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Normalized Solutions for Schrödinger Equations with Stein–Weiss Potential of Critical Exponential Growth

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Abstract

In this paper, we focus on the existence of normalized solutions to the following Schrödinger equation with the Stein–Weiss potential

$$-\Delta u + \lambda u = \left(I_\mu \times \frac{F(u)}{|x|^\alpha} \right) \frac{f(u)}{|x|^\alpha}, x \in \mathbb{R}^2,$$

where $2\alpha + \mu \leq 2$, $0 < \mu < 2$, I_μ denotes the Riesz potential and $f : \mathbb{R} \to \mathbb{R}$ has critical exponential growth which behaves like $e^{\alpha u^2}$. The solutions correspond to critical points of the underlying energy functional subject to the L^2-norm constraint, namely, $\int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = a^2$ for $a > 0$ given. Under some weak assumptions, we prove the existence of the normalized solution for the equation by developing refined variational methods. In particular, we shall establish two new approaches to estimate precisely the minimax level of the underlying energy functional. As far as we know, our result is the first one in seeking normalized solutions of nonlinear equations involving the nonlocal Stein–Weiss reaction.

Keywords Nonlinear Schrödinger equation · Stein–Weiss potential · Critical exponential growth · Trudinger–Moser inequality

Mathematics Subject Classification 35J20 · 35J62 · 35Q55

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1 Introduction

In this paper, we are concerned with the existence of normalized solutions to the following nonlinear Schrödinger equation with the Stein–Weiss reaction under the exponential critical growth case

$$\begin{cases} -\Delta u + \lambda u = \left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right) \frac{f(u)}{|x|^\alpha},\ x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = a^2, \end{cases} \tag{$\mathcal{P}_a$}$$

where $a > 0$, $0 < \mu < 2$, $2\alpha + \mu \leq 2$. I_μ denotes the Riesz potential defined by

$$I_\mu(x) = \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{2-\mu}{2})2^{2-\mu}\pi|x|^\mu} := \frac{A_\mu}{|x|^\mu},\ x \in \mathbb{R}^2 \setminus \{0\},$$

where Γ represents the gamma function, $*$ indicates the convolution operator, $F(s)$ is the primitive of $f(s)$ with that $f(s)$ has exponential critical growth in $\mathbb{R}^2$. Next, we shall introduce three typical features of this problem to set the tone for the rest of the paper.

1.1 Introduction of Three Typical Features

1.1.1 L^2-Constraint

We aim to search for solutions to ($\mathcal{P}_a$) having prescribed mass, the normalized stationary states, whose existence can be formulated as follows: given $a > 0$, we aim to find $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^2)$ solving ($\mathcal{P}_a$) together with the normalization condition

$$|u|_2^2 = \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = a,$$

and in this case $\lambda \in \mathbb{R}$ cannot be prescribed but appear as Lagrange multipliers in the variational approach. This type of problem has important physical significance in Bose–Einstein condensates and the nonlinear optics framework, and the L^2-norm of such solutions is a preserved quantity of the evolution and the corresponding variational feature contributes to analyzing the orbital stability or instability. Naturally, such problems have attracted much attention in the fields of nonlinear PDEs in the last decades. We give brief introduction of the relative progress with the most general Schrödinger equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^N,\ N \geq 2, \\ \int_{\mathbb{R}^N} u^2 \mathrm{d}x = a, \end{cases} \tag{1.1}$$

which has been investigated extensively via the variational methods. One can search for the existence of the normalized solutions of (1.1) by considering the critical points of the corresponding energy functional $\mathcal{J} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x,$$

under the constraint

$$\mathcal{S}_{a,N} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a \right\}.$$

Generally, the types of such problems are divided into three parts: L^2-subcritical case, L^2-critical case, L^2-supercritical case, after a simple stretching consideration presented by

$$u_{t,N}(x) = t^{\frac{N}{2}} u(x),$$

during which one can find there exists a new L^2-critical exponent $q^* = 2 + 4/N$. Generalizing to more general nonlinear term f, one can conclude that if f admits a L^2-subcritical growth, i.e., f has a growth u^{p-1} with $p < q^*$ at infinity, then $\mathcal{J}|_{\mathcal{S}_{a,N}}$ is bound below and in this occasion, minimization method is the conventional approach to find normalized solutions, we refer to [12, 31] and the references therein in this aspect. If f admits a L^2-supercritical growth, i.e., f has a growth u^{p-1} with $p > q^*$ at infinity, then $\mathcal{J}$ is unbound below on $\mathcal{S}_{a,N}$, which implies the traditional minimization method does not work and more efforts are always required in the study of the L^2-supercritical case.

One of the groundbreaking pieces of work in the L^2-supercritical case is accomplished by Jeanjean [21]. Jeanjean [21] obtained the normalized solution at the mountain pass level of its energy functional under the following conditions:

(H0) f is odd;

(H1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist $\alpha, \beta \in \mathbb{R}$ satisfying $(2N+4)/N < \alpha \le \beta < 2^* = 2N/(N-2)$ such that

$$0 < \alpha F(t) \le f(t)t \le \beta F(t), \quad \forall\, t \in \mathbb{R} \setminus \{0\},$$

and the existence of ground state solutions was proved if f also satisfies

(H2) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and

$$\tilde{F}'(t)t > \frac{2N+4}{N} F(t), \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

Condition (H1) is the general Ambrosetti–Rabinowitz condition, which is benefit for us to obtain bounded Palais–Smale sequences for $\mathcal{J}$ constrained on $\mathcal{S}_{a,N}$, and Jeanjean developed a novel argument related to the mountain pass geometry for the scaled functional $\tilde{\mathcal{J}}(u, s) := J(u, s)$ with $s * u(\cdot) := e^{\frac{Ns}{2}}(e^s \cdot)$, which is widely used to find normalized solutions in the L^2-supercritical case. Based on the above conditions and by applying the fountain theorem to the scaled functional $\mathcal{J}$, Bartsch and de Valerioda [5] obtained infinitely normalized solutions. Another variational approach is presented by Bartsch and Soave [6, 7], and it is based on Ghoussoub minimax principle [19]. Applying this abstract minimax theorem, Barstch and Soave established the existence and multiplicity of normalized solutions of equation (1.1) with general nonlinearities f. We also refer to Bieganowski and Mederski [8], which considered the normalized solutions under the L^2-supercritical case but Sobolev subcritical, they provided nearly optimal conditions in some ways. For more information, please see [8] and its references.

We conclude this part with a brief progress about the critical Schrödinger equation in the sense of Sobolev embedding when $N \geq 3$. In 2020, Soave [28] first considered the Schrödinger equation with Sobolev critical growth:

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2}u + |u|^{2^*-2}u, \text{ in } \mathbb{R}^N, \ N \geq 3, \\ \int_{\mathbb{R}^N} u^2 \mathrm{d}x = a. \end{cases} \tag{1.2}$$

This type of problem is more delicate since they need to analyze how the lower-order term $|u|^{q-2}u$ affects the structure of the corresponding energy functional and to solve the lack of compactness caused by the Sobolev critical growth. According to the findings in [28], Eq. (1.2) has ground state solutions in the L^2-subcritical perturbation case $2 < q < 2 + N/4$ and L^2-supercritical perturbation case $2 + N/4 < q < 2^*$, respectively, for $\mu a(1-\gamma_q)q < \alpha$, where $\alpha = \alpha(N, q)$ is a specific constant depending on N, q and $\gamma_q = N(q-2)$. We also refer to Wei and Wu [33], Jeanjean and Le [22], Jeanjean et al. [24] which settled several open questions proposed by Soave [28].

1.1.2 Critical Exponential Case

Besides the L^2-constraint, another novel feature of equation $(\mathcal{P}_a)$ is that functions $f(u)$ have critical exponential growth that is the maximal growth that allows us to treat $(\mathcal{P}_a)$ variationally in $H^1(\mathbb{R}^2)$, which was shown by Trudinger [32] and Moser [26], and it is motivated by the following Trudinger–Moser inequality [11].

Lemma 1.1 (i) *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty;$$

(ii) *if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < \infty$, and $\alpha < 4\pi$, then there exists a constant $\mathcal{C}(M, \alpha)$, which depends only on M and α, such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \mathcal{C}(M, \alpha).$$

Inspired by the Trudinger–Moser-type inequality, we can say that a function $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ possesses critical exponential growth if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0. \end{cases} \tag{F1}$$

Here, we refer Adimurthi–Yadava [36] and de Figueiredo–Miyagaki–Ruf [17] to the readers for more information. Based on the Trudinger–Moser inequalities, many authors considered the existence and multiplicity of weak solutions for the nonlinear Schrödinger equations. We refer to [2, 13, 14] and the references therein for more details about the relative progress in this direction.

1.1.3 Stein–Weiss Convolution Reaction

We briefly recall the related background and some pioneering contributions in this field, and we start with the weighted L^p estimates for the fractional integral

$$(T_\mu\varphi)(x) = \int_{\mathbb{R}^N} \frac{\varphi(y)}{|x-y|^\mu}\mathrm{d}y, \quad 0 < \mu < N,$$

which is a fundamental problem in the field of harmonic analysis and such weighted L^p estimates are generated from quite natural phenomena and have practical significance in the large wide of mathematical fields, which can be summarized as that the appearance of some suitable symmetry hypotheses, notably radial symmetry, contributes to improving the classical estimates and some embedding properties of function spaces.

Many mathematicians have studied on the weighted L^p estimates for the fractional integral T_μ. Historically, Hardy and Littlewood [20] first considered the weighted L^p estimates for the one-dimensional fractional integral operator T_μ, then Sobolev [29] extended it to the N-dimensional case. Later, Stein and Weiss [30] obtained the following two-weight extension of the Hardy–Littlewood–Sobolev inequality, which is known as the Stein–Weiss inequality.

Proposition 1.2 (Doubly weighted Hardy–Littlewood–Sobolev inequality) *Let $t, s > 1$ and $0 < \mu < N$ with $\vartheta + \beta > 0$, $1/t + (\mu + \vartheta + \beta)/N + 1/s = 2$, $\vartheta < \frac{N}{t'}$, $\beta < \frac{N}{s'}$, $g_1 \in L^t(\mathbb{R}^N)$ and $g_2 \in L^s(\mathbb{R}^N)$, where t' and s' denote the Hölder conjugate of t and s, respectively. Then there exists a constant $C(N, \mu, \vartheta, \beta, t, s)$, independent of g_1, g_2 such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^\mu |y|^\vartheta |x|^\beta}\mathrm{d}x\mathrm{d}y \le C(N, \mu, \vartheta, \beta, t, s)\|g_1\|_t\|g_2\|_s.$$

For $\vartheta = \beta = 0$, it is reduced to the Hartree type (also called the Choquard type) nonlinearity, which is driven by the classical Hardy–Littlewood–Sobolev inequality.

Integrability for integral operators can be quantified using the Stein–Weiss inequality, which is fundamentally determined by the dilation nature of integral operators. Due to its significance in applications to issues in harmonic analysis and partial differential equations, the study of and comprehension of the Stein–Weiss inequality has sparked a growing amount of attention among scholars. We now look at the applications that the Stein–Weiss term has in relation to them. The polyharmonic Kirchhoff equations involving the critical Choquard-type exponential nonlinearity with singular weights were explored by Giacomoni et al. [4]. It is important to note the beautiful work of Du et al. [18], where they investigated the equation as below,

$$-\Delta u = \frac{1}{|x|^{\alpha}}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{\alpha,\mu}}}{|x-y|^{\mu}|y|^{\alpha}}\mathrm{d}y\right)|u(x)|^{2^*_{\alpha,\mu}-2}u, \quad x\in\mathbb{R}^N,$$

where $2^*_{\alpha,\mu} = (2N-2\alpha-\mu)/(N-2)$. In order to analyze the existence of solutions, study the regularity, and symmetry of positive solutions by moving plane arguments under the critical situation, as well as the results under the subcritical situation, the authors created a nonlocal version of the concentration—compactness principle. Yang et al. [38] achieved the symmetry, regularity, and asymptotic features of the weighted nonlocal system with critical exponents associated with the Stein–Weiss inequality by employing the moving plane arguments in integral form. For other results, we refer to [9, 37, 39, 40] and the references therein.

1.2 Introduction of Our Goal and Main Results

Among the investigations into normalized solutions of the nonlinear Schrödinger equation with critical growth, an new emerging interest is seeking the normalized solutions under the critical exponential growth in the sense of the Trudinger–Moser inequality, which was recently constructed by Alves et al. [3]. Under following hypothesis,

- (F2′) $\lim_{|t|\to 0}|f(t)|/|t|^l = 0$ for some constant $l > 3$;
- (F3′) there exists a constant $\mu_0 > 4$ such that $f(t)t \geq \mu_0 F(t) > 0$ for all $t\in\mathbb{R}\backslash\{0\}$;
- (F4′) there exist constants $p > 4$ and $\gamma > 0$ such that $F(t) \geq \gamma|t|^p$ for all $t\in\mathbb{R}$;
- (F5′) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and satisfies

$$\tilde{F}'(t)t \geq 4F(t), \quad \forall\, t\in\mathbb{R},$$

they established the existence of normalized solutions to equation:

$$\begin{cases} -\Delta u + \lambda u = f(u), \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = a. \end{cases} \tag{1.3}$$

Their result reads as follows in this topic.

Theorem 1.3 ([3, Theorem 1.2]) *Assume that f possesses critical exponential growth and satisfies* (F2′)-(F4′). *If $a\in(0,1)$, then there exists $\gamma^*(a) > 0$ such that* (1.3) *has a radial solution for all $\gamma \geq \gamma^*(a)$, where γ is given by* (F4′)*, moreover, this solution can be chosen as a positive ground state solution if f also satisfies* (H0) *and* (F5′).

