2\\ &\le 4\frac{4^2}{r^2}e^{2at}\int_{B_{2r}} |\mathrm{d}f|^2\end{aligned}$$

$$\leq 4\frac{4^2}{r^2}e^{2at}2E_0.$$

Integrating from t_1 to t_2 gives the result. □

Lemma 15 *Furthermore, assume*

$$\sup_{t_1 \leq t \leq t_2} E(B_{2r}, t) < \varepsilon_1.$$

Then we have

$$\int_{t_1}^{t_2} \int_{B_{2r}} e^{2u}|f_t|^2\varphi^2 \leq 4^2\varepsilon_1 \left(1 + \frac{e^{2at_2} - e^{2at_1}}{2ar^2}\right), \tag{36}$$

$$\int_{t_1}^{t_2} \int_{B_{2r}} |f_t|^2\varphi^2 \leq e^{2at_2}4^2\varepsilon_1 \left(1 + \frac{e^{2at_2} - e^{2at_1}}{2ar^2}\right). \tag{37}$$

Proof The first equation directly comes from (34), by changing E_0 to ε_1. Also, it is easy to see that

$$\int_{t_1}^{t_2} \int_{B_{2r}} |f_t|^2\varphi^2 = \int_{t_1}^{t_2} \int_{B_{2r}} e^{-2u}e^{2u}|f_t|^2\varphi^2 \leq e^{2at_2} \int_{t_1}^{t_2} \int_{B_{2r}} e^{2u}|f_t|^2\varphi^2$$
$$\leq e^{2at_2}4^2\varepsilon_1 \left(1 + \frac{e^{2at_2} - e^{2at_1}}{2ar^2}\right).$$

□

6.2 Estimate for $\int |f_t|^2$

The next step is to get estimate for derivative of $\int_{B_{2r}} |f_t|^2\varphi^2$, which will lead to the control of itself. For the future purpose, here we introduce more general version of it. For now, we need $p = 0$.

Proposition 16 *Let* (f, u) *are solutions of* (2). *For* $p \geq 0$, *we have*

$$\begin{aligned}\frac{d}{dt} \int_{B_{2r}} e^{2u}|f_t|^{p+2}\varphi^2 \leq\ & 2a(p+1) \int_{B_{2r}} e^{2u}|f_t|^{p+2}\varphi^2 + 4(p+2) \int_{B_{2r}} |f_t|^{p+2}|\nabla\varphi|^2 \\ & - \frac{p+2}{4} \int_{B_{2r}} |\nabla f_t|^2|f_t|^p\varphi^2 \\ & + \left((p+2)C_N + \frac{(p+2)C_N^2}{2} - 2b(p+1)\right) \int_{B_{2r}} |\mathrm{d}f|^2|f_t|^{p+2}\varphi^2.\end{aligned} \tag{38}$$

Especially, we have

$$\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^{p+2}\varphi^2(t) \le \mathrm{e}^{2a(p+1)(t-t_0)} \cdot \left(\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^{p+2}\varphi^2(t_0) + 4(p+2)\int_{t_0}^{t}\int_{B_{2r}} |f_t|^{p+2}|\nabla\varphi|^2 \right). \tag{39}$$

Proof By taking time-derivative to (2a), we have

$$(\mathrm{e}^{2u} f_t)_t = \Delta f_t + A(\mathrm{d}f, \mathrm{d}f)_t.$$

Taking inner product with $f_t|f_t|^p\varphi^2$ and integrating gives

$$\begin{aligned}
\int \langle (\mathrm{e}^{2u} f_t)_t, f_t|f_t|^p\varphi^2\rangle &= \int \langle \Delta f_t, f_t|f_t|^p\varphi^2\rangle + \int \langle A(\mathrm{d}f, \mathrm{d}f)_t, f_t|f_t|^p\varphi^2\rangle \\
&= -\int |\nabla f_t|^2|f_t|^p\varphi^2 - \int \langle \nabla f_t, f_t\rangle p|f_t|^{p-2}\varphi^2\langle \nabla f_t, f_t\rangle \\
&\quad - 2\int \langle \nabla f_t, f_t\rangle |f_t|^p\varphi\nabla\varphi + \int \langle DA(\mathrm{d}f, \mathrm{d}f)\cdot f_t, f_t|f_t|^p\varphi^2\rangle \\
&\quad + \int \langle A(\mathrm{d}f_t, \mathrm{d}f), f_t|f_t|^p\varphi^2\rangle \\
&= -\int |\nabla f_t|^2|f_t|^p\varphi^2 - p\int |\langle \nabla f_t, f_t\rangle|^2|f_t|^{p-2}\varphi^2 \\
&\quad + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\mathrm{III} &\le \frac{1}{4}\int |\nabla f_t|^2\varphi^2|f_t|^p + 4\int |f_t|^{p+2}|\nabla\varphi|^2, \\
\mathrm{IV} &\le C_N \int |\mathrm{d}f|^2|f_t|^{p+2}\varphi^2, \\
\mathrm{V} &\le C_N \int |\nabla f_t||\mathrm{d}f||f_t|^{p+1}\varphi^2, \\
&\le \frac{1}{2}\int |\nabla f_t|^2\varphi^2|f_t|^p + \frac{C_N^2}{2}\int |\mathrm{d}f|^2|f_t|^{p+2}\varphi^2.
\end{aligned}$$

