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Global Smooth Solutions to the 3D Compressible Viscous Non-Isentropic Magnetohydrodynamic Flows Without Magnetic Diffusion

Yongsheng Li[1] · Huan Xu[2] · Xiaoping Zhai[3]

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Abstract

How to construct the global smooth solutions to the compressible viscous, non-isentropic, non-resistive magnetohydrodynamic equations in $\mathbb{T}^3$ appears to be unknown. In this paper, we give a positive answer to this problem. More precisely, we prove a global stability result on perturbations near a strong background magnetic field to the 3D compressible viscous, non-isentropic, non-resistive magnetohydrodynamic equations. This stability result provides a significant example of the stabilizing effect of the magnetic field on electrically conducting fluids. In addition, we obtain an explicit decay rate for the solutions to this nonlinear system.

Keywords Global smooth solutions · Non-resistive MHD · Diophantine condition

Mathematics Subject Classification 35Q35 · 76N10 · 76W05

✉ Xiaoping Zhai
pingxiaozhai@163.com

Yongsheng Li
yshli@scut.edu.cn

Huan Xu
huan.xu@utsa.edu

1 School of Mathematics, South China University of Technology, Guangzhou 510640, China

2 Department of Mathematics, The University of Texas at San Antonio, San Antonio, TX 78249, USA

3 School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510520, China

1 Introduction and Main Result

1.1 Model and Synopsis of Result

The compressible MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They consist of a coupled system of compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Besides their wide physical applicability (see, e.g., [4]), the MHD equations are also of great interest in mathematics. The motion of a compressible, viscous non-isentropic magnetohydrodynamic flows without magnetic diffusion can be described by the following equations (cf. [22, Chapter 3]):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ c_\nu (\rho \partial_t \vartheta + \rho \mathbf{u} \cdot \nabla \vartheta) - \kappa \Delta \vartheta + P \operatorname{div} \mathbf{u} = 2\mu |D(\mathbf{u})|^2 + \lambda (\operatorname{div} \mathbf{u})^2, \\ \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} \operatorname{div} \mathbf{u} = 0, \\ \operatorname{div} \mathbf{B} = 0. \end{cases} \tag{1.1}$$

Here ρ denotes the density of the fluid, $\mathbf{u}$ the velocity field, ϑ the temperature, and $\mathbf{B}$ the magnetic field, respectively. The parameters μ and λ are shear viscosity and volume viscosity coefficients, respectively, which satisfy the standard strong parabolicity assumption,

$$\mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.$$

c_ν is a positive constant and $\kappa > 0$ is the heat-conductivity coefficient. The fluid is assumed to obey the ideal polytropic law, so the pressure $P = R\rho\vartheta$ for a positive constant R. The deformation tensor $D(\mathbf{u}) = \frac{1}{2}\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\big)$.

The system (1.1) is supplemented with the initial condition

$$(\rho, \mathbf{u}, \vartheta, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{u}_0, \vartheta_0, \mathbf{B}_0).$$

Several simplified models than (1.1) have been extensively studied in the literature. If $\mathbf{B} = \mathbf{0}$, the system (1.1) reduces to the non-isentropic compressible Navier–Stokes system which has been widely studied, see [5, 7–10, 12, 13, 20, 40, 41] and the references therein. While both the effect of the density and the temperature are neglected, (1.1) reduces to the viscous non-resistive incompressible MHD system which has also been studied by many researchers, see [1, 2, 15, 21, 29–32, 35, 36, 43, 45] and the references therein. When the temperature fluctuation is neglected, (1.1) becomes the compressible viscous non-resistive MHD system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} \operatorname{div} \mathbf{u} = 0, \\ \operatorname{div} \mathbf{B} = 0, \quad P(\rho) = A\rho^\gamma. \end{cases} \tag{1.2}$$

Due to the lack of magnetic diffusion, the global well-posedness issue of (1.2) becomes quite difficult. There are satisfactory results in the simplified 1D geometry, see [17, 26]. In higher dimensions, Wu and Wu [37] presented a systematic approach to the small data global well-posedness and stability problem on the 2D compressible non-resistive MHD equations. Tan and Wang [33] obtained the global existence of smooth solutions to the 3D compressible barotropic viscous non-resistive MHD system in the horizontally infinite flat layer $\Omega = \mathbb{R}^2 \times (0, 1)$. Jiang and Jiang [16] showed the stability/instability criteria for the stratified compressible magnetic Rayleigh-Taylor problem in Lagrangian coordinates in the three-dimensional case. Assuming that the motion of the fluids takes place in the plane while the magnetic field acts on the fluids only in the vertical direction, Li and Sun [27] obtained the existence of global weak solutions with large initial data, see also [30] for an extension including density-dependent viscosity coefficient and non-monotone pressure law. Dong, Wu and Zhai [6] proved the global strong solutions to the compressible non-resistive MHD equations with small initial data. Wu and Zhai [38] obtained the global existence and stability result for smooth solutions to (1.2) near any background magnetic field satisfying a Diophantine condition. Wu and Zhu [39] investigated such a system on bounded domains and solved this problem by pure energy estimates, which helped reduce the complexity of other approaches. Zhong [46] construct local-in-time strong solutions without any Cho-Choe-Kim type compatibility conditions in $\mathbb{R}^2$. Li and Sun [27] obtained the existence of global weak solutions for the 2D non-resistive compressible MHD equations. Liu and Zhang [30] extended this global result to include density-dependent viscosity coefficient and non-monotone pressure law.

However, since the temperature is taken into account in (1.2), relevant results seem not to be so fruitful due to some mathematical challenges. Zhang and Zhao [44] established the global well-posedness of strong solutions to the one-dimensional compressible viscous heat-conducting non-resistive equations of magnetohydrodynamics on $(0, 1)$. Li [23] obtained the global strong solutions to the one-dimensional heat-conductive model for planar non-resistive magnetohydrodynamics with large data. Li and Jiang [25] studied the global weak solutions for the Cauchy problem to one-dimensional heat-conductive MHD equations of viscous non-resistive gas. In multi-dimensions, Li [24] proved the global well-posedness of the three-dimensional full compressible viscous non-resistive MHD system in an infinite slab $\mathbb{R}^2 \times (0, 1)$ with a strong background magnetic field. By assuming that the motion of fluids takes place in the plane while the magnetic field acts on the fluids only in the vertical direction, Li and Sun [28] obtained the existence of global weak solutions with large initial data. However, to our knowledge, the global well-posedness or stability result of (1.1) in the whole space $\mathbb{R}^3$ or the periodic box $\mathbb{T}^3$ is still unknown even when the initial data is small or near a steady-state solution.

The main difficulty of studying the global well-posedness of (1.1) lies in the lack of magnetic diffusion. To overcome this difficulty, we consider the background magnetic field $\mathbf{n} \in \mathbb{R}^3$ satisfying the so-called Diophantine condition: for any $\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}$,

$$|\mathbf{n} \cdot \mathbf{k}| \geq \frac{c}{|\mathbf{k}|^r}, \quad \text{for some } c > 0 \text{ and } r > 2. \tag{1.3}$$

The key point is that for $\mathbf{n}$ satisfying the Diophantine condition, it holds that the following lemma whose proof is standard by the Plancherel formula.

Lemma 1.4 *If* $\mathbf{n} \in \mathbb{R}^3$ *satisfies the Diophantine condition* (1.3)*, then it holds, for any* $s \in \mathbb{R}$, *that*

$$\|f\|_{H^s} \leq C\|\mathbf{n}\cdot\nabla f\|_{H^{s+r}}, \tag{1.5}$$

provided that $\nabla f \in H^{s+r}(\mathbb{T}^3)$ *satisfies* $\int_{\mathbb{T}^3} f\,dx = 0$.

This inequality has been used in the recent work of W. Chen, Z. Zhang and J. Zhou [3] in which they proved the global well-posedness of the 3D incompressible MHD equations without magnetic diffusion. Inspired by [3] and [38], we obtain the global small solutions of (1.1) in $\mathbb{T}^3$ when the initial magnetic field is close to a background magnetic field satisfying the Diophantine condition.

Let us denote $\mathbf{b} = \mathbf{B} - \mathbf{n}$ the perturbation around a constant background field $\mathbf{n}$. Then the perturbed equations can be rewritten as

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0, \\ \rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} + \nabla P = \mathbf{n}\cdot\nabla\mathbf{b} \\ \qquad + \mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{n}\nabla\mathbf{b} - \mathbf{b}\nabla\mathbf{b}, \\ c_v(\rho\partial_t\vartheta + \rho\mathbf{u}\cdot\nabla\vartheta) - \kappa\Delta\vartheta + P\mathrm{div}\,\mathbf{u} = 2\mu|D(\mathbf{u})|^2 + \lambda(\mathrm{div}\,\mathbf{u})^2, \\ \partial_t\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{n}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{n}\mathrm{div}\,\mathbf{u} - \mathbf{b}\mathrm{div}\,\mathbf{u}, \\ \mathrm{div}\,\mathbf{b} = 0, \\ (\rho, \mathbf{u}, \vartheta, \mathbf{b})|_{t=0} = (\rho_0, \mathbf{u}_0, \vartheta_0, \mathbf{b}_0). \end{cases} \tag{1.6}$$

Here and in what follows, the above system is considered for $(t, x) \in [0, \infty) \times \mathbb{T}^3$ with the volume of $\mathbb{T}^3$ normalized to unity:

$$\left|\mathbb{T}^3\right| = 1. \tag{1.7}$$

1.2 Main Result

Given two positive constants $\bar{\rho}$ and $\bar{\vartheta}$, we shall prove the asymptotic stability of the steady state $(\rho_s, \mathbf{u}_s, \vartheta_s, \mathbf{b}_s) \stackrel{\text{def}}{=} (\bar{\rho}, \mathbf{0}, \bar{\vartheta}, \mathbf{0})$, despite the lack of magnetic diffusion in (1.6). Owing to the conservation law obeyed by the system (1.6), we may assume that the initial data $(\rho_0, \mathbf{u}_0, \vartheta_0, \mathbf{b}_0)$ satisfies

$$\int_{\mathbb{T}^3} (\rho_0 - \bar{\rho})\,dx = \int_{\mathbb{T}^3} \rho_0\mathbf{u}_0\,dx = \int_{\mathbb{T}^3} \mathbf{b}_0\,dx = 0, \tag{1.8}$$

which implies, for sufficiently regular solutions, that

$$\int_{\mathbb{T}^3} (\rho - \bar{\rho})\,dx = \int_{\mathbb{T}^3} \rho\mathbf{u}\,dx = \int_{\mathbb{T}^3} \mathbf{b}\,dx = 0, \quad \forall\, t > 0. \tag{1.9}$$

Our main result is stated as follows:

Theorem 1.1 *For any $N \geq 4r + 7$ with $r > 2$. Assume that the initial data $(\rho_0, \mathbf{u}_0, \vartheta_0, \mathbf{b}_0)$ satisfies (1.8) and*

$$(\rho_0 - \bar{\rho}, \vartheta_0 - \bar{\vartheta}) \in H^N(\mathbb{T}^3), \quad c_0 \leq \rho_0, \vartheta_0 \leq c_0^{-1}, \quad (\mathbf{u}_0, \mathbf{b}_0) \in H^N(\mathbb{T}^3), \quad (1.10)$$

for some constant $c_0 > 0$. There exists a small constant $\varepsilon > 0$ such that if

$$\left\|(\rho_0 - \bar{\rho}, \vartheta_0 - \bar{\vartheta})\right\|_{H^N} + \|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^N} \leq \varepsilon,$$

then the system (1.6) admits a global solution $(\rho - \bar{\rho}, \mathbf{u}, \vartheta - \bar{\vartheta}, \mathbf{b}) \in C([0, \infty); H^N)$. Moreover, for any $t \geq 0$ and $r + 4 \leq \beta < N$, it holds that

$$\|(\rho - \bar{\rho})(t)\|_{H^\beta} + \|\mathbf{u}(t)\|_{H^\beta} + \left\|(\vartheta - \bar{\vartheta})(t)\right\|_{H^\beta} + \|\mathbf{b}(t)\|_{H^\beta} \leq C(1 + t)^{-\frac{3(N-\beta)}{2(N-r-4)}}.$$

Remark 1.2 As established in [3, Section 2], almost all vector fields $\mathbf{n}$ in $\mathbb{R}^3$ satisfy the Diophantine condition (1.3). Of course, there are vectors that do not satisfy this condition such as those with all components being rational numbers.

Remark 1.3 A similar result may be proved if the physical coefficients μ, λ, and κ depend smoothly on the density. Here, we assume them to be constants to avoid more technicalities.

Remark 1.4 In a forthcoming paper, [42], we will use the method developed in this paper to show the global well-posedness of the inviscid, heat-conductive and resistive compressible MHD equations.

Remark 1.5 We believe that it is a challenging problem to drop the Diophantine condition (1.3) in our main theorem.