Based on the idea introduced by Jeanjean [21], for every $a\in(0,1)$, they constructed a special (PS) sequence $\{u_n\}\subset\mathcal{S}_a^r := \mathcal{S}_a\cap H_r^1(\mathbb{R}^2)$ such that

$$\varphi(u_n)\to c_\gamma^\infty(a) > 0, \;\; \varphi|'_{\mathcal{S}_a^r}(u_n)\to 0 \;\text{ and }\; \vartheta(u_n)\to 0,$$

where the mountain pass level $c_\gamma^\infty(a)$ depends on γ and $\varphi : H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x,$$

on the constraint $\mathcal{S}_a$ which is defined by

$$\mathcal{S}_a = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 = a \right\}.$$

As a consequence of the Pohozaev identity (see [21, Lemma 2.7]), any solution u of φ exists in the Pohozaev manifold given by

$$\mathcal{M}_a = \{u \in \mathcal{S}_a : \vartheta(u) = 0\},$$

where ϑ is called the Pohozaev functional defined by

$$\vartheta(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^2} [f(u)u - 2F(u)]\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^2).$$

Next, let us discuss a few key components of their proofs in the work [3]. By establishing the crucial estimation

$$\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \to 0 \;\text{ as }\; \gamma \to \infty,$$

they could overcome the influence caused by the exponential critical growth. Indeed, as long as $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 < 1 - a$, then the following property

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^s \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x = \int_{\mathbb{R}^2} |\bar{u}|^s \left(e^{\alpha \bar{u}^2} - 1\right) \mathrm{d}x,$$

is a natural conclusion, since we have the Trudinger–Moser inequality and the compact embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for any $s > 2$. Then they were able to derive the Brezis–Lieb property

$$\int_{\mathbb{R}^2} [f(u_n)u_n - f(\bar{u})\bar{u} - f(u_n - \bar{u})(u_n - \bar{u})]\,\mathrm{d}x = o(1),$$

and also the convergence

$$\int_{\mathbb{R}^2} [f(u_n)u_n - f(\bar{u})\bar{u}]\,\mathrm{d}x = o(1),$$

if $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^2)$. In this case, the dealing process of their problem is similar to the one of nonlinearities like $f(u) \sim |u|^{q-2}u$ with $q > 2$. After this work, a series of subsequent studies have been done on the existence of different types of normalized solutions to the nonlinear equations in the planar case. Here, we refer to the work by Jeanjean and Lu [23], and Chen and Tang [15] and the references therein for more details.

Motivated by the above works, especially [3], in this paper, we discuss some refined analysis for the existence of normalized solutions to nonlinear Schrödinger equation with Stein–Weiss convolution term. As far as we know, our result is the first one in the fields of seeking normalized solutions involving the nonlocal Stein–Weiss reaction.

To state our results, besides (F1), we make the assumptions on f:

(F2) $\lim_{|t|\to 0} f(t)/t^{\frac{4-\mu-2\alpha}{2}} = 0$;
(F3) $f(t)t \geq (6-\mu-2\alpha)F(t)/2$ for all $t \in \mathbb{R}\setminus\{0\}$;
(F4) $\liminf_{|t|\to\infty} \frac{f(t)}{e^{\alpha_0 t^2}} > 0$;
(F5) there exist constants $M_0 > 0$ and $\beta_0 > 0$ such that

$$F(t) \leq M_0|f(t)|, \quad \forall\, |t| \geq \beta_0.$$

Theorem 1.4 *Assume that $a > 0$, $\alpha < \mu$ and f satisfies* (F1)–(F5). *Then equation $(\mathcal{P}_a)$ has a radial normalized solution. Moreover, for any solution the associated Lagrange multiplier λ is positive.*

Remark 1.5 We would like to point out that the purpose of both Theorem 1.4 and following Theorem 1.6 is to demonstrate that radial solutions exist for $(\mathcal{P}_a)$. As a result, we focus on the space $H_r^1(\mathbb{R}^2)$ since it compactly embeds in $L^s(\mathbb{R}^2)$ for all $s > 2$ and aids in the recovery of compactness. The solutions in $H_r^1(\mathbb{R}^2)$ are in reality solutions in whole $H^1(\mathbb{R}^2)$ according to Palais' principle of symmetric criticality [35].

Set

$$\gamma^*(a) = \frac{A^{\frac{2p-(6-2\alpha-\mu)}{4}}}{\sqrt{C(N,\mu,\alpha)}a^{\frac{4-2\alpha-\mu}{4}} C_{\frac{4p}{4-2\alpha-\mu}}^{\frac{4-2\alpha-\mu}{4}}} \left[\frac{2\alpha_0}{(4-2\alpha-\mu)\pi}\right]^{\frac{2p-(6-2\alpha-\mu)}{4}}. \tag{1.4}$$

where A is defined in (3.10).

Theorem 1.6 *Assume that $a > 0$, $\alpha < \mu$ and f satisfies* (F1)–(F3), (F4$'$) *with $\gamma > \gamma^*(a)$ and* (F5). *Then conclusions of Theorem 1.4 hold.*

Let us now outline the main strategy to prove Theorems 1.4 and 1.6. Our arguments are based on variational approaches and refined analysis techniques in order to complete the proofs of main results. It is easily seen that solutions of problem $(\mathcal{P}_a)$ can be found by looking for critical points of the energy functional $\Phi : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x.$$

By proposition 1.2, we know the latter term of the right-hand side of equation $\Phi(u)$ is well defined if $F(u) \in L^t(\mathbb{R}^2)$ for $t > 1$ given by $2/t + (\mu + 2\alpha)/2 = 2$. This means that we must require $F(u) \in L^{\frac{4}{4-\mu-2\alpha}}(\mathbb{R}^2)$, which can be guaranteed by (F1), (F2) and the continuous Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, where $p \geq 2$. Furthermore, it is standard to show that $\Phi \in \mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$. More precisely, since the exponential critical growth in $\mathbb{R}^2$ implies that the energy functional Φ is no more bounded from below on $\mathcal{S}_a$, and we shall look for a critical point satisfying a minimax characterization. Here, we introduce the following definition.

Definition 1.7 For given $a > 0$, we say that Φ possesses a mountain pass geometry on $\mathcal{S}_a$ if there exists $\rho_a > 0$ such that

$$c(a) := \inf_{g \in \Gamma_a} \max_{\tau \in [0,1]} \Phi(g(\tau)) > \max_{g \in \Gamma_a} \max\{\Phi(g(0)), \Phi(g(1))\}, \tag{1.5}$$

where $\Gamma_a := \{g \in \mathcal{C}([0,1], \mathcal{S}_a) : \|\nabla g(0)\|_2^2 \leq \rho_a, \Phi(g(1)) < 0\}$.

We intend to make use of the above definition to verify the mountain pass geometry of our problem. Besides, we highlight how the Stein–Weiss convolution term, in conjunction with the critical exponential growth, presents some new difficulties to our approach, as well as the accomplishments in our paper, which can be seen below.

(i) One of the main challenges comes after the appearance of the Stein–Weiss convolution term. We must confirm that the weak limit of the (PS) sequence of energy functional Φ is, in fact, the solution to the equation $(\mathcal{P}_a)$. However, in the case of exponential growth, the appearance of the Trudinger–Moser inequality requires that the critical exponent α_0 be less than 2π. We notice that Alves and Shen [1] provided a version of proof of such property; here, we would like to show another version of proof based on the condition that $\alpha < \mu$, which is motivated by the method explored in Qin and Tang [27, Lemma 4.8].

(ii) The work by Alves et al.[3] is significant in the field of normalized solutions in the two-dimensional critical case, and the technique previously described demonstrates that it is possible to control the energy level of the corresponding energy functional arbitrary small by simply taking a large enough value for the parameter γ. This estimation is one of the most crucial components that cannot be overlooked because we are working with the exponential critical case. The first technique we developed in this instance is to provide an exact lower bound γ, which is defined by (1.4). We shall provide a precise range of values for γ to fulfill the energy estimation in this method.

(iii) We would like to point out that, in the ordinary methods, one usually could take advantage of the Moser-type functions to pull down the critical value to a particular threshold value. However, it seems like there is no such estimation in seeking normalized solutions of the nonlinear equations except the very recent work by Zhang et al. [41], where they used the traditional Moser-type function, then to perform a stretch to satisfy the constraint mass and then gave the estimation. Here, motivated by Chen et al. [16], we will directly improve the traditional Moser-type function. By the suitable extension of the traditional Moser-type function, we can obtain the test functions in $H^1(\mathbb{R}^2)$ on the L^2-constraint $\mathcal{S}_a$. Another highlight in this progress is our assumption with the growth on f at infinity is relatively weak compared to the existing works about the Stein–Weiss convolution term.

(iv) Our results are superior than those of [3] in that we additionally change the constraint condition utilized in [3] from $a \in (0, 1)$ to $a > 0$.

Remark 1.8 We emphasize that the ground state normalized solution can be obtained using our work with some additional assumptions. However, in some ways, finding a ground state normalized solution is similar to the process explored as in our paper.

The organization of the remainder of this paper is as follows. In Sect. 2, we shall introduce some preliminary results and establish the mountain pass geometry of the associated energy functional. In Sect. 3, we shall apply two different approaches to give a precise estimation for the mountain pass energy level. In Sect. 4, we shall restore the compactness and prove the existence of normalized solutions of the equation $\mathcal{P}_a$.

Finally, we introduce some notations that will clarify what follows.

• C, C_i, c_i $(i = 1, 2, ...)$ denote positive constants which may vary from line to line.

• For any exponent $p > 1$, p' denotes the conjugate of p and is given as $p' = p/(p-1)$.

• $B_r(x)$ denotes the ball of radius r centered at $x \in \mathbb{R}^2$.

• The arrows $\rightharpoonup$ and $\to$ denote the weak convergence and strong convergence, respectively.

• $L^s(\mathbb{R}^2)(1 \leq s < +\infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^2} |u|^s \mathrm{d}x)^{1/s}$.

• $\beta : H := H^1(\mathbb{R}^2) \times \mathbb{R} \to H^1(\mathbb{R}^2)$ is a continuous map defined by

$$\beta(v, t)(x) = e^t v(e^t x) \ \text{ for } v \in H^1(\mathbb{R}^2),\ t \in \mathbb{R} \text{ and } x \in \mathbb{R}^2, \tag{1.6}$$

where H is a Banach space equipped with the scalar product

$$((v_1, s_1), (v_2, s_2))_H = (v_1, v_2) + s_1 s_2, \quad \forall\, (v_i, s_i) \in H,\ i = 1, 2,$$

and corresponding norm $\|(v, t)\|_H := \left(\|v\|^2 + |t|^2\right)^{1/2}$ for all $(v, s) \in H$.

2 Preliminary Results

In this section, we give some preliminary results which will be useful throughout the rest of the paper.

Lemma 2.1 (Gagliardo–Nirenberg inequality [34]) *Let $q > 2$. Then there exists a sharp constant $\mathcal{S}_q > 0$ such that*

$$\|u\|_q \leq \mathcal{S}_q^{1/q} \|\nabla u\|_2^{\frac{q-2}{q}} \|u\|_2^{\frac{2}{q}},$$

where $\mathcal{S}_q = \frac{q}{2\|U_q\|_2^{q-2}}$, and U_q is the ground state solution of the following equation:

$$-\Delta U + \frac{2}{q-2} U = \frac{2}{q-2} |U|^{q-2} U.$$

To deal with the nonlocal type problem $(\mathcal{P}_a)$, we also need the following inequality.

Lemma 2.2 (Cauchy–Schwarz type inequality [25, Sect. 5]) *For* $f, h \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, *there holds*

$$\int_{\mathbb{R}^2}\left(\frac{1}{|x|^{2-\mu}}\times|f|\right)|h|\mathrm{d}x \le \left[\int_{\mathbb{R}^2}\left(\frac{1}{|x|^{2-\mu}}\times|f|\right)|f|\mathrm{d}x\int_{\mathbb{R}^2}\left(\frac{1}{|x|^{2-\mu}}\times|h|\right)|h|\mathrm{d}x\right]^{\frac{1}{2}}.$$

Finding the bounded (PS) sequence relies on the mountain pass geometry based on Definition 1.7. Especially, verifying the boundedness of such a sequence is not trivial, which needs the information of L^2-Pohozaev inequality. Here, we introduce the following:

$$\Phi(tu_t)=\frac{t^2}{2}\|\nabla u\|_2^2-\frac{t^{\mu+2\alpha-4}}{2}\int_{\mathbb{R}^2}\left(I_\mu\times\frac{F(tu)}{|x|^\alpha}\right)\frac{F(tu)}{|x|^\alpha}\mathrm{d}x, \tag{2.1}$$

and

$$\begin{aligned} J(u)=\frac{\mathrm{d}}{\mathrm{d}t}\Phi(tu_t)\Big|_{t=1} &= \|\nabla u\|_2^2+\frac{4-\mu-2\alpha}{2}\int_{\mathbb{R}^2}\left(I_\mu\times\frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x \\ &\quad -\int_{\mathbb{R}^2}\left(I_\mu\times\frac{F(u)}{|x|^\alpha}\right)\frac{f(u)u}{|x|^\alpha}\mathrm{d}x. \end{aligned} \tag{2.2}$$

Following that, we show that Φ has a mountain pass geometry on the constraint $\mathcal{S}_a$, which is as follows.

Lemma 2.3 *Assume that* (F1)–(F3) *and* (F5) *hold. Then*

(i) *there exists* $K(a)>0$ *small enough such that* $\Phi(u)>0$ *and* $J(u)>0$ *if* $u\in A_{2K}$ *and*

$$0<\sup_{u\in A_K}\Phi(u)<\inf\left\{\Phi(u): u\in\mathcal{S}_a, \|\nabla u\|_2^2=2K(a)\right\}, \tag{2.3}$$

where $A_K=\left\{u\in\mathcal{S}_a:\|\nabla u\|_2^2\le K(a)\right\}$ *and* $A_{2K}=\left\{u\in\mathcal{S}_a:\|\nabla u\|_2^2\le 2K(a)\right\}$.