On the other hand, LHS becomes

$$\begin{aligned}
\int \langle (\mathrm{e}^{2u} f_t)_t, f_t|f_t|^p\varphi^2\rangle &= \frac{1}{p+2}\frac{d}{\mathrm{d}t}\int \mathrm{e}^{2u}|f_t|^{p+2}\varphi^2 + 2\frac{p+1}{p+2}\int \mathrm{e}^{2u}|f_t|^{p+2}u_t\varphi^2 \\
&= \frac{1}{p+2}\frac{d}{\mathrm{d}t}\int \mathrm{e}^{2u}|f_t|^{p+2}\varphi^2 + 2b\frac{p+1}{p+2}\int |\mathrm{d}f|^2|f_t|^{p+2}\varphi^2 \\
&\quad - 2a\frac{p+1}{p+2}\int \mathrm{e}^{2u}|f_t|^{p+2}\varphi^2.
\end{aligned}$$

All together, we have

$$\begin{aligned}\frac{d}{dt}\int e^{2u}|f_t|^{p+2}\varphi^2 \le{}& 2a(p+1)\int e^{2u}|f_t|^{p+2}\varphi^2 + 4(p+2)\int |f_t|^{p+2}|\nabla\varphi|^2\\ &-\frac{p+2}{4}\int |\nabla f_t|^2|f_t|^p\varphi^2\\ &+\left((p+2)C_N+\frac{(p+2)C_N^2}{2}-2b(p+1)\right)\int |df|^2|f_t|^{p+2}\varphi^2.\end{aligned}$$

By the choice of b, the last term is negative for all $p \ge 0$. Hence,

$$\begin{aligned}&\frac{d}{dt}\int e^{2u}|f_t|^{p+2}\varphi^2 \le 2a(p+1)\int e^{2u}|f_t|^{p+2}\varphi^2 + 4(p+2)\int |f_t|^{p+2}|\nabla\varphi|^2\\ &\int_{B_{2r}} e^{2u}|f_t|^{p+2}\varphi^2(t) \le e^{2a(p+1)(t-t_0)}\\ &\quad\cdot\left(\int_{B_{2r}} e^{2u}|f_t|^{p+2}\varphi^2(t_0) + 4(p+2)\int_{t_0}^t\int_{B_{2r}} |f_t|^{p+2}|\nabla\varphi|^2\right)\end{aligned}$$

by Gronwall's inequality. □

Lemma 17 *Let* (f, u) *are solutions of* (2)*. Assume that*

$$\sup_{T-2\delta r^2\le t\le T} E(B_{2r}, t) < \varepsilon_1.$$

Then for $t \in [T-\delta r^2, T]$*, we have*

$$\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2(t) \le C_1(r,\delta,t)C_2(r,\delta,t)\varepsilon_1 \tag{40}$$

where

$$C_1(r,\delta,t) = 4^2\left(1+e^{2at}\frac{1-e^{-2a\delta r^2}}{2ar^2}\right), \tag{41}$$

$$C_2(r,\delta,t) = e^{6a\delta r^2}\left(\frac{1}{\delta r^2}+\frac{16(4)^2}{r^2}e^{2at}\right). \tag{42}$$

Proof Suppose φ be a cut-off function supported on $B_{3r/2}$ and $\varphi \equiv 1$ on B_r and $|\nabla\varphi| \le \frac{4}{r}$. Also, let ψ be a cut-off function supported on B_{2r} and $\psi \equiv 1$ on $B_{3r/2}$ and $|\nabla\psi| \le \frac{4}{r}$. From (39) for $p = 0$ and using (37), we have

$$\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2(t) \le e^{2a(t-t_0)}\left(\int_{B_{2r}} e^{2u}|f_t|^2\varphi^2(t_0) + 8\int_{t_0}^t\int_{B_{3r/2}} |f_t|^2|\nabla\varphi|^2\right)$$

$$
\begin{aligned}
&\leq \mathrm{e}^{2a(t-t_0)}\left(\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(t_0) + \frac{8(4)^2}{r^2}\int_{t_0}^t \int_{B_{2r}} |f_t|^2\psi^2\right) \\
&\leq \mathrm{e}^{2a(t-t_0)}\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(t_0) \\
&\quad + \mathrm{e}^{2a(t-t_0)}\frac{8(4)^2}{r^2}\mathrm{e}^{2at}4^2\varepsilon_1\left(1+\frac{\mathrm{e}^{2at}-\mathrm{e}^{2at_0}}{2ar^2}\right).
\end{aligned}
$$

