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The Existence of Normalized Solutions to the Kirchhoff Equation with Potential and Sobolev Critical Nonlinearities

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Abstract

In the present paper, we study the existence of normalized solutions $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ to the following Kirchhoff problem

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u + \lambda u = g(u) + |u|^4 u \quad \text{in } \mathbb{R}^3,$$

satisfying the normalization constraint $\int_{\mathbb{R}^3} u^2 \mathrm{d}x = c$, where $a, b, c > 0$ are prescribed constants, and the nonlinearities $g(s)$ are very general and of mass super-critical. Under some suitable assumptions on $V(x)$ and $g(u)$, we will prove that the above problem has a ground state normalized solutions for any given $c > 0$, by studying a constraint problem on a Nehari–Pohozaev manifold.

Keywords Kirchhoff problem · General nonlinearities · Normalized solutions · Nehari–Pohozaev manifold

Mathematics Subject Classification 35J20 · 35J60 · 35B09

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1 Introduction

In this paper, we consider the existence of the ground state solutions to the following Kirchhoff type equations

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u+\lambda u=f(x,u)\ \text{ in } \mathbb{R}^3, \tag{1.1}$$

with the L^2-mass constraint

$$\int_{\mathbb{R}^3}|u|^2\mathrm{d}x=c, \tag{1.2}$$

where $a, b, c > 0$ are prescribed constants. If we set $V(x)+\lambda=0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega\subset\mathbb{R}^3$, (1.1) reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases}-(a+b\int_\Omega|\nabla u|^2\mathrm{d}x)\Delta u=f(x,u) & \text{in } \ \Omega,\\ u=0 & \text{on } \ \partial\Omega,\end{cases}$$

which is related to the following well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial t}|^2dx\right)\frac{\partial^2 u}{\partial x^2}=f(x,u). \tag{1.3}$$

The equation (1.3) is first proposed by G. Kirchhoff in [8], describing free vibrations of elastic strings. Because of the appearance of the term $\int_{\mathbb{R}^3}|\nabla u|^2$, (1.1) is regard as a nonlocal problem, which implies that equation (1.1) is not a pointwise identity. What's more, this phenomenon provokes some mathematical difficulties that makes the study of (1.1) more interesting. So after the pioneer work of J.L. Lions [10], where a functional analysis approach is proposed, the Kirchhoff type equations began to call attention of many researchers.

In (1.1), if $\lambda\in\mathbb{R}$ is fixed, then we call (1.1) the *fixed frequency problem*. There are various mathematical skills to find critical points of the corresponding energy functional $I_{\lambda,V}(u)$, including traditional constrained variational method, fixed point theorem and Lyapunov-Schmidt reduction, where

$$\begin{aligned}I_{\lambda,V}(u)\ &=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}(V(x)+\lambda)u^2dx-\int_{\mathbb{R}^3}F(x,u)dx\end{aligned} \tag{1.4}$$

and $F(x,s)=\int_0^s f(x,t)\mathrm{d}t$. In this respect, researchers have done a lot of research and obtained many results about the existence, multiplicity and concentration behavior of solutions of (1.1) (see [1, 3–7, 9, 12, 13, 18, 20] and the references therein).

Nowadays, physicists are more interested in solutions satisfying the L^2-mass constraint (1.2). From such a point of view, the mass $c>0$ is prescribed, while the frequency λ is unknown and will appear as a Lagrange multiplier. Hence, we call (1.1)–(1.2) *fixed mass problem* and the solution (u, λ) is called a normalized solution. Normalized solutions of (1.1) can be searched as the critical points of $I_V(u)$ constrained on S_c, where

$$I_V(u) := \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx, \tag{1.5}$$

and

$$S_c := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c\right\}. \tag{1.6}$$

As we know, the first work about normalized solutions to equation (1.1) is due to Ye. Specifically, in [14], Ye considered the normalized solutions to (1.1) with $V(x) \equiv 0$ and $f(x,u) = |u|^{p-2}u$, i,e, the following prolem

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + \lambda u = |u|^{p-2}u \ \text{ in } \mathbb{R}^3, \tag{1.7}$$

and searched for minimizers to the following minimization problem:

$$E_c := \inf_{u \in S_c} I_0(u), \tag{1.8}$$

where $I_0(u) := I_V(u)|_{V\equiv 0}$. By a scaling technique and applying the concentration-compactness principle, she proved that there exists $c_p^* \geq 0$, such that E_c is attained if and only if $c > c_p^*$ with $0 < p \leq 2 + \frac{4}{N}$, or $c \geq c_p^*$ with $2 + \frac{4}{N} < p < 2 + \frac{8}{N}$. The author also showed that there is no minimizers for problem (1.8) if $p \geq 2 + \frac{8}{N}$. In particular, for the case of $2 + \frac{8}{N} < p < 2^*$, $E_c = -\infty$. However, the author could find a mountain pass critical point for the functional $I_0(u)$ constrained on S_c. Later on, Ye [15] studied (1.7) for the case of $p = 2 + \frac{8}{N}$ and proved that there is a mountain pass critical point for the functional $I_0(u)$ on S_c if $c > c^*$. Also, if $0 < c < c^*$, the existence of minimizers for problem (1.8) was obtained by adding a new perturbation functional on the functional $I_0(u)$. Zeng and Zhang in [19] proved the existence and uniqueness of the minimizers of E_c, by means of some simple energy estimates rather than using the concentration-compactness principles. In [11], Luo and Wang studied the multiplicity of normalized solutions of equation (1.7) with $\frac{14}{3} < p < 6$. Very recently, Li, Luo and Yang [16] considered the existence and asymptotic properties of normalized solutions to the following Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + \lambda u = |u|^{p-2}u + \mu|u|^{q-2}u \ \text{ in } \mathbb{R}^3, \tag{1.9}$$

where $a, b, c, \mu > 0$, $2 < q < \frac{14}{3} < p \leq 6$ or $\frac{14}{3} < q < p \leq 6$, and proved a multiplicity result for the case of $2 < q < \frac{10}{3}$ and $\frac{14}{3} < p < 6$, and the existence of ground state normalized solutions for $2 < q < \frac{10}{3} < p = 6$ or $\frac{14}{3} < q < p \leq 6$. They also gave some asymptotic results on the obtained normalized solutions. In [24], He et al. established the existence of ground state normalized solutions to the following problem

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+\lambda u=g(u)\ \text{ in }\ \mathbb{R}^N,\,(N=1,2,3), \tag{1.10}$$

for any given $c>0$, by using fiber maps and establishing some mini-max structure, where $g(u)$ satisfies the assumptions:

(G1) $g:\mathbb{R}\to\mathbb{R}$ is continuous and odd;
(G2) There exists some $(\alpha,\beta)\in\mathbb{R}^2_+$ satisfying $2+\frac{8}{N}<\alpha\leq\beta<2^*:=\frac{2N}{N-2}$, such that

$$0<\alpha G(s)\leq g(s)s\leq\beta G(s)\text{ for s }\neq 0,\ \text{ where } G(s)=\int_0^s g(t)dt.$$

(G3) The function defined by $\tilde{G}(s):=\frac{1}{2}g(s)s-G(s)$ is of class C^1 and

$$\tilde{G}'(s)s\geq(2+\frac{8}{N})\tilde{G}(s),\,\forall s\in\mathbb{R}.$$

Zeng et al. [34] showed the existence, nonexistence and multiplicity of the normalized solutions to (1.10), based on the scaling skills and the results about the existence, nonexistence and multiplicity of the normalized solutions to Schödinger equation in [28]. Recently, Cui, He, Lv and Zhong in [25] studied the existence of ground state normalized solutions to the following Kirchhoff equation with potential and general nonlinear term

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u+\lambda u=g(u)\ \text{ in }\ \mathbb{R}^3, \tag{1.11}$$

and showed that if g and $V(x)$ satisfy $(G1)$, $(G2)$, $(G3)$, $(V1)$, $(V2)$ and $(V3)$, then problem (1.11) has a ground state normalized solutions for any $c>0$, which extends the results, proved by Ding and Zhong [22], on the semi-linear Schödinger equation to that about the Kirchhoff equation and where $(V1)$, $(V2)$ and $(V3)$ are defined as follows:

(V1) $\lim\limits_{|x|\to+\infty}V(x)=\sup\limits_{x\in\mathbb{R}^3}V(x)=0$ and there exists some $\sigma_1\in[0,\frac{3(\alpha-2)-4}{3(\alpha-2)}a)$ such that

$$\left|\int_{\mathbb{R}^3}V(x)u^2\mathrm{d}x\right|\leq\sigma_1\|\nabla u\|_2^2,\ \text{ for all } u\in H^1(\mathbb{R}^3).$$

(V2) $\nabla V(x)$ exists for a.e. $x\in\mathbb{R}^3$. putting $W(x):=\frac{1}{2}\langle\nabla V(x),x\rangle$, there exists some $\sigma_2\in[0,\frac{3(\alpha-2)(a-\sigma_1)}{4}-a]$ such that

$$\left|\int_{\mathbb{R}^3}W(x)u^2\mathrm{d}x\right|\leq\sigma_2\|\nabla u\|_2^2,\ \text{ for all } u\in H^1(\mathbb{R}^3).$$

(V3) $\nabla W(x)$ exists for a.e. $x \in \mathbb{R}^3$. Letting $\Upsilon(x) := 4W(x) + \langle \nabla W(x), x\rangle$, there exists some $\sigma_3 \in [0, 2a)$ such that

$$\int_{\mathbb{R}^3} \Upsilon_+(x)u^2 \mathrm{d}x \le \sigma_3 \|\nabla u\|_2^2, \text{ for all } u \in H^1(\mathbb{R}^3).$$

Inspired by the above mentioned results, we want to study the existence of ground state normalized solutions to the following problem

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u + \lambda u = g(u) + u^5 \text{ in } \mathbb{R}^3, \tag{1.12}$$

where $a, b, c > 0$, g and $V(x)$ satisfy $(G1)$, $(G2)$, $(G3)$, $(V1)$, $(V2)$ and $(V3)$. Compared with the above problem, we encounter with some new difficulties, for example, we can not carry on in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ as in [24], and the critical term u^5 will bring much more difficulty in showing the compactness than that in [22] and [25]. So the presence of the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2\, dx$, the potential term $V(x)u$ and the critical term u^5 makes this problem much more interesting.

It is easy to see that normalized solutions of (1.12) can be searched as critical points of $J_V(u)$ constrained on S_c, where

$$J_V(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \tag{1.13}$$

and S_c has been defined in (1.6). Ones can see that

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + \lambda u = g(u) + u^5 \quad \text{in } \mathbb{R}^3, \tag{1.14}$$

is a special case, corresponding to $V(x) \equiv 0$, of (1.12), whose functional can be defined as

$$J_0(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x. \tag{1.15}$$

We use the preceding notation and, to be short, we write below $J_0(u)$ for $J_V(u)|_{V\equiv 0}$. If we need to discuss about the functional of $V(x) \ne 0$ and $V(x) \equiv 0$, then we use J_V and J_0 respectively.

Before stating our results, we give a definition of the ground state normalized solution:

Definition 1.1 For any $c > 0$, a solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ to (1.12)–(1.2) is called a ground state normalized solution, or least energy normalized solution, if

$$J_V(u) = \min\left\{J_V(v) : v \in S_c \text{ and it solves(1.12)for some} \lambda \in \mathbb{R}\right\}.$$

Before studying the existence of ground state normalized solution to (1.12), we need to consider the existence of ground state normalized solution of its limiting problem. So we show a result on (1.14) as follows:

Theorem 1.2 *Let $a, b > 0$. If g satisfies (G1)-(G3), then, for any given $c > 0$, the Kirchhoff problem (1.14)–(1.2) has a ground state normalized solution $(\tilde{u}, \lambda_c)$ with $\lambda_c > 0$ and $\tilde{u} \in S_c$.*

Remark 1.3 Similar to [24], in present paper, we firstly introduce one more constraint, denoted by $\mathcal{P}_{V,c}$, see (2.5). Next, We shall prove the new constraint $\mathcal{P}_{V,c}$ is natural, see Lemma 3.5. Then we can devote to search for the critical point of $J_V(u)$ on $\mathcal{P}_{V,c}$. We shall prove that the functional $J_V(u)$ possesses a mini-max structure, and the infimum of $J_V(u)$ constrained on $\mathcal{P}_{V,c}$ denoted by $m_V(c)$, coincides to the mini-max value , i.e.,

$$m_V(c) := \inf_{u \in \mathcal{P}_{V,c}} J_V(u) = \inf_{u \in S_c} \max_{t>0} J_V(t \star u) \quad \text{and}$$
$$m_0(c) := \inf_{u \in \mathcal{P}_{V,c}} J_0(u) = \inf_{u \in S_c} \max_{t>0} J_0(t \star u),$$

where the fiber map $t \mapsto (t \star u)(x) \in H^1(\mathbb{R}^3)$ is defined by $(t \star u)(x) := t^{\frac{3}{2}} u(tx)$, which preserves the L^2-norm. In the meantime, we need some subtle energy estimates under the L^2-constraint to recover compactness in the Sobolev critical case.