(ii) $\Gamma_a=\{g\in\mathcal{C}([0,1],\mathcal{S}_a):\|\nabla g(0)\|_2^2\le K(a), \Phi(g(1))<0\}\neq\emptyset$ *and*

$$\begin{aligned} c(a)=\inf_{g\in\Gamma_a}\max_{t\in[0,1]}\Phi(g(t)) &\ge \inf\left\{\Phi(u): u\in\mathcal{S}_a, \|\nabla u\|_2^2=2K(a)\right\} \\ &> \max_{g\in\Gamma_a}\max\{\Phi(g(0)),\Phi(g(1))\}. \end{aligned}$$

Proof **(i)** Fixing $\alpha>\alpha_0$, by (F1) and (F2), we know that for any $\varepsilon>0$ and any $q\ge 1$, there exists $C_{\alpha,\varepsilon,q}>0$ such that

$$|f(t)|\le\varepsilon|t|^{\frac{4-\mu-2\alpha}{2}}+C_{\alpha,\varepsilon,q}|t|^q\left(e^{\alpha t^2}-1\right), \quad \forall\, t\in\mathbb{R}, \tag{2.4}$$

moreover, using (2.4), we deduce that for any $\varepsilon > 0$, there exists $C_{\alpha,\varepsilon} > 0$ such that

$$|F(t)| \leq \varepsilon |t|^{\frac{6-\mu-2\alpha}{2}} + C_{\alpha,\varepsilon} |t|^{\frac{10-\mu-2\alpha}{4}} \left(e^{\alpha t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{2.5}$$

Thus, by Proposition 1.2, Lemma 2.1, and (2.5) and by selecting ε small enough, we have

$$\begin{aligned}
\Phi(v) &\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{C(2,\mu,\alpha)}{2}\|F(v)\|^2_{\frac{4}{4-\mu-2\alpha}} \\
&\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{C(2,\mu,\alpha)}{2}\Bigg(\int_{\mathbb{R}^2} C_1 \varepsilon^{\frac{4}{4-\mu-2\alpha}} |v|^{\frac{2(6-\mu-2\alpha)}{4-\mu-2\alpha}} \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} C_1 C_{\alpha,\varepsilon}^{\frac{4}{4-\mu-2\alpha}} \left(e^{\alpha v^2} - 1\right)^{\frac{4}{4-\mu-2\alpha}} |v|^{\frac{10-\mu-2\alpha}{4-\mu-2\alpha}} \mathrm{d}x\Bigg)^{\frac{4-\mu-2\alpha}{2}} \\
&\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{C(2,\mu,\alpha)}{2} C_2 C_1^{\frac{4-\mu-2\alpha}{2}} \varepsilon^2 \|v\|^{6-\mu-2\alpha}_{\frac{2(6-\mu-2\alpha)}{4-\mu-2\alpha}} \\
&\quad - \frac{C(2,\mu,\alpha)}{2} C_3 C_1^{\frac{4-\mu-2\alpha}{2}} C^2_{\alpha,\varepsilon} \\
&\quad \left(\int_{\mathbb{R}^2}\left[e^{\frac{8\alpha v^2}{4-\mu-2\alpha}} - 1\right]\mathrm{d}x\right)^{\frac{4-\mu-2\alpha}{4}} \left(\int_{\mathbb{R}^2} |v|^{\frac{2(10-\mu-2\alpha)}{4-\mu-2\alpha}}\right)^{\frac{4-\mu-2\alpha}{4}} \\
&\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{C(2,\mu,\alpha)}{2} C_2 C_1^{\frac{4-\mu-2\alpha}{2}} \varepsilon^2 \mathcal{S}^{-\frac{4-\mu-2\alpha}{2}}_{\frac{2(6-\mu-2\alpha)}{4-\mu-2\alpha}} a^{4-\mu-2\alpha} \|\nabla v\|_2^2 \\
&\quad - \frac{C(2,\mu,\alpha)}{2} C_3 C_1^{\frac{4-\mu-2\alpha}{2}} C^2_{\alpha,\varepsilon} \left(\int_{\mathbb{R}^2}\left[e^{\frac{8\alpha v^2}{4-\mu-2\alpha}} - 1\right]\mathrm{d}x\right)^{\frac{4-\mu-2\alpha}{4}} \\
&\quad \mathcal{S}^{-\frac{4-\mu-2\alpha}{4}}_{\frac{2(10-\mu-2\alpha)}{4-\mu-2\alpha}} a^{\frac{4-\mu-2\alpha}{2}} \|\nabla v\|_2^3 \\
&= \frac{1}{2}\|\nabla v\|_2^2 - C_4 a^{4-\mu-2\alpha} \varepsilon^2 \|\nabla v\|_2^2 - C_5 a^{\frac{4-\mu-2\alpha}{2}} \|\nabla v\|_2^3 \\
&\geq \frac{1}{4}\|\nabla v\|_2^2 - C_5 a^{\frac{4-\mu-2\alpha}{2}} \|\nabla v\|_2^3.
\end{aligned}$$

Now, let $0 < K$ be arbitrary but fixed and suppose $u, u_0, v, v_0 \in \mathcal{S}_a$ satisfy that $\|\nabla u\|_2^2 \leq K$, $\|\nabla v\|_2^2 \leq 2K$, and $\|\nabla v_0\|_2^2 = 2K$. From above, when $K > 0$ is small enough, we can say that $\Phi(v) > 0$. Similarly, we can obtain that $J(v) > 0$ and again by selecting ε, K small enough, we have

$$\begin{aligned}
\Phi(v_0) - \Phi(u) &= \frac{1}{2}\|\nabla v_0\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(v_0)}{|x|^\alpha}\right)\frac{F(v_0)}{|x|^\alpha}\mathrm{d}x - \frac{1}{2}\|\nabla u\|_2^2 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x \\
&\geq \frac{1}{2}K - C_6 a^{4-\mu-2\alpha}\varepsilon^2 K - C_7 a^{\frac{4-\mu-2\alpha}{2}} (2K)^{3/2}
\end{aligned}$$

$$= \frac{1}{4}K - C_8 a^{\frac{4-\mu-2\alpha}{2}}(2K)^{3/2} \geq \frac{1}{8}K.$$

From above, there exists $K = K(a) > 0$ sufficiently small such that $\Phi(u) > 0$ and $J(u) > 0$ if $u \in A_{2k}$, and (2.3) holds.

(ii) We first prove that $\Gamma_a \neq \emptyset$. Using (F3) and (F5), it is easy to see that

$$F(t) \geq F(\beta_0 sign(t)) e^{(|t|-\beta_0)/M_0}, \quad \forall\, |t| \geq \beta_0. \tag{2.6}$$

For any given $w \in \mathcal{S}_a$, we have $\|tw_t\|_2 = \|w\|_2$, and so $tw_t \in \mathcal{S}_a$ for every $t > 0$. Then (2.1) and (2.6) yield

$$\Phi(tw_t) \to -\infty \text{ as } t \to +\infty. \tag{2.7}$$

Thus we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\|\nabla(t_1 w_{t_1})\|_2^2 = t_1^2\|\nabla w\|_2^2 \leq K(a), \quad \|\nabla(t_2 w_{t_2})\|_2^2 = t_2^2\|\nabla w\|_2^2 > 2K(a) \text{ and } \Phi(t_2 w_{t_2}) < 0.$$

Let $g_0(t) := (t_1 + (t_2 - t_1)t)w_{t_1+(t_2-t_1)t}$. Then $g_0 \in \Gamma_a$, and so $\Gamma_a \neq \emptyset$. Now using the intermediate value theorem, for any $g \in \Gamma_a$, there exists $t_0 \in (0, 1)$, depending on g, such that $\|\nabla g(t_0)\|_2^2 = 2K(a)$ and

$$\max_{t\in[0,1]} \Phi(g(t)) \geq \Phi(g(t_0)) \geq \inf\left\{\Phi(u) : u \in \mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\},$$

which, together with the arbitrariness of $g \in \Gamma_a$, implies

$$c(a) = \inf_{g\in\Gamma_a} \max_{t\in[0,1]} \Phi(g(t)) \geq \inf\left\{\Phi(u) : u \in \mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\}. \tag{2.8}$$

Hence, (ii) follows directly from (2.3) and (2.8), and the proof is completed. □

We recall that any solution of $(\mathcal{P}_a)$ lives in the L^2-Pohozaev manifold given by

$$\mathcal{M}_a = \left\{u \in \mathcal{S}_a : J(u) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(tu_t)\Big|_{t=1} = 0\right\}.$$

Remark 2.4 From $J(v) > 0$ when $v \in \mathcal{S}_c$ and $\|\nabla v\|_2^2 \leq 2K$, we can deduce that for any $a > 0$, there exists a constant $\rho(a) > 0$, just depending on $a > 0$, such that $\|\nabla u\|_2 \geq \rho(a)$ for all $u \in \mathcal{M}_a$.

Next, inspired by [21], we consider the following auxiliary functional:

$$\tilde{\Phi}(v, t) = \Phi(\beta(v, t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 - \frac{e^{(\mu+2\alpha-4)t}}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha}\right)\frac{F(e^t v)}{|x|^\alpha}\mathrm{d}x. \tag{2.9}$$

We shall show that $\tilde{\Phi}$ also possesses a kind of mountain pass geometrical structure on $\mathcal{S}_a \times \mathbb{R}$. Since the proof is standard, we omit it here.

Lemma 2.5 *Assume that* (F1)–(F3) *and* (F5) *hold. Let* $v \in \mathcal{S}_a$ *be arbitrary but fixed. Then we have*

(i) $\|\nabla \beta(v,t)\|_2 \to 0$ *and* $\Phi(\beta(v,t)) \to 0$ *as* $t \to -\infty$;
(ii) $\|\nabla \beta(v,t)\|_2 \to +\infty$ *and* $\Phi(\beta(v,t)) \to -\infty$ *as* $t \to +\infty$;
(iii) *there exist* $s_1 < 0$ *and* $s_2 > 0$, *depending on* a *and* v, *such that the functions* $\tilde{v}_1 = \beta(v, s_1)$ *and* $\tilde{v}_2 = \beta(v, s_2)$ *satisfy*

$$\|\nabla \tilde{v}_1\|_2^2 \le K(a), \quad \|\nabla \tilde{v}_2\|_2^2 > 2K(a) \ \text{ and } \ \Phi(\tilde{v}_2) < 0.$$

Lemma 2.6 *Assume that* (F1)–(F3) *and* (F5) *hold. Then*

$$c(a) = \tilde{c}(a) := \inf_{\tilde{g} \in \tilde{\Gamma}_a} \max_{\tau \in [0,1]} \tilde{\Phi}(\tilde{g}(\tau)) > \max_{\tilde{g} \in \tilde{\Gamma}_a} \max \left\{ \tilde{\Phi}(\tilde{g}(0)), \tilde{\Phi}(\tilde{g}(1)) \right\},$$

where

$$\tilde{\Gamma}_a := \{\tilde{g} \in \mathcal{C}([0,1], \mathcal{S}_a \times \mathbb{R}) : \tilde{g}(0) = (\tilde{g}_1(0), 0), \|\nabla \tilde{g}_1(0)\|_2^2 \le K(a), \tilde{\Phi}(\tilde{g}(1)) < 0\}.$$

By the argument explored as in [35], we know that for any $a > 0$, $\mathcal{S}_a$ is a submanifold of $H^1(\mathbb{R}^2)$ with codimension 1 and the tangent space at $\mathcal{S}_a$ is defined as

$$T_u = \left\{ v \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} uv \mathrm{d}x = 0 \right\}.$$

The norm of the $\mathcal{C}^1$ restriction functional $\Phi|'_{\mathcal{S}_a}(u)$ is defined by

$$\|\Phi|'_{\mathcal{S}_a}(u)\| = \sup_{v \in T_u, \|v\|=1} \left\langle \Phi'(u), v \right\rangle.$$

As in Jeanjean [21], for every $(u,t) \in \mathcal{S}_a \times \mathbb{R}$, we define the following linear space

$$\tilde{T}_{u,t} = \left\{ (v,s) \in H : \int_{\mathbb{R}^2} uv \mathrm{d}x = 0 \right\}.$$

We see that $\tilde{\Phi}(v,t)$ is of class $\mathcal{C}^1$ and for any $(w,s) \in H$,

$$\begin{aligned} \left\langle \tilde{\Phi}'(v,t), (w,s) \right\rangle &= e^{2t} \int_{\mathbb{R}^2} \nabla v \cdot \nabla w \mathrm{d}x + e^{2t} s \|\nabla v\|_2^2 \\ &\quad - e^{(\mu+2\alpha-4)t} \int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha} \right) \frac{f(e^t v) e^t w}{|x|^\alpha} \mathrm{d}x \\ &\quad + \frac{(4-\mu-2\alpha)s}{2e^{(4-\mu-2\alpha)t}} \int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha} \right) \frac{F(e^t v)}{|x|^\alpha} \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&- \frac{s}{e^{(4-\mu-2\alpha)t}} \int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha} \right) \frac{f(e^t v) e^t v}{|x|^\alpha} \mathrm{d}x \\
&= \left\langle \Phi'(\beta(v,t)), \beta(w,t) \right\rangle + sJ(\beta(v,t)).
\end{aligned}
$$

The norm of the derivative of the $\mathcal{C}^1$ restriction functional $\tilde{\Phi}|_{\mathcal{S}_a \times \mathbb{R}}$ is defined by

$$
\|\tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(u,t)\| = \sup_{(v,s) \in \tilde{T}_{u,t}, \|(v,s)\|_H = 1} \left\langle \tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(u,t), (v,s) \right\rangle.
$$

In the same way as [21, Proposition 2.2], we have the following proposition.