Now take $t_0 \in [t-\delta r^2, t]$ such that

$$
\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(t_0) = \min_{t-\delta r^2 \leq s \leq t} \int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(s).
$$

Then by (36),

$$
\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(t_0) \leq \frac{1}{\delta r^2}\int_{t-\delta r^2}^t \int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2 \leq \frac{1}{\delta r^2}4^2\varepsilon_1\left(1+\frac{\mathrm{e}^{2at}-\mathrm{e}^{2a(t-\delta r^2)}}{2ar^2}\right).
$$

Therefore,

$$
\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2(t) \leq 4^2\varepsilon_1\left(1+\frac{\mathrm{e}^{2at}-\mathrm{e}^{2at-2a\delta r^2}}{2ar^2}\right)\left(\frac{1}{\delta r^2}+\frac{8(4)^2}{r^2}\mathrm{e}^{2at}\right)\mathrm{e}^{2a\delta r^2}.
$$

This completes the proof. □

Corollary 18 *Under the same assumption as above, we also have*

$$
\int_{t-\delta r^2}^t \int_{B_{2r}} |\nabla f_t|^2\varphi^2 \leq CC_1(r,\delta,t)C_2(r,\delta,t)\varepsilon_1, \tag{43}
$$

$$
\int_{t-\delta r^2}^t \int_{B_{2r}} |\mathrm{d}f|^2|f_t|^2\varphi^2 \leq CC_1(r,\delta,t)C_2(r,\delta,t)\varepsilon_1. \tag{44}
$$

Proof From (38) with $p=0$, we can integrate from $t-\delta r^2$ to t.

$$
\begin{aligned}
\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2\Big|_{t-\delta r^2}^t &\leq 2a\int_{t-\delta r^2}^t \int_{B_{2r}} \mathrm{e}^{2u}|f_t|^2\varphi^2 + 8\int_{t-\delta r^2}^t \int_{B_{2r}} |f_t|^2|\nabla\varphi|^2 \\
&\quad - \frac{1}{2}\int_{t-\delta r^2}^t \int_{B_{2r}} |\nabla f_t|^2\varphi^2 \\
&\quad + \left(2C_N + C_N^2 - 2b\right)\int_{t-\delta r^2}^t \int_{B_{2r}} |\mathrm{d}f|^2|f_t|^2\varphi^2.
\end{aligned}
$$

Hence, we have

$$
\frac{1}{2}\int_{t-\delta r^2}^t \int_{B_{2r}} |\nabla f_t|^2\varphi^2 \leq 2C_1(r,\delta,t)C_2(r,\delta,t)\varepsilon_1 + 2aC_1(r,\delta,t)\varepsilon_1
$$

$$+ 8\frac{4^2}{r^2}e^{2at}C_1(r,\delta,t)\varepsilon_1 \le CC_1(r,\delta,t)C_2(r,\delta,t)\varepsilon_1.$$

The other inequality is similar. □

6.3 Higher Estimate for Time Derivatives

In this subsection we will get estimate for $e^{2u}|f_t|^4$. We first build up a $(p+2)$-version of (34).

Proposition 19 *For solutions* (f,u) *of* (2) *and for* $p \ge 1$, *we have*

$$\int_{t_1}^{t_2}\int_{B_{2r}} e^{2u}|f_t|^{p+2}\varphi^2 \le C\int_{t_1}^{t_2}\int_{B_{2r}} |f_{ti}|^2|f_t|^{p-1}\varphi^2 + C\int_{t_1}^{t_2}\int_{B_{2r}} |f_t|^{p+1}|\nabla\varphi|^2 + C\int_{t_1}^{t_2}\int_{B_{2r}} |\mathrm{d}f|^2|f_t|^{p+1}\varphi^2. \tag{45}$$

Proof First note that for any $p \ge 1$, $\nabla_i|f_t|^p = p|f_t|^{p-2}\langle f_{ti}, f_t\rangle$. Also, for simplicity, denote $\int\int = \int_{t_1}^{t_2}\int_{B_{2r}}$. Multiplying $\tau(f)$ to (2a) gives

$$2e^{2u}|f_t|^2 = -2\langle f_{ti}, f_i\rangle + \nabla_i(2\langle f_t, f_i\rangle).$$

Multiplying $|f_t|^p\varphi^2$ for $p \ge 1$ and integrating gives

$$\begin{aligned} 2\int\int e^{2u}|f_t|^{p+2}\varphi^2 &= -2\int\int\langle f_{ti}, f_i\rangle|f_t|^p\varphi^2 - 4\int\int\langle f_t, f_i\rangle|f_t|^p\varphi\nabla_i\varphi \\ &\quad - 2p\int\int\langle f_t, f_i\rangle\varphi^2|f_t|^{p-2}\langle f_{ti}, f_t\rangle \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{I} &\le C\int\int|f_{ti}|^2|f_t|^{p-1}\varphi^2 + C\int\int|\mathrm{d}f|^2|f_t|^{p+1}\varphi^2, \\ \mathrm{II} &\le C\int\int|f_t|^{p+1}|\nabla\varphi|^2 + C\int\int|\mathrm{d}f|^2|f_t|^{p+1}\varphi^2, \\ \mathrm{III} &\le C\int\int|f_{ti}|^2|f_t|^{p-1}\varphi^2 + C\int\int|\mathrm{d}f|^2|f_t|^{p+1}\varphi^2. \end{aligned}$$

This completes the proof. □

Now we will show the desired estimate.