Theorem 1.4 *Assume that $g(s)$ satisfies (G1)-(G3) and $V(x)$ satisfies (V1)-(V3). Then for any $c > 0$, problem (1.12)–(1.2) admits a ground state normalized solutions $(\bar{u}, \bar{\lambda}_c) \in S_c \times \mathbb{R}$.*

The paper is organized as follows. Some notations and preliminaries will be introduced in Sect. 2. In the Sect. 3, we shall prove the nimi-max structure of $J_V(u)$ and the fact that $\mathcal{P}_{V,c}$ is a natural constraint. We give the subtle energy estimates of $J_0(u)$ and prove that $m_0(c)$ is attained by some $u_c \in H^1(\mathbb{R}^N)$ in the Sect. 4,which is a positive decreasing function and the corresponding Lagrange multiplier λ_c is also positive. The proof of Theorem 1.4 will be put into the Sect. 5.

Throughout the paper we use the notation $\|u\|_p$ to denote the L^p-norm. The notation $\rightharpoonup$ denotes weak convergence in $H^1(\mathbb{R}^N)$. Capital latter C stands for positive constant, which may depend on some parameters and whose precise value can change from line to line.

2 Notation and Preliminaries

In this section, we introduce some notations and collect some useful preliminaries. Firstly, we recall the well-known Gagliardo-Nirenberg inequality with the best constant (see [32]): Let $p \in [2, 6)$. Then for any $u \in H^1(\mathbb{R}^3)$, we have

$$\|u\|_p^p \leq \frac{p}{2\|Q\|_2^{p-2}} \|\nabla u\|_2^{\frac{3p-6}{2}} \|u\|_2^{\frac{6-p}{2}}, \tag{2.1}$$

and, up to translations, Q is the unique positive radial solution (we refer to [29] for the uniqueness) of

$$-\frac{3p-6}{4}\Delta Q + \left(\frac{6-p}{4}\right) Q = |Q|^{p-2}Q \text{ in } \mathbb{R}^3.$$

Secondly, following from the assumptions (G1) and (G2), we deduce that for all $t \in \mathbb{R}$ and $s \geq 0$,

$$\begin{cases} s^\beta G(t) \leq G(ts) \leq s^\alpha G(t), & \text{if } s \leq 1, \\ s^\alpha G(t) \leq G(ts) \leq s^\beta G(t), & \text{if } s \geq 1. \end{cases} \tag{2.2}$$

Moreover, there exist some constants $C_1, C_2 > 0$ such that for all $s \in \mathbb{R}$,

$$C_1 \min\{|s|^\alpha, |s|^\beta\} \leq G(s) \leq C_2 \max\{|s|^\alpha, |s|^\beta\} \leq C_2(|s|^\alpha + |s|^\beta), \tag{2.3}$$

and

$$\left(\frac{\alpha}{2} - 1\right) G(s) \leq \frac{1}{2} g(s)s - G(s) \leq \left(\frac{\beta}{2} - 1\right) G(s) \leq \left(\frac{\beta}{2} - 1\right) C_2(|s|^\alpha + |s|^\beta). \tag{2.4}$$

As usually, we introduce the fiber map

$$u(x) \mapsto (t\star u)(x) := t^{\frac{3}{2}} u(tx) \quad x \in \mathbb{R}^3,$$

for $(t, u) \in \mathbb{R}^+ \times S_c$. Of course, one can easily check that for any $u \in H^1(\mathbb{R}^3)$,

$$\|t\star u\|_2^2 = \|u\|_2^2 \text{ and } \|\nabla(t\star u)\|_2^2 = t^2\|\nabla u\|_2^2.$$

So $t\star u \in S_c$ for any $u \in S_c$. Define

$$J_{V,u}(t) := J_V(t\star u) \quad \text{and} \quad J_{0,u}(t) := J_0(t\star u).$$

Then we introduce the so-called Pohozaev manifold. Denote

$$\mathcal{P}_V := \{u \in H^1(\mathbb{R}^3) : P_V(u) = 0\} \text{ and } \mathcal{P}_0 := \{u \in H^1(\mathbb{R}^3) : P_0(u) = 0\}, \tag{2.5}$$

where

$$P_V(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)u^2(x)\mathrm{d}x - 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \|u\|_6^6, \tag{2.6}$$

and

$$P_0(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \|u\|_6^6.$$

We also define the Pohozaev sub-manifold as follows:

$$\mathcal{P}_{V,c} := S_c \cap \mathcal{P}_V \quad \text{and} \quad \mathcal{P}_{0,c} := S_c \cap \mathcal{P}_0. \tag{2.7}$$

Set

$$m_V(c) := \inf_{u \in \mathcal{P}_{V,c}} J_V(u), \quad \text{and} \quad m_0(c) := \inf_{u \in \mathcal{P}_{0,c}} J_0(u).$$

To be much better at distinguishing the types of some critical points for $J_0\big|_{S_c}$ (J_0 is defined in (1.15)) and $J_V\big|_{S_c}$ (J_V is defined in (1.13)),we decide to decompose $\mathcal{P}_{0,c}$ and $\mathcal{P}_{V,c}$ into the disjoint unions $\mathcal{P}_{0,c} = \mathcal{P}^+_{0,c} \cup \mathcal{P}^-_{0,c} \cup \mathcal{P}^0_{0,c}$, $\mathcal{P}_{V,c} = \mathcal{P}^+_{V,c} \cup \mathcal{P}^-_{V,c} \cup \mathcal{P}^0_{V,c}$ respectively, where

$$\begin{aligned} \mathcal{P}^+_{0,c} &:= \{u \in \mathcal{P}_{0,c} : (J_{0,u})''(1) > 0\}, & \mathcal{P}^+_{V,c} &:= \{u \in \mathcal{P}_{V,c} : (J_{V,u})''(1) > 0\}, \\ \mathcal{P}^-_{0,c} &:= \{u \in \mathcal{P}_{0,c} : (J_{0,u})''(1) < 0\}, & \mathcal{P}^-_{V,c} &:= \{u \in \mathcal{P}_{V,c} : (J_{V,u})''(1) < 0\}, \\ \mathcal{P}^0_{0,c} &:= \{u \in \mathcal{P}_{0,c} : (J_{0,u})''(1) = 0\}, & \mathcal{P}^0_{V,c} &:= \{u \in \mathcal{P}_{V,c} : (J_{V,u})''(1) = 0\}. \end{aligned}$$

3 Mini–Max Structure and Pohozaev Manifold

Lemma 3.1 *Suppose that $u \in H^1(\mathbb{R}^3)$ is a weak solution of (1.12), then $u \in \mathcal{P}_V$.*

Proof Assume that $u \in H^1(\mathbb{R}^3)$ is a weak solution of (1.12). By the standard regularity theory, we obtain that $u \in C^2(\mathbb{R}^3)$. So we have

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 + \int_{\mathbb{R}^3} (V(x)+\lambda)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} g(u)u\mathrm{d}x - \int_{\mathbb{R}^3} |u|^6 dx = 0. \tag{3.1}$$

Additionally, invoking by the Pohozaev identity, we also deduce that

$$\begin{aligned} &(a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4) + 3\textstyle\int_{\mathbb{R}^3} (V(x)+\lambda)u^2 \mathrm{d}x \\ &-6\textstyle\int_{\mathbb{R}^3} G(u)\mathrm{d}x + \int_{\mathbb{R}^3} \langle \nabla V(x), x\rangle u^2 \mathrm{d}x - \int_{\mathbb{R}^3} |u|^6 dx = 0. \end{aligned} \tag{3.2}$$

Eliminating the parameter λ from the above equalities (3.1)–(3.2), we conclude that

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)u^2 \mathrm{d}x - 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \int_{\mathbb{R}^3} |u|^6 dx = 0,$$

which implies that $u \in \mathcal{P}_V$. □

Proposition 3.2 *Let $u \in S_c$. Then, $t \in \mathbb{R}^+$ is a critical point for $J_{V,u}(t) = J_V(t \star u)$ if and only if $t \star u \in \mathcal{P}_{V,c}$.*

Proof By direct calculations, it yields $(J_{V,u})'(t) = \frac{1}{t} P_V(t \star u)$, which implies that $(J_{V,u})'(t) = 0$ is equivalent to $t \star u \in \mathcal{P}_{V,c}$. In other words, $t \in \mathbb{R}^+$ is a critical point of $J_{V,u}(t) = J_V(t \star u)$ if and only if $t \star u \in \mathcal{P}_{V,c}$. □

Proposition 3.3 *For any critical point of $J_V\big|_{\mathcal{P}_{V,c}}$, if $(J_{V,u})''(1) \neq 0$, then there exists some $\lambda \in \mathbb{R}$ satisfying*

$$J_V'(u) + \lambda u = 0 \quad in \quad H^{-1}(\mathbb{R}^3).$$

Proof Let u be a critical point of $J_V(u)$ restricted to $\mathcal{P}_{V,c}$, then by the Lagrange multipliers rule there exist $\lambda, \mu \in \mathbb{R}$ such that

$$J_V'(u) + \lambda u + \mu P_V'(u) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3). \tag{3.3}$$

It remains to verify $\mu = 0$.

We claim that if u solves (3.3), then u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi(t\star u)\big)\big|_{t=1} = 0,$$

where

$$\Phi(u) := J_V(u) + \frac{1}{2}\lambda\|u\|_2^2 + \mu P_V(u).$$

In fact, we observe that

$$\begin{aligned}\Phi(u) &= J_0(u) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x + \frac{1}{2}\lambda\|u\|_2^2 + \mu P_0(u) - \mu\int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x\\ &= \Phi_0(u) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x - \mu\int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x,\end{aligned}$$

where

$$\Phi_0(u) := J_0(u) + \frac{1}{2}\lambda\|u\|_2^2 + \mu P_0(u).$$

After that, it is not difficult to verify that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi(t\star u)\big)\Big|_{t=1} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi_0(t\star u)\big)\Big|_{t=1} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}^3} V(x)t^3u^2(tx)\mathrm{d}x\right)\Big|_{t=1}\\ - \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3} W(x)t^3u^2(tx)\mathrm{d}x\right)\Big|_{t=1}.\end{aligned}$$