1.3 Scheme of the Proof and Organization of the Paper

Now let us outline the main points of the study and explain some of the major difficulties and techniques presented in this article. We shall use five subsections to complete the proof of Theorem 1.1. By the continuity argument, the existence of the global solutions can be proven by combining the local existence and the *a priori* estimates. The local well-posedness can be proved by a standard energy method. The key point is to obtain the *a priori* estimates of the solutions. Due to the lack of dissipation on the equations of the density and the magnetic field, the situation here is more complicated than the incompressible case in [3]. The first step is to make the basic energy estimate, see Proposition 2.1. In the second step, we shall obtain the high-order energy estimate, see Proposition 2.3. In the third step, we capture the dissipation of the magnetic field, see Proposition 2.4. In the fourth step, we introduce the so-called effective velocity to capture the hidden dissipation of the combined quantity $a + \mathbf{n} \cdot \mathbf{b}$, see Proposition 2.5. In the fifth step, we succeed in estimating the nonlinear terms and get the Lyapunov-type inequality in time for energy norms, see Proposition 2.6. Finally, we use the continuity argument to prove the global solutions of (2.3). Simultaneously, we use the interpolation inequality to get an algebraic decay of the high-order norm of the solutions.

2 Proof of the Main Theorem

Given the initial data $(\rho_0 - \bar{\rho}, \mathbf{u}_0, \vartheta_0 - \bar{\vartheta}, \mathbf{b}_0) \in H^N(\mathbb{T}^3)$ with N suitably large, the local well-posedness of (1.6) is nowadays standard. On the one hand, we could directly prove the local well-posedness to (1.6) based on linearization, construction of approximating solutions, and application of compactness argument. On the other hand, one can also refer to [19] for a similar result. Thus, we may assume that there exists $T > 0$ such that the system (1.6) has a unique solution $(\rho - \bar{\rho}, \mathbf{u}, \vartheta - \bar{\vartheta}, \mathbf{b}) \in C([0, T]; H^N)$. Moreover, it holds that

$$\frac{1}{2}c_0 \le \rho(t, x), \vartheta(t, x) \le 2c_0^{-1}, \quad \text{for any } t \in [0, T]. \tag{2.1}$$

Therefore, by a continuity argument, to prove Theorem 1.1, it suffices to derive the a priori estimates. To do this, we may assume that

$$\sup_{t\in[0,T]} \left(\left\|(\rho - \bar{\rho}, \vartheta - \bar{\vartheta})\right\|_{H^N} + \|(\mathbf{u}, \mathbf{b})\|_{H^N}\right) \le \delta, \tag{2.2}$$

for some $0 < \delta < 1$ to be determined later.

To simplify the notation, we assume $\bar{\rho} = \bar{\vartheta} = 1$ and define

$$a \stackrel{\text{def}}{=} \rho - 1, \quad \theta \stackrel{\text{def}}{=} \vartheta - 1, \quad I(a) \stackrel{\text{def}}{=} \frac{a}{1+a}, \quad \text{and} \quad J(a) = \ln(1 + a).$$

Then the system (1.6) can be reformulated as

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{u} = f_1, \\ \partial_t \mathbf{u} - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\operatorname{div} \mathbf{u} + \nabla a + \nabla\theta = \mathbf{n} \cdot \nabla\mathbf{b} - \nabla(\mathbf{n} \cdot \mathbf{b}) + f_2, \\ \partial_t\theta - \kappa\Delta\theta + \operatorname{div} \mathbf{u} = f_3, \\ \partial_t \mathbf{b} = \mathbf{n} \cdot \nabla\mathbf{u} - \mathbf{n}\operatorname{div} \mathbf{u} + f_4, \\ \operatorname{div} \mathbf{b} = 0, \\ (a, \mathbf{u}, \theta, \mathbf{b})|_{t=0} = (a_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} f_1 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla a - a\operatorname{div} \mathbf{u}, \\ f_2 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla\mathbf{u} + \mathbf{b} \cdot \nabla\mathbf{b} + \mathbf{b}\nabla\mathbf{b} + I(a)\nabla a - \theta\nabla J(a) \\ &\quad -I(a)(\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\operatorname{div} \mathbf{u}) - I(a)(\mathbf{n} \cdot \nabla\mathbf{b} + \mathbf{b} \cdot \nabla\mathbf{b} - \mathbf{n}\nabla\mathbf{b} - \mathbf{b}\nabla\mathbf{b}), \\ f_3 &\stackrel{\text{def}}{=} -\operatorname{div} (\theta\mathbf{u}) - \kappa I(a)\Delta\theta + \frac{2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div} \mathbf{u})^2}{1 + a}, \\ f_4 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla\mathbf{b} + \mathbf{b} \cdot \nabla\mathbf{u} - \mathbf{b}\operatorname{div} \mathbf{u}. \end{aligned} \tag{2.4}$$

In what follows, we divide the proof of Theorem 1.1 into five subsections, which we shall admit for the time being.

2.1 Basic Energy Estimates

The first subsection is concerned with the basic energy estimates. We will prove the following proposition.

Proposition 2.1 *Let $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0, T]; H^N)$ be a solution to the system (2.3). There holds the following basic energy inequality.*

$$\|(a, \mathbf{u}, \theta, \mathbf{b})\|_{L^2}^2 + \mu \|\nabla \mathbf{u}\|_{L^2}^2 + (\lambda + \mu) \int_0^t \|\operatorname{div} \mathbf{u}\|_{L^2}^2 \, dt' + \kappa \int_0^t \|\nabla \theta\|_{L^2}^2 \, dt' \\ \leq C \|(a_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)\|_{L^2}^2 . \tag{2.5}$$

2.2 High-Order Energy Estimates

In this subsection, we derive the high-order energy estimates.

Proposition 2.2 *Let $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0, T]; H^N)$ be a solution to the system (2.3). For any $0 \leq m \leq N$, there holds*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\|(\Lambda^m a, \Lambda^m \mathbf{u}, \Lambda^m \theta, \Lambda^m \mathbf{b})\right\|_{L^2}^2 + \mu \left\|\Lambda^m \nabla \mathbf{u}\right\|_{L^2}^2 \\ &\quad + (\lambda + \mu) \left\|\Lambda^m \operatorname{div} \mathbf{u}\right\|_{L^2}^2 + \kappa \left\|\Lambda^m \nabla \theta\right\|_{L^2}^2 \\ &\leq C\left|\int_{\mathbb{T}^3} \Lambda^m f_1 \cdot \Lambda^m a \, dx\right| + C\left|\int_{\mathbb{T}^3} \Lambda^m f_2 \cdot \Lambda^m \mathbf{u} \, dx\right| \\ &\quad + C\left|\int_{\mathbb{T}^3} \Lambda^m f_3 \cdot \Lambda^m \theta \, dx\right| + C\left|\int_{\mathbb{T}^3} \Lambda^m f_4 \cdot \Lambda^m \mathbf{b} \, dx\right|. \end{aligned} \tag{2.6}$$

Proposition 2.3 *Let $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0, T]; H^N)$ be a solution to the system (2.3). For any $0 \leq \ell \leq N$, there holds*

$$\frac{1}{2} \frac{d}{dt} \|(a, \mathbf{u}, \theta, \mathbf{b})\|_{H^\ell}^2 + \mu \|\nabla \mathbf{u}\|_{H^\ell}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}\|_{H^\ell}^2 + \kappa \|\nabla \theta\|_{H^\ell}^2 \\ \leq C Y_\infty(t) \|(a, \mathbf{u}, \theta, \mathbf{b})\|_{H^\ell}^2 \tag{2.7}$$

with

$$\begin{aligned} Y_\infty(t) \stackrel{\text{def}}{=} &\|(\Delta \mathbf{u}, \Delta \theta)\|_{L^\infty} + (1 + \|a\|_{L^\infty}^2) \|(a, \theta, \mathbf{b})\|_{L^\infty}^2 \\ &+ (1 + \|a\|_{L^\infty}) \|(\nabla a, \nabla \mathbf{u}, \nabla \theta, \nabla \mathbf{b})\|_{L^\infty} \\ &+ (1 + \|(a, \mathbf{b})\|_{L^\infty}^2 + \|\nabla \mathbf{u}\|_{L^\infty}^2) \|(\nabla a, \nabla \mathbf{u}, \nabla \theta, \nabla \mathbf{b})\|_{L^\infty}^2 . \end{aligned} \tag{2.8}$$

2.3 The Dissipativity of the Magnetic Field

We shall find the hidden dissipativity of the magnetic field in this subsection.

Proposition 2.4 *Let* $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0, T]; H^N)$ *be a solution to the system* (2.3). *There holds*

$$\begin{aligned}&\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 - \frac{d}{dt}\sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx\\&\quad\le C\,\|(\mathbf{u},\theta)\|_{H^{r+5}}^2 + C\,\|a+\mathbf{n}\cdot\mathbf{b}\|_{H^{r+4}}^2\,.\end{aligned}\tag{2.9}$$

2.4 The Dissipativity of the Combined Quantity $a + \mathbf{n}\cdot\mathbf{b}$

In the fourth subsection, we shall find the hidden dissipativity of the combined quantity $a + \mathbf{n}\cdot\mathbf{b}$. In order to do so, we introduce two unknown good functions

$$d \stackrel{\text{def}}{=} a + \mathbf{n}\cdot\mathbf{b}, \qquad \text{and} \quad \mathbf{G} \stackrel{\text{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{\nu}\Delta^{-1}\nabla d. \tag{2.10}$$

Then, we will prove the following crucial proposition.

Proposition 2.5 *Let* $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0, T]; H^N)$ *be a solution to the system* (2.3). *For any* $0 \le m \le N$*, there holds*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^m d\right\|_{L^2}^2 + \left\|\Lambda^m\mathbf{G}\right\|_{L^2}^2\right) + \frac{1}{\nu}\left\|\Lambda^m d\right\|_{L^2}^2 + \nu\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2\\&\quad\le C\left(\|(\mathbf{u},\theta)\|_{H^{m+1}}^2 + \|(f_1, f_4)\|_{H^m}^2 + \|f_2\|_{H^{m-1}}^2 + \|f_2\|_{L^2}^2\right).\end{aligned}\tag{2.11}$$

2.5 The Derivation of the Differential Inequality for the Energy

In this subsection, we shall use the product laws in Sobolev spaces to bound the nonlinear terms involved f_1, f_2, f_3, and f_4 in Propositions 2.2, and 2.5. Moreover, we aim to prove the following differential inequality from which we can get the Lyapunov-type inequality in time for energy norms. We define the energy as

$$\mathscr{E}(t) = \widetilde{c}\left(\|(a, \mathbf{u}, \theta, \mathbf{b}, d, \mathbf{G})\|_{H^{r+4}}^2\right) - \sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx,$$

and the dissipation as

$$\begin{aligned}\mathscr{D}(t) =&\widetilde{c}\Big(\frac{1}{\nu}\,\|d\|_{H^{r+4}}^2 + \mu\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2 + (\lambda+\mu)\,\|\mathrm{div}\,\mathbf{u}\|_{H^{r+4}}^2\\&+ \kappa\,\|\nabla\theta\|_{H^{r+4}}^2 + \nu\,\|\nabla\mathbf{G}\|_{H^{r+4}}^2\Big) + \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\,.\end{aligned}$$

Then, we will prove the following proposition.

Proposition 2.6 *Let* $(a, \mathbf{u}, \theta, \mathbf{b}) \in C([0,T]; H^N)$ *be a solution to the system (2.3) and* $\widetilde{c}$ *be a suitable large constant determined later, then there holds*

$$\frac{d}{dt}\mathscr{E}(t) + \mathscr{D}(t) \le C\delta^2 \|(\nabla\mathbf{u}, \nabla\theta)\|^2_{H^{r+4}} + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{b}\|^2_{H^{r+3}} + C\delta^2 \|d\|^2_{H^{r+4}}. \tag{2.12}$$

With the above six propositions in hand, we now begin to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Thanks to $a = d - \mathbf{n}\cdot\mathbf{b}$ and

$$\Big|\sum_{0\le s\le r+3}\int_{\mathbb{T}^3} \Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx\Big| \le C\|\mathbf{u}\|_{H^{r+3}}\|\mathbf{b}\|_{H^{r+4}}, \tag{2.13}$$

we can take $\widetilde{c} > 1$ such that

$$\mathscr{E}(t) \ge \|(\mathbf{u}, \theta, \mathbf{b}, d, \mathbf{G})\|^2_{H^{r+4}}.$$