Proposition 20 *Let* (f,u) *are solutions of* (2). *Assume that*

$$\sup_{T-2\delta r^2\le t\le T} E(B_{2r},t) < \varepsilon_1.$$

Then for $t \in [T - \delta r^2, T]$, *we have*

$$\int_{B_{2r}} e^{2u} |f_t|^4 \varphi^2(t) \leq C_3, \tag{46}$$

where

$$C_3 = CC_1(r, \delta, t) C_2(r, \delta, t)^3 \varepsilon_1. \tag{47}$$

Note that C_3 depends on r, t, δ.

Proof For simplicity, denote $C_1 = C_1(r, \delta, t)$, $C_2 = C_2(r, \delta, t)$. Also, denote C for any number appeared in computations. Suppose φ be a cut-off function supported on $B_{3r/2}$ and $\varphi \equiv 1$ on B_r and $|\nabla \varphi| \leq \frac{4}{r}$. Also, let ψ be a cut-off function supported on B_{2r} and $\psi \equiv 1$ on $B_{3r/2}$ and $|\nabla \psi| \leq \frac{4}{r}$. Let $t_1 = t - \delta r^2$ and $t_2 = t$.

The proof consists of several steps, increasing power of $|f_t|$.

Step 1. Estimate for $\int \int e^{2u} |f_t|^3 \varphi^2$.

From (45) with $p = 1$ and using (37), (43) and (44), we have

$$\int_{t-\delta r^2}^{t} \int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2 \leq CC_1 C_2 \varepsilon_1 \tag{48}$$

and

$$\int_{t-\delta r^2}^{t} \int_{B_{2r}} |f_t|^3 \varphi^2 \leq e^{2at} CC_1 C_2 \varepsilon_1. \tag{49}$$

Step 2. Estimate for $\int e^{2u} |f_t|^3 \varphi^2$.

Now let $t_0 \in [t - \delta r^2, t]$ be such that

$$\int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2(t_0) = \min_{t-\delta r^2 \leq s \leq t} \int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2(s).$$

From (39) with $p = 1$ and using (48) and (49), we have

$$\begin{aligned}
\int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2(t) &\leq e^{4a\delta r^2} \left(\int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2(t_0) + 12 \int_{t_0}^{t} \int_{B_{3r/2}} |f_t|^3 |\nabla \varphi|^2 \right) \\
&\leq e^{4a\delta r^2} \left(\frac{1}{\delta r^2} \int_{t-\delta r^2}^{t} \int_{B_{2r}} e^{2u} |f_t|^3 \varphi^2 + 12 \frac{4^2}{r^2} \int_{t-\delta r^2}^{t} \int_{B_{2r}} |f_t|^3 \psi^2 \right) \\
&\leq e^{4a\delta r^2} \left(\frac{1}{\delta r^2} CC_1 C_2 \varepsilon_1 + 12 \frac{4^2}{r^2} e^{2at} CC_1 C_2 \varepsilon_1 \right) \\
&= CC_1 C_2 \varepsilon_1 \left(\frac{1}{\delta r^2} + \frac{12(4)^2}{r^2} e^{2at} \right) e^{4a\delta r^2}.
\end{aligned}$$

So, simply,

$$\int_{B_{2r}} e^{2u}|f_t|^3\varphi^2(t) \leq CC_1C_2^2\varepsilon_1. \tag{50}$$

Step 3. Estimate for $\int\int |\nabla f_t|^2|f_t|\varphi^2$ and $\int\int |\mathrm{d}f|^2|f_t|^3\varphi^2$.
From (38) with $p=1$, we can integrate from $t-\delta r^2$ to t.

$$\begin{aligned}\int_{B_{2r}} e^{2u}|f_t|^3\varphi^2\Big|_{t-\delta r^2}^{t} \leq{}& 4a\int_{t-\delta r^2}^{t}\int_{B_{2r}} e^{2u}|f_t|^3\varphi^2 + 12\int_{t-\delta r^2}^{t}\int_{B_{2r}} |f_t|^3|\nabla\varphi|^2\\ &-\frac{3}{4}\int_{t-\delta r^2}^{t}\int_{B_{2r}} |\nabla f_t|^2|f_t|\varphi^2\\ &+\left(3C_N+\frac{3C_N^2}{2}-4b\right)\int_{t-\delta r^2}^{t}\int_{B_{2r}} |\mathrm{d}f|^2|f_t|^3\varphi^2.\end{aligned}$$