By direct calculations, it is easy to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi_0(t\star u)\big)\big|_{t=1} \;=\; & \big(a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - 3\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x - \|u\|_6^6\big)\\ & +\mu\big(2a\|\nabla u\|_2^2 + 4b\|\nabla u\|_2^4 + 9\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x\\ & -\frac{9}{2}\int_{\mathbb{R}^3}\widetilde{G}'(u)u\mathrm{d}x - 6\|u\|_6^6\big),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}^3}V(x)t^3u^2(tx)\mathrm{d}x\right)\Big|_{t=1} &\overset{y=tx}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}^3}V(\frac{y}{t})u^2(y)\mathrm{d}y\right)\Big|_{t=1}\\ &= \left(\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(\frac{y}{t}),-\frac{y}{t^2}\rangle u^2(y)\mathrm{d}y\right)\Big|_{t=1}\\ &= \left(\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),-\frac{x}{t}\rangle(t\star u)^2\mathrm{d}x\right)\Big|_{t=1}\\ &= -\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle u^2(x)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3}W(x)t^3u^2(tx)\mathrm{d}x\right)\Big|_{t=1} &\overset{y=tx}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3}W(\frac{y}{t})u^2(y)\mathrm{d}y\right)\Big|_{t=1}\\ &= \left(\int_{\mathbb{R}^3}\langle\nabla W(\frac{y}{t}),-\frac{y}{t^2}\rangle u^2(y)\mathrm{d}y\right)\Big|_{t=1}\\ &= -\int_{\mathbb{R}^3}\langle\nabla W(x),x\rangle u^2(x)\mathrm{d}x.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi(t\star u)\big)\Big|_{t=1} &= \big(a\|\nabla u\|_2^2+b\|\nabla u\|_2^4-3\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x-\|u\|_6^6\big)\\ &\quad+\mu\big(2a\|\nabla u\|_2^2+4b\|\nabla u\|_2^4+9\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x\\ &\quad-\frac{9}{2}\int_{\mathbb{R}^3}\widetilde{G}'(u)u\mathrm{d}x-6\|u\|_6^6\big)\\ &\quad-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle u^2(x)\mathrm{d}x+\mu\int_{\mathbb{R}^3}\langle\nabla W(x),x\rangle u^2(x)\mathrm{d}x.\end{aligned}$$

On the other hand, a solution to (3.3) must satisfy the so-called Pohozaev identity

$$\begin{aligned}&a\|\nabla u\|_2^2+b\|\nabla u\|_2^4+\mu\big(2a\|\nabla u\|_2^2+4b\|\nabla u\|_2^4\big)\\ &=-\mu\left(9\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x-\frac{9}{2}\int_{\mathbb{R}^3}\widetilde{G}'(u)u\mathrm{d}x-6\|u\|_6^6\right)+3\int_{\mathbb{R}^3}\widetilde{G}(u)\mathrm{d}x+\|u\|_6^6\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle u^2(x)\mathrm{d}x-\mu\int_{\mathbb{R}^3}\langle\nabla W(x),x\rangle u^2(x)\mathrm{d}x,\end{aligned}$$

which implies that $\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi(t\star u)\big)|_{t=1}=0$. This completes the proof of the claim.

Now we deduce by direct computations that

$$\Phi(t\star u)=J_V(t\star u)+\frac{1}{2}\lambda\|u\|_2^2+\mu P_V(t\star u)=J_{V,u}(t)+\frac{1}{2}\lambda\|u\|_2^2+\mu t(J_{V,u})'(t),$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t\star u) = (1+\mu)(J_{V,u})'(t) + \mu t (J_{V,u})''(t).$$

Since $u \in \mathcal{P}_{V,c}$, $P_V(u) = 0$. we deduce by $P_V(u) = \frac{\mathrm{d}}{\mathrm{d}t} J_V(t\star u)\big|_{t=1}$ that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi(t\star u)\big)\big|_{t=1} \\ &= (1+\mu)(J_{V,u})'(1) + \mu (J_{V,u})''(1) \\ &= (1+\mu)P_V(u) + \mu(J_{V,u})''(1) = \mu(J_{V,u})''(1). \end{aligned}$$

Finally, by the fact that $(J_{V,u})''(1) \neq 0$, we get $\mu = 0$, which implies that

$$J_V'(u) + \lambda u = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3).$$

□

Lemma 3.4 *Assume that the assumptions (G1)–(G2) and (V1)–(V2) hold. Then for any* $c > 0$*, there exists some* $\bar{\delta}_c > 0$ *such that*

$$\inf_{u\in\mathcal{P}_{V,c}} \|\nabla u\|_2 \geq \bar{\delta}_c. \tag{3.4}$$

Proof 1) Since $u \in \mathcal{P}_{V,c}$, we have the following Pohozaev identity

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)u^2(x)\mathrm{d}x - \|u\|_6^6 = 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x. \tag{3.5}$$

By the assumption (V2), we observe that for any $u \in H^1(\mathbb{R}^3)$,

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)u^2(x)\mathrm{d}x \geq (a-\sigma_2)\|\nabla u\|_2^2 + b\|\nabla u\|_2^4. \tag{3.6}$$

By the assumption (G2) and the Gagliardo-Nirenberg inequality (2.1), we deduce that

$$3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x \leq C\int_{\mathbb{R}^3} (|u|^\alpha + |u|^\beta)\mathrm{d}x \leq C(\|\nabla u\|_2^{\frac{3(\alpha-2)}{2}} + \|\nabla u\|_2^{\frac{3(\beta-2)}{2}}),$$

which, together with (3.5) and (3.6), implies that

$$\begin{aligned} (a-\sigma_2)\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 &\leq 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x + \|u\|_6^6 \\ &\leq C(\|\nabla u\|_2^{\frac{3(\alpha-2)}{2}} + \|\nabla u\|_2^{\frac{3(\beta-2)}{2}}) + \frac{1}{S^3}\|\nabla u\|_2^6 \end{aligned} \tag{3.7}$$

The assumptions (G2) and (V2) give $\frac{3(\alpha-2)}{2}, \frac{3(\beta-2)}{2} > 2$ and $a - \sigma_2 > 0$, and then we conclude from (3.7) that there exists some $\bar{\delta}_c > 0$ such that

$$\|\nabla u\|_2 \geq \bar{\delta}_c.$$

We complete the proof. □

Lemma 3.5 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. Then* $\mathcal{P}^-_{V,c} = \mathcal{P}_{V,c}$ *is closed in* $H^1(\mathbb{R}^3)$ *and it is a natural constraint of* $J_V\big|_{S_c}$.

Proof For any $u \in \mathcal{P}_{V,c}$, we have

$$a\|\nabla u\|_2^2 = -b\|\nabla u\|_2^4 + \int_{\mathbb{R}^3} W(x)u^2 \mathrm{d}x + 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x + \|u\|_6^6. \tag{3.8}$$

By virtue of equality (3.8), the assumptions (V3), (G3) and Lemma 3.4, we obtain that

$$\begin{aligned}
(J_{V,u})''(1) &= a\|\nabla u\|_2^2 + 3b\|\nabla u\|_2^4 + \int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x + \int_{\mathbb{R}^3} \langle \nabla W(x), x\rangle u^2 \mathrm{d}x \\
&\quad + 12\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \frac{9}{2}\int_{\mathbb{R}^3} \widetilde{G}'(u)u\mathrm{d}x - 5\|u\|_6^6 \\
&= 2b\|\nabla u\|_2^4 + 2\int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x + \int_{\mathbb{R}^3} \langle \nabla W(x), x\rangle u^2 \mathrm{d}x \\
&\quad + 15\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \frac{9}{2}\int_{\mathbb{R}^3} \widetilde{G}'(u)u\mathrm{d}x - 4\|u\|_6^6 \\
&\leq 2b\|\nabla u\|_2^4 + 2\int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x + \int_{\mathbb{R}^3} \langle \nabla W(x), x\rangle u^2 \mathrm{d}x \\
&\quad - 6\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - 4\|u\|_6^6 \\
&= -2a\|\nabla u\|_2^2 + 4\int_{\mathbb{R}^3} W(x)u^2\mathrm{d}x + \int_{\mathbb{R}^3} \langle \nabla W(x), x\rangle u^2 \mathrm{d}x - 2\|u\|_6^6 \\
&\leq -2a\|\nabla u\|_2^2 + \int_{\mathbb{R}^3} \Upsilon_+ u^2 \leq (-2a+\sigma_3)\|\nabla u\|_2^2 < 0,
\end{aligned} \tag{3.9}$$

which implies that $\mathcal{P}^+_{V,c} = \mathcal{P}^0_{V,c} = \varnothing$. Hence, $\mathcal{P}^-_{V,c} = \mathcal{P}_{V,c}$ is closed in $H^1(\mathbb{R}^3)$. By Proposition 3.3, we can obtain that $\mathcal{P}_{V,c}$ is a natural constraint of $J_V\big|_{S_c}$.

The proof of Lemma 3.5 is completed. □

Remark 3.6 Let $\{w_n\} \subseteq \mathcal{P}^-_{V,c}$ be such that $J_V(w_n) \to m_V(c)$. Therefore, there exist two sequences $\{\lambda_n\}, \{\mu_n\} \subseteq \mathbb{R}$ such that, as $n \to +\infty$,

$$J'_V(w_n) + \lambda_n w_n + \mu_n P'_V(w_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3).$$

Using a similar argument as Proposition 3.3, we can get that, as $n \to +\infty$,

$$\mu_n (J_{V,w_n})''(1) \to 0. \tag{3.10}$$

By Lemma 3.4 and (3.9), we have

$$(J_{V,w_n})''(1) \leq (-2a + \sigma_3)\bar{\delta}_c^2 < 0,$$

which, together with (3.10), implies that $\mu_n \to 0$ as $n \to +\infty$.

Hence, if furthermore $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then we obtain that, as $n \to +\infty$,

$$J_V'(w_n) + \lambda_n w_n \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3).$$

Lemma 3.7 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. Then for every $u \in S_c$ with $c > 0$, there exists a unique $t_u \in \mathbb{R}^+$ such that $t_u \star u \in \mathcal{P}_{V,c}$. Moreover, t_u is the unique critical point of the function $J_{V,u}(t)$, and satisfies $J_{V,u}(t_u) = \max_{t>0} J_V(t \star u)$.*

Proof Let $u \in S_c$. Since $u \in H^1(\mathbb{R}^3)$, we have $\|\nabla u\|_2 > 0$. By the assumption (V2) and direct computations, we have

$$\begin{aligned}(J_{V,u})'(t) &= at\|\nabla u\|_2^2 + bt^3\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)t^2u^2(tx)\mathrm{d}x \\ &\quad - 3\int_{\mathbb{R}^3} \widetilde{G}(t^{\frac{3}{2}}u(x))\mathrm{d}x t^{-4} - t^5\|u\|_6^6 \\ &\geq (a - \sigma_2)\|\nabla u\|_2^2 t + b\|\nabla u\|_2^4 t^3 - C\left(t^{\frac{3}{2}\alpha - 4}\|u\|_\alpha^\alpha + t^{\frac{3}{2}\beta - 4}\|u\|_\beta^\beta\right) \\ &\quad - \frac{t^5}{S^3}\|\nabla u\|_2^6,\end{aligned}$$

where $\beta > \alpha > \frac{14}{3}$ and $a - \sigma_2 > 0$. It yields that $(J_{V,u})'(t) > 0$ for $t > 0$ small enough. Therefore, there exists some $t_1 > 0$ such that $J_{V,u}(t)$ increases in $t \in (0, t_1)$.

On the other hand, according to the assumption (V1), we obtain

$$\begin{aligned}J_{V,u}(t) &\leq \frac{a}{2}t^2\|\nabla u\|_2^2 + \frac{b}{4}t^4\|\nabla u\|_2^4 + \frac{1}{2}\sigma_1 t^2\|\nabla u\|_2^2 - \int_{\mathbb{R}^3} G(t^{\frac{3}{2}}u(tx))\mathrm{d}x - \frac{1}{6}t^6\|u\|_6^6 \\ &\leq \frac{a}{2}t^2\|\nabla u\|_2^2 + \frac{b}{4}t^4\|\nabla u\|_2^4 + \frac{\sigma_1}{2}t^2\|\nabla u\|_2^2 - t^{\frac{3}{2}\alpha - 3}\|u\|_\alpha^\alpha - \frac{1}{6}t^6\|u\|_6^6.\end{aligned}$$

Since $\alpha > \frac{14}{3}$, we can infer that $\lim_{t \to +\infty} J_{V,u}(t) = -\infty$. Hence, there exists some $t_2 > t_1$ such that

$$J_{V,u}(t_2) = \max_{t>0} J_V(t \star u).$$

It is clear that $(J_{V,u})'(t_2) = 0$ and $t_2 \star u \in \mathcal{P}_{V,c}$ by Proposition 3.2. We suppose to the contrary that there exists another $t_3 > 0$ such that $t_3 \star u \in \mathcal{P}_{V,c}$. Without loss of generality, we may assmue $t_3 > t_2$. Following from Lemma 3.5, we observe that both

t_2 and t_3 are strict local maximum points of $J_{V,u}(t)$, which implies that there exists some $t_4 \in (t_2, t_3)$ such that

$$J_{V,u}(t_4) = \min_{t\in[t_2,t_3]} J_{V,u}(t).$$

It follows that $(J_{V,u})'(t_3) = 0$ and $(J_{V,u})''(t_3) \geq 0$, which allows us to conclude that $t_4 \star u \in \mathcal{P}^+_{V,c} \cup \mathcal{P}^0_{V,c}$, a contradiction to Lemma 3.5.