Proposition 2.7 *Assume that $\tilde{\Phi}$ has a mountain pass geometry on the constraint $\mathcal{S}_a \times \mathbb{R}$. Let $a > 0$ and $\{\tilde{g}_n\} \subset \tilde{\Gamma}_a$ be such that*

$$
\max_{\tau \in [0,1]} \tilde{\Phi}(\tilde{g}_n(\tau)) \le \tilde{c}(a) + \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.
$$

Then there exists a sequence $\{(v_n, t_n)\} \subset \mathcal{S}_a \times \mathbb{R}$ such that

1. $\tilde{\Phi}(v_n, t_n) \in \left[\tilde{c}(a) - \frac{1}{n}, \tilde{c}(a) + \frac{1}{n}\right]$;
2. $\min_{\tau \in [0,1]} \|(v_n, t_n) - \tilde{g}_n(\tau)\|_H \le \frac{1}{\sqrt{n}}$;
3. $\|\tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(v_n, t_n)\| \le \frac{2}{\sqrt{n}}$, *i.e.,*

$$
|\langle \tilde{\Phi}'(v_n, t_n), (v,s) \rangle| \le \frac{2}{\sqrt{n}} \|(v,s)\|_H, \quad \forall\, (v,s) \in \tilde{T}_{v_n, t_n}.
$$

Note that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\Phi}(v,t) &= \left\langle \tilde{\Phi}'(v,t), (0,1) \right\rangle \\
&= e^{2t} \|\nabla v\|_2^2 + \frac{(4-\mu-2\alpha)}{2e^{(4-\mu-2\alpha)t}} \int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha} \right) \frac{F(e^t v)}{|x|^\alpha} \mathrm{d}x \\
&\quad - \frac{1}{e^{(4-\mu-2\alpha)t}} \int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(e^t v)}{|x|^\alpha} \right) \frac{f(e^t v) e^t v}{|x|^\alpha} \mathrm{d}x \\
&= J(\beta(v,t)), \quad \forall\, (v,t) \in H.
\end{aligned} \tag{2.10}
$$

With the aforementioned lemmas, we can get the desired sequence as follows.

Lemma 2.8 *Assume that* (F1)–(F3) *and* (F5) *hold. Then there exists a bounded sequence $\{u_n\} \subset \mathcal{S}_a$ such that*

$$
\Phi(u_n) \to c(a) > 0, \quad \Phi|'_{\mathcal{S}_a}(u_n) \to 0 \text{ and } J(u_n) \to 0. \tag{2.11}
$$

Proof Let

$$
u_n = \beta(v_n, t_n) \text{ and } g_n(\tau) = \beta(\tilde{g}_n(\tau)) \text{ for } \tau \in [0,1], \tag{2.12}
$$

where β is defined by (1.6), v_n, t_n, and $\tilde{g}_n$ are given in Proposition 2.7. Then $u_n \in \mathcal{S}_a$ and $g_n \in \Gamma_a$ by (ii) of Lemma 2.3. Moreover, by (2.9), (2.10), Lemma 2.6, and Proposition 2.7, we have

$$\Phi(u_n) = \tilde{\Phi}(v_n, t_n) \in \left[c(a) - \frac{1}{n}, c(a) + \frac{1}{n}\right], \tag{2.13}$$

and

$$J(u_n) = \left\langle \tilde{\Phi}'(v_n, t_n), (0, 1)\right\rangle \to 0. \tag{2.14}$$

By (F3), we have

$$\begin{aligned} c(a) + o(1) &= \Phi(u_n) - \frac{1}{2}J(u_n) \\ &= \frac{-6 + \mu + 2\alpha}{4}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{f(u)u}{|x|^\alpha}\mathrm{d}x \\ &\geq \frac{6 - \mu - 2\alpha}{4}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x, \end{aligned}$$

which implies that $\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{F(u)}{|x|^\alpha}\mathrm{d}x$ is bounded. Using (F3) and (F4), we know that for any $\delta > 0$ there exists $R_\delta > 0$ such that

$$f(t)t \geq \delta F(t) > 0, \quad \forall\, |t| \geq R_\delta.$$

Then we have

$$\begin{aligned} & c(a) + o(1) \\ &\quad = \Phi(u_n) - \frac{1}{4}J(u_n) \\ &\quad = \frac{1}{4}\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{|u_n|<R_4}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{\left[f(u)u - \frac{8-\mu-2\alpha}{2}F(u)\right]}{|x|^\alpha}\mathrm{d}x \\ &\qquad + \frac{1}{4}\int_{|u_n|\geq R_4}\left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right)\frac{\left[f(u)u - \frac{8-\mu-2\alpha}{2}F(u)\right]}{|x|^\alpha}\mathrm{d}x \\ &\quad \geq \frac{1}{4}\|\nabla u_n\|_2^2 - \frac{1}{4}\left[\int_{\mathbb{R}^2}\left(I_\mu \times \frac{\left[\frac{8-\mu-2\alpha}{2}F(u) - f(u)u\right]\chi_{|u_n|<R_4}}{|x|^\alpha}\right)\right. \\ &\qquad \left.\frac{\left[\frac{8-\mu-2\alpha}{2}F(u) - f(u)u\right]\chi_{|u_n|<R_4}}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\times \left[\int_{\mathbb{R}^2} \left(I_\mu \times \frac{F(u)}{|x|^\alpha}\right) \frac{F(u)}{|x|^\alpha} \mathrm{d}x\right]^{\frac{1}{2}} \\
&\geq \frac{1}{4}\|\nabla u_n\|_2^2 - \frac{C}{4}\left[\int_{\mathbb{R}^2} \left(I_\mu \times \frac{\left[C|u_n|^{\frac{4-\mu-2\alpha}{2}}\right]}{|x|^\alpha}\right) \frac{\left[C|u_n|^{\frac{4-\mu-2\alpha}{2}}\right]}{|x|^\alpha} \mathrm{d}x\right]^{\frac{1}{2}} \\
&\geq \frac{1}{4}\|\nabla u_n\|_2^2 - C\|u_n\|_2^{(4-\mu-2\alpha)/2} \quad \text{by taking } \delta = 4,
\end{aligned}
$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. To finish the proof, it remains to prove that $\Phi|'_{\mathcal{S}_a}(u_n) \to 0$, i.e., $\langle \Phi'(u_n), w\rangle \to 0$ for all $w \in T_{u_n}$. For this, we just need to show that $\{(\beta(w, -t_n), 0)\} \subset T_{v_n,t_n}$ and $\{(\beta(w, -t_n), 0)\}$ is bounded in H since

$$
\langle \Phi'(u_n), w\rangle = \left\langle \tilde{\Phi}'(v_n, t_n), (\beta(w, -t_n), 0)\right\rangle \leq \frac{2}{\sqrt{n}}\|(\beta(w, -t_n), 0)\|_H, \quad \forall\, w \in T_{u_n}.
$$

Indeed, for any $w \in T_{u_n}$, i.e.,

$$
\int_{\mathbb{R}^2} u_n w \mathrm{d}x = \int_{\mathbb{R}^2} e^{t_n} v_n(e^{t_n}x) w(x) \mathrm{d}x = 0,
$$

we have

$$
\int_{\mathbb{R}^2} v_n(x)\beta(w, -t_n)(x)\mathrm{d}x = \int_{\mathbb{R}^2} v_n(x) e^{-t_n} w(e^{-t_n}x)\mathrm{d}x = \int_{\mathbb{R}^2} e^{t_n} v_n(e^{t_n}x) w(x)\mathrm{d}x = 0,
$$

which implies

$$
(\beta(w, -t_n), 0) \in T_{v_n,t_n}. \tag{2.15}
$$

Moreover, by (ii) of Proposition 2.7, we have

$$
|t_n| \leq \min_{\tau\in[0,1]} \|(v_n, t_n) - \tilde{g}_n(\tau)\|_H \leq 1 \text{ for large } n \in \mathbb{N},
$$

which leads to

$$
\begin{aligned}
\|(\beta(w, -t_n), 0)\|_H^2 &= \|\beta(w, -t_n)\|^2 \\
&= e^{-2t_n}\|\nabla w\|_2^2 + \|w\|_2^2 \leq e^2\|w\|^2 \text{ for large } n \in \mathbb{N}.
\end{aligned}
$$

This shows that $\{(\beta(w, -t_n), 0)\} \subset T_{v_n,t_n}$ is bounded in H. Jointly with (2.15), we get $\Phi|'_{\mathcal{S}_a}(u_n) \to 0$. From this, (2.13) and (2.14), we conclude that $\{u_n\}$, defined by (2.12), is bounded, and satisfies (2.11). The proof is completed. □

3 Energy Estimates for Minimax Level

In this subsection, we give a precise estimation for the energy level $c(a)$ given by (2.8), which helps us to restore the compactness in the critical exponential case in next subsection.

Let $\kappa := \liminf_{|t|\to\infty} \frac{f(t)}{e^{\alpha_0 t^2}}$. By (F4), we know that $\kappa > 0$. Then we can choose $d > 0$ such that

$$\kappa > \frac{(2-\mu)(3-\mu)(4-\mu)[(4-2\alpha-\mu)(1+\varepsilon)\pi]^{\frac{6-2\alpha-\mu}{2}}}{2e\pi^2 d^{4-2\alpha-\mu}\alpha_0^{\frac{4-2\alpha-\mu}{2}}}. \tag{3.1}$$

For large $n \in \mathcal{N}$, let $R_n \geq d$ be such that

$$a = \frac{d^2}{16\log n}\left(1 + 2\log 2 + 2\log^2 2 - \frac{4}{n^2} - \frac{8}{n^2}\log n\right) + \frac{\log^2 2}{48(2R_n - d)\log n}\left(8R_n^3 + 4R_n^2 d - 10R_n d^2 + 3d^3\right).$$

Then one has

$$\lim_{n\to\infty} \frac{R_n^2}{\log n} = \frac{12a}{\log^2 2}.$$

Now we define the following new Moser-type functions $w_n(x)$ supported in $B_d := B_d(0)$

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq d/n; \\ \frac{\log(d/|x|)}{\sqrt{\log n}}, & d/n \leq |x| \leq d/2; \\ \frac{2(R_n - |x|)\log 2}{(2R_n - d)\sqrt{\log n}}, & d/2 \leq |x| \leq R_n; \\ 0, & |x| \geq R_n. \end{cases} \tag{3.2}$$

Computing directly, we get that for large $n \in \mathcal{N}$,

$$\|\nabla w_n\|_2^2 = \int_{\mathbb{R}^2} |\nabla w_n|^2 \mathrm{d}x = 1 - \frac{\log 2}{\log n} + \frac{(2R_n + d)\log^2 2}{2(2R_n - d)\log n} \leq 1, \tag{3.3}$$

and

$$\|w_n\|_2^2 = \int_{\mathbb{R}^2} |w_n|^2 \mathrm{d}x = \int_0^{d/n} (\log n) r\,\mathrm{d}r + \int_{d/n}^{d/2} \frac{\log^2(d/r)}{\log n} r\,\mathrm{d}r + \int_{d/2}^{R_n} \frac{4(R_n - r)^2 \log^2 2}{(2R_n - d)^2 \log n} r\,\mathrm{d}r$$

$$
\begin{aligned}
&= \frac{d^2}{16\log n}\left(1 + 2\log 2 + 2\log^2 2 - \frac{4}{n^2} - \frac{8}{n^2}\log n\right) \\
&\quad + \frac{\log^2 2}{48(2R_n - d)\log n}\left(8R_n^3 + 4R_n^2 d - 10R_n d^2 + 3d^3\right) \\
&= a.
\end{aligned}
$$

We also give the observation on the estimation of convolution term.

$$
\begin{aligned}
\int_{B_{\rho/n}}\int_{B_{\rho/n}} \frac{1}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}x\mathrm{d}y &\geq \left(\frac{\rho}{n}\right)^{-2\alpha} \int_{B_{\rho/n}}\int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&\geq \frac{4\pi^2}{(2-\mu)(3-\mu)(4-\mu)}\left(\frac{\rho}{n}\right)^{4-2\alpha-\mu}.
\end{aligned}
$$

Lemma 3.1 *Assume that* (F1)–(F4) *hold. Then there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$
\sup_{t>0} \Phi(t(w_{\bar{n}})_t) < \frac{(4-2\alpha-\mu)\pi}{2\alpha_0}. \tag{3.4}
$$

Proof By (F4), we may choose $\varepsilon > 0$ small and $t_\varepsilon > 0$ such that

$$
f(t) \geq (\kappa - \varepsilon)e^{\alpha_0 t^2}, \quad tF(t) \geq \frac{\kappa - \varepsilon}{2\alpha_0} e^{\alpha_0 t^2}, \quad \forall\, |t| \geq t_\varepsilon. \tag{3.5}
$$

Using (3.3), we have

$$
\begin{aligned}
\Phi(t(w_n)_t) &= \frac{t^2}{2}\|\nabla w_n\|_2^2 - \frac{1}{2t^{4-2\alpha-\mu}} \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \frac{F(tw_n(y))}{|x-y|^\mu |y|^\alpha}\right)\frac{F(tw_n(x))}{|x|^\alpha}\mathrm{d}x \\
&\leq \frac{t^2}{2} - \frac{1}{2t^{4-2\alpha-\mu}} \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \frac{F(tw_n(y))}{|x-y|^\mu |y|^\alpha}\right)\frac{F(tw_n(x))}{|x|^\alpha}\mathrm{d}x, \\
&\quad \forall\, t > 0, \text{ for large } n \in \mathbb{N}.
\end{aligned}
$$

There are three cases to distinguish. Without mentioning, all inequalities hold for large $n \in \mathbb{N}$ in the rest of the Lemma.