Hence, by choosing $\delta > 0$ small enough, we can get from (2.12) that

$$\frac{d}{dt}\mathscr{E}(t) + \frac{1}{2}\mathscr{D}(t) \le 0. \tag{2.14}$$

For any $N \ge 4r + 7$, by the interpolation inequality, we have

$$\|\mathbf{b}\|^2_{H^{r+4}} \le \|\mathbf{b}\|^{\frac{3}{2}}_{H^3}\|\mathbf{b}\|^{\frac{1}{2}}_{H^N} \le C\delta^{\frac{1}{2}}\|\mathbf{n}\cdot\nabla\mathbf{b}\|^{\frac{3}{2}}_{H^{r+3}}$$

which further implies that

$$\begin{aligned}
\mathscr{E}(t) \le& C(\|d\|^2_{H^{r+4}} + \|(\mathbf{u}, \theta, \mathbf{G})\|^2_{H^{r+4}} + \|\mathbf{b}\|^2_{H^{r+4}})\\
\le& C\|d\|^{\frac{3}{2}}_{H^{r+4}}\|d\|^{\frac{1}{2}}_{H^{r+4}} + C\|(\mathbf{u}, \theta, \mathbf{G})\|^{\frac{3}{2}}_{H^3}\|(\mathbf{u}, \theta, \mathbf{G})\|^{\frac{1}{2}}_{H^N} + C\|\mathbf{b}\|^{\frac{3}{2}}_{H^3}\|\mathbf{b}\|^{\frac{1}{2}}_{H^N}\\
\le& C\|d\|^{\frac{3}{2}}_{H^{r+4}}\|d\|^{\frac{1}{2}}_{H^N} + C\|(\mathbf{u}, \theta, \mathbf{G})\|^{\frac{3}{2}}_{H^3}\|(\mathbf{u}, \theta, \mathbf{G})\|^{\frac{1}{2}}_{H^N} + C\|\mathbf{b}\|^{\frac{3}{2}}_{H^3}\|\mathbf{b}\|^{\frac{1}{2}}_{H^N}\\
\le& C\delta^{\frac{1}{2}}\|d\|^{\frac{3}{2}}_{H^{r+4}} + C\delta^{\frac{1}{2}}\|\nabla(\mathbf{u}, \theta, \mathbf{G})\|^{\frac{3}{2}}_{H^{r+4}} + C\delta^{\frac{1}{2}}\|\mathbf{n}\cdot\nabla\mathbf{b}\|^{\frac{3}{2}}_{H^{r+3}}\\
\le& (\mathscr{D}(t))^{\frac{3}{4}}.
\end{aligned}$$

So, we get a Laputa-type inequality

$$\frac{d}{dt}\mathscr{E}(t) + c(\mathscr{E}(t))^{\frac{4}{3}} \le 0.$$

Solving this inequality yields

$$\mathscr{E}(t) \le C(1+t)^{-3}. \tag{2.15}$$

Now, taking $\ell = N$ in (2.7) and using the embedding relation give

$$\frac{d}{dt}\|(a,\mathbf{u},\theta,\mathbf{b})\|_{H^N}^2+\mu\|\nabla\mathbf{u}\|_{H^N}^2+(\lambda+\mu)\|\mathrm{div}\,\mathbf{u}\|_{H^N}^2+\kappa\|\nabla\theta\|_{H^N}^2 \\ \leq CZ(t)\|(a,\mathbf{u},\theta,\mathbf{b})\|_{H^N}^2 \tag{2.16}$$

with

$$Z(t)\stackrel{\text{def}}{=}\|(a,\mathbf{u},\theta,\mathbf{b})\|_{H^4}+(1+\|(a,\mathbf{b})\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2)\|(a,\mathbf{u},\theta,\mathbf{b})\|_{H^3}^2.$$

It follows from (2.15) that

$$\int_0^t Z(\tau)\,d\tau \leq C, \tag{2.17}$$

thus, exploiting the Gronwall inequality to (2.16) implies

$$\|(a,\mathbf{u},\theta,\mathbf{b})\|_{H^N}^2 \leq C\|(a_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0)\|_{H^N}^2 \leq C\varepsilon^2. \tag{2.18}$$

Taking ε small enough so that $C\varepsilon \leq \delta/2$, we deduce from a continuity argument that the local solution can be extended to a global one in time.

Moreover, from (2.15), we have the following decay rate

$$\|a(t)\|_{H^{r+4}}+\|\mathbf{u}(t)\|_{H^{r+4}}+\|\theta(t)\|_{H^{r+4}}+\|\mathbf{b}(t)\|_{H^{r+4}} \leq C(1+t)^{-\frac{3}{2}}.$$

Thus, for any $\beta > r+4$, choosing $N > \beta$ and using the following interpolation inequality

$$\|f(t)\|_{H^\beta} \leq \|f(t)\|_{H^{r+4}}^{\frac{N-\beta}{N-r-4}}\|f(t)\|_{H^N}^{\frac{\beta-r-4}{N-r-4}},$$

we can get the decay rate for the higher-order energy

$$\|a(t)\|_{H^\beta}+\|\mathbf{u}(t)\|_{H^\beta}+\|\theta(t)\|_{H^\beta}+\|\mathbf{b}(t)\|_{H^\beta} \leq C(1+t)^{-\frac{3(N-\beta)}{2(N-r-4)}}.$$

This completes the proof of Theorem 1.1. □

3 Proof of the Propositions

The left of the work is to prove Propositions 2.1–2.6.

3.1 Proof of Proposition 2.1

Proof First, we can reformulate the mass equation $(1.6)_1$ as

$$\rho\mathrm{div}\,\mathbf{u} = -\rho(\partial_t \ln\rho + \mathbf{u}\cdot\nabla\ln\rho).$$

Multiplying the above equation by R and integrating by parts, we get

$$R\int_{\mathbb{T}^3}\rho\mathrm{div}\,\mathbf{u}\,dx = -\frac{d}{dt}R\int_{\mathbb{T}^3}\rho\ln\rho\,dx = -\frac{d}{dt}R\int_{\mathbb{T}^3}(\rho\ln\rho-\rho+1)\,dx, \quad (3.1)$$

where in the second equality we used the fact that $\int_{\mathbb{T}^3}\rho\,dx$ is conserved over time. While multiplying the momentum equation $(1.6)_2$ by $\mathbf{u}$ and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\rho|\mathbf{u}|^2\,dx + \mu\int_{\mathbb{T}^3}|\nabla\mathbf{u}|^2\,dx + (\lambda+\mu)\int_{\mathbb{T}^3}|\mathrm{div}\,\mathbf{u}|^2\,dx - R\int_{\mathbb{T}^3}\rho\vartheta\mathrm{div}\,\mathbf{u}\,dx\\ &= \int_{\mathbb{T}^3}\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\,dx - \int_{\mathbb{T}^3}\mathbf{b}\nabla\mathbf{b}\cdot\mathbf{u}\,dx + \int_{\mathbb{T}^3}\mathbf{n}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\,dx - \int_{\mathbb{T}^3}\mathbf{n}\nabla\mathbf{b}\cdot\mathbf{u}\,dx.\end{aligned} \quad (3.2)$$

Next, integrating the energy equation $(1.6)_3$, multiplying the mass equation $(1.6)_1$ by $c_\nu\vartheta$ and integrating over $\mathbb{T}^3$, and then summing up the resultants, we get

$$\frac{d}{dt}c_\nu\int_{\mathbb{T}^3}\rho\vartheta\,dx + R\int_{\mathbb{T}^3}\rho\vartheta\mathrm{div}\,\mathbf{u}\,dx = \int_{\mathbb{T}^3}\left(\frac{\mu}{2}|\nabla\mathbf{u}+(\nabla\mathbf{u})^\top|^2+\lambda(\mathrm{div}\,\mathbf{u})^2\right)dx. \quad (3.3)$$

Multiplying the energy equation $(1.6)_3$ by ϑ^{-1} and then integrating by parts, using $(1.6)_1$ again, we get

$$\begin{aligned}&-\frac{d}{dt}c_\nu\int_{\mathbb{T}^3}\rho\ln\vartheta\,dx + \kappa\int_{\mathbb{T}^3}\frac{|\nabla\vartheta|^2}{|\vartheta|^2}\,dx - R\int_{\mathbb{T}^3}\rho\mathrm{div}\,\mathbf{u}\,dx\\ &= -\int_{\mathbb{T}^3}\frac{1}{\vartheta}\left(\frac{\mu}{2}|\nabla\mathbf{u}+(\nabla\mathbf{u})^\top|^2+\lambda(\mathrm{div}\,\mathbf{u})^2\right)dx.\end{aligned} \quad (3.4)$$

Similarly, multiplying the magnetic equation $(1.6)_4$ by $\mathbf{b}$ and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\mathbf{b}|^2\,dx + \int_{\mathbb{T}^3}\mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,dx + \int_{\mathbb{T}^3}\mathbf{b}\mathrm{div}\,\mathbf{u}\cdot\mathbf{b}\,dx\\ &= \int_{\mathbb{T}^3}\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,dx + \int_{\mathbb{T}^3}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,dx - \int_{\mathbb{T}^3}\mathbf{n}\mathrm{div}\,\mathbf{u}\cdot\mathbf{b}\,dx.\end{aligned} \quad (3.5)$$

Since $\mathbf{b}$ is divergence-free, it is easy to check that

$$\int_{\mathbb{T}^3} (\mathbf{n}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{b})\cdot\mathbf{u}\,dx+\int_{\mathbb{T}^3} (\mathbf{n}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{u})\cdot\mathbf{b}\,dx=0, \tag{3.6}$$

$$\int_{\mathbb{T}^3} (\mathbf{n}\nabla\mathbf{b}+\mathbf{b}\nabla\mathbf{b})\cdot\mathbf{u}\,dx+\int_{\mathbb{T}^3} \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,dx+\int_{\mathbb{T}^3} (\mathbf{b}\mathrm{div}\,\mathbf{u}+\mathbf{n}\mathrm{div}\,\mathbf{u})\cdot\mathbf{b}\,dx=0, \tag{3.7}$$

$$\int_{\mathbb{T}^3} |\nabla\mathbf{u}+(\nabla\mathbf{u})^\top|^2\,dx=2\int_{\mathbb{T}^3} (|\nabla\mathbf{u}|^2+(\mathrm{div}\,\mathbf{u})^2)\,dx. \tag{3.8}$$

Thus, putting (3.1)–(3.5) together gives

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{T}^3}\rho|\mathbf{u}|^2\,dx+R\int_{\mathbb{T}^3}(\rho\ln\rho-\rho+1)\,dx\\&+c_\nu\int_{\mathbb{T}^3}\rho(\vartheta-\ln\vartheta-1)\,dx+\frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{b}|^2\,dx\Big)\\&+C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\vartheta\|_{L^2}^2\right)\le 0.\end{aligned} \tag{3.9}$$

By the Taylor expansion, for fixed positive constant c_0, if $c_0\le\rho\le c_0^{-1}$ one has

$$\begin{aligned}&\rho\ln\rho-\rho+1\sim(\rho-1)^2,\ \text{ and }\ \rho(\vartheta-\ln\vartheta-1)\sim(\vartheta-1)^2\\&\text{as}\quad \rho\to 1\ \text{ and }\ \vartheta\to 1.\end{aligned} \tag{3.10}$$

Then, we can infer from (3.9) that there holds (2.5). □

3.2 Proof of Proposition 2.2

Proof For any $0\le m\le N$, denote $\Lambda^m f=(-\Delta)^{\frac{m}{2}}f$, especially, for $m=0$, we define $\Lambda^m f\overset{\text{def}}{=}f$. Applying Λ^m on both hand side of (2.3) and then taking L^2 inner product with $\Lambda^m a$, $\Lambda^m\mathbf{u}$, $\Lambda^m\theta$, $\Lambda^m\mathbf{b}$ respectively, we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|(\Lambda^m a,\Lambda^m\mathbf{u},\Lambda^m\theta,\Lambda^m\mathbf{b})\right\|_{L^2}^2+\mu\left\|\Lambda^m\nabla\mathbf{u}\right\|_{L^2}^2\\&\quad+(\lambda+\mu)\left\|\Lambda^m\mathrm{div}\,\mathbf{u}\right\|_{L^2}^2+\kappa\left\|\Lambda^m\nabla\theta\right\|_{L^2}^2\\&=\int_{\mathbb{T}^3}\Lambda^m f_1\cdot\Lambda^m a\,dx+\int_{\mathbb{T}^3}\Lambda^m\mathrm{div}\,\mathbf{u}\cdot\Lambda^m a\,dx+\int_{\mathbb{T}^3}\Lambda^m f_2\cdot\Lambda^m\mathbf{u}\,dx\\&\quad+\int_{\mathbb{T}^3}\Lambda^m\nabla a\cdot\Lambda^m\mathbf{u}\,dx+\int_{\mathbb{T}^3}\Lambda^m\nabla\theta\cdot\Lambda^m\mathbf{u}\,dx+\int_{\mathbb{T}^3}\Lambda^m(\mathbf{n}\cdot\nabla\mathbf{b})\cdot\Lambda^m\mathbf{u}\,dx\\&\quad-\int_{\mathbb{T}^3}\Lambda^m\nabla(\mathbf{n}\cdot\mathbf{b})\cdot\Lambda^m\mathbf{u}\,dx+\int_{\mathbb{T}^3}\Lambda^m f_3\cdot\Lambda^m\theta\,dx+\int_{\mathbb{T}^3}\Lambda^m\mathrm{div}\,\mathbf{u}\cdot\Lambda^m\theta\,dx\\&\quad+\int_{\mathbb{T}^3}\Lambda^m f_4\cdot\Lambda^m\mathbf{b}\,dx+\int_{\mathbb{T}^3}\Lambda^m(\mathbf{n}\cdot\nabla\mathbf{u})\cdot\Lambda^m\mathbf{b}\,dx-\int_{\mathbb{T}^3}\Lambda^m(\mathbf{n}\mathrm{div}\,\mathbf{u})\cdot\Lambda^m\mathbf{b}\,dx.\end{aligned} \tag{3.11}$$

