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Inverse Problem for a Structural Acoustic System with Variable Coefficients

Song-Ren Fu[1] **· Peng-Fei Yao**[2]

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Abstract

We consider stability in an inverse problem of determining the material coefficient matrix for a coupled system that describes acoustic interactions, by the Riemannian geometrical approach. The stability is proved by the Carleman estimates and observability inequalities.

Keywords Inverse problem · Riemannian geometry · Carleman estimate · Observability inequality · Stability

Mathematics Subject Classification 74K20 (primary) · 74B20 (secondary)

1 Introduction and Main Results

Let Ω be a connected open bounded domain of IR^3 with boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$, where Γ_0 and Γ_1 are open and nonempty. Moreover, Γ_1 is assumed to be convex and of class C^2, and $\Gamma_0 \subset \mathrm{IR}^2$ to be flat with smooth boundary $\partial\Gamma_0$. For a possible geometric graphics of the structural acoustic chamber Ω, we refer to [23, 25].

✉ Peng-Fei Yao
pfyao@iss.ac.cn

1 Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

2 School of Mathematical Sciences, Shanxi University, Taiyuan 030006, People's Republic of China

We consider the following coupling system on the finite time interval $(0, T)$:

$$\begin{cases} z_{tt} - \mathcal{A}z = 0 & \text{in } \Omega \times (0, T), \\ \frac{\partial z}{\partial \nu_{\mathcal{A}}} = 0 & \text{on } \Gamma_1 \times (0, T), \\ \frac{\partial z}{\partial \nu_{\mathcal{A}}} = v_t & \text{on } \Gamma_0 \times (0, T), \\ v_{tt} + \mathcal{A}_0^2 v = -z_t & \text{on } \Gamma_0 \times (0, T), \\ v = \frac{\partial v}{\partial n_0} = 0 & \text{on } \partial\Gamma_0 \times (0, T), \\ (z(x,0), z_t(x,0)) = (z_0, 0) & \text{in } \Omega, \\ (v(x,0), v_t(x,0)) = (v_0, 0) & \text{on } \Gamma_0, \end{cases} \tag{1.1}$$

where $\mathcal{A}z = \mathrm{div} A(x)\nabla z$ and $\mathcal{A}_0 v = \mathrm{div}_0 A_0(x)\nabla v$. In (1.1), $A(x)$ and $A_0(x)$ are symmetric, positive matrices satisfying

$$A(x) = A_0(x) \quad \text{for} \quad x \in \Gamma_0.$$

Moreover, in (1.1), z denotes the acoustic velocity potential in Ω, which is a wave-type equation with the Neumann boundary condition and v describes the vertical displacement of the flat Γ_0 . In addition, ν, n_0, div, and ∇ are the outward unit normal vector of Ω along Γ, the outward unit normal vector of Γ_0 along $\partial\Gamma_0$, the divergence, and the gradient, respectively, in the Euclidean metric. Finally, $\frac{\partial z}{\partial \nu_{\mathcal{A}}} = \langle \nabla z, A(x)\nu \rangle$ and $\frac{\partial v}{\partial n_0} = \langle \nabla v, A_0(x) n \rangle$.

We assume that the matrix $A_0(x)$ is given but the matrix $A(x)$ is unknown which needs to be determined. Note that the map $A(x) \to \{z, v\}$ is nonlinear. Thus the inverse map $\{z, v\} \to A(x)$ is also nonlinear. We have taken the initial data $z_t(x, 0) = v_t(x, 0) = 0$ in order to make the even extensions of the solutions z and v to $\Omega \times [-T, T]$. The extended solutions retain the same regularity in the domain $\Omega \times [-T, T]$. The explicit regularity needed in our inverse problems will be specified in Sect. 2. Therefore, here and after, we consider all the PDE systems in the domain $Q = \Omega \times [-T, T]$ with the lateral boundary $\Sigma = \Gamma \times [-T, T]$.

As for the nonlinear inverse problem $\{z, v\} \to A(x)$ of system (1.1), we view z_0 and v_0 as the input, and the acceleration of the elastic plate $v_{tt}|_{\Sigma_0}$, a physically measurable quantity, as the output (observation). More precisely, we consider the following inverse problem:

- **Uniqueness of the inverse problem for system** (1.1)

Can the principal coefficients matrix $A(x)$ be uniquely determined by the acceleration of the elastic plate $v_{tt}|_{\Sigma_0}$ by finite many times changing initial values suitably? In other words, do finitely many $v_{tt}|_{\Sigma_0} = 0$ imply $A_1(x) = A_2(x)$, a.e. $x \in \Omega$?

- **Stability of the inverse problem for system** (1.1)

For a matrix $A(x) = (a_{ij}(x))_{1\le i,j\le 3}$, we define the following norm:

$$||A||^2_{H^1(\Omega)} = \sum_{i,j=1}^{3} ||a_{ij}(x)||^2_{H^1(\Omega)}.$$

Is it possible to estimate $||A_1 - A_2||_{H^1(\Omega)}$ by some suitable norms of the difference of the corresponding plate accelerations $(v_{2k} - v_{1k})_{tt}|_{\Sigma_0}$?