The proof of Lemma 3.7 is complete. □

Corollary 3.8 *Under the assumptions (G1)-(G3), for any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, let t_u be given by Lemma 3.7, then we have that*

$$t_u = (>, <)1 \Leftrightarrow (J_{V,u})'(1) = (>, <)0 \Leftrightarrow P_V(u) = (>, <)0.$$

Proof By Lemma 3.7, we have that

$$J_{V,u}(t_u) = \max_{t>0} J_{V,u}(t).$$

Furthermore,

$$(J_{V,u})'(t) > 0 \text{ for } 0 < t < t_u \text{ and } (J_{V,u})'(t) < 0 \text{ for } t > t_u.$$

On the other hand, we recall that $P[t\star u] = t(J_{V,u})'(t)$.

Hence, the conclusion holds. □

Lemma 3.9 *Under the assumptions (G1)–(G2) and (V1)–(V2), $J_V\big|_{\mathcal{P}_{V,c}}$ is coercive, that is,*

$$\lim_{u\in\mathcal{P}_{V,c},\|\nabla u\|_2\to\infty} J_V(u) = +\infty.$$

Proof Since $u \in \mathcal{P}_{V,c}$, we deduce by the assumptions (G2) and (V2) that

$$\begin{aligned}(a+\sigma_2)\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \|u\|_6^6 &\geq a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} W(x)u^2 \mathrm{d}x - \|u\|_6^6 \\ &= 3\int_{\mathbb{R}^3}\left(\frac{1}{2}g(u)u - G(u)\right)\mathrm{d}x \\ &\geq \frac{3(\alpha-2)}{2}\int_{\mathbb{R}^3} G(u)\mathrm{d}x,\end{aligned}$$

which, together with (V1), implies that, as $\|\nabla u\|_2 \to +\infty$,

$$\begin{aligned} J_V(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\|u\|_6^6 \\ &\geq \frac{1}{2}(a-\sigma_1)\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\|u\|_6^6 \\ &\geq \left(\frac{1}{2}(a-\sigma_1) - \frac{2(a+\sigma_2)}{3(\alpha-2)}\right)\|\nabla u\|_2^2 + \left(\frac{b}{4} - \frac{2b}{3(\alpha-2)}\right)\|\nabla u\|_2^4 \\ &\quad + (\frac{2}{3(\alpha-2)} - \frac{1}{6})\|u\|_6^6 \\ &\geq \left(\frac{1}{2}(a-\sigma_1) - \frac{2(a+\sigma_2)}{3(\alpha-2)}\right)\|\nabla u\|_2^2 + \left(\frac{b}{4} - \frac{2b}{3(\alpha-2)}\right)\|\nabla u\|_2^4 \to +\infty. \end{aligned} \tag{3.11}$$

Hence,

$$\lim_{u \in \mathcal{P}_{V,c}, \|\nabla u\|_2 \to +\infty} J_V(u) = +\infty.$$

The proof of Lemma 3.9 is complete. □

Lemma 3.10 *There holds the following mini-max structure*

$$m_V(c) := \inf_{u \in \mathcal{P}_{V,c}} J_V(u) = \inf_{u \in \mathcal{S}_c} \max_{t>0} J_V(t\star u) > 0. \tag{3.12}$$

Proof For any $u \in \mathcal{P}_{V,c}$, by Lemma 3.7, we have

$$J_V(u) = J_{V,u}(1) = \max_{t>0} J_V(t\star u) \geq \inf_{u \in \mathcal{S}_c} \max_{t>0} J_V(t\star u),$$

which implies that

$$\inf_{u \in \mathcal{P}_{V,c}} J_V(u) \geq \inf_{u \in \mathcal{S}_c} \max_{t>0} J_V(t\star u). \tag{3.13}$$

On the other hand, for any $u \in \mathcal{S}_c$, by Lemma 3.7 again, we obtain that there exists t_u such that $t_u\star u \in \mathcal{P}_{V,c}$ and $J_V(t_u\star u) = \max_{t>0} J_V(t\star u)$. Therefore,

$$\inf_{u \in \mathcal{P}_{V,c}} J_V(u) \leq J_V(t_u\star u) = \max_{t>0} J_V(t\star u),$$

which implies that

$$\inf_{u \in \mathcal{P}_{V,c}} J_V(u) \leq \inf_{u \in \mathcal{S}_c} \max_{t>0} J_V(t\star u). \tag{3.14}$$

(3.13) and (3.14) imply that

$$\inf_{u \in \mathcal{P}_{V,c}} J_V(u) = \inf_{u \in \mathcal{S}_c} \max_{t>0} J_V(t\star u).$$

By (3.11), (3.4) and (V2), we see

$$J_V(u) \geq C_1 \overline{\delta}_c^2 + C_2 \overline{\delta}_c^4 > 0, \quad \text{for any } u \in \mathcal{P}_{V,c}.$$

Hence

$$m_V(c) \geq C_1 \overline{\delta}_c^2 + C_2 \overline{\delta}_c^4 > 0.$$

The proof of Lemma 3.10 is complete. □

Remark 3.11 Since $V(x) \equiv 0$ is a special function satisfying the assumptions $(V1)$, $(V2)$ and $(V3)$, the V in this Section can take 0. That is, all the conclusions in this Section are true, even if we replace V with 0.

4 Energy Estimates and Compactness Analysis

Lemma 4.1 *Under the assumptions (G1)-(G3), for any $c > 0$, we have $m_0(c) < \frac{a\mathcal{S}\Lambda}{3} + \frac{b\mathcal{S}^2\Lambda^2}{12}$, where $\Lambda = \frac{b\mathcal{S}^2}{2} + \sqrt{a\mathcal{S} + \frac{b^2\mathcal{S}^4}{4}}$.*

Proof The idea of the proof is similar to the Lemma 5.5 of [16], we shall imitate and revise it. By Theorem 1.42 of [17], we know that $\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}$ is attained by

$$U_\varepsilon(x) := 3^{\frac{1}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{1}{2}}, \forall \varepsilon > 0.$$

Furthermore, we have $\|\nabla U_\varepsilon\|_2^2 = \|U_\varepsilon\|_6^6 = \mathcal{S}^{\frac{3}{2}}$. Take a radially decreasing cut-off function $\eta \in C_c^\infty\left(\mathbb{R}^3\right)$ such that $\eta \equiv 1$ in $B_1(0)$, $\eta \equiv 0$ in $B_2^c(0) := \mathbb{R}^3 \setminus B_2(0)$, and let

$$u_\varepsilon(x) := \eta(x) U_\varepsilon(x), \quad \text{and} \quad v_\varepsilon(x) := c\frac{u_\varepsilon(x)}{\|u_\varepsilon\|_2}, \quad \forall \varepsilon \in (0,1).$$

Clearly, $v_\varepsilon \in S_c$, by Lemma 3.7 and Remark 3.11, there exists a unique $t_{v_\varepsilon} \in \mathbb{R}$ such that

$$m_0(c) := \inf_{u \in \mathcal{P}_{V,c}} J_0[u] = \inf_{u \in S_c} \max_{t>0} J_0[t \star u] \leq \max_{t>0} J_0[t \star v_\varepsilon] = J_0[t_{v_\varepsilon} \star v_\varepsilon], \quad \forall \varepsilon > 0. \tag{4.1}$$

So, it is sufficient to prove $\max_{t>0} J_0[t \star v_\varepsilon] = J_0[t_{v_\varepsilon} \star v_\varepsilon] < \frac{a\mathcal{S}\Lambda}{3} + \frac{b\mathcal{S}^2\Lambda^2}{12}$. By Lemmas 3.4, 3.7 and Remark 3.11, we notice that $t_{v_\varepsilon} \star v_\varepsilon \in \mathcal{P}_{0,c}$ and $t_{v_\varepsilon} > 0$.

To this end, we need some integral estimates. Similar to Lemma 1.46 in [17], we can derive that

$$\|\nabla u_\varepsilon\|_2^2 = \mathcal{S}^{\frac{3}{2}} + O(\varepsilon), \quad \|u_\varepsilon\|_6^6 = \mathcal{S}^{\frac{3}{2}} + O\left(\varepsilon^3\right), \quad \|u_\varepsilon\|_2^2 = O(\varepsilon),$$
$$\|\nabla u_\varepsilon\|_2^2 \geq C_1, \quad \frac{1}{C_2} \geq \|u_\varepsilon\|_6^6 \geq C_2, \quad \|u_\varepsilon\|_2^2 \geq C_3\varepsilon,$$

for some constants $C_i > 0 (i = 1, 2, 3)$, which are independent of ε, c and μ. Since $t > 0$ and (G2), we have

$$\int_{\mathbb{R}^3} G(t^{\frac{3}{2}}u)dxt^{-3} > \int_{\mathbb{R}^3} G(u)dxt^4 > 0,$$

which implies that

$$\begin{aligned}
J_{0,v_\varepsilon}(t) &= \frac{a}{2}\|\nabla v_\varepsilon\|_2^2 t^2 + \frac{b}{4}\|\nabla v_\varepsilon\|_2^4 t^4 - \int_{\mathbb{R}^3} G(t^{\frac{3}{2}}v_\varepsilon)dxt^{-3} - \frac{1}{6}t^6 \|v_\varepsilon\|_6^6 \\
&< \frac{a}{2}\|\nabla v_\varepsilon\|_2^2 t^2 + \frac{b}{4}\|\nabla v_\varepsilon\|_2^4 t^4 - \frac{1}{6}t^6 \|v_\varepsilon\|_6^6 .
\end{aligned}$$

Now, we set $\tilde{J}_{0,v_\varepsilon}(t) := \frac{a}{2}\|\nabla v_\varepsilon\|_2^2 t^2 + \frac{b}{4}\|\nabla v_\varepsilon\|_2^4 t^4 - \frac{1}{6}t^6 \|v_\varepsilon\|_6^6$. It is obviously that $\tilde{J}_{0,v_\varepsilon}(t)$ has a unique maximum point $\tilde{t}_{v_\varepsilon}$ such that

$$\tilde{t}_{v_\varepsilon}^2 = \frac{b\|\nabla v_\varepsilon\|_2^4}{2\|v_\varepsilon\|_6^6} + \sqrt{\frac{a\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^6} + \frac{b^2\|\nabla v_\varepsilon\|_2^8}{4\|v_\varepsilon\|_6^{12}}}.$$