It's easy to check that

$$\begin{aligned}
&\int_{\mathbb{T}^3} \Lambda^m \operatorname{div} \mathbf{u} \cdot \Lambda^m a \, dx + \int_{\mathbb{T}^3} \Lambda^m \nabla a \cdot \Lambda^m \mathbf{u} \, dx = 0; \\
&\int_{\mathbb{T}^3} \Lambda^m \nabla \theta \cdot \Lambda^m \mathbf{u} \, dx + \int_{\mathbb{T}^3} \Lambda^m \operatorname{div} \mathbf{u} \cdot \Lambda^m \theta \, dx = 0; \\
&\int_{\mathbb{T}^3} \Lambda^m (\mathbf{n} \cdot \nabla \mathbf{b}) \cdot \Lambda^m \mathbf{u} \, dx + \int_{\mathbb{T}^3} \Lambda^m (\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \Lambda^m \mathbf{b} \, dx = 0; \\
&\int_{\mathbb{T}^3} \Lambda^m \nabla (\mathbf{n} \cdot \mathbf{b}) \cdot \Lambda^m \mathbf{u} \, dx + \int_{\mathbb{T}^3} \Lambda^m (\mathbf{n} \operatorname{div} \mathbf{u}) \cdot \Lambda^m \mathbf{b} \, dx = 0,
\end{aligned}$$

which and (3.11) imply (2.6). We complete the proof of Proposition 2.2. □

3.3 Proof of Proposition 2.3

Proof For $\ell = 0$, the basic energy inequality implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|(a, \mathbf{u}, \mathbf{b}, \theta)\|_{L^2}^2 + \mu \|\nabla \mathbf{u}\|_{L^2}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \\
&\quad = \int_{\mathbb{T}^3} f_1 \cdot a \, dx + \int_{\mathbb{T}^3} f_2 \cdot \mathbf{u} \, dx + \int_{\mathbb{T}^3} f_3 \cdot \theta \, dx + \int_{\mathbb{T}^3} f_4 \cdot \mathbf{b} \, dx
\end{aligned} \tag{3.12}$$

where we have used the cancelations in (3.6) and (3.7).

By using the Hölder inequality, we can get directly that

$$\begin{aligned}
&\int_{\mathbb{T}^3} f_1 \cdot a \, dx \leq C(\|\nabla a\|_{L^\infty} + \|\nabla \mathbf{u}\|_{L^\infty}) \|(a, \mathbf{u})\|_{L^2}^2, \\
&\int_{\mathbb{T}^3} f_2 \cdot \mathbf{u} \, dx \leq C(\|\nabla a\|_{L^\infty} + \|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty} + \|\Delta \mathbf{u}\|_{L^\infty}) \|(a, \mathbf{u}, \theta, \mathbf{b})\|_{L^2}^2, \\
&\int_{\mathbb{T}^3} f_3 \cdot \theta \, dx \leq C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \theta\|_{L^\infty} + \|\Delta \theta\|_{L^\infty}) \|(\mathbf{u}, \theta)\|_{L^2}^2 \\
&\qquad\qquad + C(1 + \|a\|_{L^\infty}) \|\nabla \mathbf{u}\|_{L^\infty} \|(\nabla \mathbf{u}, \theta)\|_{L^2}^2, \\
&\int_{\mathbb{T}^3} f_4 \cdot \mathbf{b} \, dx \leq C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty}) \|(\mathbf{u}, \mathbf{b})\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (3.12), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|(a, \mathbf{u}, \theta, \mathbf{b})\|_{L^2}^2 + \mu \|\nabla \mathbf{u}\|_{L^2}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \\
&\quad \leq C(\|(\Delta \mathbf{u}, \Delta \theta)\|_{L^\infty} + (1 + \|a\|_{L^\infty}) \|\nabla \mathbf{u}\|_{L^\infty} \\
&\qquad + \|(\nabla a, \nabla \mathbf{u}, \nabla \theta, \nabla \mathbf{b})\|_{L^\infty})(\|(a, \theta, \mathbf{b})\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2).
\end{aligned} \tag{3.13}$$

To obtain the high order energy estimates, we need to reformulate (1.6) into another new form. So, we define

$$\bar{\mu}(\rho) \overset{\text{def}}{=} \frac{\mu}{\rho}, \quad \bar{\lambda}(\rho) \overset{\text{def}}{=} \frac{\lambda+\mu}{\rho}, \quad \bar{\kappa}(\rho) \overset{\text{def}}{=} \frac{\kappa}{\rho}, \quad I(a) \overset{\text{def}}{=} \frac{a}{1+a}, \quad \text{and} \quad J(a)=\ln(1+a),$$

then direct calculation implies that

$$\begin{cases} a_t + \operatorname{div}\mathbf{u} = F_1, \\ \partial_t\mathbf{u} - \operatorname{div}(\bar{\mu}(\rho)\nabla\mathbf{u}) - \nabla(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}) + \nabla a + \nabla\theta = \mathbf{n}\cdot\nabla\mathbf{b} - \nabla(\mathbf{n}\cdot\mathbf{b}) + F_2, \\ \partial_t\theta - \operatorname{div}(\bar{\kappa}(\rho)\nabla\theta) + \operatorname{div}\mathbf{u} = F_3, \\ \partial_t\mathbf{b} = \mathbf{n}\cdot\nabla\mathbf{u} - \mathbf{n}\operatorname{div}\mathbf{u} + F_4, \\ \operatorname{div}\mathbf{b} = 0, \\ (a, \mathbf{u}, \theta, \mathbf{b})|_{t=0} = (a_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0) \end{cases} \tag{3.14}$$

where

$$\begin{aligned} F_1 &\overset{\text{def}}{=} -\mathbf{u}\cdot\nabla a - a\operatorname{div}\mathbf{u}, \\ F_2 &\overset{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \mathbf{b}\nabla\mathbf{b} + I(a)\nabla a - \theta\nabla J(a) + \mu(\nabla I(a))\nabla\mathbf{u} \\ &\quad +(\lambda+\mu)(\nabla I(a))\operatorname{div}\mathbf{u} - I(a)(\mathbf{n}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{n}\nabla\mathbf{b} - \mathbf{b}\nabla\mathbf{b}), \\ F_3 &\overset{\text{def}}{=} -\operatorname{div}(\theta\mathbf{u}) - \kappa(\nabla I(a))\nabla\theta + \frac{2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2}{1+a}, \\ F_4 &\overset{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{b}\operatorname{div}\mathbf{u}. \end{aligned}$$

Throughout we make the assumption that

$$\sup_{t\in\mathbb{R}_+,\, x\in\mathbb{T}^3} |a(t,x)| \le \frac{1}{2} \tag{3.15}$$

which will enable us to use freely the following composition estimate

$$\|G(a)\|_{H^s} \le C\|a\|_{H^s}, \quad \text{for } G(0)=0 \text{ and any } s>0. \tag{3.16}$$

Note that as $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$, Condition (3.15) will be ensured by the fact that the constructed solution has a small norm in $H^2(\mathbb{T}^3)$.

For any $1 \le s \le \ell$, applying Λ^s on both hand side of (3.14) and then taking L^2 inner product with $\Lambda^s a$, $\Lambda^s\mathbf{u}$, $\Lambda^s\theta$, $\Lambda^s\mathbf{b}$ respectively gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|(\Lambda^s a, \Lambda^s\mathbf{u}, \Lambda^s\theta, \Lambda^s\mathbf{b})\right\|_{L^2}^2 - \int_{\mathbb{T}^3} \Lambda^s\operatorname{div}(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\Lambda^s\mathbf{u}\,dx \\ &\quad - \int_{\mathbb{T}^3} \Lambda^s\nabla(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u})\cdot\Lambda^s\mathbf{u}\,dx - \int_{\mathbb{T}^3} \Lambda^s\operatorname{div}(\bar{\kappa}(\rho)\nabla\theta)\cdot\Lambda^s\theta\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^3} \Lambda^s F_1 \cdot \Lambda^s a \, dx + \int_{\mathbb{T}^3} \Lambda^s F_2 \cdot \Lambda^s \mathbf{u} \, dx \\
&\quad + \int_{\mathbb{T}^3} \Lambda^s F_3 \cdot \Lambda^s \theta \, dx + \int_{\mathbb{T}^3} \Lambda^s F_4 \cdot \Lambda^s \mathbf{b} \, dx.
\end{aligned} \tag{3.17}
$$

For the second term of the left-hand side, we have

$$
\begin{aligned}
&- \int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\, (\bar{\mu}(\rho) \nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u} \, dx \\
&= \int_{\mathbb{T}^3} \Lambda^s (\bar{\mu}(\rho) \nabla \mathbf{u}) \cdot \nabla \Lambda^s \mathbf{u} \, dx \\
&= \int_{\mathbb{T}^3} \bar{\mu}(\rho) \nabla \Lambda^s \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx + \int_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho)] \nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx.
\end{aligned} \tag{3.18}
$$

Due to (2.1), we have for any $t \in [0, T]$ that

$$
\int_{\mathbb{T}^3} \bar{\mu}(\rho) \nabla \Lambda^s \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx \geq c_0^{-1} \mu \left\| \Lambda^{s+1} \mathbf{u} \right\|_{L^2}^2 . \tag{3.19}
$$

For the last term in (3.18), we first rewrite this term into

$$
\begin{aligned}
\int_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho)] \nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx &= \int_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho) - \mu + \mu] \nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx \\
&= - \int_{\mathbb{T}^3} [\Lambda^s, \mu I(a)] \nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx,
\end{aligned}
$$

then, with the aid of (3.16), we have

$$
\begin{aligned}
\left| \int_{\mathbb{T}^3} [\Lambda^s, \mu I(a)] \nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} \, dx \right| \leq & C \left\| \nabla \Lambda^s \mathbf{u} \right\|_{L^2} \left(\|\nabla I(a)\|_{L^\infty} \left\| \Lambda^s \mathbf{u} \right\|_{L^2} \right. \\
& \left. + \|\nabla \mathbf{u}\|_{L^\infty} \left\| \Lambda^s I(a) \right\|_{L^2} \right) \\
\leq & \frac{c_0^{-1}}{2} \mu \left\| \Lambda^{s+1} \mathbf{u} \right\|_{L^2}^2 + C \left(\|\nabla a\|_{L^\infty}^2 \left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2 \right. \\
& \left. + \|\nabla \mathbf{u}\|_{L^\infty}^2 \left\| \Lambda^s a \right\|_{L^2}^2 \right)
\end{aligned} \tag{3.20}
$$

where we have used the following lemmas for commutators and composite functions.

Lemma 3.21 ([18]) *Let $s > 0$, for any $f \in H^s(\mathbb{T}^3) \cap W^{1,\infty}(\mathbb{T}^3)$, $g \in H^{s-1}(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, there holds*

$$
\left\| [\Lambda^s, f \cdot \nabla] g \right\|_{L^2} \leq C \left(\|\nabla f\|_{L^\infty} \left\| \Lambda^s g \right\|_{L^2} + \left\| \Lambda^s f \right\|_{L^2} \|\nabla g\|_{L^\infty} \right).
$$

Lemma 3.22 ([34]) *Let M be a smooth function on $\mathbb{R}$ with $M(0) = 0$. For any $s > 0$, and $f \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, we have*

$$\|M(f)\|_{H^s} \leq C(1 + \|f\|_{L^\infty})^{[s]+1} \|f\|_{H^s}$$

where the constant C depends on $\sup_{k \leq [s]+2, |t| \leq \|f\|_{L^\infty}} \|M^k(t)\|_{L^\infty}$.

Putting (3.19) and (3.20) into (3.18) leads to

$$\begin{aligned}-\int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\bar{\mu}(\rho)\nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u}\, dx \geq & \frac{c_0^{-1}}{2} \mu \left\| \Lambda^{s+1} \mathbf{u} \right\|_{L^2}^2 \\ & - C \left(\|\nabla a\|_{L^\infty}^2 \left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty}^2 \left\| \Lambda^s a \right\|_{L^2}^2 \right).\end{aligned}$$

In the same manner, there holds

$$\begin{aligned}-\int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\bar{\kappa}(\rho)\nabla \theta) \cdot \Lambda^s \theta\, dx \geq & \frac{c_0^{-1}}{2} \kappa \left\| \Lambda^{s+1} \theta \right\|_{L^2}^2 \\ & - C \left(\|\nabla a\|_{L^\infty}^2 \left\| \Lambda^s \theta \right\|_{L^2}^2 + \|\nabla \theta\|_{L^\infty}^2 \left\| \Lambda^s a \right\|_{L^2}^2 \right).\end{aligned}$$

The last term on the left-hand side of (3.17) can be dealt with similarly, hence we can get that

$$\begin{aligned}& \frac{1}{2} \frac{d}{dt} \left\| (\Lambda^s a, \Lambda^s \mathbf{u}, \Lambda^s \theta, \Lambda^s \mathbf{b}) \right\|_{L^2}^2 + c_0^{-1} \mu \left\| \Lambda^{s+1} \mathbf{u} \right\|_{L^2}^2 + c_0^{-1} \kappa \left\| \Lambda^{s+1} \theta \right\|_{L^2}^2 \\ & \quad \leq C \left(\|\nabla a\|_{L^\infty}^2 \left\| (\Lambda^s \mathbf{u}, \Lambda^s \theta) \right\|_{L^2}^2 + \|(\nabla \mathbf{u}, \nabla \theta)\|_{L^\infty}^2 \left\| \Lambda^s a \right\|_{L^2}^2 \right) \\ & \qquad + C \left| \int_{\mathbb{T}^3} \Lambda^s F_1 \cdot \Lambda^s a\, dx \right| \\ & \qquad + C \left| \int_{\mathbb{T}^3} \Lambda^s F_2 \cdot \Lambda^s \mathbf{u}\, dx \right| + C \left| \int_{\mathbb{T}^3} \Lambda^s F_3 \cdot \Lambda^s \theta\, dx \right| + C \left| \int_{\mathbb{T}^3} \Lambda^s F_4 \cdot \Lambda^s \mathbf{b}\, dx \right|.\end{aligned} \tag{3.23}$$

In the following, we estimate successively each of terms on the right-hand side of (3.23).