Case i $t \in \left[0, \sqrt{(4-2\alpha-\mu)\pi/2\alpha_0}\right]$. Then by (F3), we have

$$
\begin{aligned}
\Phi(t(w_n)_t) &\leq \frac{t^2}{2} - \frac{1}{2t^{4-2\alpha-\mu}} \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \frac{F(tw_n(y))}{|x-y|^\mu |y|^\alpha}\right)\frac{F(tw_n(x))}{|x|^\alpha}\mathrm{d}x \\
&\leq \frac{t^2}{2} \leq \frac{(4-2\alpha-\mu)\pi}{4\alpha_0},
\end{aligned}
$$

which yields the existence of $\bar{n} \in \mathbb{N}$ satisfying (3.4).

Case ii $t \in \left[\sqrt{(4-2\beta-\mu)\pi/2\alpha_0}, \sqrt{(4-2\alpha-\mu)(1+\varepsilon)\pi/\alpha_0}\right]$. In this case, $tw_n(x) \geq t_\varepsilon$ for $x \in B_{d/n}$ and $n \in \mathbb{N}$ large. Then it follows from (3.2) and (3.5) that

$$\begin{aligned}
&\frac{1}{2t^{4-2\alpha-\mu}}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(tw_n(y))}{|x-y|^{\mu}|y|^{\alpha}}\right)\frac{F(tw_n(x))}{|x|^{\alpha}}\mathrm{d}x\\
&\geq \frac{1}{2t^{4-2\alpha-\mu}}\int_{B_{d/n}}\left(\int_{B_{d/n}}\frac{F(tw_n(y))}{|x-y|^{\mu}|y|^{\alpha}}\right)\frac{F(tw_n(x))}{|x|^{\alpha}}\mathrm{d}x\\
&\geq \frac{\pi^3 d^{4-2\alpha-\mu}(\kappa-\varepsilon)^2 e^{\alpha_0\pi^{-1}t^2\log n}}{(2-\mu)(3-\mu)(4-\mu)n^{4-2\alpha-\mu}\alpha_0^2\log n t^{6-2\alpha-\mu}},
\end{aligned}$$

then

$$\begin{aligned}
\Phi(t(w_n)_t) &\leq \frac{t^2}{2}-\frac{1}{2t^{4-2\alpha-\mu}}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(tw_n(y))}{|x-y|^{\mu}|y|^{\alpha}}\mathrm{d}y\right)\frac{F(tw_n(x))}{|x|^{\alpha}}\mathrm{d}x\\
&\leq \frac{t^2}{2}-\frac{\pi^3 d^{4-2\alpha-\mu}(\kappa-\varepsilon)^2 e^{\alpha_0\pi^{-1}t^2\log n}}{(2-\mu)(3-\mu)(4-\mu)n^{4-2\alpha-\mu}\alpha_0^2\log n t^{6-2\alpha-\mu}}\\
&\leq \frac{t^2}{2}-\frac{\pi^3 d^{4-2\alpha-\mu}\alpha_0^{\frac{2-2\alpha-\mu}{2}}}{(2-\mu)(3-\mu)(4-\mu)[(4-2\alpha-\mu)(1+\varepsilon)\pi]^{\frac{6-2\alpha-\mu}{2}}}\\
&\quad\frac{(\kappa-\varepsilon)^2 e^{\alpha_0\pi^{-1}t^2\log n}}{n^{4-2\alpha-\mu}\log n}\\
&:=\varphi_n(t).
\end{aligned}$$

Choosing $t_n>0$ be such that $\varphi_n'(t_n)=0$, then we have

$$1=\frac{2\pi^2 d^{4-2\alpha-\mu}\alpha_0^{\frac{4-2\alpha-\mu}{2}}}{(2-\mu)(3-\mu)(4-\mu)[(4-2\alpha-\mu)(1+\varepsilon)\pi]^{\frac{6-2\alpha-\mu}{2}}}\frac{(\kappa-\varepsilon)^2 e^{\alpha_0\pi^{-1}t_n^2\log n}}{n^{4-2\alpha-\mu}}.$$

Let

$$\begin{aligned}
B_1 &= 2\pi^2 d^{4-2\alpha-\mu}\alpha_0^{\frac{4-2\alpha-\mu}{2}},\\
B_2 &= (2-\mu)(3-\mu)(4-\mu)[(4-2\alpha-\mu)(1+\varepsilon)\pi]^{\frac{6-2\alpha-\mu}{2}},
\end{aligned}$$

then it follows that

$$\begin{aligned}
t_n^2 &= \frac{(4-2\alpha-\mu)\pi}{\alpha_0}\left[1+\frac{\log B_2-\log[B_1(\kappa-\varepsilon)^2]}{(4-2\alpha-\mu)\log n}\right]\\
&= \frac{(4-2\alpha-\mu)\pi}{\alpha_0}-\frac{\pi}{\alpha_0\log n}\log\frac{[B_1(\kappa-\varepsilon)^2]}{B_2},
\end{aligned}\tag{3.6}$$

and

$$\varphi_n(t) \le \varphi_n(t_n) = \frac{t_n^2}{2} - \frac{\pi}{2\alpha_0 \log n}, \quad \forall\, t \ge 0. \tag{3.7}$$

Using (3.6) and (3.7), we are led to

$$\begin{aligned}
\varphi_n(t) &\le \frac{t_n^2}{2} - \frac{\pi}{2\alpha_0 \log n} \\
&= \frac{(4-2\alpha-\mu)\pi}{2\alpha_0} - \frac{\pi}{2\alpha_0 \log n} \log \frac{B_1(\kappa-\varepsilon)^2}{B_2} - \frac{\pi}{2\alpha_0 \log n} \\
&= \frac{(4-2\alpha-\mu)\pi}{2\alpha_0} - \frac{\pi}{2\alpha_0 \log n}\left[1 + \log \frac{B_1(\kappa-\varepsilon)^2}{B_2}\right],
\end{aligned}$$

where from (3.1), we know that

$$1 + \log \frac{2\pi^2 d^{4-2\alpha-\mu} \alpha_0^{\frac{4-2\alpha-\mu}{2}} (\kappa-\varepsilon)^2}{(2-\mu)(3-\mu)(4-\mu)[(4-2\alpha-\mu)(1+\varepsilon)\pi]^{\frac{6-2\alpha-\mu}{2}}} > 0,$$

thus we have

$$\Phi(t(w_n)_t) \le \frac{(4-2\alpha-\mu)\pi}{2\alpha_0} - \frac{\pi}{2\alpha_0 \log n}\left[1 + \log \frac{B_1(\kappa-\varepsilon)^2}{B_2}\right] < \frac{(4-2\alpha-\mu)\pi}{2\alpha_0}.$$

Then we deduce that (3.4) holds for some $\bar{n} \in \mathbb{N}$.

Case iv $t \in (\sqrt{(4-2\beta-\mu)(1+\varepsilon)\pi/\alpha_0}, \infty)$. Since $tw_n(x) \ge t_\varepsilon$ for $x \in B_{d/n}$ and $\bar{n} \in \mathbb{N}$ large, we deduce from (3.2) that

$$\begin{aligned}
\Phi(t(w_n)_t) &\le \frac{t^2}{2} - \frac{1}{2t^{4-2\alpha-\mu}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{F(tw_n(y))}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \frac{F(tw_n(x))}{|x|^\alpha} \mathrm{d}x \\
&\le \frac{t^2}{2} - \frac{\pi^3 d^{4-2\alpha-\mu}}{(2-\mu)(3-\mu)(4-\mu)} \frac{(\kappa-\varepsilon)^2 e^{\alpha_0 \pi^{-1} t^2 \log n}}{n^{4-2\alpha-\mu} \alpha_0^2 \log n t^{6-2\alpha-\mu}} \\
&\le \frac{(4-2\beta-\mu)(1+\varepsilon)\pi}{2\alpha_0} \\
&\quad - \frac{\pi^3 d^{4-2\alpha-\mu}}{(2-\mu)(3-\mu)(4-\mu)} \frac{(\kappa-\varepsilon)^2 e^{(4-2\beta-\mu)\varepsilon \log n}}{\alpha_0^2 \log n [(4-2\beta-\mu)(1+\varepsilon)\pi/\alpha_0]^{\frac{6-2\alpha-\mu}{2}}} \\
&\le \frac{(4-2\beta-\mu)(1+\varepsilon)\pi}{3\alpha_0},
\end{aligned}$$

where we have used the fact that the function

$$\varphi_n(t) := \frac{t^2}{2} - \frac{\pi^3 d^{4-2\alpha-\mu}}{(2-\mu)(3-\mu)(4-\mu)} \frac{(\kappa-\varepsilon)^2 e^{\alpha_0 \pi^{-1} t^2 \log n}}{n^{4-2\alpha-\mu} \alpha_0^2 \log n t^{6-2\alpha-\mu}},$$

is decreasing on $t \in \left(\sqrt{\frac{(4-2\alpha-\mu)\pi}{\alpha_0}}(1+\varepsilon), +\infty\right)$ for large n. In fact,

$$\varphi_n'(t) = t - \frac{\pi^3 d^{4-2\alpha-\mu}}{(2-\mu)(3-\mu)(4-\mu)} \frac{(\kappa-\varepsilon)^2}{n^{4-2\alpha-\mu}\alpha_0^2 \log n} \cdot \frac{e^{\alpha_0 \pi^{-1} t^2 \log n}}{t^{7-2\alpha-\mu}} \left(\frac{2\alpha_0 \log n t^2}{\pi} - (6-2\alpha-\mu)\right).$$

Assume that $s_n \geq \sqrt{\frac{(4-2\alpha-\mu)\pi}{\alpha_0}}(1+\varepsilon)$ such that $\varphi_n'(s_n) = 0$ for large n. Then

$$s_n^{8-2\alpha-\mu} = \frac{\pi^2 d^{4-2\alpha-\mu}}{(2-\mu)(3-\mu)(4-\mu)} \frac{(\kappa-\varepsilon)^2}{n^{4-2\alpha-\mu}\alpha_0^2} \cdot \left(2\alpha_0 s_n^2 - \frac{(6-2\alpha-\mu)\pi}{\log n}\right) e^{\alpha_0 \pi^{-1} s_n^2 \log n},$$

which yields

$$\begin{aligned} s_n^2 = &\frac{(4-2\alpha-\mu)\pi}{\alpha_0} \Bigg[1+ \\ &\frac{\log[(2-\mu)(3-\mu)(4-\mu)\alpha_0^2 s_n^{8-2\alpha-\mu}] - \log\left(\pi^2 d^{4-2\alpha-\mu}(\kappa-\varepsilon)^2 \left(2\alpha_0 s_n^2 - \frac{(6-2\alpha-\mu)\pi}{\log n}\right)\right)}{(4-2\alpha-\mu)\log n}\Bigg] \\ = &\frac{(4-2\alpha-\mu)\pi}{\alpha_0} + \frac{\pi}{\alpha_0 \log n} \log \frac{[(2-\mu)(3-\mu)(4-\mu)\alpha_0^2 s_n^{8-2\alpha-\mu}]}{\pi^2 d^{4-2\alpha-\mu}(\kappa-\varepsilon)^2 \left(2\alpha_0 s_n^2 - \frac{(6-2\alpha-\mu)\pi}{\log n}\right)}. \end{aligned}$$

This implies that $\lim_{n\to\infty} s_n^2 = \frac{(4-2\alpha-\mu)\pi}{\alpha_0}$, a contradiction. So $\varphi_n(t)$ is decreasing for large n when $t \in \left(\sqrt{\frac{(4-2\alpha-\mu)\pi}{\alpha_0}}(1+\varepsilon), +\infty\right)$. Thus (3.4) holds for some $\bar{n} \in \mathbb{N}$. Till now, we have completed the proof. □

Lemma 3.2 *Assume that* (F1)–(F5) *hold. Then* $c(a) < 2\pi/\alpha_0$ *for any* $a > 0$.

Proof Let $w_{\bar{n}}$ be given by Lemma 3.1. Since $\|\nabla t(w_{\bar{n}})_t\|_2^2 = t^2\|\nabla w_{\bar{n}}\|_2^2$, we know that there exists $t_w > 0$ small enough and $T_w > 0$ large enough such that $\|\nabla t_w(w_{\bar{n}})_{t_w}\|_2^2 \leq K(a)$ and $\Phi(T_w(w_{\bar{n}})_{T_w}) < 0$ by (2.7). Set

$$g_0(\tau) = [(1-\tau)t_w + \tau T_w](w_{\bar{n}})_{(1-\tau)t_w+\tau T_w}, \quad \forall\, \tau \in [0, 1].$$

Then $g_0 \in \Gamma_a$. Jointly with the definition of $c(a)$, we have $c(a) < 2\pi/\alpha_0$ for any $a > 0$. □

Lemma 3.3 *Assume that* f *satisfies* (F1)–(F3) *and* (F4′) *with* $\gamma > \gamma^*(a)$. *Then* $c(a) < [(4-2\mu-\beta)\pi]/(2\alpha_0)$, *where* $\gamma^*(a)$ *is given by* (1.4).