Note that $3C_N+\frac{3C_N^2}{2}-4b<0$. Now, from (48), (49), and (50), we have

$$\begin{aligned}\frac{3}{4}\int_{t-\delta r^2}^{t}\int_{B_{2r}} |\nabla f_t|^2|f_t|\varphi^2 &\leq 2CC_1C_2^2\varepsilon_1 + 4aCC_1C_2\varepsilon_1 + 12\frac{4^2}{r^2}e^{2at}CC_1C_2\varepsilon_1\\ &\leq CC_1C_2^2\varepsilon_1.\end{aligned}$$

So, we have

$$\int_{t-\delta r^2}^{t}\int_{B_{2r}} |\nabla f_t|^2|f_t|\varphi^2 \leq CC_1C_2^2\varepsilon_1. \tag{51}$$

Similarly,

$$\int_{t-\delta r^2}^{t}\int_{B_{2r}} |\mathrm{d}f|^2|f_t|^3\varphi^2 \leq CC_1C_2^2\varepsilon_1. \tag{52}$$

Step 4. Estimate for $\int\int e^{2u}|f_t|^4\varphi^2$.
From (45) with $p=2$ and using (49), (51) and (52), we have

$$\int_{t-\delta r^2}^{t}\int_{B_{2r}} e^{2u}|f_t|^4\varphi^2 \leq CC_1C_2^2\varepsilon_1 \tag{53}$$

and

$$\int_{t-\delta r^2}^{t}\int_{B_{2r}} |f_t|^4\varphi^2 \leq e^{2at}CC_1C_2^2\varepsilon_1. \tag{54}$$

Step 5. Estimate for $\int e^{2u}|f_t|^4\varphi^2$.

Now let $t_0 \in [t - \delta r^2, t]$ be such that

$$\int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2(t_0) = \min_{t-\delta r^2 \le s \le t} \int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2(s).$$

From (39) with $p = 2$ and using (53) and (54), we have

$$\begin{aligned}
\int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2(t) &\le \mathrm{e}^{6a\delta r^2} \left(\int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2(t_0) + 16 \int_{t_0}^{t} \int_{B_{3r/2}} |f_t|^4 |\nabla \varphi|^2 \right) \\
&\le \mathrm{e}^{6a\delta r^2} \left(\frac{1}{\delta r^2} \int_{t-\delta r^2}^{t} \int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2 + 16 \frac{4^2}{r^2} \int_{t-\delta r^2}^{t} \int_{B_{2r}} |f_t|^3 \psi^2 \right) \\
&\le \mathrm{e}^{6a\delta r^2} \left(\frac{1}{\delta r^2} C C_1 C_2^2 \varepsilon_1 + 16 \frac{4^2}{r^2} \mathrm{e}^{2at} C C_1 C_2^2 \varepsilon_1 \right) \\
&= C C_1 C_2^2 \varepsilon_1 \left(\frac{1}{\delta r^2} + \frac{16(4)^2}{r^2} \mathrm{e}^{2at} \right) \mathrm{e}^{6a\delta r^2}.
\end{aligned}$$

So, simply,

$$\int_{B_{2r}} \mathrm{e}^{2u} |f_t|^4 \varphi^2(t) \le C C_1 C_2^3 \varepsilon_1.$$

□

Remark 1 We can keep going on to get bounds for $\int_{B_{2r}} \mathrm{e}^{2u} |f_t|^n \varphi^2(t) \le C_3(n)$ for any n. However, these bounds blow up to infinity as $n \to \infty$.

7 $W^{2,2}$ and Gradient Estimate

In this section we will get $W^{2,2}$ estimate and gradient estimate for the solution f of (2a). For simplicity, denote $\|\cdot\|_{k,p} = \|\cdot\|_{W^{k,p}(B_{2r})}$ and $\|\cdot\|_p = \|\cdot\|_{0,p}$. First observe the following.

Lemma 21 *Let u be a solution of* (2b). *For $p > 2$ and for any $r > 0$,*

$$\int_{B_{2r}} \mathrm{e}^{pu} \varphi^r(t) \le \int_{B_{2r}} \mathrm{e}^{pu} \varphi^r(t_0) + \frac{2b^2(p-2)}{pa} \int_{t_0}^{t} \int_{B_{2r}} |\mathrm{d}f|^p \varphi^r. \tag{55}$$

Proof Note that

$$\partial_t(\mathrm{e}^{pu}) = p\mathrm{e}^{pu} u_t = pb\mathrm{e}^{(p-2)u} |\mathrm{d}f|^2 - pa\mathrm{e}^{pu}.$$