Then, we drive that

$$\begin{aligned}
\frac{c^2\tilde{t}_{v_\varepsilon}^2}{\|u_\varepsilon\|_2^2} &= \frac{b\|\nabla u_\varepsilon\|_2^4}{2\|u_\varepsilon\|_6^6} + \sqrt{\frac{a\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_6^6} + \frac{b^2\|\nabla u_\varepsilon\|_2^8}{4\|u_\varepsilon\|_6^{12}}} \\
&= \frac{b\left(\mathcal{S}^{\frac{3}{2}} + O(\varepsilon)\right)^2}{2\left(\mathcal{S}^{\frac{3}{2}} + O\left(\varepsilon^3\right)\right)} + \sqrt{\frac{a\left(\mathcal{S}^{\frac{3}{2}} + O(\varepsilon)\right)}{\mathcal{S}^{\frac{3}{2}} + O\left(\varepsilon^3\right)} + \frac{b^2\left(\mathcal{S}^{\frac{3}{2}} + O(\varepsilon)\right)^4}{4\left(\mathcal{S}^{\frac{3}{2}} + O\left(\varepsilon^3\right)\right)^2}} \\
&= \frac{b\mathcal{S}^{\frac{3}{2}}}{2} + \sqrt{a + \frac{b^2\mathcal{S}^3}{4} + O(\varepsilon)} + O(\varepsilon) \\
&\leq \frac{b\mathcal{S}^{\frac{3}{2}}}{2} + \sqrt{a + \frac{b^2\mathcal{S}^3}{4}} + O\left(\varepsilon^{\frac{1}{2}}\right) = \frac{\Lambda}{\sqrt{\mathcal{S}}} + O\left(\varepsilon^{\frac{1}{2}}\right),
\end{aligned}$$

where $\Lambda := \frac{b\mathcal{S}^2}{2} + \sqrt{a\mathcal{S} + \frac{b^2\mathcal{S}^4}{4}}$. This leads to that

$$\tilde{J}_{0,v_\varepsilon}(\tilde{t}_{v_\varepsilon}) = \frac{a}{2}\frac{c^2\tilde{t}_{v_\varepsilon}^2}{\|u_\varepsilon\|_2^2}\|\nabla u_\varepsilon\|_2^2 + \frac{b}{4}\frac{c^4\tilde{t}_{v_\varepsilon}^4}{\|u_\varepsilon\|_2^4}\|\nabla u_\varepsilon\|_2^4 - \frac{1}{6}\frac{c^6\tilde{t}_{v_\varepsilon}^6}{\|u_\varepsilon\|_2^6}\|u_\varepsilon\|_6^6$$

$$
\begin{aligned}
&= \frac{a}{2}\frac{c^2\tilde{t}_{v_\varepsilon}^2}{\|u_\varepsilon\|_2^2}\left(\mathcal{S}^{\frac{3}{2}}+O(\varepsilon)\right)+\frac{b}{4}\frac{c^4\tilde{t}_{v_\varepsilon}^4}{\|u_\varepsilon\|_2^4}\left(\mathcal{S}^{\frac{3}{2}}+O(\varepsilon)\right)^2-\frac{c^6\tilde{t}_{v_\varepsilon}^6}{\|u_\varepsilon\|_2^6}\frac{\left(\mathcal{S}^{\frac{3}{2}}+O\left(\varepsilon^3\right)\right)}{6}\\
&\le \frac{a}{2}\left(\frac{\Lambda}{\sqrt{\mathcal{S}}}+O\left(\varepsilon^{\frac{1}{2}}\right)\right)\left(\mathcal{S}^{\frac{3}{2}}+O(\varepsilon)\right)+\frac{b}{4}\left(\frac{\Lambda}{\sqrt{\mathcal{S}}}+O\left(\varepsilon^{\frac{1}{2}}\right)\right)^2\left(\mathcal{S}^3+O(\varepsilon)\right)\\
&\quad-\left(\frac{b\mathcal{S}^{\frac{3}{2}}}{2}+\sqrt{a+\frac{b^2\mathcal{S}^3}{4}+O(\varepsilon)}+O(\varepsilon)\right)^3\frac{\left(\mathcal{S}^{\frac{3}{2}}+O\left(\varepsilon^3\right)\right)}{6}\\
&\le \frac{a\Lambda\mathcal{S}}{2}+\frac{b\Lambda^2\mathcal{S}^2}{4}+O\left(\varepsilon^{\frac{1}{2}}\right)-\left(\frac{b\mathcal{S}^{\frac{3}{2}}}{2}+\sqrt{a+\frac{b^2\mathcal{S}^3}{4}}\right)^3\frac{\mathcal{S}^{\frac{3}{2}}}{6}\\
&= \frac{a\Lambda\mathcal{S}}{2}+\frac{b\Lambda^2\mathcal{S}^2}{4}-\frac{\Lambda^3}{6}+O\left(\varepsilon^{\frac{1}{2}}\right)=\frac{a\mathcal{S}\Lambda}{3}+\frac{b\mathcal{S}^2\Lambda^2}{12}+O\left(\varepsilon^{\frac{1}{2}}\right).
\end{aligned}
$$

From (4.1), we obtain that

$$
m_0(c)\le J_0(t_{v_\varepsilon}\star u)=J_{0,v_\varepsilon}(t_{v_\varepsilon})<\tilde{J}_{0,v_\varepsilon}(t_{v_\varepsilon})=\frac{a\mathcal{S}\Lambda}{3}+\frac{b\mathcal{S}^2\Lambda^2}{12}+O\left(\varepsilon^{\frac{1}{2}}\right).
$$

□

Lemma 4.2 *Assume that* $\{u_n\}\subset\mathcal{P}_{0,c}$ *is a minimizing sequence of* $m_0(c)$*. There is a sequence* $\{x_n\}\subset\mathbb{R}^3$ *and* $R>0,\ \kappa>0$ *such that*

$$
\int_{B_R(x_n)} {u_n}^2 \ge \kappa,
$$

Proof Assuming the contrary that the lemma does not hold. By the Vanishing Theorem, it follows that

$$
\int_{\mathbb{R}^3}|u_n|^p dx \to 0\ as\ n\to\infty,\quad for\ 2<p<6.
$$

Following from (2.4) and $P_0(u_n)=o_n(1)$, we have

$$
\int_{\mathbb{R}^3}\tilde{G}(u_n)dx \to 0\ as\ n\to\infty,
$$

and

$$
\ell := \lim_{n\to\infty}\|u_n\|_6^6=\lim_{n\to\infty}\left(a\|\nabla u_n\|_2^2+b\|\nabla u_n\|_2^4\right).
$$

Thus, we obtain $\lim_{n\to\infty}\|\nabla u_n\|_2^2=\sqrt{\frac{\ell}{b}+\frac{a^2}{4b^2}}-\frac{a}{2b}$. According to the Sobolev inequality, we have $\ell\ge b\mathcal{S}^2\ell^{\frac{2}{3}}+a\mathcal{S}\ell^{\frac{1}{3}}$. Two possible cases may occur: (i) $\ell\ge\Lambda^3$ and $\lim_{n\to\infty}\|\nabla u_n\|_2^2\ge\mathcal{S}\Lambda$, (ii) $\ell=0=\lim_{n\to\infty}\|\nabla u_n\|_2^2$, where $\Lambda=\frac{b\mathcal{S}^2}{2}+\sqrt{a\mathcal{S}+\frac{b^2\mathcal{S}^4}{4}}$.

If alternative (i) holds, we have

$$\begin{aligned} m_0(c) = \lim_{n\to+\infty} J_0(u_n) &= \lim_{n\to+\infty}\left[\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6\right] \\ &= \frac{\ell}{12} + \frac{a}{4}\left(\sqrt{\frac{\ell}{b} + \frac{a^2}{4b^2}} - \frac{a}{2b}\right) \\ &\geq \frac{\Lambda^3}{12} + \frac{a}{4}\sqrt{\frac{\Lambda^3}{b} + \frac{a^2}{4b^2}} - \frac{a^2}{8b} \\ &= \frac{\Lambda^3}{12} + \frac{a\mathcal{S}\Lambda}{4} = \frac{a\mathcal{S}\Lambda}{3} + \frac{b\mathcal{S}^2\Lambda^2}{12}, \end{aligned}$$

which contradicts to Lemma 4.1.

If alternative (ii) holds, we have

$$\begin{aligned} m_0(c) &= \lim_{n\to+\infty} J_0(u_n) \\ &= \lim_{n\to+\infty}\left[\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6\right] = 0, \end{aligned}$$

which contradicts to Lemma 3.10. Thus, we obtain that $\int_{B_R(x_n)} {u_n}^2 \geq \kappa$. □

The proof of Theorem 1.2: Let $\{u_n\} \subset \mathcal{P}_{0,c}$ be a minimizing sequence for $J_0|_{\mathcal{P}_{0,c}}$ at a positive level $m_0(c)$. Denote $\tilde{u}_n(x) = u_n(x + x_n)$, where $\{x_n\}$ is the sequence given in Lemma 4.2. By Lemma 3.9, we see that $\{\tilde{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Using standard argument, up to a subsequence, we may assume that there is a $\tilde{u} \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} & \text{in } H^1(\mathbb{R}^3), \\ \tilde{u}_n \to \tilde{u} & \text{in } L^p_{\text{loc}}(\mathbb{R}^3)\ 1 \leq p < 6, \\ \tilde{u}_n \to \tilde{u} & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

By Lemma 4.2, we see that $\tilde{u}$ is nontrivial. Moreover, $\tilde{u}$ satisfies

$$-(a + bA)\,\Delta\tilde{u} = g(\tilde{u}) + |\tilde{u}|^4\tilde{u},$$

where $A := \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla\tilde{u}_n|^2$ and $\int_{\mathbb{R}^3} |\nabla\tilde{u}|^2 \leq A$. Hence, we have the following Pohozaev identity

$$\begin{aligned} P_A(\tilde{u}) :&= (a + Ab)\|\nabla\tilde{u}\|_2^2 - 3\int_{\mathbb{R}^3} \tilde{G}(\tilde{u})dx - \|\tilde{u}\|_6^6 = 0 \\ &\geq a\|\nabla\tilde{u}\|_2^2 + b\|\nabla\tilde{u}\|_2^4 - 3\int_{\mathbb{R}^3} \tilde{G}(\tilde{u})dx - \|\tilde{u}\|_6^6 = P_0(\tilde{u}). \end{aligned}$$

Now, we prove that $P_0(\tilde{u}) = 0$. Just suppose $P_0(\tilde{u}) < 0$, then there exists a unique $0 < \tilde{t} < 1$ such that $P_0(\tilde{t}\star\tilde{u})$=0 by Corollary 3.8.

We note that the assumption (G3) implies that $s^{-2}F(s) := \frac{3}{14}g(s)s^{-1} - G(s)s^{-2}$ increases in $(0, +\infty)$ and decreases in $(-\infty, 0)$. Therefore

$$(\tilde{t}^{\frac{3}{2}}\tilde{u})^{-2}F(\tilde{t}^{\frac{3}{2}}\tilde{u}) \le (\tilde{u})^{-2}F(\tilde{u}), \quad x \in \{x \in \mathbb{R}^3 : \tilde{u} \neq 0\}$$

where we have used that $\tilde{t} \in (0, 1)$. Then we have

$$\tilde{t}^{-3}F(\tilde{t}^{\frac{3}{2}}\tilde{u}) \le F(\tilde{u}), \quad x \in \{x \in \mathbb{R}^3 : \tilde{u} \neq 0\}. \tag{4.2}$$

It is easy to see that $F(0) = 0$, which implies that

$$\tilde{t}^{-3}F(\tilde{t}^{\frac{3}{2}}\tilde{u}) = 0 = F(\tilde{u}), \quad x \in \{x \in \mathbb{R}^3 : \tilde{u} = 0\}. \tag{4.3}$$

According to the definition of $F(s)$, combing (4.2) with (4.3), we obtain

$$\tilde{t}^{-3}\big(\frac{3}{14}g(\tilde{t}^{\frac{3}{2}}\tilde{u})\tilde{t}^{\frac{3}{2}}\tilde{u} - G(\tilde{t}^{\frac{3}{2}}\tilde{u})\big) \le \frac{3}{14}g(\tilde{u})\tilde{u} - G(\tilde{u}), x \in \mathbb{R}^3. \tag{4.4}$$