Proof Since

$$\mathcal{C}_{\frac{4p}{4-2\alpha-\mu}}^{-\frac{4-2\alpha-\mu}{2}} = \inf_{u\in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|\nabla u\|_2^{2p-(4-2\alpha-\mu)}\|u\|_2^{4-2\alpha-\mu}}{\|u\|_{\frac{4p}{4-2\alpha-\mu}}^{2p}},$$

we can choose $v_n \in \mathcal{S}_a$ such that

$$\mathcal{C}_{\frac{4p}{4-2\alpha-\mu}}^{-\frac{4-2\alpha-\mu}{2}} \le \frac{\|\nabla v_n\|_2^{2p-(4-2\alpha-\mu)} a^{\frac{4-2\alpha-\mu}{2}}}{\|u\|_{\frac{4p}{4-2\alpha-\mu}}^{2p}} < \mathcal{C}_{\frac{4p}{4-2\alpha-\mu}}^{-\frac{4-2\alpha-\mu}{2}} + \frac{1}{n}, \quad \forall\, n \in \mathbb{N}. \tag{3.8}$$

Note that

$$\begin{aligned}\Phi(t(v_n)_t) &= \frac{t^2}{2}\|\nabla v_n\|_2^2 - \frac{1}{2t^{4-2\alpha-\mu}}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(tv_n(y))}{|x-y|^{\mu}|y|^{\alpha}}\right)\frac{F(tv_n(x))}{|x|^{\alpha}}\mathrm{d}x\\ &\le \frac{t^2}{2}\|\nabla v_n\|_2^2 - \frac{1}{2t^{4-2\alpha-\mu}}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{\gamma|tv_n(y)|^p}{|x-y|^{\mu}|y|^{\alpha}}\right)\frac{\gamma|tv_n(x)|^p}{|x|^{\alpha}}\mathrm{d}x\\ &\le \frac{t^2}{2}\|\nabla v_n\|_2^2 - \frac{\gamma^2\mathcal{C}(2,\mu,\alpha)t^{2p-(4-2\alpha-\mu)}}{2}\|v_n\|_{\frac{4p}{4-2\alpha-\mu}}^{2p}\\ &:= g_n(t), \quad \forall\, t>0,\ n\in\mathbb{N}.\end{aligned} \tag{3.9}$$

Let $g_n'(t_n) = 0$, then one has

$$t_n^{2p-(6-2\alpha-\mu)} = \frac{2\|\nabla v_n\|_2^2}{[2p-(4-2\alpha-\mu)]\gamma^2\mathcal{C}(2,\mu,\alpha)\|v_n\|_{\frac{4p}{4-2\alpha-\mu}}^{2p}}.$$

It is easy to see that $g_n(t) \le g_n(t_n)$ for all $t > 0$. We define that

$$A := \frac{2^{\frac{(8-2\alpha-\mu)-2p}{2p-(6-2\alpha-\mu)}}[2p-(6-2\alpha-\mu)]}{[2p-(4-2\alpha-\mu)]^{\frac{2p-(4-2\alpha-\mu)}{2p-(6-2\alpha-\mu)}}}. \tag{3.10}$$

Then it follows from (3.8) and (3.9) that

$$\Phi(t(v_n)_t) \le g_n(t_n) = A\left(\frac{1}{\gamma^2\mathcal{C}(2,\mu,\alpha)}\right)^{\frac{2}{2p-(6-2\alpha-\mu)}} \left(\frac{\|\nabla v_n\|_2^{2p-(4-2\alpha-\mu)}}{\|v_n\|_{\frac{4p}{4-2\alpha-\mu}}^{2p}}\right)^{\frac{2}{2p-(6-2\alpha-\mu)}}$$

$$\leq A\left(\frac{1}{\gamma^2\mathcal{C}(2,\mu,\alpha)}\right)^{\frac{2}{2p-(6-2\alpha-\mu)}}\left(\frac{\mathcal{C}^{-\frac{4-2\alpha-\mu}{2}}_{\frac{4p}{4-2\alpha-\mu}}+\frac{1}{n}}{a^{\frac{4-2\alpha-\mu}{2}}}\right)^{\frac{2}{2p-(6-2\alpha-\mu)}},$$

$$\forall\, t>0,\ n\in\mathbb{N}. \tag{3.11}$$

Since $p>\frac{6-2\alpha-\mu}{2}$ and $\gamma>\gamma^*(a)$, then there exists $\epsilon_0>0$ such that

$$\gamma=\gamma^*(a)(1-\epsilon_0)^{[(6-2\alpha-\mu)-2p]/4}$$
$$=\frac{A^{\frac{2p-(6-2\alpha-\mu)}{4}}}{\sqrt{\mathcal{C}(2,\mu,\alpha)}a^{\frac{4-2\alpha-\mu}{4}}C^{\frac{4-2\alpha-\mu}{4}}_{\frac{4p}{4-2\alpha-\mu}}}\left[\frac{2\alpha_0}{(4-2\alpha-\mu)\pi(1-\varepsilon_0)}\right]^{\frac{2p-(6-2\alpha-\mu)}{4}},$$

which together with (3.11) imply that

$$\Phi(t(v_n)_t)\leq\left(\frac{\mathcal{C}^{-\frac{4-2\alpha-\mu}{2}}_{\frac{4p}{4-2\alpha-\mu}}+\frac{1}{n}}{\mathcal{C}^{-\frac{4-2\alpha-\mu}{2}}_{\frac{4p}{4-2\alpha-\mu}}}\right)^{\frac{2}{2p-(6-2\alpha-\mu)}}\frac{(4-2\alpha-\mu)\pi(1-\epsilon_0)}{2\alpha_0},\quad \forall\, t>0,\ n\in\mathbb{N},$$

which implies that there exists $\bar{n}\in\mathbb{N}$ large enough such that

$$\max_{t>0}\Phi(t(v_{\bar{n}})_t)<\frac{(4-2\alpha-\mu)\pi}{2\alpha_0}. \tag{3.12}$$

Replacing $w_{\bar{n}}$ by $v_{\bar{n}}$ in the proof of Lemma 3.2, we can get $c(a)\leq\max_{t>0}\Phi(t(v_{\bar{n}})_t)$ for any $\gamma>\gamma^*(a)$. From this and (3.12), we derived the desired conclusion, and so the proof is completed. □

4 Restore the Compactness

Let us first establish the following two convergence results which contribute to the final proof.

Lemma 4.1 *Assume that* $\alpha<\mu$, $u_n\rightharpoonup u$ *in* $H_r^1(\mathbb{R}^2)$ *and*

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\leq\mathcal{K},$$

for some constant $\mathcal{K}>0$. *Then for every* $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$, *we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))\varphi(x)}{|x|^\alpha}\mathrm{d}x$$

$$= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u(x))\varphi(x)}{|x|^\alpha}\mathrm{d}x.$$

Proof By the Fatou's Lemma we have

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u(x))u(x)}{|x|^\alpha}\mathrm{d}x \le \mathcal{K}.$$

Take $\Omega = \operatorname{supp}\varphi$, for any given $\varepsilon > 0$, let $M_\varepsilon := \mathcal{K}\|\varphi\|_\infty\varepsilon^{-1}$, then it follows that for n large enough,

$$\begin{aligned}&\int_{(|u_n|\ge M_\varepsilon)\cup(|u|=M_\varepsilon)}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))\varphi(x)|}{|x|^\alpha}\mathrm{d}x\\ &\le \frac{2\varepsilon}{\mathcal{K}}\int_{|u_n|\ge\frac{M_\varepsilon}{2}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x \le 2\varepsilon,\end{aligned}$$

and similarly

$$\int_{|u|\ge M_\varepsilon}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u(x))\varphi(x)|}{|x|^\alpha}\mathrm{d}x \le \varepsilon.$$

Since $|f(u_n)|\chi_{|u_n|\le M_\varepsilon} \to |f(u)|\chi_{|u|\le M_\varepsilon}$ a.e. in $\Omega\backslash D_\varepsilon$, where $D_\varepsilon = \{x \in \Omega : |u(x)| = M_\varepsilon\}$, and

$$|f(u_n)|\chi_{|u_n|\le M_\varepsilon} \le \max_{|t|\le M_\varepsilon}|f(t)| < \infty, \quad \forall\, x \in \Omega,$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty}\int_{(\Omega\backslash D_\varepsilon)\cup\{|u_n|\le M_\varepsilon\}}|f(u_n)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x = \int_{(\Omega\backslash D_\varepsilon)\cup\{|u|\le M_\varepsilon\}}|f(u)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x.$$

Here, we choose $K_\varepsilon > t_0$ such that

$$\|\varphi\|_\infty\left(\frac{M_0\mathcal{K}}{K_\varepsilon}\right)^{\frac{1}{2}}\left[2C(2,\mu,\alpha)\int_\Omega|f(u)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x\right]^{\frac{4-2\alpha-\mu}{4}} < \varepsilon,$$

and

$$\int_{|u|\le M_\varepsilon}\left[\frac{F(u(y))\chi_{|u|\ge K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right]\frac{|f(u(x))\varphi|}{|x|^\alpha}\mathrm{d}x < \varepsilon.$$

With the help of Lemma 2.2, we have

$$\int_{(|u_n|\le M_\varepsilon)\cap(|u|\ne M_\varepsilon)}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))\chi_{|u_n|\ge K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))\varphi|}{|x|^\alpha}\mathrm{d}x$$

$$\leq \|\varphi\|_\infty \left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))\chi_{|u_n|\geq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{|u_n|\geq K_\varepsilon}}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|f(u_n(y))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}},$$

then from (F5) and Proposition 1.2, one has

$$\begin{aligned}
&\int_{(|u_n|\leq M_\varepsilon)\cap(|u|\neq M_\varepsilon)}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))\chi_{|u_n|\geq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))\varphi|}{|x|^\alpha}\mathrm{d}x\\
&\leq \|\varphi\|_\infty\left[\int_{|u_n|\geq K_\varepsilon}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}}\\
&\quad\times\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|f(u_n(y))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}}\\
&\leq \|\varphi\|_\infty\left[\int_{|u_n|\geq K_\varepsilon}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}}\\
&\quad\times\left[C(2,\mu,\alpha)\int_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}|f(u_n)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x\right]^{\frac{4-2\alpha-\mu}{4}}\\
&\leq \|\varphi\|_\infty\left[\frac{M_0}{K_\varepsilon}\int_{|u_n|\geq K_\varepsilon}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\right]^{\frac{1}{2}}\\
&\quad\times\left[2C(2,\mu,\alpha)\int_\Omega|f(u)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x+o(1)\right]^{\frac{4-2\alpha-\mu}{4}}\\
&\leq \|\varphi\|_\infty\left(\frac{M_0\mathcal{K}}{K_\varepsilon}\right)^{\frac{1}{2}}\left[2C(2,\mu,\alpha)\int_\Omega|f(u)|^{\frac{4}{4-2\alpha-\mu}}\mathrm{d}x\right]^{\frac{4-2\alpha-\mu}{4}}+o(1)<\varepsilon+o(1).
\end{aligned}$$

For any $x\in\mathbb{R}^2$, define $\zeta_n(x)$ and $\bar{\zeta}(x)$ as follows:

$$\zeta_n(x):=\int_{\mathbb{R}^2}\frac{|F(u_n(y))|\chi_{|u_n|\leq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y,$$

and

$$\bar{\zeta}(x):=\int_{\mathbb{R}^2}\frac{|F(u(y))|\chi_{|u|\leq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y.$$

Let us first point out some relationships here. For fixed $x\in\mathbb{R}^2$, we consider the term

$$\int_{|x-y|\leq R}\frac{1}{|y|^{\alpha p_1}|x-y|^{\mu p_1}}\mathrm{d}y.$$

When $x \in \mathbb{R}^2/B_{2R}(0)$, $y \in B_R(x)$, thus $|x-y| < |y|$. Select p_1 such that $(\mu+\alpha)p_1 < 2$, and thus we have,

$$\int_{|x-y|\le R} \frac{1}{|y|^{\alpha p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \int_{|x-y|\le R} \frac{1}{|x-y|^{(\mu+\alpha)p_1}}\mathrm{d}y = \mathcal{O}\left(R^{2-(\mu+\alpha)p_1}\right).$$

When $x \in B_{2R}(0)$, one has

$$\int_{|x-y|\le R} \frac{1}{|y|^{\alpha p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \int_{|y|\le R} \frac{1}{|y|^{(\mu+\alpha)p_1}}\mathrm{d}y + \int_{|x-y|\le 3R} \frac{1}{|x-y|^{(\alpha+\mu)p_1}}\mathrm{d}y = \mathcal{O}\left(R^{2-(\mu+\alpha)p_1}\right).$$

That is

$$\int_{|x-y|\le R} \frac{1}{|y|^{\alpha p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \mathcal{O}\left(R^{2-(\mu+\alpha)p_1}\right).$$

Choosing q such that $\alpha q < 2 < \mu q$, one has

$$\begin{aligned}
&\int_{|x-y|\ge R} \frac{1}{|y|^{\alpha q}|x-y|^{\mu q}}\mathrm{d}y \\
&= \int_{(\mathbb{R}^2\setminus B_R(x))\cap B_R(0)} \frac{1}{|y|^{\alpha q}|x-y|^{\mu q}}\mathrm{d}y + \int_{(\mathbb{R}^2\setminus B_R(x))\cap(\mathbb{R}^2\setminus B_R(0))} \frac{1}{|y|^{\alpha q}|x-y|^{\mu q}}\mathrm{d}y \\
&\le \frac{1}{R^{\mu q}}\int_{|y|\le R}\frac{1}{|y|^{\alpha q}}\mathrm{d}y + \frac{1}{R^{\alpha q}}\int_{\mathbb{R}^2\setminus B_R(x)}\frac{1}{|x-y|^{\mu q}}\mathrm{d}y = \mathcal{O}\left(R^{2-(\alpha+\mu)q}\right).
\end{aligned}$$