So, multiplying φ^r and integrating over B_{2r} gives

$$\frac{d}{\mathrm{d}t} \int_{B_{2r}} \mathrm{e}^{pu} \varphi^r = pb \int_{B_{2r}} \mathrm{e}^{(p-2)u} |\mathrm{d}f|^2 \varphi^r - pa \int_{B_{2r}} \mathrm{e}^{pu} \varphi^r$$

$$
\begin{aligned}
&\leq b\lambda(p-2)\int_{B_{2r}} \mathrm{e}^{pu}\varphi^r + 2b\lambda^{-1}\int_{B_{2r}} |\mathrm{d}f|^p\varphi^r - pa\int_{B_{2r}} \mathrm{e}^{pu}\varphi^r \\
&= \frac{2b^2(p-2)}{pa}\int_{B_{2r}} |\mathrm{d}f|^p\varphi^r
\end{aligned}
$$

by Young's inequality with weight $\lambda = \frac{pa}{b(p-2)}$. Hence, by integrating, we obtain the result. □

Lemma 22 *Let f be any smooth function and let $\varphi \in C_0^\infty(B_{2r})$ be a cut-off function. Then for any $r > 1$ and $p \geq 2$, we have*

$$
\| \, |\mathrm{d}f|^r\varphi\|_p \leq C\| \, |\mathrm{d}f|^{r-1}\|_p\|f\varphi\|_{2,2}. \tag{56}
$$

Proof Let $1 \leq s < 2$ be such that $p = 2s(2-s)$. By Sobolev embedding,

$$
\begin{aligned}
\| \, |\mathrm{d}f|^r\varphi\|_p &\leq C\|\nabla(|\mathrm{d}f|^r\varphi)\|_s \\
&\leq C\||\mathrm{d}f|^{r-1}\nabla(|\mathrm{d}f|\varphi)\|_s \\
&\leq C\| \, |\mathrm{d}f|^{r-1}\|_p\|f\varphi\|_{2,2}.
\end{aligned}
$$

□

Next, we will show $W^{2,2}$ estimate.

Proposition 23 *Let (f, u) are solutions of (2). Then there exists $\varepsilon_1 > 0$ such that the following holds*:

Assume that

$$
\sup_{T-2\delta r^2 \leq t \leq T} E(B_{2r}, t) \leq \varepsilon_1, \quad \int_{B_{2r}\times\{T-2\delta r^2\}} \mathrm{e}^{6u} \leq \varepsilon_1.
$$

Then for $t \in [T-\delta r^2, T]$, we have

$$
\|f\varphi\|_{2,2} \leq C_4 = C_4(r, \delta, T, \varepsilon_1, C_N), \tag{57}
$$

where

$$
C_4 = \left(CC_3\varepsilon_1 + C\varepsilon_1^4\right)^{1/4}\left(1 + C_3\varepsilon_1\delta r^2\exp(C_3\varepsilon_1\delta r^2)\right)^{1/4}.
$$

Proof Suppose φ be a cut-off function supported on $B_{3r/2}$ and $\varphi \equiv 1$ on B_r and $|\nabla\varphi| \leq \frac{4}{r}$. Also, let ψ be a cut-off function supported on B_{2r} and $\psi \equiv 1$ on $B_{3r/2}$ and $|\nabla\psi| \leq \frac{4}{r}$. Let $t_0 = T - 2\delta r^2$.

Without loss of generality, assume $\int_\Omega f = 0$. Then we have, by Poincare,

$$
\|f\|_p \leq C_p\|\mathrm{d}f\|_p.
$$

From the equation $\Delta f + A(\mathrm{d}f, \mathrm{d}f) = \mathrm{e}^{2u} f_t$, multiplying φ and arranging terms gives

$$\begin{aligned}|\Delta(f\varphi)| &\leq |A(\mathrm{d}f, \mathrm{d}f)\varphi| + |\mathrm{e}^{2u} f_t\varphi| + k(\varphi)(|f| + |\mathrm{d}f|)\\ &\leq C_N|\,|\mathrm{d}f|^2\varphi| + |\mathrm{e}^{2u} f_t\varphi| + k(\varphi)(|f| + |\mathrm{d}f|).\end{aligned}$$

By the L^p estimate, we have

$$\|f\varphi\|_{2,p} \leq C\left(C_N\|\,|\mathrm{d}f|^2\varphi\|_p + \|\mathrm{e}^{2u}|f_t|\varphi\|_p + \|\mathrm{d}f\|_p\right), \tag{58}$$

where the constant C only depends on p and r.

Now let $p = 2$. Note that, by (46) and (55),

$$\begin{aligned}\|\mathrm{e}^{2u}|f_t|\varphi\|_2^4 &= \left(\int_{B_{2r}} \mathrm{e}^{4u}|f_t|^2\varphi^2\right)^2\\ &\leq \left(\int_{B_{3r/2}} \mathrm{e}^{2u}|f_t|^4\right)\left(\int_{B_{2r}} \mathrm{e}^{6u}\varphi^4\right)\\ &\leq \left(\int_{B_{2r}} \mathrm{e}^{2u}|f_t|^4\psi^2\right)\left(\int_{B_{2r}} \mathrm{e}^{6u}\varphi^4\right)\\ &\leq C_3\left(\int_{B_{2r}} \mathrm{e}^{6u}\varphi^4(t_0) + \frac{8b^2}{6a}\int_{t_0}^t\int_{B_{2r}} |\mathrm{d}f|^6\varphi^4\right)\\ &\leq C_3\varepsilon_1 + C_3\int_{t_0}^t\int_{B_{2r}} |\mathrm{d}f|^6\varphi^4.\end{aligned}$$