For the first term in F_1, we rewrite it into

$$\begin{aligned}\int_{\mathbb{T}^3} \Lambda^s (\mathbf{u} \cdot \nabla a) \cdot \Lambda^s a\, dx = & \int_{\mathbb{T}^3} (\Lambda^s (\mathbf{u} \cdot \nabla a) - \mathbf{u} \cdot \nabla \Lambda^s a) \cdot \Lambda^s a\, dx \\ & + \int_{\mathbb{T}^3} \mathbf{u} \cdot \nabla \Lambda^s a \cdot \Lambda^s a\, dx \\ \stackrel{\text{def}}{=} & A_1 + A_2.\end{aligned} \tag{3.24}$$

By Lemma 3.21, we have

$$|A_1| \leq C \left\| [\Lambda^s, \mathbf{u} \cdot \nabla] a \right\|_{L^2} \left\| \Lambda^s a \right\|_{L^2}$$

$$\leq C\left(\|\nabla \mathbf{u}\|_{L^\infty}\left\|\Lambda^s a\right\|_{L^2}+\left\|\Lambda^s \mathbf{u}\right\|_{L^2}\|\nabla a\|_{L^\infty}\right)\left\|\Lambda^s a\right\|_{L^2}$$
$$\leq C\left(\|\nabla \mathbf{u}\|_{L^\infty}+\|\nabla a\|_{L^\infty}\right)\left(\left\|\Lambda^s a\right\|_{L^2}^2+\left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2\right). \tag{3.25}$$

For the term A_2, using the integration by part directly we get

$$|A_2|=\left|-\int_{\mathbb{T}^3} \operatorname{div} \mathbf{u}|\Lambda^s a|^2\, dx\right| \leq C\|\nabla \mathbf{u}\|_{L^\infty}\left\|\Lambda^s a\right\|_{L^2}^2. \tag{3.26}$$

To bound the second term in F_1, we need the following product laws in Sobolev spaces.

Lemma 3.27 ([18]) *Let $s \geq 0$, $f, g \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, it holds that*

$$\|fg\|_{H^s} \leq C(\|f\|_{L^\infty}\|g\|_{H^s}+\|g\|_{L^\infty}\|f\|_{H^s}). \tag{3.28}$$

Now, it follows from Lemma 3.27 that

$$\int_{\mathbb{T}^3} \Lambda^s(a \operatorname{div} \mathbf{u}) \cdot \Lambda^s a\, dx \leq C\left(\|\operatorname{div} \mathbf{u}\|_{L^\infty}\|a\|_{H^s}+\|\operatorname{div} \mathbf{u}\|_{H^s}\|a\|_{L^\infty}\right)\left\|\Lambda^s a\right\|_{L^2}$$
$$\leq \frac{\mu}{16}\|\Lambda^{s+1} \mathbf{u}\|_{H^s}^2+C(\|\nabla \mathbf{u}\|_{L^\infty}+\|a\|_{L^\infty}^2)\left\|\Lambda^s a\right\|_{L^2}^2 \tag{3.29}$$

where we have used the fact that

$$\|a\|_{H^s} \leq C\left\|\Lambda^s a\right\|_{L^2}, \quad \|\operatorname{div} \mathbf{u}\|_{H^s} \leq C\|\Lambda^{s+1} \mathbf{u}\|_{L^2}.$$

Collecting (3.25), (3.26) and (3.29), we get

$$\left|\int_{\mathbb{T}^3} \Lambda^s F_1 \cdot \Lambda^s a\, dx\right| \leq \frac{\mu}{16}\|\Lambda^{s+1} \mathbf{u}\|_{H^s}^2$$
$$+C(\|\nabla \mathbf{u}\|_{L^\infty}+\|\nabla a\|_{L^\infty}+\|a\|_{L^\infty}^2)\left(\left\|\Lambda^s a\right\|_{L^2}^2+\left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2\right). \tag{3.30}$$

For the first term in F_4, we can get by a similar derivation of (3.25), (3.26) that

$$\int_{\mathbb{T}^3} \Lambda^s(\mathbf{u} \cdot \nabla \mathbf{b}) \cdot \Lambda^s \mathbf{b}\, dx \leq C(\|\nabla \mathbf{u}\|_{L^\infty}+\|\nabla \mathbf{b}\|_{L^\infty})\left(\left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2+\left\|\Lambda^s \mathbf{b}\right\|_{L^2}^2\right). \tag{3.31}$$

For the last two terms in F_4, we get by a similar derivation of (3.29) that

$$\int_{\mathbb{T}^3} \Lambda^s(\mathbf{b} \cdot \nabla \mathbf{u}-\mathbf{b} \operatorname{div} \mathbf{u}) \cdot \Lambda^s a\, dx \leq \frac{\mu}{16}\|\Lambda^{s+1} \mathbf{u}\|_{H^s}^2+C\left(\|\nabla \mathbf{u}\|_{L^\infty}+\|\mathbf{b}\|_{L^\infty}^2\right)\left\|\Lambda^s \mathbf{b}\right\|_{L^2}^2. \tag{3.32}$$

Consequently, we have

$$\left|\int_{\mathbb{T}^3} \Lambda^s F_4 \cdot \Lambda^s \mathbf{b}\,dx\right| \le \frac{\mu}{16}\|\Lambda^{s+1}\mathbf{u}\|_{H^s}^2 + C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty}^2)\left(\left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2 + \left\|\Lambda^s \mathbf{b}\right\|_{L^2}^2\right). \tag{3.33}$$

In the following, we bound the terms in F_2. To do so, we write

$$\int_{\mathbb{T}^3} \Lambda^s F_2 \cdot \Lambda^s \mathbf{u}\,dx = \sum_{i=3}^{10} A_i \tag{3.34}$$

with

$$\begin{aligned}
A_3 &\overset{\text{def}}{=} -\int_{\mathbb{T}^3} \Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Lambda^s\mathbf{u}\,dx, \qquad A_4 \overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\Lambda^s\mathbf{u}\,dx,\\
A_5 &\overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(I(a)\nabla a)\cdot\Lambda^s\mathbf{u}\,dx, \qquad A_6 \overset{\text{def}}{=} -\int_{\mathbb{T}^3} \Lambda^s(\theta\nabla J(a))\cdot\Lambda^s\mathbf{u}\,dx,\\
A_7 &\overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(\mu(\nabla I(a))\nabla\mathbf{u})\cdot\Lambda^s\mathbf{u}\,dx,\\
A_8 &\overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s((\lambda+\mu)(\nabla I(a))\mathrm{div}\,\mathbf{u})\cdot\Lambda^s\mathbf{u}\,dx,\\
A_9 &\overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(I(a)(\mathbf{n}\cdot\nabla\mathbf{b}-\mathbf{n}\nabla\mathbf{b}))\cdot\Lambda^s\mathbf{u}\,dx,\\
A_{10} &\overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(I(a)(\mathbf{b}\cdot\nabla\mathbf{b}-\mathbf{b}\nabla\mathbf{b}))\cdot\Lambda^s\mathbf{u}\,dx.
\end{aligned}$$

The term A_3 can be bounded in the same way as (3.24) so that

$$|A_3| \le C\,\|\nabla\mathbf{u}\|_{L^\infty}\left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2. \tag{3.35}$$

We next deal with the term A_4. In view of $\mathrm{div}\,\mathbf{b}=0$, we have

$$\begin{aligned}
|A_4| &= \left|\int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\mathbf{b}\otimes\mathbf{b})\cdot\Lambda^s\mathbf{u}\,dx\right|\\
&\le C\|\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^s}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\
&\le \frac{\mu}{16}\|\Lambda^{s+1}\mathbf{u}\|_{H^s}^2 + C\|\mathbf{b}\|_{L^\infty}^2\left\|\Lambda^s\mathbf{b}\right\|_{L^2}^2.
\end{aligned} \tag{3.36}$$

We now turn to bound the terms involving composition functions in F_2.

For A_5 and A_6, by Lemmas 3.22 and 3.27, we have

$$\begin{aligned}
|A_5| &\le C\big(\|\nabla a\|_{L^\infty}\|I(a)\|_{H^{s-1}} + \|\nabla a\|_{H^{s-1}}\|I(a)\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\
&\le \frac{\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} + C(\|\nabla a\|_{L^\infty}^2 + \|a\|_{L^\infty}^2)\left\|\Lambda^s a\right\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}|A_6| \le & C\big(\|\nabla J(a)\|_{L^\infty}\|\theta\|_{H^{s-1}} + \|\nabla J(a)\|_{H^{s-1}}\|\theta\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\ \le & \frac{\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} + C(\|\nabla a\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2)\left(\left\|\Lambda^s a\right\|_{L^2}^2 + \left\|\Lambda^s\theta\right\|_{L^2}^2\right).\end{aligned}$$

Similarly, for the terms A_7, A_8 and A_9, there hold

$$\begin{aligned}|A_7| + |A_8| \le & C\big(\|\nabla I(a)\|_{L^\infty}\left\|\Lambda^s\mathbf{u}\right\|_{L^2} + \|\nabla I(a)\|_{H^{s-1}}\|\nabla\mathbf{u}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\ \le & \frac{\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\left(\|\nabla a\|_{L^\infty}^2\left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2\left\|\Lambda^s a\right\|_{L^2}^2\right),\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}|A_9| \le & C\big(\|I(a)\|_{L^\infty}\|\nabla\mathbf{b}\|_{H^{s-1}} + \|I(a)\|_{H^{s-1}}\|\nabla\mathbf{b}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\ \le & \frac{\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\left(\|a\|_{L^\infty}^2\left\|\Lambda^s\mathbf{b}\right\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^\infty}^2\left\|\Lambda^s a\right\|_{L^2}^2\right).\end{aligned} \tag{3.38}$$

For the last term A_{10}, we use Lemmas 3.22 and 3.27 again to get

$$|A_{10}| \le C\big(\|I(a)\|_{L^\infty}\|\mathbf{b}\nabla\mathbf{b}\|_{H^{s-1}} + \|I(a)\|_{H^{s-1}}\|\mathbf{b}\nabla\mathbf{b}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}. \tag{3.39}$$

Noting that

$$\|\mathbf{b}\nabla\mathbf{b}\|_{H^{s-1}} \le C\|\mathbf{b}\|_{L^\infty}^2\|\mathbf{b}\|_{H^s}^2 \le C\|\mathbf{b}\|_{L^\infty}^2\left\|\Lambda^s\mathbf{b}\right\|_{L^2}^2,$$

putting it into (3.39) leads to

$$|A_{10}| \le \frac{\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\left(\|a\|_{L^\infty}^2\|\mathbf{b}\|_{L^\infty}^2\left\|\Lambda^s\mathbf{b}\right\|_{L^2}^2 + \|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^\infty}^2\left\|\Lambda^s a\right\|_{L^2}^2\right). \tag{3.40}$$

Inserting $A_3 - A_{10}$ into (3.34), we get

$$\begin{aligned}\left|\int_{\mathbb{T}^3}\Lambda^s F_2\cdot\Lambda^s\mathbf{u}\,dx\right| \le & \frac{3\mu}{8}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C(\|\nabla\mathbf{u}\|_{L^\infty}\\ & + \|(\nabla a, \nabla\mathbf{u}, \nabla\mathbf{b})\|_{L^\infty}^2 + \|(a, \mathbf{b})\|_{L^\infty}^2\\ & + \|a\|_{L^\infty}^2\|\mathbf{b}\|_{L^\infty}^2 + \|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^\infty}^2)\left\|(\Lambda^s a, \Lambda^s\mathbf{u}, \Lambda^s\mathbf{b})\right\|_{L^2}^2.\end{aligned} \tag{3.41}$$