Using (4.4), together with $P_0(\tilde{t}\star\tilde{u})$=0, we obtain

$$\begin{aligned}
m_0(c) &\le J_0(\tilde{t}\star\tilde{u}) - \frac{1}{4}P_0(\tilde{t}\star\tilde{u}) \\
&= \frac{a}{4}\tilde{t}^2\|\nabla\tilde{u}\|_2^2 + \frac{1}{12}\tilde{t}^6\|\tilde{u}\|_6^6 + \frac{7}{4}\int_{\mathbb{R}^3}\tilde{t}^{-3}\big(\frac{3}{14}g(\tilde{t}^{\frac{3}{2}}\tilde{u})\tilde{t}^{\frac{3}{2}}\tilde{u} - G(\tilde{t}^{\frac{3}{2}}\tilde{u})\big)dx \\
&< \frac{a}{4}\|\nabla\tilde{u}\|_2^2 + \frac{1}{12}\|\tilde{u}\|_6^6 + \frac{7}{4}\int_{\mathbb{R}^3}\big(\frac{3}{14}g(\tilde{u})\tilde{u} - G(\tilde{u})\big)dx \\
&\le \varliminf_{n\to\infty}[\frac{a}{4}\|\nabla\tilde{u}_n\|_2^2 + \frac{1}{12}\|\tilde{u}_n\|_6^6 + \frac{7}{4}\int_{\mathbb{R}^3}\big(\frac{3}{14}g(\tilde{u}_n)\tilde{u}_n - G(\tilde{u}_n)\big)dx] \\
&= \varliminf_{n\to\infty}[I(\tilde{u}_n) - \frac{1}{4}P_0(\tilde{u}_n)] = m_0(c).
\end{aligned} \tag{4.5}$$

which cause a contradiction. Thus, we obtain that $A = \int_{\mathbb{R}^3}|\nabla\tilde{u}|^2dx$ and $\tilde{t} = 1$. Using (4.5) again with $\tilde{t} = 1$, we deduce that $J_0(\tilde{u}) = m_0(c)$. By Lemma 3.5 and Remark 3.11, we can see that there exists a $\lambda_c \in \mathbb{R}$ such that $(\tilde{u}, \lambda_c)$ is a normalized solution of Problem (1.14)–(1.2). □

5 Proof of Theorem 1.4

Lemma 5.1 *$m_0(c)$ is strictly decreasing with respect to $c \in (0, +\infty)$.*

Proof A similar argument, as the proof of Theorem 1.1 of [33], can be used to show this Lemma. So we omit it here. □

Lemma 5.2 *Assume that $V(x) \neq 0$ satisfies $(V1)$, $(V2)$ and $(V3)$. For any $c > 0$, there holds*

$$m_V(c) < m_0(c). \tag{5.1}$$

Proof In Theorem 1.2, we have shown the following fact: $m_0(c)$ can be attained. Thus, we may let $\tilde{u}(x) \in \mathcal{P}_{0,c}$ attain $m_0(c)$. Following from the standard potential theory and maximum principle, we can see that $\tilde{u}(x) > 0$ in $\mathbb{R}^3$. By Lemma 3.7, we can see that there exists $t_{\tilde{u}} > 0$ such that $t_{\tilde{u}}\star\tilde{u} \in \mathcal{P}_{V,c}$, which, combining the fact that $V(x) \not\equiv 0$ and $\sup\limits_{x\in\mathbb{R}^3} V(x) = 0$, implies that

$$\begin{aligned} m_V(c) \le J_V(t_{\tilde{u}}\star\tilde{u}) &= J_0(t_{\tilde{u}}\star\tilde{u}) + \frac{1}{2}\int_{\mathbb{R}^3} V(x) t_{\tilde{u}}^3 \tilde{u}^2(t_{\tilde{u}}x)\mathrm{d}x \\ &< J_0(t_{\tilde{u}}\star\tilde{u}) \le \max_{t>0} J_0(t\star\tilde{u}) = J_0(\tilde{u}) = m_0(c). \end{aligned}$$

The proof of Lemma 5.2 is complete. □

Proposition 5.3 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. For any $c > 0$, let $\{\bar{u}_n\} \subseteq \mathcal{P}_{V,c}$ be a minimizing sequence for $J_V\big|_{\mathcal{P}_{V,c}}$ at a positive level $m_V(c)$. Then there exist a subsequence of $\{\bar{u}_n\}$ (still denoted by $\{\bar{u}_n\}$), a $\bar{u} \in H^1(\mathbb{R}^3)$ satisfying*

$$-\left(a + bB^2\right)\Delta\bar{u} + V(x)\bar{u} + \lambda\bar{u} = g(\bar{u}) + |\bar{u}|^4\bar{u} \text{ in } \mathbb{R}^3, \tag{5.2}$$

$k_0 \in \mathbb{N}\cup\{0\}$, nontrivial solutions $w^1, \ldots, w^{k_0}$ of the following problem

$$-(a + B^2 b)\Delta u + \lambda u = |u|^4 u + g(u) \tag{5.3}$$

$m_0 \in \mathbb{N}\cup\{0\}$, nontrivial solutions $\hat{u}^1, \hat{u}^2, \ldots, \hat{u}^{m_0}$ of the following problems

$$-(a + B^2 b)\Delta u = |u|^4 u, \tag{5.4}$$

such that

$$\begin{aligned} &m_V(c) + \frac{bB^4}{4} = J_{V,B}(\bar{u}) + \sum_{i=1}^{k_0} J_{0,B}(w^i) + \sum_{j=1}^{m_0} \hat{J}_B(\hat{u}^j), \\ &||\nabla\bar{u}_n||_2^2 \to ||\nabla\bar{u}||_2^2 + \sum_{i=1}^{k_0} ||\nabla w^i||_2^2 + \sum_{j=1}^{m_0} ||\nabla\hat{u}^j||_2^2 \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} &\lambda = \lim_{n\to+\infty} \lambda_n, \quad B^2 = \lim_{n\to\infty} \|\nabla\bar{u}_n\|_2^2, \\ &J_{V,B}(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{bB^2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x \end{aligned} \tag{5.6}$$

$$-\int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\mathrm{d}x, \tag{5.7}$$

$$J_{0,B}(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{bB^2}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\mathrm{d}x. \tag{5.8}$$

and

$$\hat{J}_B(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{bB^2}{2}\|\nabla u\|_2^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\mathrm{d}x. \tag{5.9}$$

Proof We divide the proof into three steps:

step 1: By Lemma 3.9, Remark 3.6 and the fact that $\{\bar{u}_n\} \subseteq \mathcal{P}_{V,c}$, it is easy to see that $\{\bar{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and

$$J_V'(\bar{u}_n) + \lambda_n \bar{u}_n \to 0 \text{ in } H^{-1}(\mathbb{R}^3), \tag{5.10}$$

which implies that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Up to a subsequence, we may assume that there is $\bar{u} \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} \bar{u}_n \rightharpoonup \bar{u} & \text{in } H^1(\mathbb{R}^3), \\ \bar{u}_n \to \bar{u} & \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \le p < 6, \\ \bar{u}_n \to \bar{u} & \text{a.e. in } \mathbb{R}^3, \end{cases}$$

which, together with (5.10), conclude that $\bar{u}$ is a solution of

$$-\left(a + bB^2\right)\Delta\bar{u} + V(x)\bar{u} + \lambda\bar{u} = g(\bar{u}) + |\bar{u}|^4\bar{u} \text{ in } \mathbb{R}^3.$$

We claim that $\bar{u} \neq 0$. In fact, if $\bar{u} = 0$, then the Brézis-Lieb Lemma and (V1) lead to

$$\int_{\mathbb{R}^3} V(x)\bar{u}_n^2\mathrm{d}x = \int_{\mathbb{R}^3} V(x)\bar{u}^2\mathrm{d}x + \int_{\mathbb{R}^3} V(x)(\bar{u}_n - \bar{u})^2\mathrm{d}x + o_n(1) = o_n(1),$$

which implies that $m_V(c) + o_n(1) = J_0(\bar{u}_n)$. Furthermore, we obtain $(J_{0,\bar{u}_n})'(1) = (J_{V,\bar{u}_n})'(1) + o_n(1) = o_n(1)$. It follows from the uniqueness of the critical point of $J_{0,\bar{u}_n}(t)$ (See Corollary 3.9, [24]) that there exists $t_n = 1 + o_n(1)$ such that $t_n \star \bar{u}_n \in \mathcal{P}_{0,c}$. Hence

$$m_0(c) \le J_0(t_n \star \bar{u}_n) = J_0(\bar{u}_n) + o_n(1) = m_V(c) + o_n(1),$$

which contradicts to Lemma 5.2. Therefore $\bar{u} \neq 0$.

Step 2: If the vanishing case occurs, then we go to step 3. So we may assume that the vanishing does not occur. Let $\bar{u}_n^1 = \bar{u}_n - \bar{u}$. Since

$$J_{V,B}(\bar{u}_n) \to m_V(c) + \frac{bB^4}{4}, \quad J_{V,B}'(\bar{u}_n) + \lambda_n\bar{u}_n \to 0 \; in \; H^{-1}(\mathbb{R}^3), \tag{5.11}$$

the Brezis-Lieb Lemma implies that

$$||\nabla \bar{u}_n||_2^2 - ||\nabla \bar{u}_n^1||_2^2 \to ||\nabla \bar{u}||_2^2, \tag{5.12}$$

$$J_{0,B}(\bar{u}_n^1) \to m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u}), \tag{5.13}$$

and

$$J'_{0,B}(\bar{u}_n^1) + \lambda \bar{u}_n^1 \to 0 \; in \; H^{-1}(\mathbb{R}^3), \tag{5.14}$$

which implies that $\{\bar{u}_n^1\}$ is bounded in $H^1(\mathbb{R}^3)$. Since the vanishing does not occur, there exists $\{y_n^1\}$ with $y_n^1 \to +\infty$ as $n \to +\infty$ and $\omega^1 \neq 0$ such that

$$\begin{cases} \bar{u}_n^1(x + y_n^1) \rightharpoonup \omega^1 & \text{in } H^1(\mathbb{R}^3), \\ \bar{u}_n^1(x + y_n^1) \to \omega^1 & \text{in } L^p_{\text{loc}}(\mathbb{R}^3), \; 1 \le p < 6, \\ \bar{u}_n^1(x + y_n^1) \to \omega^1 & \text{a.e. in } \mathbb{R}^3, \end{cases}$$

which, combining with (5.14), implies that ω^1 is a nontrivial solution of (5.3).

Let $\bar{u}_n^2 = \bar{u}_n - \bar{u} - \omega^1(x - y_n^1)$. If the vanishing occurs, then we stop and go to step 3. We may assume that $\lim\limits_{n\to+\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B(y,1)} |\bar{u}_n - \bar{u} - \omega^1(x - y_n^1)|^2 \, dy \neq 0$. By the Brezis-Lieb Lemma, we have that

$$\begin{aligned} &||\nabla \bar{u}_n||_2^2 - ||\nabla \bar{u}_n^2||_2^2 \to ||\nabla \bar{u}||_2^2 + ||\nabla \omega^1||_2^2, \\ &J_{0,B}(\bar{u}_n^2) \to m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u}) - J_{0,B}(\omega^1), \\ &\;J'_{0,B}(\bar{u}_n^2) + \lambda \bar{u}_n^2 \to 0 \; in \; H^{-1}(\mathbb{R}^3), \end{aligned} \tag{5.15}$$

which implies that $\{\bar{u}_n^2\}$ is bounded in $H^1(\mathbb{R}^3)$. Since the vanishing does not occur, there exists $\{y_n^2\}$ with $y_n^1 \to +\infty$ as $n \to +\infty$ and $\omega^2 \neq 0$ such that

$$\begin{cases} \bar{u}_n^2(x + y_n^2) \rightharpoonup \omega^2 & \text{in } H^1(\mathbb{R}^3), \\ \bar{u}_n^2(x + y_n^2) \to \omega^2 & \text{in } L^p_{\text{loc}}(\mathbb{R}^3), \; 1 \le p < 6, \\ \bar{u}_n^2(x + y_n^2) \to \omega^2 & \text{a.e. in } \mathbb{R}^3, \end{cases}$$

which, together with (5.15), implies that ω^2 is a nontrivial solution of (5.3). Going on as above, the vanishing must occur after finite steps, since $||\nabla \omega^i||_2^2 \ge \delta_c > 0$ (See Lemma 3.4). We may assume that the vanishing occurs after k_0 steps, which implies that $\omega^1, \omega^2, \cdots, \omega^{k_0}$ are nontrivial solutions of (5.3),

$$\int_{\mathbb{R}^3} |\bar{u}_n^{k_0+1}|^s \to 0 (2 < s < 2^*), \tag{5.16}$$

$$||\nabla \bar{u}_n||_2^2 - ||\nabla \bar{u}_n^{k_0+1}||_2^2 \to ||\nabla \bar{u}||_2^2 + \sum_{i=1}^{k_0} ||\nabla \omega^i||_2^2, \tag{5.17}$$

and

$$J_{0,B}(\bar{u}_n^{k_0+1}) \to m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u}) - \sum_{i=1}^{k_0} J_{0,B}(\omega^i),$$
$$J'_{0,B}(\bar{u}_n^{k_0+1}) + \lambda \bar{u}_n^{k_0+1} \to 0 \; in \; H^{-1}(\mathbb{R}^3), \tag{5.18}$$

where $\bar{u}_n^{k_0+1} = \bar{u}_n - \bar{u} - \sum_{i=1}^{k_0} \omega^i(x - y_n^i)$.