Now applying Lemma 22 with $r = 3/2, q = 4$ gives

$$\begin{aligned}\left(\int_{B_{2r}} |\mathrm{d}f|^6\varphi^4\right)^{1/4} &= \|\,|\mathrm{d}f|^{3/2}\varphi\|_4 \leq C\|\,|\mathrm{d}f|^{1/2}\|_4\|f\varphi\|_{2,2}\\ &\leq C\varepsilon_1^{1/4}\|f\varphi\|_{2,2}.\end{aligned}$$

On the other hand, applying Lemma 22 with $r = 2$, $q = 2$ gives

$$\|\,|\mathrm{d}f|^2\varphi\|_2 \leq C\|\mathrm{d}f\|_2\|f\varphi\|_{2,2} \leq C\varepsilon_1^{1/2}\|f\varphi\|_{2,2}.$$

All together, we have

$$\|f\varphi\|_{2,2}^4 \leq CC_N^4\varepsilon_1^2\|f\varphi\|_{2,2}^4 + CC_3\varepsilon_1 + CC_3\varepsilon_1\int_{t_0}^t \|f\varphi\|_{2,2} + C\varepsilon_1^4.$$

Let $X = \|f\varphi\|_{2,2}^4$. Then the above equation becomes

$$(1 - CC_N^4\varepsilon_1^2)X \le CC_3\varepsilon_1 + C\varepsilon_1^4 + C_3\varepsilon_1 \int_{t_0}^t X.$$

So, if ε_1 is small enough so that $1 - CC_N^4\varepsilon_1^2 > 1/2$, then by Gronwall's inequality, we have

$$\begin{aligned}\|f\varphi\|_{2,2}^4 &\le \left(CC_3\varepsilon_1 + C\varepsilon_1^4\right)(1 + C_3\varepsilon_1(t-t_0)\exp(C_3\varepsilon_1(t-t_0)))\\ &\le \left(CC_3\varepsilon_1 + C\varepsilon_1^4\right)\left(1 + C_3\varepsilon_1\delta r^2\exp(C_3\varepsilon_1\delta r^2)\right).\end{aligned}$$

This completes the proof. □

From Sobolev embedding, we now have, for $t \in [T - \delta r^2, T]$,

$$\|f\varphi\|_{1,p} \le C_4 \tag{59}$$

for any $p > 1$.

Now we will show gradient estimate. This can be achieved by obtaining better estimate than $W^{2,2}$, say $W^{2,3}$.

Proposition 24 *Assume the same as in Proposition 23. In addition, we assume that*

$$\int_{B_{2r}\times\{T-2\delta r^2\}} e^{18u} \le \varepsilon_1.$$

Then for $t \in [T - \delta r^2, T]$, *we have*

$$\|f\varphi\|_{2,3} \le C_5 = C_5(r, \delta, T, \varepsilon_1, C_N) \tag{60}$$

where

$$C_5 = C\left(C_N C_4^2 + C_3^{1/12}\varepsilon_1^{1/12} + C_3^{1/12}\delta^{1/12}r^{1/6}C_4^{3/2} + C_4\right).$$

In particular,

$$\sup_{B_r} |\mathrm{d}f| \le C_5. \tag{61}$$

Proof By (59), we have uniform bound for $|\mathrm{d}f|^p$ for any p. Now from Eq. (58), we have

$$\|f\varphi\|_{2,p} \le C\left(C_N\|\mathrm{d}f\|_{2p}^2 + \|e^{2u}|f_t|\varphi\|_p + \|\mathrm{d}f\|_p\right).$$

Now let $p = 3$ and $t_0 = T - 2\delta r^2$. Then we have, using (55) and (59),

$$\begin{aligned}\|e^{2u}|f_t|\varphi\|_3^{12} &\le \|e^{u/2}|f_t|\varphi\|_4^{12}\|e^{3u/2}\|_{12}^{12} \\ &\le C_3\int_{B_{2r}} e^{18u} \\ &\le C_3\left(\int_{B_{2r}} e^{18u}(t_0) + C\int_{t_0}^{t}\int_{B_{2r}}|df|^{18}\right) \\ &\le C_3\varepsilon_1 + CC_3\delta r^2 C_4^{18}.\end{aligned}$$

Applying (59) completes the proof. □

8 Global Weak Solution

In this section, we will prove the main Theorem 1.

Lemma 25 *There exists $\varepsilon_1 > 0$ such that if (f, u) be a smooth solution of (2) on $B_{2r} \times [T - 2\delta r^2, T]$ and*

$$\sup_{T-2\delta r^2\le t\le T} E(B_{2r}, t) \le \varepsilon_1 \quad \text{and} \quad \int_{B_{2r}\times\{T-2\delta r^2\}} e^{18u} \le \varepsilon_1, \tag{62}$$

then Hölder norms of f, u and their derivatives are all bounded by constants only depending on $T, r, \delta, \varepsilon_1, C_N$.