Then from (2.4), we have

$$\begin{aligned}
|\zeta_n(x)-\bar{\zeta}(x)| &\le \int_{\mathbb{R}^2} \frac{\left||F(u_n(y))|\chi_{|u_n|\le K_\varepsilon} - |F(u(y))|\chi_{|u|\le K_\varepsilon}\right|}{|y|^\alpha|x-y|^\mu}\mathrm{d}y \\
&\le \left[\int_{|x-y|\le R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}} \\
&\quad\times\left(\int_{|x-y|\le R}\frac{1}{|y|^{\alpha p_1}|x-y|^{\mu p_1}}\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\quad+\left[\int_{|x-y|>R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{q'}\mathrm{d}y\right]^{\frac{1}{q'}} \\
&\quad\times\left(\int_{|x-y|>R}\frac{1}{|y|^{\alpha q}|x-y|^{\mu q}}\mathrm{d}y\right)^{\frac{1}{q}} \\
&\le \mathcal{O}\left(R^{2/p_1-\alpha-\mu}\right)\left[\int_{|x-y|\le R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}} \\
&\quad+\mathcal{O}\left(R^{2/q-\alpha-\mu}\right)\left(\int_{|x-y|>R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{q'}\mathrm{d}y\right)^{\frac{1}{q'}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}\left(R^{2/p_1-\alpha-\mu}\right)\left[\int_{|x-y|\leq R}\left||F(u_n)|\chi_{|u_n|\leq K_\varepsilon}-|F(\bar{u})|\chi_{|\bar{u}|\leq K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}}\\
&\quad+\mathcal{O}\left(R^{2/q-\alpha-\mu}\right)\left[\|u_n\|_{\frac{(4-2\alpha-\mu)pq'}{4}}^{\frac{(4-2\alpha-\mu)p}{4}}+\|\bar{u}\|_{\frac{(4-2\alpha-\mu)pq'}{4}}^{\frac{(4-2\alpha-\mu)p}{4}}\right]\\
&\leq \mathcal{O}\left(R^{2/p_1-\alpha-\mu}\right)o_n(1)+\mathcal{O}\left(R^{2/q-\alpha-\mu}\right),\ \ \forall\ x\in\mathbb{R}^2,
\end{aligned}
$$

which implies that for any $x \in \mathbb{R}^2$, we have $\zeta_n(x) \to \bar{\zeta}(x)$. Similarly, by choosing suitable p_2 and p_3, then for any $x \in \mathbb{R}^2$, we know that

$$
\begin{aligned}
|\zeta_n(x)| &\leq \int_{\mathbb{R}^2}\frac{|F(u_n(x))|\chi_{|u_n|\leq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\\
&\leq \left[\int_{|x-y|\leq R}\left|F(u_n(x))\chi_{|u_n|\leq K_\varepsilon}\right|^{p_2'}\mathrm{d}y\right]^{\frac{1}{p_2'}}\left[\int_{|x-y|\leq R}\frac{1}{|y|^{\alpha p_2}|x-y|^{\mu p_2}}\mathrm{d}y\right]^{\frac{1}{p_2}}\\
&\quad+\left[\int_{|x-y|>R}\left|F(u_n(x))\chi_{|u_n|\leq K_\varepsilon}\right|^{p_3'}\mathrm{d}y\right]^{\frac{1}{p_3'}}\left[\int_{|x-y|>R}\frac{1}{|y|^{\alpha p_3}|x-y|^{\mu p_3}}\mathrm{d}y\right]^{\frac{1}{p_3}}\\
&\leq \left(\pi R^2\right)^{\frac{1}{p_2'}}\mathcal{O}\left(R^{2/p_2-\alpha-\mu}\right)\max_{|t|\leq K_\varepsilon}|F(t)|+\mathcal{O}\left(R^{2/p_3-\alpha-\mu}\right)\|u_n\|_{\frac{(4-2\alpha-\mu)pp_3'}{4}}^{\frac{(4-2\alpha-\mu)p}{4}}\\
&\leq C.
\end{aligned}
$$

It follows that

$$
\left|\frac{\zeta_n(x)f(u_n(x))\varphi(x)\chi_{|u_n|\leq M_\varepsilon}}{|x|^\alpha}\right|\leq C\left|\frac{\varphi(x)\max_{|t|\leq M_\varepsilon}|f(t)|}{|x|^\alpha}\right|\leq\frac{C'}{|x|^\alpha},\quad \forall\, x\in\Omega.
$$

By $\alpha < 2$, it is easy to verify that $\frac{1}{|x|^\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Therefore, together with $\zeta_n(x) \to \bar{\zeta}(x)$ and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\int_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|\neq M_\varepsilon)}\left(\int_{\mathbb{R}^2}\frac{F(u_n(x))\chi_{|u_n|\leq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))\varphi(x)|}{|x|^\alpha}\mathrm{d}x\\
&\quad\to\int_{|\bar{u}|<M_\varepsilon}\left(\int_{\mathbb{R}^2}\frac{F(u(x))\chi_{|u|\leq K_\varepsilon}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u(x))\varphi(x)|}{|x|^\alpha}\mathrm{d}x.
\end{aligned}
$$

From the arguments above all and by the arbitrariness of $\varepsilon > 0$, we can conclude this Lemma. □

Lemma 4.2 *Assume that $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^2)$, $u_n \rightharpoonup u$ in $H^1_r(\mathbb{R}^2)$ and*

$$
\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\leq C. \tag{4.1}
$$

Then we have

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\to\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\mathrm{d}x.$$

Proof In view of $u_n\rightharpoonup\bar{u}$ in $H_r^1(\mathbb{R}^2)$, we know $u_n\to u$ in $L^q(\mathbb{R}^2)$ with $q>2$. By [35, Theorem A.1], there exists $g\in L^q(\mathbb{R}^2)$ such that

$$|u_n(x)|\le g(x),\quad |u(x)|\le g(x),\quad \text{a.e. } x\in\mathbb{R}^2.$$

For any given $\varepsilon\in(0,M_0/t_0)$, it follows from (F5) that

$$\begin{aligned}&\int_{|u_n|\ge M_0\varepsilon^{-1}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\\ &\le M_0\int_{|u_n|\ge M_0\varepsilon^{-1}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|f(u_n(x))|}{|x|^\alpha}\mathrm{d}x\\ &\le \varepsilon\int_{|u_n|\ge M_0\varepsilon^{-1}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\le C\varepsilon.\end{aligned}$$

Similarly, one has

$$\int_{|u|\ge M_0\varepsilon^{-1}}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\mathrm{d}x\le C\varepsilon.$$

Now, we can choose $R_\varepsilon>0$ such that

$$\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left|\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\right|\mathrm{d}x<\varepsilon,$$

$$\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left|\left(\int_{\mathbb{R}^2}\frac{|u(y)|^q}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u(x)|^q}{|x|^\alpha}\right|\mathrm{d}x<\varepsilon,$$

and

$$\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left|\left(\int_{\mathbb{R}^2}\frac{g^{\tilde{q}+1}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{g^{\tilde{q}+1}}{|x|^\alpha}\right|\mathrm{d}x<\varepsilon.$$

Let C be the constant in (4.1) and choose $K\ge\max\{CM_0/\varepsilon,t_0\}$ such that

$$\int_{|u|\le M_0\varepsilon^{-1}}\left(\int_{|u|\ge K}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\mathrm{d}x<\varepsilon. \tag{4.2}$$

By (F5), one has

$$
\begin{aligned}
&\int_{|u_n|\le M_0\varepsilon^{-1}}\left(\int_{|u_n|\ge K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x\\
&\quad\le\frac{1}{K}\int_{|u_n|\le M_0\varepsilon^{-1}}\left(\int_{|u_n|\ge K}\frac{F(u_n(y))u_n(y)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x\\
&\quad\le\frac{M_0}{K}\int_{|u_n|\le M_0\varepsilon^{-1}}\left(\int_{|u_n|\ge K}\frac{f(u_n(y))u_n(y)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x\\
&\quad\le\frac{M_0}{K}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\le\varepsilon.
\end{aligned}
\tag{4.3}
$$

By (F2), we know that there exist $C>0$ and $\tilde{q}>\frac{(4-2\alpha-\mu)}{2}$ such that for $|t|\le K$,

$$
|F(t)|\le C|t|^{\tilde{q}+1}. \tag{4.4}
$$

Thus we have

$$
\begin{aligned}
&\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left(\int_{|u_n|\le K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\\
&\quad\le C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(\int_{|u_n|\le K}\frac{u_n^{\tilde{q}+1}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{u_n^{\tilde{q}+1}}{|x|^\alpha}\mathrm{d}x\\
&\quad\le C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(\int_{|u_n|\le K}\frac{g^{\tilde{q}+1}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{g^{\tilde{q}+1}}{|x|^\alpha}\mathrm{d}x\le C\varepsilon,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\left|\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left[\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\right.\right.\\
&\qquad\left.\left.-\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\right]\mathrm{d}x\right|\\
&\le\left|\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\right|\\
&\quad+\left|\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\mathrm{d}x\right|\\
&<\varepsilon+\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left(\int_{|u_n|\le K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x)}{|x|^\alpha}\mathrm{d}x\\
&\quad+\int_{\{\mathbb{R}^2\setminus B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}}\left(\int_{|u_n|\ge K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x<(2+C)\varepsilon.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left|\int_{B_{R_\varepsilon}}\left[\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}-\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\right]\mathrm{d}x\right|\\
&\leq\ 2C\varepsilon+\left|\int_{B_{R_\varepsilon}\cap\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\frac{F(u_n(y))}{|y|^\alpha|x-y|^\alpha}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\right.\\
&\quad\left.-\int_{B_{R_\varepsilon}\cap\{|u|\leq M_0\varepsilon^{-1}\}}\left(\frac{F(u(y))}{|y|^\alpha|x-y|^\alpha}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\mathrm{d}x\right|.
\end{aligned}
$$

It remains to prove that as $n\to\infty$,

$$
\begin{aligned}
&\int_{\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x\\
&\quad\to\int_{\{|u|\leq M_0\varepsilon^{-1}\}}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x. \qquad (4.5)
\end{aligned}
$$

Combining (4.2) with (4.3), we can see that

$$
\begin{aligned}
&\left|\int_{|u_n|\leq M_0\varepsilon^{-1}}\left\{\int_{|u_n|\geq K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\right.\right.\\
&\quad\left.\left.-\int_{|u|\geq K}\left(\int_{\mathbb{R}^2}\frac{F(u(y))}{|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\right\}\mathrm{d}x\right|\leq 2\varepsilon.
\end{aligned}
$$

In order to prove (4.5), it remains to verify that as $n\to+\infty$ there holds

$$
\begin{aligned}
&\int_{\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\int_{|u_n|\leq K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x\\
&\quad\to\int_{\{|u|\leq M_0\varepsilon^{-1}\}}\left(\int_{|u|\leq K}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))\chi_{B_{R_\varepsilon}}}{|x|^\alpha}\mathrm{d}x.
\end{aligned}
$$

Indeed, it can be easily verified that as $n\to\infty$,

$$
\begin{aligned}
&\left(\int_{|u_n|\leq K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\chi_{\{B_{R_\varepsilon}\cap|u_n|\leq M_0\varepsilon^{-1}\}}\\
&\to\left(\int_{|u|\leq K}\frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u(x))}{|x|^\alpha}\chi_{\{B_{R_\varepsilon}\cap|u|\leq M_0\varepsilon^{-1}\}}\quad\text{pointwise a.e.}
\end{aligned}
$$

From (4.4), we have

$$
\int_{B_{R_\varepsilon}\cap|u_n|\leq M_0\varepsilon^{-1}}\left(\int_{|u_n|\leq K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x
$$

$$
\begin{aligned}
&\le C\int_{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}}\left(\int_{|u_n|\le K}\frac{|u_n(y)|^{\tilde q+1}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\tilde q+1}}{|x|^\alpha}\mathrm{d}x\\
&\le C\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|u_n(y)|^{\tilde q+1}(y)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\tilde q+1}}{|x|^\alpha}\mathrm{d}x\\
&\le C\cdot C(\mu,\alpha)\|u_n\|^{2(\tilde q+1)}_{\frac{4(\tilde q+1)}{4-2\alpha-\mu}}\to C\cdot C(\mu,\alpha)\|u\|^{2(\tilde q+1)}_{\frac{4(\tilde q+1)}{4-2\alpha-\mu}},\quad \text{as } n\to\infty.
\end{aligned}
$$

From [10, Theorem 4.9], there exists $\mathcal{F}\in L^1(\mathbb{R}^2)$ such that up to a subsequence, still denoted by $\{u_n\}$, for each $n\in\mathbb{N}$, we have

$$
\left|\left(\int_{|u_n|\le K}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\chi_{\{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}\}}\right|\le|\mathcal{F}(x)|.
$$

Using the Lebesgue dominated convergence theorem, we can conclude this Lemma. □

Lemma 4.3 *Assume that* (F1)–(F3) *hold. If there exist $u\in H^1(\mathbb{R}^2)$ and $\lambda\in\mathbb{R}$ such that*

$$
-\Delta u+\lambda u=\left(I_\mu\times\frac{F(u)}{|x|^\alpha}\right)\frac{f(u)}{|x|^\alpha},\qquad x\in\mathbb{R}^2,
$$

then $J(u)=0$, where J is defined by (2.2).

The proof of lemma is standard, so we omit it. Hereafter, we are ready to prove the Theorem 1.4.