Step 3: The vanishing occurs. If the vanishing occurs, then, following from (5.16) and (5.18), we have that

$$\hat{J}_{0,B}(\bar{u}_n^{k_0+1}) \to m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u}) - \sum_{i=1}^{k_0} J_{V,B}(\omega^i), \quad \hat{J}'_B(\bar{u}_n^{k_0+1}) \to 0 \; in \; H^{-1}(\mathbb{R}^3). \tag{5.19}$$

If $m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u}) - \sum_{i=1}^{k_0} J_{V,B}(\omega^i) = 0$, we complete the proof. Otherwise, going on as the proof of Theorem 2.5 in [2], we can see that there exist $m_0 \in \mathbb{N} \cup \{0\}$, $\{\sigma_n^j\}_{j=1}^{m_0}$, $\{z_n^j\}_{j=1}^{m_0}$, nontrivial solutions $\hat{u}^1, \hat{u}^2, ..., \hat{u}^{m_0}$ of the following problems

$$-(a + B^2 b)\Delta u = |u|^4 u,$$

such that

$$m_V(c) + \frac{bB^4}{4} - J_B(\bar{u}) - \sum_{i=1}^{k_0} I_B(\omega^i) = \sum_{j=1}^{m_0} \hat{I}_B(\hat{u}_j),$$

and

$$||\nabla \bar{u}_n^{k_0+1}||_2^2 \to \sum_{j=1}^{m_0} ||\nabla \hat{u}_j||_2^2,$$

which, together with (5.17), implies that

$$||\nabla \bar{u}_n||_2^2 \to ||\nabla \bar{u}||_2^2 + \sum_{i=1}^{k_0} ||\nabla \omega^i||_2^2 + \sum_{j=1}^{m_0} ||\nabla \hat{u}_j||_2^2.$$

We complete the proof. □

Lemma 5.4 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. If $\bar{u} \in H^1(\mathbb{R}^3)$ is a nontrivial solution of (5.2), then*

$$J_{V,B}(\bar{u}) \geq \frac{bB^2}{4} \|\nabla \bar{u}\|_2^2, \tag{5.20}$$

where $J_{V,B}(u)$ *has been defined in* (5.7).

Proof The argument is similar to [25], for readers's convenience, we give a detailed proof.

Since $\bar{u}$ is a nontrivial solution of (5.2), $\bar{u}$ satisfies the corresponding Pohozaev identity $P_B(\bar{u}) = 0$, where

$$P_B(\bar{u}) := a\|\nabla \bar{u}\|_2^2 + bB^2\|\nabla \bar{u}\|_2^2 - \int_{\mathbb{R}^3} W(x)\bar{u}^2 \mathrm{d}x - 3\int_{\mathbb{R}^3} \widetilde{G}(\bar{u})\mathrm{d}x - \int_{\mathbb{R}^3} |\bar{u}|^6 \mathrm{d}x \tag{5.21}$$

By the assumptions (G2), (V2) and the Pohozaev identity (5.21), we can deduce by the assumption (V2) that

$$\begin{aligned}(a+\sigma_2)\|\nabla \bar{u}\|_2^2 + bB^2\|\nabla \bar{u}\|_2^2 - \|\bar{u}\|_6^6 &\geq a\|\nabla \bar{u}\|_2^2 + bB^2\|\nabla \bar{u}\|_2^2 - \int_{\mathbb{R}^3} W(x)\bar{u}^2 \mathrm{d}x - \|\bar{u}\|_6^6 \\ &= 3\int_{\mathbb{R}^3} \left(\frac{1}{2} g(\bar{u})\bar{u} - G(\bar{u})\right) \mathrm{d}x \\ &\geq \frac{3(\alpha-2)}{2}\int_{\mathbb{R}^3} G(\bar{u})\mathrm{d}x.\end{aligned} \tag{5.22}$$

Hence by the assumptions (V1)–(V2) and (5.22), we conclude that

$$\begin{aligned}& J_{V,B}(\bar{u}) - \frac{bB^2}{4}\|\nabla \bar{u}\|_2^2 \\ &= \frac{a}{2}\|\nabla \bar{u}\|_2^2 + \frac{bB^2}{4}\|\nabla \bar{u}\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)\bar{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(\bar{u})\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |\bar{u}|^6 \mathrm{d}x \\ &\geq \left(\frac{1}{2}(a-\sigma_1) - \frac{2(a+\sigma_2)}{3(\alpha-2)}\right)\|\nabla \bar{u}\|_2^2 + \left(\frac{b}{4} - \frac{2b}{3(\alpha-2)}\right) B^2\|\nabla \bar{u}\|_2^2 \\ &\quad + \left(\frac{2}{3(\alpha-2)} - \frac{1}{6}\right)\|\bar{u}\|_6^6 \\ &\geq 0.\end{aligned}$$

□

Lemma 5.5 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. If* $\omega^i, i = 1, 2, \cdots, k_0$, *is a nontrivial solution of* (5.3)*, then*

$$J_{0,B}(w^i) \geq m_0(\|w^i\|_2^2) + \frac{bB^2}{4}\|\nabla w^i\|_2^2 \tag{5.23}$$

where $J_{0,B}(u)$ *has been defined in* (5.8).

Proof Since w^i is a weak solution to (5.3), it satisfies the corresponding Pohozaev identity $P_{0,B}(w^i) = 0$, where

$$P_{0,B}(u) := a\|\nabla u\|_2^2 + bB^2\|\nabla u\|_2^2 - 3\int_{\mathbb{R}^3} \widetilde{G}(u)\mathrm{d}x - \|u\|_6^6. \tag{5.24}$$

It follows that

$$\begin{aligned} J_{0,B}(w^i) &= \frac{a}{4}\|\nabla w^i\|_2^2 + \frac{a}{4}\|\nabla w^i\|_2^2 + \frac{bB^2}{2}\|\nabla w^i\|_2^2 - \int_{\mathbb{R}^3} G(w^i)\mathrm{d}x - \frac{1}{6}\|w^i\|_6^6 \\ &= \frac{a}{4}\|\nabla w^i\|_2^2 + \frac{bB^2}{4}\|\nabla w^i\|_2^2 + \frac{7}{4}\int_{\mathbb{R}^3}\left(\frac{3}{14}g(w^i)w^i - G(w^i)\right)\mathrm{d}x + \frac{1}{12}\|w^i\|_6^6. \end{aligned} \tag{5.25}$$

By (5.6), we can deduce that

$$\begin{aligned} P_0(w^i) &= a\|\nabla w^i\|_2^2 + b\|\nabla w^i\|_2^4 - 3\int_{\mathbb{R}^3} \widetilde{G}(w^i)\mathrm{d}x - \|w^i\|_6^6 \\ &< a\|\nabla w^i\|_2^2 + bB^2\|\nabla w^i\|_2^2 - 3\int_{\mathbb{R}^3} \widetilde{G}(w^i)\mathrm{d}x - \|w^i\|_6^6 \\ &= P_{0,B}(w^i) = 0. \end{aligned}$$

According to (2.4), for $0 < t < 1$ sufficiently small, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \widetilde{G}(t\star w^i)\mathrm{d}x &= \int_{\mathbb{R}^3}\left(\frac{1}{2}g(t\star w^i)(t\star w^i) - G(t\star w^i)\right)\mathrm{d}x \\ &\le \left(\frac{\beta}{2}-1\right)C_2\int_{\mathbb{R}^3}(|t\star w^i|^\alpha + |t\star w^i|^\beta)\mathrm{d}x \\ &= \left(\frac{\beta}{2}-1\right)C_2\int_{\mathbb{R}^3}(t^{\frac{3\alpha}{2}-3}|w^i|^\alpha + t^{\frac{3\beta}{2}-3}|w^i|^\beta)\mathrm{d}x. \end{aligned}$$

By Corollary 3.8 ,we obtain that there exists a $t_{w^i} \in (0, 1)$ such that $P_0(t_{w^i}\star w^i) = 0$. Therefore, we deduce from Proposition 3.2 that t_{w^i} is the unique critical point of $I_{w^i}(t) = I(t\star w^i)$ and

$$J_0(t_{w^i}\star w^i) = \max_{t>0} J_0(t\star w^i).$$

Hence

$$\begin{aligned} J_0(t_{w^i}\star w^i) &= \frac{at_{w^i}^2}{2}\|\nabla w^i\|_2^2 + \frac{bt_{w^i}^4}{4}\|\nabla w^i\|_2^4 - \int_{\mathbb{R}^3} G(t_{w^i}\star w^i)\mathrm{d}x - \frac{t_{w^i}^6}{6}\|w^i\|_6^6 \\ &= \frac{at_{w^i}^2}{4}\|\nabla w^i\|_2^2 + \frac{7}{4}\int_{\mathbb{R}^3}\left(\frac{3}{14}g(t_{w^i}\star w^k)(t_{w^i}\star w^i) - G(t_{w^i}\star w^i)\right)\mathrm{d}x \\ &\quad + \frac{t_{w^i}^6}{12}\|w^i\|_6^6 \\ &< \frac{a}{4}\|\nabla w^i\|_2^2 + \frac{1}{12}\|w^i\|_6^6 + \frac{7}{4}\int_{\mathbb{R}^3}\left(\frac{3}{14}g(w^i)w^i - G(w^i)\right)\mathrm{d}x \\ &= J_{0,B}(w^i) - \frac{bB^2}{4}\|\nabla w^i\|_2^2. \end{aligned} \tag{5.26}$$

So combining with (5.25)–(5.26), we have

$$J_{0,B}(w^i) \geq J_0(t_{w^i} \star w^i) + \frac{bB^2}{4}\|\nabla w^i\|_2^2 \geq m_0(\|w^i\|_2^2) + \frac{bB^2}{4}\|\nabla w^i\|_2^2. \tag{5.27}$$

□

Lemma 5.6 *Assume that the assumptions (G1)–(G3) and (V1)–(V3) hold. For any $c > 0$, let $\{\bar{u}_n\} \subseteq \mathcal{P}_{V,c}$ be a minimizing sequence for $J_V\big|_{\mathcal{P}_{V,c}}$ at a positive level $m_V(c)$. Then there exist a subsequence of $\{\bar{u}_n\}$ (still denoted by $\{\bar{u}_n\}$) and a $\bar{u} \in H^1(\mathbb{R}^3)$ satisfying*

$$-\big(a + bB^2\big)\Delta\bar{u} + V(x)\bar{u} + \lambda\bar{u} = g(\bar{u}) + |\bar{u}|^4\bar{u} \text{ in } \mathbb{R}^3,$$

such that, $n \to +\infty$,

$$\bar{u}_n \to \bar{u} \quad \text{in } H^1(\mathbb{R}^3).$$

Proof We claim that $k_0 = 0$. To check this, we may suppose that $k_0 \neq 0$. If $m_0 \neq 0$, according to (5.5), (5.6), (5.20), (5.23) and (5.27), we deduce that

$$\begin{aligned}
m_V(c) + \frac{bB^4}{4} &= J_{V,B}(\bar{u}) + \sum_{i=1}^{k_0} J_{0,B}(w^i) + \sum_{j=1}^{m_0} \hat{J}_B(\hat{u}^j) \\
&\geq \frac{bB^2}{4}\|\nabla\bar{u}\|_2^2 + k_0 m_0(\|w^i\|_2^2) + \frac{bB^2}{4}\sum_{i=1}^{k_0}\|\nabla w^i\|_2^2 + \frac{a+bB^2}{3}\sum_{j=1}^{m_0}\|\nabla\hat{u}^j\|_2^2 \\
&\geq \frac{bB^2}{4}\|\nabla\bar{u}\|_2^2 + k_0 m_0(c) + \frac{bB^2}{4}\sum_{i=1}^{k_0}\|\nabla w^i\|_2^2 + \frac{a+bB^2}{3}\sum_{j=1}^{m_0}\|\nabla\hat{u}^j\|_2^2 \\
&\geq m_0(c) + \frac{bB^4}{4} + (\frac{a}{3} + \frac{bB^2}{12})\sum_{j=1}^{m_0}\|\nabla\hat{u}^j\|_2^2 \\
&> m_V(c) + \frac{bB^4}{4},
\end{aligned}$$

which is impossible. If $m_0 = 0$, we can complete the proof of $k_0 = 0$ by means of similar method, the rest being standard.