Finally, we have to bound the terms in F_3. We first rewrite

$$\int_{\mathbb{T}^3}\Lambda^s F_3\cdot\Lambda^s\theta\,dx = A_{11} + A_{12} + A_{13} \tag{3.42}$$

with

$$A_{11} \overset{\text{def}}{=} -\int_{\mathbb{T}^3} \Lambda^s(\operatorname{div}(\theta\mathbf{u})) \cdot \Lambda^s\theta\, dx,$$
$$A_{12} \overset{\text{def}}{=} -\kappa\int_{\mathbb{T}^3} \Lambda^s((\nabla I(a))\nabla\theta) \cdot \Lambda^s\theta\, dx,$$
$$A_{13} \overset{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s\Big(\frac{2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2}{1+a}\Big) \cdot \Lambda^s\theta\, dx. \tag{3.43}$$

The term A_{11} can be bound the same as (3.30)

$$|A_{11}| \le \frac{\mu}{16}\|\Lambda^{s+1}\mathbf{u}\|_{H^s}^2 + C(\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\theta\|_{L^\infty}^2)\left(\left\|\Lambda^s\theta\right\|_{L^2}^2 + \left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2\right). \tag{3.44}$$

The term A_{12} can be bound the same as A_7, A_8 such that

$$|A_{12}| \le \frac{\kappa}{16}\left\|\Lambda^{s+1}\theta\right\|_{L^2}^2 + C\left(\|\nabla a\|_{L^\infty}^2\left\|\Lambda^s\theta\right\|_{L^2}^2 + \|\nabla\theta\|_{L^\infty}^2\left\|\Lambda^s a\right\|_{L^2}^2\right). \tag{3.45}$$

For the last term A_{13}, we get that

$$\begin{aligned} A_{13} &= \int_{\mathbb{T}^3} \Lambda^s\Big((1+I(a))(2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2)\Big) \cdot \Lambda^s\theta\, dx \\ &= \int_{\mathbb{T}^3} \Lambda^s\Big(2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2\Big) \cdot \Lambda^s\theta\, dx \\ &\quad + \int_{\mathbb{T}^3} \Lambda^s\Big(I(a)(2\mu|D(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2)\Big) \cdot \Lambda^s\theta\, dx \\ &\overset{\text{def}}{=} A_{13}^{(1)} + A_{13}^{(2)}. \end{aligned} \tag{3.46}$$

In view of Lemma 3.27, we have

$$\begin{aligned} \left|A_{13}^{(1)}\right| &\le C\big(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^{s-1}}\big)\left\|\Lambda^{s+1}\theta\right\|_{L^2} \\ &\le \frac{\kappa}{16}\|\Lambda^{s+1}\theta\|_{H^s}^2 + C\|\nabla\mathbf{u}\|_{L^\infty}^2\left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \left|A_{13}^{(2)}\right| &\le C\|I(a)\|_{L^\infty}\||\nabla\mathbf{u}|^2\|_{H^{s-1}} + \||\nabla\mathbf{u}|^2\|_{L^\infty}\|I(a)\|_{H^{s-1}}\left\|\Lambda^{s+1}\theta\right\|_{L^2} \\ &\le \frac{\kappa}{16}\|\Lambda^{s+1}\theta\|_{H^s}^2 + C\big(\|a\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}\|_{H^s}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^4\|a\|_{H^s}^2\big). \end{aligned} \tag{3.48}$$

Inserting the above two estimates into (3.46), we obtain

$$\left|A_{13}\right| \le \frac{\kappa}{8}\|\Lambda^{s+1}\theta\|_{H^s}^2 + C(1 + \|a\|_{L^\infty}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2)\|\nabla\mathbf{u}\|_{L^\infty}^2(\|\mathbf{u}\|_{H^s}^2 + \|a\|_{H^s}^2). \tag{3.49}$$

Collecting (3.44), (3.45), and (3.49), we have

$$\begin{aligned}\left|\int_{\mathbb{T}^3} \Lambda^s F_3 \cdot \Lambda^s \theta\, dx\right| \leq & \frac{\mu}{16}\|\Lambda^{s+1}\mathbf{u}\|_{H^s}^2 + \frac{3\kappa}{16}\left\|\Lambda^{s+1}\theta\right\|_{L^2}^2 \\ & + C(1+\|a\|_{L^\infty}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2)\|\nabla\mathbf{u}\|_{L^\infty}^2\|(a,\mathbf{u})\|_{H^s}^2 \\ & + C(\|(\nabla\mathbf{u},\nabla\theta)\|_{L^\infty} + \|(\theta,\nabla a,\nabla\theta)\|_{L^\infty}^2)\|(a,\mathbf{u},\theta)\|_{H^s}^2.\end{aligned} \tag{3.50}$$

Plugging (3.30), (3.33), (3.41), and (3.50) into (3.23) and then summing up (3.13) over $1 \leq s \leq \ell$, we arrive at the desired estimate (2.7). Consequently, we complete the proof of proposition 2.3. □

3.4 Proof of Proposition 2.4.

In this subsection, we shall reveal the hidden dissipativity of the magnetic field. Define the projector operator

$$\mathbb{P} = \mathbb{I} - \mathbb{Q} = \mathbb{I} - \nabla\Delta^{-1}\mathrm{div}\,.$$

It's straightforward to check that

$$\begin{aligned}\mathbb{P}\,(\mathbf{n}\cdot\nabla\mathbf{b}) =& \mathbf{n}\cdot\nabla\mathbf{b}, \quad \text{thus} \quad \mathbb{Q}\,(\mathbf{n}\cdot\nabla\mathbf{b}) = 0, \\ \mathbb{P}\,(\nabla(\mathbf{n}\cdot\mathbf{b})) =& 0, \quad \text{thus} \quad \mathbb{Q}\,(\nabla(\mathbf{n}\cdot\mathbf{b})) = \nabla(\mathbf{n}\cdot\mathbf{b}).\end{aligned} \tag{3.51}$$

Applying the projector operator $\mathbb{P}$ to both hand sides of the second equation in (2.3) gives

$$\partial_t\mathbb{P}\mathbf{u} - \mu\Delta\mathbb{P}\mathbf{u} = \mathbf{n}\cdot\nabla\mathbf{b} + \mathbb{P}f_2. \tag{3.52}$$

Applying $\Lambda^s(0 \leq s \leq r+3)$ to (3.52), and multiplying it by $\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})$ then integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 =& \int_{\mathbb{T}^3}\Lambda^s\partial_t\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \\ & -\mu\int_{\mathbb{T}^3}\Lambda^s\Delta\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx - \int_{\mathbb{T}^3}\Lambda^s(\mathbb{P}f_2)\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx.\end{aligned} \tag{3.53}$$

Thanks to the Hölder inequality, Young's inequality, and the embedding relation, we have

$$\begin{aligned}\int_{\mathbb{T}^3}\Lambda^s\Delta\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \leq & \frac{1}{8}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 + C\left\|\Lambda^{s+2}\mathbf{u}\right\|_{L^2}^2. \\ \int_{\mathbb{T}^3}\Lambda^s(\mathbb{P}f_2)\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \leq & \frac{1}{8}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 + C\left\|\Lambda^s f_2\right\|_{L^2}^2.\end{aligned} \tag{3.54}$$

Next, we have to bound the first term on the right-hand side of (3.53). In fact, exploiting the third equation in (2.3), we can rewrite this term into

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \Lambda^s \partial_t \mathbb{P}\mathbf{u} \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \\
&= \frac{d}{dt}\int_{\mathbb{T}^3} \Lambda^s \mathbb{P}\mathbf{u} \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx - \int_{\mathbb{T}^3} \Lambda^s \mathbb{P}\mathbf{u} \cdot \Lambda^s(\mathbf{n}\cdot\nabla\partial_t\mathbf{b})\,dx \\
&= \frac{d}{dt}\int_{\mathbb{T}^3} \Lambda^s \mathbb{P}\mathbf{u} \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx + \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s\partial_t\mathbf{b}\,dx \\
&= \frac{d}{dt}\int_{\mathbb{T}^3} \Lambda^s \mathbb{P}\mathbf{u} \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx + \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\,dx \\
&\quad - \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s(\mathbf{n}\operatorname{div}\mathbf{u})\,dx + \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s f_4\,dx. \quad (3.55)
\end{aligned}
$$

Thanks to the Hölder inequality, Young's inequality again, the last three terms in (3.55) can be controlled as

$$
\int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\,dx - \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s(\mathbf{n}\operatorname{div}\mathbf{u})\,dx \le C\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2, \quad (3.56)
$$

and

$$
\int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u}) \cdot \Lambda^s f_4\,dx \le C\left(\|\Lambda^s f_4\|_{L^2}^2 + \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2\right). \quad (3.57)
$$

Hence, collecting the above estimates, we can infer from (3.53) that

$$
\begin{aligned}
&\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 - \frac{d}{dt}\sum_{0\le s\le r+3}\int_{\mathbb{T}^3} \Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \\
&\le C\left(\left\|\Lambda^{s+2}\mathbf{u}\right\|_{L^2}^2 + \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + \left\|\Lambda^s f_4\right\|_{L^2}^2 + \left\|\Lambda^s f_2\right\|_{L^2}^2\right). \quad (3.58)
\end{aligned}
$$

By Lemma 3.27, there holds

$$
\begin{aligned}
\left\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 &\le C\big(\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{H^s}^2 + \|\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{b}\|_{L^\infty}^2\big) \\
&\le C\big(\|\mathbf{u}\|_{H^3}^2\|\mathbf{b}\|_{H^N}^2 + \|\mathbf{u}\|_{H^N}^2\|\mathbf{b}\|_{H^3}^2\big) \\
&\le C\delta^2(\|\mathbf{u}\|_{H^3}^2 + \|\mathbf{b}\|_{H^3}^2). \quad (3.59)
\end{aligned}
$$

Similarly,

$$
\left\|\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{b}\operatorname{div}\mathbf{u})\right\|_{L^2}^2 \le C\delta^2(\|\mathbf{u}\|_{H^3}^2 + \|\mathbf{b}\|_{H^3}^2). \quad (3.60)
$$

Moreover, from Lemma 1.4, there holds

$$
\|\mathbf{b}\|_{H^3}^2 \le C\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2,
$$

from which we get

$$\left\|\Lambda^s f_4\right\|_{L^2}^2 \leq C\delta^2 \left\|\mathbf{u}\right\|_{H^3}^2 + C\delta^2 \left\|\mathbf{n}\cdot\nabla\mathbf{b}\right\|_{H^{r+3}}^2 . \tag{3.61}$$

We now deal with the terms in f_2. The term $\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}^2$ can be bounded the same as (3.59)

$$\left\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\right\|_{L^2}^2 \leq C\delta^2 \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 . \tag{3.62}$$

Thanks to Lemma 3.27 again,

$$\begin{aligned}\left\|\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b}+\mathbf{b}\nabla\mathbf{b})\right\|_{L^2}^2 \leq& C(\|\mathbf{b}\|_{L^\infty}^2 \|\nabla\mathbf{b}\|_{H^s}^2 + \|\nabla\mathbf{b}\|_{L^\infty}^2 \|\mathbf{b}\|_{H^s}^2)\\ \leq& C(\|\mathbf{b}\|_{H^2}^2 \|\mathbf{b}\|_{H^{s+1}}^2 + \|\nabla\mathbf{b}\|_{H^2}^2 \|\mathbf{b}\|_{H^s}^2)\\ \leq& C\|\mathbf{b}\|_{H^{s+1}}^2 \|\mathbf{b}\|_{H^3}^2\\ \leq& C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 .\end{aligned} \tag{3.63}$$

The term $\mathbb{P}(I(a)\nabla a) = 0$. It follows from Lemma 3.27 that

$$\begin{aligned}\left\|\Lambda^s(\theta\nabla J(a))\right\|_{L^2}^2 \leq& C(\|\theta\|_{L^\infty}^2 \|\nabla J(a)\|_{H^s}^2 + \|\nabla J(a)\|_{L^\infty}^2 \|\theta\|_{H^s}^2)\\ \leq& C(\|\theta\|_{H^2}^2 \|a\|_{H^{s+1}}^2 + \|a\|_{H^3}^2 \|\theta\|_{H^s}^2)\\ \leq& C\delta^2 \|\theta\|_{H^{s+2}}^2 ,\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}\left\|\Lambda^s(I(a)\Delta\mathbf{u})\right\|_{L^2}^2 \leq& C(\|I(a)\|_{L^\infty}^2 \|\Delta\mathbf{u}\|_{H^s}^2 + \|\Delta\mathbf{u}\|_{L^\infty}^2 \|I(a)\|_{H^s}^2)\\ \leq& C(\|a\|_{H^2}^2 \|\mathbf{u}\|_{H^{s+2}}^2 + \|\mathbf{u}\|_{H^3}^2 \|a\|_{H^s}^2)\\ \leq& C\delta^2 \|\mathbf{u}\|_{H^{s+2}}^2 + C\delta^2 \|\mathbf{u}\|_{H^3}^2 .\end{aligned} \tag{3.65}$$