Proof of Theorem 1.4: Let $\mathcal{S}_a^r=\mathcal{S}_a\cap H_r^1(\mathbb{R}^2)$. In the same way as Lemmas 2.8 and 3.2, we can deduce that for any $a>0$, there exists a bounded sequence $\{u_n\}\subset\mathcal{S}_a^r$ such that

$$
\Phi(u_n)\to c_r(a)\in(0,2\pi/\alpha_0),\quad \Phi|'_{\mathcal{S}_a^r}(u_n)\to 0\ \text{ and }\ J(u_n)\to 0,\tag{4.6}
$$

and

$$
c_r(a)=\inf_{g\in\Gamma_{r,a}}\max_{t\in[0,1]}\Phi(g(t))>\max_{g\in\Gamma_{r,a}}\max\{\Phi(g(0)),\Phi(g(1))\},
$$

where $\Gamma_{r,a}=\{g\in\mathcal{C}([0,1],\mathcal{S}_a^r):\|\nabla g(0)\|_2^2\le K(a),\Phi(g(1))<0\}$ and $K(a)$ is given in Lemma 2.3. Then there exists $\bar u\in H_r^1(\mathbb{R}^2)$ such that, passing to a subsequence,

$$
u_n\rightharpoonup\bar u\ \text{ in } H_r^1(\mathbb{R}^2),\quad u_n\to\bar u\ \text{ in } L^s(\mathbb{R}^2)\text{ for } s>2,\quad u_n\to\bar u \text{ a.e. in } \mathbb{R}^2.
$$

Arguing similar as Lemma 2.8, we know that

$$
\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}\mathrm{d}x\le\mathcal{K}.
$$

Then, it follows that Lemma 4.2 holds, that is

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\\&=\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(\bar u(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(\bar u(x))}{|x|^\alpha}\mathrm{d}x+o(1).\end{aligned}\tag{4.7}$$

Next, we claim that $\bar u\neq 0$. Otherwise, we suppose that $u_n\rightharpoonup 0$ in $H_r^1(\mathbb{R}^2)$. Then one has

$$\begin{aligned}\|\nabla u_n\|^2&=2\Phi(u_n)+\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(u_n(x))}{|x|^\alpha}\mathrm{d}x\\&=2c_r(a)+o(1):=\frac{(4-2\mu-\alpha)\pi}{\alpha_0}(1-3\tilde\varepsilon)+o(1)\ \text{ for some constant } \bar\varepsilon>0.\end{aligned}\tag{4.8}$$

Choosing $q\in(1,2)$ be such that

$$\frac{(1+\bar\varepsilon)(1-3\bar\varepsilon)q}{1-\bar\varepsilon}<1,$$

using (F1), we get

$$|f(t)|^q\le C_1\left[e^{\alpha_0(1+\bar\varepsilon)qt^2}-1\right],\quad \forall\,|t|\ge 1,$$

and using (ii) of Lemma 1.1, we get

$$\int_{|u_n|\ge1}|f(u_n)|^{\frac{4q}{4-2\alpha-\mu}}\mathrm{d}x\le\int_{|u_n|\ge1}\left(e^{\frac{4\alpha_0(1+\bar\varepsilon)q\|\nabla u_n\|^2}{4-2\alpha-\mu}\left(\frac{u_n}{\|\nabla u_n\|}\right)^2}-1\right)\mathrm{d}x\le C.$$

Thus,

$$\begin{aligned}&\int_{|u_n|\ge1}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}dx\\&\le C_2\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{F(u_n(x))}{|x|^\alpha}dx\right]^{1/2}\\&\quad\times\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{f(u_n(y))u_n(y)\chi_{|u_n|\ge1}}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))(u_n(x))\chi_{|u_n|\ge1}}{|x|^\alpha}dx\right]^{1/2}\\&\le C_3\left[\int_{|u_n|\ge1}|f(u_n)|^{\frac{4q}{4-2\alpha-\mu}}dx\right]^{\frac{4-2\alpha-\mu}{4q}}\left[\int_{|u_n|\ge1}|u_n|^{\frac{4q}{(q-1)(4-2\alpha-\mu)}}dx\right]^{\frac{(4-2\alpha-\mu)(q-1)}{4q}}\\&\le C_4\|u_n\|_{\frac{4q}{(q-1)(4-2\alpha-\mu)}}=o(1).\end{aligned}$$

Similarly, we have

$$\int_{|u_n|\le 1}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))u_n(x)}{|x|^\alpha}dx \le C_5\|u_n\|_2^{(4-2\alpha-\mu)/2}=o(1). \tag{4.9}$$

Then it follows from (4.9) and (4.9) that

$$c_r(a)+o(1)=\Phi(u_n)-\frac{1}{2}J(u_n)=\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{\left[f(u_n)u_n-\frac{6-\mu-2\alpha}{2}F(u_n)\right](x)}{2|x|^\alpha}dx=o(1),$$

which is a contradiction due to $c_r(a)>0$ for any $a>0$. This shows that $\bar{u}\neq 0$ as claimed.

By (4.6) and the boundedness of the sequence $\{u_n\}$, one can easily verify that there exist a bounded sequence $\{\lambda_n\}\subset\mathbb{R}$ and $\bar{\lambda}$ such that, up to a subsequence,

$$\lambda_n\to\bar{\lambda}\in\mathbb{R}, \tag{4.10}$$

$$-\Delta u_n+\lambda_n u_n-\left(I_\mu\times\frac{F(u_n)}{|x|^\alpha}\right)\frac{f(u_n)}{|x|^\alpha}\to 0 \quad\text{in}\quad (H_r^1(\mathbb{R}^2))^*, \tag{4.11}$$

and

$$-\Delta u_n+\bar{\lambda} u_n-\left(I_\mu\times\frac{F(u_n)}{|x|^\alpha}\right)\frac{f(u_n)}{|x|^\alpha}\to 0 \quad\text{in}\quad (H_r^1(\mathbb{R}^2))^*.$$

Again, in view of the conclusion of Lemma 4.1, we can see that

$$-\Delta \bar{u}+\bar{\lambda}\bar{u}-\left(I_\mu\times\frac{F(\bar{u})}{|x|^\alpha}\right)\frac{f(\bar{u})}{|x|^\alpha}=0 \quad\text{in}\quad (H_r^1(\mathbb{R}^2))^*. \tag{4.12}$$

Hereafter, the only thing we need to verify is that $\|\bar{u}\|_2^2=a$, and next our goal is to prove that $u_n\to\bar{u}$ in $H_r^1(\mathbb{R}^2)$. Note that (4.11) yields

$$\|\nabla u_n\|_2^2+\lambda_n\|u_n\|_2^2-\int_{\mathbb{R}^2}\left(\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\right)\frac{f(u_n)u_n}{|x|^\alpha}\mathrm{d}x\to 0, \tag{4.13}$$

and

$$\int_{\mathbb{R}^2}(\nabla u_n\cdot\nabla\bar{u}+\lambda_n u_n\bar{u})\,\mathrm{d}x-\int_{\mathbb{R}^2}\left(\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\right)\frac{f(u_n)\bar{u}}{|x|^\alpha}\mathrm{d}x\to 0. \tag{4.14}$$

By (4.13) minus $J(u_n) \to 0$, and using (4.10) and (4.7), we have

$$\bar{\lambda}a + o(1) = \lambda_n \|u_n\|_2^2 = \frac{6-\mu-2\alpha}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u_n)}{|x|^\alpha}\right)\frac{F(u_n)}{|x|^\alpha}\mathrm{d}x + o(1)$$
$$= \frac{6-\mu-2\alpha}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(\bar{u})}{|x|^\alpha}\right)\frac{F(\bar{u})}{|x|^\alpha}\mathrm{d}x + o(1),$$

which, together with $F(t) > 0$ for $t \neq 0$, yields $\bar{\lambda} > 0$. By (4.13) and (4.14), we have

$$\int_{\mathbb{R}^2}[\nabla u_n \cdot \nabla(u_n - \bar{u}) + \lambda_n u_n(u_n - \bar{u})]\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)(u_n-\bar{u})}{|x|^\alpha}\mathrm{d}x.$$

Next, we claim that $\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)(u_n-\bar{u})}{|x|^\alpha}\mathrm{d}x = o(1)$. By (4.12) and the Lemma 4.3, we have $J(\bar{u}) = 0$. This, jointly with (F3) implies

$$\Phi(\bar{u}) = \Phi(\bar{u}) - \frac{1}{2}J(\bar{u}) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(\bar{u}(y))}{|y|^\alpha|x-y|^\mu}\right)\frac{\left[f(\bar{u})\bar{u} - \frac{6-\mu-2\alpha}{2}F(\bar{u})\right](x)}{|x|^\alpha}\mathrm{d}x \geq 0.$$

Thus,

$$c_r(a) + o(1) = \Phi(u_n) = \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^2}\left(I_\mu \times \frac{F(u_n)}{|x|^\alpha}\right)\frac{F(u_n)}{|x|^\alpha}\mathrm{d}x$$
$$= \frac{1}{2}\left(\|\nabla(u_n-\bar{u})\|_2^2 + \|\nabla\bar{u}\|_2^2\right)$$
$$- \frac{1}{2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(\bar{u}(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{F(\bar{u}(x))}{|x|^\alpha}\mathrm{d}x + o(1)$$
$$= \frac{1}{2}\|\nabla(u_n-\bar{u})\|_2^2 + \Phi(\bar{u}) + o(1)$$
$$\geq \frac{1}{2}\|\nabla(u_n-\bar{u})\|_2^2 + o(1).$$

Since $0 < c_r(a) < \frac{(4-2\mu-\beta)\pi}{2\alpha_0}$ for any $a > 0$, similarly as in (4.8), it follows that there exists $\bar{\varepsilon} > 0$ such that

$$\|\nabla(u_n-\bar{u})\|_2^2 \leq \frac{(1-3\bar{\varepsilon})^2(4-2\mu-\beta)\pi}{\alpha_0} \quad \text{for large } n \in \mathbb{N}.$$

Nothing that $q/(q-1) > 2$, by using the Hölder inequality, we have

$$\int_{|u_n|\geq 1}|f(u_n)|^{\frac{4q}{4-2\alpha-\mu}}\mathrm{d}x \leq C_6\int_{|u_n|\geq 1}\left[e^{\frac{\alpha_0(1+\bar{\varepsilon})4qu_n^2}{4-2\alpha-\mu}} - 1\right]\mathrm{d}x$$
$$\leq C_6\int_{|u_n|\geq 1}\left[e^{\frac{4\alpha_0(1+\bar{\varepsilon})^2\bar{\varepsilon}^{-1}q\bar{u}^2}{4-2\alpha-\mu}}e^{\frac{4\alpha_0(1+\bar{\varepsilon})^2q(u_n-\bar{u})^2}{4-2\alpha-\mu}} - 1\right]\mathrm{d}x$$

$$
\begin{aligned}
&\le \frac{(q-1)C_6}{q}\int_{|u_n|\ge 1}\left[e^{\frac{\alpha_0(1+\bar\varepsilon)^2\bar\varepsilon^{-1}q^2(q-1)^{-1}\bar u^2}{4-2\alpha-\mu}}-1\right]\mathrm{d}x\\
&\quad+\frac{C_6}{q}\int_{|u_n|\ge 1}\left[e^{\frac{4\alpha_0(1+\bar\varepsilon)^2q^2(u_n-\bar u)^2}{4-2\alpha-\mu}}-1\right]\mathrm{d}x\\
&\le \frac{(q-1)C_6}{q}\int_{\mathbb{R}^2}\left[e^{\frac{4\alpha_0(1+\bar\varepsilon)^2\bar\varepsilon^{-1}q^2(q-1)^{-1}\bar u^2}{4-2\alpha-\mu}}-1\right]\mathrm{d}x\\
&\quad+\frac{C_6}{q}\int_{\mathbb{R}^2}\left[e^{\frac{4\alpha_0(1+\bar\varepsilon)^2q^2(u_n-\bar u)^2}{4-2\alpha-\mu}}-1\right]\mathrm{d}x\\
&\le C_7+\frac{C_6}{q}\int_{\mathbb{R}^2}\left[e^{\frac{4\alpha_0(1+\bar\varepsilon)^2q^2\|\nabla(u_n-\bar u)\|_2^2}{4-2\alpha-\mu}\cdot\frac{(u_n-\bar u)^2}{\|\nabla(u_n-\bar u)\|_2^2}}-1\right]\mathrm{d}x\le C_5,
\end{aligned}
$$

where $\frac{4\alpha_0(1+\bar\varepsilon)^2q^2\|\nabla(u_n-\bar u)\|_2^2}{4-2\alpha-\mu}<4\pi(1-3\bar\varepsilon)^2(1+\bar\varepsilon)^2q^2<4\pi$. Moreover, we have

$$
\begin{aligned}
&\int_{|u_n|\ge 1}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))(u_n-\bar u)(x)}{|x|^\alpha}dx\\
&\le C_8\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{F(u_n(x))}{|x|^\alpha}dx\right]^{1/2}\\
&\quad\times\left[\int_{|u_n|\ge 1}\left(\int_{|u_n|\ge 1}\frac{f(u_n(y))(u_n-\bar u)(y)}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))(u_n-\bar u)(x)}{|x|^\alpha}dx\right]^{1/2}\\
&\le C_9\left[\int_{|u_n|\ge 1}|f(u_n)|^{\frac{4q}{4-2\alpha-\mu}}dx\right]^{\frac{4-2\alpha-\mu}{4q}}\left[\int_{|u_n|\ge 1}|u_n-\bar u|^{\frac{4q}{(q-1)(4-2\alpha-\mu)}}dx\right]^{\frac{(4-2\alpha-\mu)(q-1)}{4q}}\\
&\le C_{10}\|u_n-\bar u\|_{\frac{4q}{(q-1)(4-2\alpha-\mu)}}=o(1),
\end{aligned}
$$

and similarly one has

$$
\int_{|u_n|\le 1}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}dy\right)\frac{f(u_n(x))(u_n-\bar u)(x)}{|x|^\alpha}dx=o(1).
$$

Till now, we have finished the Claim. Then, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^2}[\nabla u_n\cdot\nabla(u_n-\bar u)+\lambda_n u_n(u_n-\bar u)]\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)(u_n-\bar u)}{|x|^\alpha}\mathrm{d}x,
\end{aligned}
$$

which, together with $u_n\rightharpoonup\bar u$ in $H_r^1(\mathbb{R}^2)$ and $\lambda\to\bar\lambda>0$, implies that $u_n\to\bar u$ in $H_r^1(\mathbb{R}^2)$. Next, using Palais' principle of symmetric criticality [35], the above function $\bar u\in H_r^1(\mathbb{R}^2)\backslash\{0\}$ is in fact a radial solution of $(\mathcal{P}_a)$ in $H^1(\mathbb{R}^2)$, and so the proof of Theorem 1.4 is completed.

Remark 4.4 We omit here the proof of Theorem 1.6, since the difference between of Theorems 1.4 and 1.6 has been presented in Sect. 3. To prove Theorem 1.6, we just need to replace Lemma 3.2 by Lemma 3.3.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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