Now, we consider the case $k_0 = 0$. In this case, Lemma 5.3 allows us to obtain that if the vanishing case of $\{\bar{u}_n^1\}$ occurs, from (5.11), (5.12) and (5.13), it follows that

$$\begin{aligned}
\lim_{n\to n} J_{0,B}(\bar{u}_n^1) &= \lim_{n\to n} \hat{J}_B(\bar{u}_n^1) \\
&= \lim_{n\to n}\Big(\frac{a+bB^2}{2}\|\nabla\bar{u}_n^1\|_2^2 - \frac{1}{6}\int_{\mathbb{R}^3}|\bar{u}_n^1|^6 \mathrm{d}x\Big) \\
&= m_V(c) + \frac{bB^4}{4} - J_{V,B}(\bar{u})
\end{aligned}$$

$$
\begin{aligned}
&\leq m_V(c) + \frac{bB^4}{4} - \frac{bB^2}{4}\|\nabla \bar{u}^1\|_2^2 \\
&= m_V(c) + \frac{bB^2}{4}\big(\lim_{n\to n}\|\nabla \bar{u}_n^1\|_2^2\big) \\
&< m_0(c) + \frac{bB^2}{4}\big(\lim_{n\to n}\|\nabla \bar{u}_n^1\|_2^2\big). \qquad (5.28)
\end{aligned}
$$

Using (5.14), we see that

$$
\int_{\mathbb{R}^3}(a+B^2b)\nabla \bar{u}_n^1 \nabla \varphi \mathrm{d}x = \int_{\mathbb{R}^3}|\bar{u}_n^1|^4 \bar{u}_n^1 \varphi \mathrm{d}x + o_n(1), \quad \forall \varphi \in H^1(\mathbb{R}^3),
$$

which implies that

$$
\begin{aligned}
\gamma := \lim_{n\to\infty}\|\bar{u}_n^1\|_6^6 &= \lim_{n\to\infty}\left(a + bB^2\|\nabla \bar{u}_n^1\|_2^2\right) \\
&= \lim_{n\to\infty}\left((a + b\|\nabla \bar{u}\|_2^2)\|\nabla \bar{u}_n^1\|_2^2 + b\|\nabla \bar{u}_n^1\|_2^4\right) \\
&= \lim_{n\to\infty}\left(\bar{a}\|\nabla \bar{u}_n^1\|_2^2 + b\|\nabla \bar{u}_n^1\|_2^4\right),
\end{aligned}
$$

where $\bar{a} := a + b\|\nabla \bar{u}\|_2^2 > a$. Similar to Lemma 4.2 , we have $\lim_{n\to\infty}\left\|\nabla \bar{u}_n^1\right\|_2^2 = \sqrt{\frac{\gamma}{b} + \frac{\bar{a}^2}{4b^2}} - \frac{\bar{a}}{2b}$. According to the Sobolev inequality, we have $\gamma \geq b\mathcal{S}^2\gamma^{\frac{2}{3}} + \bar{a}\mathcal{S}\gamma^{\frac{1}{3}}$. Two possible cases may occur: either $\gamma \geq \bar{\Lambda}^3$ and $\lim_{n\to\infty}\left\|\nabla \bar{u}_n^1\right\|_2^2 \geq \mathcal{S}\bar{\Lambda}$, or $\gamma = 0 = \lim_{n\to\infty}\left\|\nabla \bar{u}_n^1\right\|_2^2$, where $\bar{\Lambda} = \frac{b\mathcal{S}^2}{2} + \sqrt{\bar{a}\mathcal{S} + \frac{b^2\mathcal{S}^4}{4}} > \Lambda = \frac{b\mathcal{S}^2}{2} + \sqrt{a\mathcal{S} + \frac{b^2\mathcal{S}^4}{4}}$.

Assume $\gamma \geq \bar{\Lambda}^3$. We notice that

$$
\sqrt{\frac{\bar{\Lambda}^3}{b} + \frac{\bar{a}^2}{4b^2}} = \frac{\bar{a} + 2b\mathcal{S}\bar{\Lambda}}{2b} = \frac{a + 2b\mathcal{S}\Lambda}{2b} + \frac{\bar{a} - a}{2b} + \frac{2b\mathcal{S}(\bar{\Lambda} - \Lambda)}{2b}. \qquad (5.29)
$$

Then we deduce from (5.28) and (5.29) that

$$
\begin{aligned}
m_0(c) &> \lim_{n\to n}\big(\frac{a}{2}\|\nabla \bar{u}_n^1\|_2^2 + \frac{bB^2}{4}\|\nabla \bar{u}_n^1\|_2^2 - \frac{1}{6}\int_{\mathbb{R}^3}|\bar{u}_n^1|^6 \mathrm{d}x\big) \\
&= \frac{\gamma}{4} - \frac{\gamma}{6} + \frac{a}{4}\lim_{n\to\infty}\left\|\nabla \bar{u}_n^1\right\|_2^2 \\
&\geq \frac{\bar{\Lambda}^3}{12} + \frac{a}{4}\left(\sqrt{\frac{\bar{\Lambda}^3}{b} + \frac{\bar{a}^2}{4b^2}} - \frac{\bar{a}}{2b}\right) \\
&> \frac{\Lambda^3}{12} + \frac{a}{4}\sqrt{\frac{\Lambda^3}{b} + \frac{a^2}{4b^2}} - \frac{a^2}{8b} + \frac{a}{4}\mathcal{S}(\bar{\Lambda} - \Lambda)
\end{aligned}
$$

$$> \frac{\Lambda^3}{12} + \frac{a\mathcal{S}\Lambda}{4} = \frac{a\mathcal{S}\Lambda}{3} + \frac{b\mathcal{S}^2\Lambda^2}{12},$$

which contradicts to Lemma 4.1. Hence, $\gamma = 0 = \lim_{n\to\infty} \|\nabla \bar{u}_n^1\|_2^2$, which implies that $m_0 = 0$ and $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$. □

The proof of Theorem 1.4: According to Lemma 5.1, Proposition 5.3 and Lemma 5.6, we can see that, under the assumptions of Theorem 1.4, we obtain $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$. So $J_V(\bar{u}) = m_V(c)$ and $\bar{u} \in \mathcal{P}_{V,c}$, which, together with Lemma 3.5 and Remark 3.11, implies that Problem (1.12)–(1.2) admits a ground state normalized solutions $(\bar{u}, \bar{\lambda}_c) \in S_c \times \mathbb{R}$. We complete the proof. □

References

1. Alves, C.O., Crrea, F.J.S.A.: On existence of solutions for a class of problem involving a nonlinear operator. Commun. Appl. Nonlinear Anal. **8**, 43–56 (2001)
2. Benci, V., Cerami, G.: Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$. J. Funct. Anal. **88**, 90–117 (1990)
3. He, X.M., Zou, W.M.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differ. Equ. **252**, 1813–1834 (2012)
4. Figueiredo, G.M., Ikoma, N., Santos Junior, J.R.: Existence and concentration result for the Kirchhoff type equations with general nonlinearities. Arch. Ration. Mech. Anal. **213**, 931–979 (2014)
5. Guo, Z.J.: Ground states for Kirchhoff equations without compact condition. J. Differ. Equ. **259**, 2884–2902 (2015)
6. He, Y., Li, G.B.: Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Calc. Var. **54**, 3067–3106 (2015)
7. He, Y., Li, G.B., Peng, S.J.: Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Adv. Nonlinear Stud. **14**, 441–468 (2014)
8. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
9. Li, G.B., Ye, H.Y.: Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$. J. Differ. Equ. **257**, 566–600 (2014)
10. Lions, J.L.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proceedings of International Symposium. Inst. Mat., Univ. Fed. Rio de Janeino, 1977, in: North-Holl. Math. Stud., vol. 30, North-Hollad, Amsterdam, pp. 284-346 (1978)
11. Luo, X., Wang, Q.F.: Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in $\mathbb{R}^3$. Nonlinear Anal. **33**, 19–32 (2017)
12. Wang, J., Tian, L., Xu, J., Zhang, F.: Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. J. Differ. Equ. **253**, 2314–2351 (2012)
13. Wu, X.: Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^3$. Nonlinear Anal. Real World Appl. **12**, 1278–1287 (2011)
14. Ye, H.Y.: The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations. Math. Methods Appl. Sci. **38**, 2663–2679 (2015)
15. Ye, H.Y.: The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations. Z. Angew. Math. Phys. **66**, 1483–1497 (2015)
16. Li, G.B., Luo, X., Yang, T.: Normalized solutions to a class of Kirchhoff equations with Sobolev critical expoent. Ann. Fenn. Math. **47**, 895–925 (2022)
17. Willem, M.: Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, Vol.24, Birkhäuser Boston, Inc., Boston (1996)
18. Wang, Z.Z., Zeng, X.Y., Zhang, Y.M.: Multi-peak solutions of Kirchhoff equations involving subcritical or critical Sobolev exponents. Math. Meth. Appl. Sci. **43**, 5151–5161 (2020)
19. Zeng, X.Y., Zhang, Y.M.: Existence and uniqueness of normalized solutions for the Kirchhoff equation. Appl. Math. Lett. **74**, 52–59 (2017)

20. Zeng, Y.L., Chen, K.S.: Remarks on normalized solutions for L^2-critical Kirchhoff problems. Taiwan. J. Math. **20**, 617–627 (2016)
21. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)
22. Ding, Y.H., Zhong, X.X.: Normalized solution to the Schödinger equation with potential and general nonlinear term: Mass super-critical case. J. Differ. Equ. **334**, 194–215 (2022)
23. Figueiredo, G.M., Ikoma, N., Santos Junior, J.R.: Existence and concentration result for the Kirchhoff type equations with general nonlinearities. Arch. Ration. Mech. Anal. **213**, 931–979 (2014)
24. He, Q.H., Lv, Z.Y., Zhang, Y.M., Zhong, X.X.: Positive normalized solution to the Kirchhoff equation with general nonlinearities of mass super-critical. J. Differ. Equ. **356**, 375–406 (2023)
25. Cui, L.L., He, Q. H., Lv, Z.Y., Zhong, X.X.: The existence of normalized solutions for a Kirchhoff type equations with potential in $\mathbb{R}^3$. arXiv:2304.07194 (2023)
26. He, Y.: Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity. J. Differ. Equ. **261**, 6178–6220 (2016)
27. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. Theory T. M. A. **28**, 1633–1659 (1997)
28. Jeanjean, L., Zhang, J.J., Zhong, X.X.: A global branch approach to normalized solutions for the Schrödinger equation. arXiv:2112.05869 (2021)
29. Kwong, M.K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. Arch. Ration. Mech. Anal. **105**, 243–266 (1989)
30. Li, G.B., Luo, P., Peng, S.J., Wang, C.H., Xiang, C.-L.: A singularly perturbed Kirchhoff problem revisited. J. Differ. Equ. **268**, 541–589 (2020)
31. Luo, P., Peng, S.J., Wang, C.H., Xiang, C.-L.: Multi-peak positive solutions to a class of Kirchhoff equations. Proc. R. Soc. Edinb. A **149**, 1097–1122 (2019)
32. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys. **87**, 567–576 (1983)
33. Yang, Z.: A new observation for the normalized solution of the Schrödinger equation. Arch. Math. **115**, 329–338 (2020)
34. Zeng, X.Y., Zhang, J.J., Zhang, Y.M., Zhong, X.X.: Positive normalized solution to the Kirchhoff equation with general nonlinearities. arXiv:2112.10293 (2021)