The term $I(a)\nabla\mathrm{div}\,\mathbf{u}$ can be treated similarly. The last term in f_2 can be bounded similarly to (3.38) and (3.39), so, we have

$$\begin{aligned}\left\|\Lambda^s(I(a)\mathbf{n}\nabla\mathbf{b})\right\|_{L^2}^2 \leq& C(\|I(a)\|_{L^\infty}^2 \|\mathbf{n}\nabla\mathbf{b}\|_{H^s}^2 + \|\mathbf{n}\nabla\mathbf{b}\|_{L^\infty}^2 \|I(a)\|_{H^s}^2)\\ \leq& C(\|a\|_{H^3}^2 \|\mathbf{n}\nabla\mathbf{b}\|_{H^s}^2 + \|\mathbf{b}\|_{H^3}^2 \|a\|_{H^s}^2)\\ \leq& C(\|d-\mathbf{n}\cdot\mathbf{b}\|_{H^3}^2 \|\mathbf{b}\|_{H^N}^2 + \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 \|a\|_{H^N}^2)\\ \leq& C((\|d\|_{H^3}^2 + \|\mathbf{b}\|_{H^3}^2) \|\mathbf{b}\|_{H^N}^2 + \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 \|a\|_{H^N}^2)\\ \leq& C\delta^2 \|d\|_{H^{r+4}}^2 + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 ,\end{aligned} \tag{3.66}$$

and

$$\left\|\Lambda^s(I(a)(\mathbf{n}\cdot\nabla\mathbf{b})\right\|_{L^2}^2 + \left\|\Lambda^s(I(a)(\mathbf{b}\cdot\nabla\mathbf{b}-\mathbf{b}\nabla\mathbf{b})\right\|_{L^2}^2 \leq C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 . \tag{3.67}$$

Combining with (3.62), (3.63)–(3.67) gives

$$\left\|\Lambda^s f_2\right\|_{L^2}^2 \leq C\delta^2 \left\|(\mathbf{u},\theta)\right\|_{H^{r+5}}^2 + C\delta^2 \left\|d\right\|_{H^{r+4}}^2 + C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2. \tag{3.68}$$

Inserting (3.61) and (3.68) into (3.58) and taking δ small enough, we finally get

$$\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 - \frac{d}{dt}\sum_{0\leq s\leq r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\,dx \leq C\left\|(\mathbf{u},\theta)\right\|_{H^{r+5}}^2 + C\left\|d\right\|_{H^{r+4}}^2. \tag{3.69}$$

3.5 Proof of Proposition 2.5

Proof In this subsection, we shall introduce the so-called effective velocity to reveal the hidden dissipativity of the combination quantity $a+\mathbf{n}\cdot\mathbf{b}$. We first deduce from the third equation in (2.3) that

$$\partial_t(\mathbf{n}\cdot\mathbf{b}) = \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{n}\mathrm{div}\,\mathbf{u} + f_4\cdot\mathbf{n}. \tag{3.70}$$

Combining it with the density equation gives

$$\partial_t(a+\mathbf{n}\cdot\mathbf{b}) = \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} - (|\mathbf{n}|^2+1)\mathrm{div}\,\mathbf{u} + f_1 + f_4\cdot\mathbf{n}. \tag{3.71}$$

Applying the projector operator $\mathbb{Q}$ to both hand sides of the second equation of (2.3) implies

$$\partial_t\mathbb{Q}\mathbf{u} - \nu\Delta\mathbb{Q}\mathbf{u} + \nabla(a+\mathbf{n}\cdot\mathbf{b}) + \nabla\theta = \mathbb{Q}f_2, \tag{3.72}$$

where $\nu \stackrel{\text{def}}{=} \lambda + 2\mu$. Recalling the notation in (2.10), direct calculations imply that

$$\begin{cases} d_t + \frac{1}{\nu}(|\mathbf{n}|^2+1)d + (|\mathbf{n}|^2+1)\mathrm{div}\,\mathbf{G} = \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} + f_1 + f_4\cdot\mathbf{n}, \\ \partial_t\mathbf{G} - \nu\Delta\mathbf{G} = \frac{1}{\nu}(|\mathbf{n}|^2+1)\mathbb{Q}\mathbf{u} - \frac{1}{\nu}\Delta^{-1}\nabla(\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}) - \nabla\theta \\ \quad + \mathbb{Q}f_2 - \frac{1}{\nu}\Delta^{-1}\nabla(f_1 + f_4\cdot\mathbf{n}). \end{cases} \tag{3.73}$$

On the one hand, for any $m\geq 0$, applying Λ^m to the first equation in (3.73), and then multiplying the resultant by $\Lambda^m d$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\Lambda^m d\right\|_{L^2}^2 + \frac{1}{\nu}(|\mathbf{n}|^2+1)\left\|\Lambda^m d\right\|_{L^2}^2 &= \int_{\mathbb{T}^3}\Lambda^m(\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n})\cdot\Lambda^m d\,dx \\ -(|\mathbf{n}|^2+1)\int_{\mathbb{T}^3}\Lambda^m\mathrm{div}\,\mathbf{G}\cdot\Lambda^m d\,dx &+ \int_{\mathbb{T}^3}\Lambda^m(f_1+f_4\cdot\mathbf{n})\cdot\Lambda^m d\,dx \\ \leq C(\left\|\Lambda^m\nabla\mathbf{u}\right\|_{L^2}\left\|\Lambda^m d\right\|_{L^2} &+ \left\|\Lambda^m\mathrm{div}\,\mathbf{G}\right\|_{L^2}\left\|\Lambda^m d\right\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{T}^3} \Lambda^m (f_1 + f_4 \cdot \mathbf{n}) \cdot \Lambda^m d\, dx) \\
&\le \frac{1}{2\nu} \left\| \Lambda^m d \right\|_{L^2}^2 + C\Big(\left\| \Lambda^{m+1} \mathbf{u} \right\|_{L^2}^2 + \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2}^2 \\
&+ \left\| \Lambda^m f_1 \right\|_{L^2}^2 + \left\| \Lambda^m f_4 \right\|_{L^2}^2 \Big).
\end{aligned} \tag{3.74}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left\| \Lambda^m d \right\|_{L^2}^2 + \frac{1}{2\nu}(|\mathbf{n}|^2 + 1) \left\| \Lambda^m d \right\|_{L^2}^2 \\
&\le C \left(\left\| \Lambda^{m+1} \mathbf{u} \right\|_{L^2}^2 + \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2}^2 + \left\| \Lambda^m f_1 \right\|_{L^2}^2 + \left\| \Lambda^m f_4 \right\|_{L^2}^2 \right).
\end{aligned} \tag{3.75}
$$

On the other hand, for the second equation in (3.73) and for any $m \ge 0$, there holds similarly that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left\| \Lambda^m \mathbf{G} \right\|_{L^2}^2 + \nu \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2}^2 \\
&= \frac{1}{\nu}(|\mathbf{n}|^2 + 1) \int_{\mathbb{T}^3} \Lambda^m \mathbb{Q}\mathbf{u} \cdot \Lambda^m \mathbf{G}\, dx - \frac{1}{\nu} \int_{\mathbb{T}^3} \Lambda^m (\Delta^{-1} \nabla (\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n})) \cdot \Lambda^m \mathbf{G}\, dx \\
&- \int_{\mathbb{T}^3} \Lambda^m \nabla\theta \cdot \Lambda^m \mathbf{G}\, dx \\
&+ \int_{\mathbb{T}^3} \Lambda^m \mathbb{Q} f_2 \cdot \Lambda^m \mathbf{G}\, dx + \frac{1}{\nu} \int_{\mathbb{T}^3} \Lambda^m \Delta^{-1} \nabla (f_1 + f_4 \cdot \mathbf{n}) \cdot \Lambda^m G\, dx.
\end{aligned} \tag{3.76}
$$

For $m = 0$, we get by the Young inequality and the Poincaré inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|\mathbf{G}\|_{L^2}^2 + \nu \|\nabla \mathbf{G}\|_{L^2}^2 \\
&= \frac{1}{\nu}(|\mathbf{n}|^2 + 1) \int_{\mathbb{T}^3} \mathbb{Q}\mathbf{u} \cdot \mathbf{G}\, dx - \frac{1}{\nu} \int_{\mathbb{T}^3} (\Delta^{-1} \nabla (\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n})) \cdot \mathbf{G}\, dx \\
&\quad - \int_{\mathbb{T}^3} \nabla\theta \cdot \mathbf{G}\, dx + \int_{\mathbb{T}^3} \mathbb{Q} f_2 \cdot \mathbf{G}\, dx + \frac{1}{\nu} \int_{\mathbb{T}^3} \Delta^{-1} \nabla (f_1 + f_4 \cdot \mathbf{n}) \cdot G\, dx \\
&\le C(\|\mathbf{u}\|_{L^2} + \|\nabla\theta\|_{L^2} + \|f_2\|_{L^2} + \left\| \Delta^{-1} \nabla (f_1 + f_4 \cdot \mathbf{n}) \right\|_{L^2}) \|\mathbf{G}\|_{L^2} \\
&\le \frac{\nu}{2} \|\nabla \mathbf{G}\|_{L^2}^2 + C(\|(\mathbf{u}, \theta)\|_{H^1}^2 + \|(f_1, f_4)\|_{H^{-1}}^2 + \|f_2\|_{L^2}^2).
\end{aligned} \tag{3.77}
$$

For $1 \le m \le N$, we get by the integration by parts and the Young inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left\| \Lambda^m \mathbf{G} \right\|_{L^2}^2 + \nu \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2}^2 \\
&\le C \left\| \Lambda^{m-1} \mathbf{u} \right\|_{L^2} \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2} + C \left\| \Lambda^m \theta \right\|_{L^2} \left\| \Lambda^{m+1} \mathbf{G} \right\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\left\|\Lambda^{m-1} f_2\right\|_{L^2}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}+C\left\|\Lambda^{m-2}(f_1+f_4\cdot\mathbf{n})\right\|_{L^2}\left\|\Lambda^{m+1}G\right\|_{L^2}\\
\leq&\ \frac{\nu}{4}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2+C\left\|\Lambda^{m-1}\mathbf{u}\right\|_{L^2}^2+C\left\|\Lambda^{m}\theta\right\|_{L^2}^2\\
&+C\left\|\Lambda^{m-2} f_1\right\|_{L^2}^2+C\left\|\Lambda^{m-2} f_4\right\|_{L^2}^2+C\left\|\Lambda^{m-1} f_2\right\|_{L^2}^2\\
\leq&\ \frac{\nu}{4}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2+C\left\|\Lambda^{m-1}\mathbf{u}\right\|_{L^2}^2+C\left\|\Lambda^{m-2} f_1\right\|_{L^2}^2\\
&+C\left\|\Lambda^{m-2} f_4\right\|_{L^2}^2+C\left\|\Lambda^{m-1} f_2\right\|_{L^2}^2.
\end{aligned}
\tag{3.78}
$$

Thus, combining with (3.77) and (3.77), we get for any $0 \leq m \leq N$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^m\mathbf{G}\right\|_{L^2}^2+\frac{\nu}{2}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2\\
&\quad\leq C(\|(\mathbf{u},\theta)\|_{H^{m+1}}^2+\|(f_1,f_4)\|_{H^m}^2+\|f_2\|_{H^{m-1}}^2+\|f_2\|_{L^2}^2).
\end{aligned}
\tag{3.79}
$$

Then, multiplying by a suitable large constant on both sides of (3.79) and then adding to (3.75), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^m d\right\|_{L^2}^2+\left\|\Lambda^m\mathbf{G}\right\|_{L^2}^2\right)+\frac{1}{\nu}\left\|\Lambda^m d\right\|_{L^2}^2+\nu\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2\\
&\quad\leq C(\|(\mathbf{u},\theta)\|_{H^{m+1}}^2+\|(f_1,f_4)\|_{H^m}^2+\|f_2\|_{H^{m-1}}^2+\|f_2\|_{L^2}^2).
\end{aligned}
\tag{3.80}
$$

Therefore, we complete the proof of proposition 2.5. □

3.6 Proof of Proposition 2.6

Proof First, we get by multiplying by a suitable large constant on both sides of (2.6) and then adding to (2.11) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\theta,\mathbf{b},d,\mathbf{G})\|_{H^m}^2+\mu\|\nabla\mathbf{u}\|_{H^m}^2+(\lambda+\mu)\|\mathrm{div}\,\mathbf{u}\|_{H^m}^2\\
&\quad+\kappa\|\nabla\theta\|_{H^m}^2+\frac{1}{\nu}\|d\|_{H^m}^2+\nu\|\nabla\mathbf{G}\|_{H^m}^2\\
&\leq C(\|(f_1,f_4)\|_{H^m}^2+\|f_2\|_{H^{m-1}}^2+\|f_2\|_{L^2}^2)+C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_1\cdot\Lambda^\alpha a\,dx\Big|\\
&\quad+C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_4\cdot\Lambda^\alpha\mathbf{b}\,dx\Big|+C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_2\cdot\Lambda^\alpha\mathbf{u}\,dx\Big|\\
&\quad+C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_3\cdot\Lambda^\alpha\theta\,dx\Big|.
\end{aligned}
\tag{3.81}
$$