Proof By the sup bound of $|df|$, we have $e^{-2u(f)} \le e^{2aT}$ and

$$\begin{aligned}e^{-2u(f)} &= \frac{e^{2at}}{1 + 2b\int_0^t e^{2as}|df|^2(x,s)ds} \ge \frac{e^{2at}}{1 + 2bM'^2\frac{e^{2at}-1}{2a}} \\ &\ge \frac{1}{1 + \frac{b}{a}M'^2}.\end{aligned}$$

Hence the operator $\partial_t - e^{-2u}\Delta$ is uniformly parabolic on $[0, T_0)$.

Similar in proof of Theorem 13, we conclude the desired estimate. □

Proof (Proof of Theorem 1) First consider f_0 is smooth. By Theorem 13, there exists a smooth solution in $(\Sigma \times [0, T))$ for some $T > 0$. Let T_1 be the maximal existence time. If $T_1 = \infty$ then we obtain global solution which is smooth everywhere. So suppose $T_1 < \infty$.

If we have $\limsup_{t\nearrow T_1} E(B_{2r}(x), t) \le \varepsilon_1$ for any $x \in \Sigma$ and $r > 0$, then by above lemma Hölder norms of f, u and their derivatives are all bounded, hence f, u can be extended beyond the time T_1. This contradicts with maximality of T_1. So there should be a point $x \in \Sigma$ such that

$$\limsup_{t\nearrow T_1} E(B_{2r}(x), t) > \varepsilon_1.$$

Since the total energy is finite, there are at most finitely many such points $\{x_1, \ldots, x_{k_1}\}$. Then by above lemma, we get smooth solution (f_1, u_1) on $\Sigma \times [0, T] \setminus \{(x_i^1, T_1)\}_{i=1,\ldots,k_1}$. If we denote $f(x, T_1)$ and $u(x, T_1)$ as the weak limit of $f(x, t)$ and $u(x, t)$ as $t \nearrow T_1$, then $f(t), u(t)$ converges to $f(T_1), u(T_1)$ strongly in $W^{1,2}_{\mathrm{loc}}(\Sigma \setminus \{x_i^1\})$.

Next, denote $g_1 = \mathrm{e}^{2u_1(x,T_1)} g_0$ and consider the flow (2) with initial map f_1 and initial metric g_1. As above, there is a smooth solution (f_2, u_2) on $\Sigma \times [0, T_2] \setminus \{(x_i^2, T_2)\}_{i=1,\ldots,k_2}$. From these we can set up a smooth solution (f, u) on $\Sigma \times [0, T_1 + T_2]$ which is smooth except $\{(x_i^1, T_1)\} \cup \{(x_i^2, T_2)\}$. Iterate this process to obtain global solution with exception points, which are at most finitely many because the total energy is finite.

□

9 Finite Time Singularity

As the conformal heat flow is developed to postpone the finite time singularity, it is expected to have no finite time singularity. In this section we will discuss few remarks about finite time singularity.

Recall the following

Lemma 26 ([23]) *There exist a compact target manifold N, a smooth map $f_0 : D \to N$ and $\varepsilon > 0$ such that every smooth map $f : D \to N$ homotopic to v_0 fails to be harmonic. If furthermore $E(f) \le E(f_0)$, then*

$$\int_D |\tau(f)|^2 \ge \varepsilon.$$

Together with energy decreasing property of harmonic map heat flow $f(t)$, the above lemma implies that no heat flow starting with initial map f_1 homotopic to f_0 above can be smooth after the time $t = \frac{E(f_1)}{\varepsilon}$.

This argument can be avoided in conformal heat flow. From (4), we have

$$E(0) - E(t) = \int_0^t \int_D \mathrm{e}^{-2u} |\tau(f(t))|^2.$$

So, if u is large, $\int_D \mathrm{e}^{-2u} |\tau(f(t))|^2$ can be smaller than ε even if $\int_D |\tau(f(t))|^2 > \varepsilon$.

The proof of the above lemma relies on no-neck property of approximate harmonic map with $\|\tau\|_{L^2} \to 0$. And the assumption $\|\tau\|_{L^2} \to 0$ is essential in the no-neck property as there is a counter example of Parker [7] where $\|\tau(f_i)\|_{L^1}$ is uniformly bounded. In fact, the energy identity and no-neck property of approximate harmonic map with $\|\tau(f_i)\|_{L^p}$ for some $p > 1$ uniformly bounded was proved in Wang–Wei–Zhang [43]. The conformal heat flow makes the tension field converge to zero with different scale. Hence the information about the converging scale of the tension field will play an important role in the property of the flow.

Acknowledgements The author would like to thanks Armin Schikorra and Thomas Parker for valuable comments and advice. The author also thank to the referee for his careful reading and valuable suggestions.

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