Thanks to the Young inequality and the Poincaré inequality, for $\alpha = 0$, the last two terms in (3.81) can be bounded as

$$\Big|\int_{\mathbb{T}^3} f_2 \cdot \mathbf{u}\,dx\Big| \le \frac{\mu}{8}\|\mathbf{u}\|_{L^2}^2 + C\|f_2\|_{L^2}^2 \le \frac{\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2 + C\|f_2\|_{L^2}^2,$$
$$\Big|\int_{\mathbb{T}^3} f_3 \cdot \theta\,dx\Big| \le \frac{\kappa}{8}\|\theta\|_{L^2}^2 + C\|f_3\|_{L^2}^2 \le \frac{\kappa}{8}\|\nabla\theta\|_{L^2}^2 + C\|f_3\|_{L^2}^2.$$

Similarly, for $1 \le \alpha \le m$, we have

$$\begin{aligned}\Big|\sum_{\alpha=1}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_2\cdot\Lambda^\alpha\mathbf{u}\,dx\Big| &\le \frac{\mu}{8}\|\nabla\mathbf{u}\|_{H^m}^2 + C\left\|\Lambda^{m-1}f_2\right\|_{L^2}^2\\ &\le \frac{\mu}{8}\|\nabla\mathbf{u}\|_{H^m}^2 + C\|f_2\|_{H^{m-1}}^2,\\ \Big|\sum_{\alpha=1}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_3\cdot\Lambda^\alpha\theta\,dx\Big| &\le \frac{\kappa}{8}\|\nabla\theta\|_{H^m}^2 + C\left\|\Lambda^{m-1}f_3\right\|_{L^2}^2\\ &\le \frac{\kappa}{8}\|\nabla\theta\|_{H^m}^2 + C\|f_3\|_{H^{m-1}}^2.\end{aligned}$$

Inserting the above inequalities into (3.81) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\theta,\mathbf{b},d,\mathbf{G})\|_{H^m}^2 + \mu\|\nabla\mathbf{u}\|_{H^m}^2 + (\lambda+\mu)\|\mathrm{div}\,\mathbf{u}\|_{H^m}^2\\ &\quad+\kappa\|\nabla\theta\|_{H^m}^2 + \frac{1}{\nu}\|d\|_{H^m}^2 + \nu\|\nabla\mathbf{G}\|_{H^m}^2\\ &\le C(\|(f_1,f_4)\|_{H^m}^2 + \|(f_2,f_3)\|_{H^{m-1}}^2 + \|(f_2,f_3)\|_{L^2}^2)\\ &\quad+C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_1\cdot\Lambda^\alpha a\,dx\Big| + C\Big|\sum_{\alpha=0}^{m}\int_{\mathbb{T}^3}\Lambda^\alpha f_4\cdot\Lambda^\alpha\mathbf{b}\,dx\Big|. \end{aligned}\tag{3.82}$$

Now, taking $m = r + 4$ in the above estimate gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\theta,\mathbf{b},d,\mathbf{G})\|_{H^{r+4}}^2 + \mu\|\nabla\mathbf{u}\|_{H^{r+4}}^2 + (\lambda+\mu)\|\mathrm{div}\,\mathbf{u}\|_{H^{r+4}}^2\\ &\quad+\kappa\|\nabla\theta\|_{H^{r+4}}^2 + \frac{1}{\nu}\|d\|_{H^{r+4}}^2 + \nu\|\nabla\mathbf{G}\|_{H^{r+4}}^2\\ &\le C(\|(f_1,f_4)\|_{H^{r+4}}^2 + \|(f_2,f_3)\|_{H^{r+3}}^2)\\ &\quad+C\Big|\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^\alpha f_1\cdot\Lambda^\alpha a\,dx\Big| + C\Big|\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^\alpha f_4\cdot\Lambda^\alpha\mathbf{b}\,dx\Big|. \end{aligned}\tag{3.83}$$

Multiplying by a suitable large constant $\widetilde{c}$ which will be determined later on both sides of (3.83) and then adding to (2.9) give rise to

$$\frac{d}{dt}\mathscr{E}(t)+\mathscr{D}(t)\leq C(\|(f_1,f_4)\|_{H^{r+4}}^2+\|(f_2,f_3)\|_{H^{r+3}}^2)$$
$$+C\Big|\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^{\alpha}f_1\cdot\Lambda^{\alpha}a\,dx\Big|+C\Big|\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^{\alpha}f_4\cdot\Lambda^{\alpha}\mathbf{b}\,dx\Big|. \tag{3.84}$$

Now, we estimate the terms on the right-hand side of (3.84). First of all, by Lemma 3.27, there holds

$$\begin{aligned}\|f_1\|_{H^{r+4}}^2&\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\nabla a\|_{H^{r+4}}^2+\|a\|_{H^{r+4}}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\&\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}^2\|a\|_{H^N}^2\\&\leq C\delta^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2.\end{aligned}\tag{3.85}$$

Similarly,

$$\begin{aligned}\|f_4\|_{H^{r+4}}^2&\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\nabla\mathbf{b}\|_{H^{r+4}}^2+\|\mathbf{b}\|_{H^{r+4}}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\&\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\mathbf{b}\|_{H^N}^2+\|\mathbf{b}\|_{H^N}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\&\leq C(\|\nabla\mathbf{u}\|_{H^{r+4}}^2\|\mathbf{b}\|_{H^N}^2)\\&\leq C\delta^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2.\end{aligned}\tag{3.86}$$

For the first integral in the last line of (3.84), we use Lemma 3.21 to get

$$\begin{aligned}&\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}(\Lambda^{\alpha}(\mathbf{u}\cdot\nabla a)-\mathbf{u}\cdot\nabla\Lambda^{\alpha}a)\cdot\Lambda^{\alpha}a\,dx+\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\mathbf{u}\cdot\nabla\Lambda^{\alpha}a\cdot\Lambda^{\alpha}a\,dx\\&\leq C\sum_{\alpha=0}^{r+4}(\|\nabla\mathbf{u}\|_{L^\infty}\left\|\Lambda^{\alpha}a\right\|_{L^2}+\left\|\Lambda^{\alpha}\mathbf{u}\right\|_{L^2}\|\nabla a\|_{L^\infty})\|a\|_{H^{r+4}}+C\|\nabla\mathbf{u}\|_{L^\infty}\|a\|_{H^{r+4}}^2\\&\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}\|a\|_{H^{r+4}}^2.\end{aligned}\tag{3.87}$$

By Lemma 3.27, there holds

$$\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^{\alpha}(a\mathrm{div}\,\mathbf{u})\cdot\Lambda^{\alpha}a\,dx\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}\|a\|_{H^{r+4}}^2.\tag{3.88}$$

As a result, we have

$$\Big|\sum_{\alpha=0}^{r+4}\int_{\mathbb{T}^3}\Lambda^{\alpha}f_1\cdot\Lambda^{\alpha}a\,dx\Big|\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}\|a\|_{H^{r+4}}^2$$

$$\leq \frac{\mu}{8} \|\nabla \mathbf{u}\|_{H^{r+4}}^2 + C \|d\|_{H^{r+4}}^4 + C \|\mathbf{b}\|_{H^{r+4}}^4$$
$$\leq \frac{\mu}{8} \|\nabla \mathbf{u}\|_{H^{r+4}}^2 + C\delta^2 \|d\|_{H^{r+4}}^2 + C \|\mathbf{b}\|_{H^{r+4}}^4 . \tag{3.89}$$

Similarly, the last term in (3.84) can be bounded as

$$\Big|\sum_{\alpha=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^\alpha f_4 \cdot \Lambda^\alpha \mathbf{b}\, dx\Big| \leq \frac{\mu}{8} \|\nabla \mathbf{u}\|_{H^{r+4}}^2 + C \|\mathbf{b}\|_{H^{r+4}}^4 . \tag{3.90}$$

For any $N \geq 2r + 5$, from Lemma 1.4, we have

$$\|\mathbf{b}\|_{H^3}^2 \leq C \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^2 , \quad \text{and} \quad \|\mathbf{b}\|_{H^{r+4}}^2 \leq C \|\mathbf{b}\|_{H^3} \|\mathbf{b}\|_{H^N} \leq C\delta \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}} . \tag{3.91}$$

This gives

$$\|\mathbf{b}\|_{H^{r+4}}^4 \leq C\delta^2 \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^2 . \tag{3.92}$$

Thus, we get

$$\Big|\sum_{\alpha=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^\alpha f_1 \cdot \Lambda^\alpha a\, dx\Big| + \Big|\sum_{\alpha=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^\alpha f_4 \cdot \Lambda^\alpha \mathbf{b}\, dx\Big|$$
$$\leq \frac{\mu}{8} \|\nabla \mathbf{u}\|_{H^{r+4}}^2 + C\delta^2 (\|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^2 + \|d\|_{H^{r+4}}^2). \tag{3.93}$$

Finally, we have to control the term $\|f_2\|_{H^{r+3}}^2$. We start with the first term $\|\mathbf{u} \cdot \nabla \mathbf{u}\|_{H^{r+3}}^2$. By Lemma 3.27, we have

$$\|\mathbf{u} \cdot \nabla \mathbf{u}\|_{H^{r+3}}^2 \leq C \|\mathbf{u}\|_{H^{r+4}}^2 \|\nabla \mathbf{u}\|_{H^{r+4}}^2$$
$$\leq C \|\mathbf{u}\|_{H^N}^2 \|\nabla \mathbf{u}\|_{H^{r+4}}^2$$
$$\leq C\delta^2 \|\nabla \mathbf{u}\|_{H^{r+4}}^2 . \tag{3.94}$$

Similarly,

$$\|\mathbf{b} \cdot \nabla \mathbf{b} + \mathbf{b} \nabla \mathbf{b}\|_{H^{r+3}}^2 \leq C \|\mathbf{b}\|_{H^3}^2 \|\nabla \mathbf{b}\|_{H^{r+3}}^2$$
$$\leq C \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^2 \|\mathbf{b}\|_{H^N}^2$$
$$\leq C\delta^2 \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^2 , \tag{3.95}$$

in which we have used Lemma 1.4.

With the aid of Lemma 3.27 again, we can deduce that

$$\|I(a) \nabla a\|_{H^{r+3}}^2 \leq C \|\nabla a\|_{H^{r+3}}^2 \|I(a)\|_{L^\infty}^2 + \|I(a)\|_{H^{r+3}}^2 \|\nabla a\|_{L^\infty}^2$$

$$
\begin{aligned}
&\leq C\,\|a\|_{H^3}^2\,\|a\|_{H^N}^2\\
&\leq C\,\|d-\mathbf{n}\cdot\mathbf{b}\|_{H^3}^2\,\|a\|_{H^N}^2\\
&\leq C(\|d\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^2)\,\|a\|_{H^N}^2\\
&\leq C\delta^2\,\|d\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\,,
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
\|\theta\nabla J(a)\|_{H^{r+3}}^2 &\leq C\,\|\nabla J(a)\|_{H^{r+3}}^2\,\|\theta\|_{L^\infty}^2+\|\theta\|_{H^{r+3}}^2\,\|\nabla J(a)\|_{L^\infty}^2\\
&\leq C\delta^2\,\|\theta\|_{H^{r+5}}^2\,,
\end{aligned}
\tag{3.97}
$$

and

$$
\begin{aligned}
\|I(a)(\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u})\|_{H^{r+3}}^2 &\leq C\,\|a\|_{H^N}^2\,\|\Delta\mathbf{u}\|_{H^{r+3}}^2\\
&\leq C\delta^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\,.
\end{aligned}
\tag{3.98}
$$

It follows from Lemmas 3.27, 3.22 and 3.16 that

$$
\begin{aligned}
\|I(a)(\mathbf{n}\cdot\nabla\mathbf{b}-\mathbf{n}\nabla\mathbf{b})\|_{H^{r+3}}^2 &\leq C(\|I(a)\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{H^{r+3}}^2+\|\nabla\mathbf{b}\|_{L^\infty}^2\|I(a)\|_{H^{r+3}}^2)\\
&\leq C(\|\mathbf{b}\|_{H^N}^2\,\|a\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^2\,\|a\|_{H^{r+4}}^2)\\
&\leq C\delta^2\,\|d-\mathbf{n}\cdot\mathbf{b}\|_{H^3}^2+C\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\,\|a\|_{H^N}^2\\
&\leq C\delta^2(\|d\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^2)+C\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\,\|a\|_{H^N}^2\\
&\leq C\delta^2\,\|d\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\,.
\end{aligned}
\tag{3.99}
$$

The last term in $\|f_2\|_{H^{r+3}}^2$ can be treated in the same way as (3.99). Hence, by collecting the above estimates we can get

$$
\|f_2\|_{H^{r+3}}^2\leq C\delta^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2+C\delta^2\,\|d\|_{H^{r+4}}^2\,. \tag{3.100}
$$

The first term in $\|f_3\|_{H^{r+3}}^2$ can be estimated the same as (3.85), the second term can be estimated analogous to (3.98) and the third term can be estimated similarly to (3.49), so

$$
\|f_3\|_{H^{r+3}}^2\leq C\delta^2\,\|(\nabla\mathbf{u},\nabla\theta)\|_{H^{r+4}}^2\,. \tag{3.101}
$$

Inserting (3.85), (3.86), (3.93) and (3.100) into (3.84) leads to (2.12). Therefore, we complete the proof of Proposition 2.6. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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