For our purposes, we shall first consider the linearized inverse problems in the following setting. Let

$$z_{ik}(x,t) = z(A_i(x), a_k) \quad \text{and} \quad v_{ik}(x,t) = v_{ik}(A_i(x), a_k),$$

respectively, solve (1.1) with respect to the coefficient matrices $A_i(x)$ and the initial values

$$[z_{ik}(x,0), \partial_t z_{ik}(x,0); v_{ik}(x,0), \partial_t v_{ik}(x,0)] = [a_k, 0; v_0, 0],$$

where v_0 is a fixed function, for $1 \le i \le 2$ and $1 \le k \le 9$. Denote

$$\begin{aligned} &B(x) = (b_{ij})_{3\times 3} = A_2(x) - A_1(x), \quad w_k(x,t) = z_{2k}(x,t) - z_{1k}(x,t),\\ &R_k(x,t) = z_{2k}(x,t) \ \text{ in } Q, \quad \text{and} \quad u_k(x,t) = v_{2k}(x,t) - v_{1k}(x,t) \ \text{ in } \Sigma_0. \end{aligned}$$

For the sake of simplicity, for $i = 1, 2$, we denote

$$\begin{aligned} &Z_i(x,t) = (z_{i1}, \ldots, z_{i9})^{\mathrm{T}}, \quad V_i(x,t) = (v_{i1}, \ldots, v_{i9})^{\mathrm{T}},\\ &Z_0(x,t) = (a_1, \ldots, a_9)^{\mathrm{T}}, \quad V_0(x,t) = (v_0, \ldots, v_0)^{\mathrm{T}},\\ &W(x,t) = (w_1, \ldots, w_9)^{\mathrm{T}}, \quad U(x,t) = (u_1, \ldots, u_9)^{\mathrm{T}}, \quad \text{and}\\ &R(x,t) = (z_{21}, \ldots, z_{29})^{\mathrm{T}}, \end{aligned}$$

where the superscript T denotes the transpose. Moreover, we let $\mathcal{A}_i = \mathrm{div} A_i(x)\nabla$ and $\frac{\partial z}{\partial \nu_{\mathcal{A}_i}} = \langle A_i(x)\nabla z, \nu\rangle$ for $i = 1, 2$. Clearly, the couple $\{W, U\}$ satisfies the following system.

$$\begin{cases} W_{tt} - \operatorname{div} A_1(x)\nabla W = \operatorname{div} B(x)\nabla R(x,t) & \text{in } \Omega \times (-T,T), \\ \frac{\partial W}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{on } \Gamma_1 \times (-T,T), \\ \frac{\partial W}{\partial \nu_{\mathcal{A}_1}} = U_t & \text{on } \Gamma_0 \times (-T,T), \\ U_{tt} + \mathcal{A}_0^2 U = -W_t & \text{on } \Gamma_0 \times (-T,T), \\ U = \frac{\partial U}{\partial n_0} = 0 & \text{on } \partial\Gamma_0 \times (-T,T), \\ W(x,0) = W_t(x,0) = 0 & \text{in } \Omega, \\ U(x,0) = U_t(x,0) = 0 & \text{on } \Gamma_0, \end{cases} \tag{1.2}$$

where $\operatorname{div} A_1 \nabla W = (\operatorname{div} A_1 \nabla w_1, \ldots, \operatorname{div} A_1 \nabla w_9)$T.

We introduce

$$g = A^{-1}(x) \quad \text{for } x \in \mathrm{IR}^3,$$

as a Riemannian metric on IR^3 and consider (IR^3, g) as a Riemannian manifold. Let

$$g(X, Y) = \langle X, Y \rangle_g = \left\langle A^{-1}(x)X,\ Y \right\rangle \quad \text{for } X, Y \in \mathrm{IR}^3_x, \quad x \in \mathrm{IR}^3,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean product of IR^3. Let D be the Levi–Civita connection in the metric g, and we have

$$DH(X, Y) = \langle D_Y H, X \rangle_g \quad \text{for } X,\ Y,\ H \in \mathrm{IR}^3_x, \quad x \in \Omega. \tag{1.3}$$

We need the following main assumptions.

Assumption (A.1) on the metric $g = A^{-1}(x)$: Assume that there exists a strictly convex function $\upsilon : \overline{\Omega} \to (0, +\infty)$ of class C^3, such that the following three properties hold.

(i) $\left.\frac{\partial \upsilon}{\partial \nu_{\mathcal{A}}}\right|_{\Gamma_1} = 0$;

(ii) There exists a positive constant $\alpha > 0$, such that

$$D^2\upsilon(X, X) \geq \alpha |X|_g^2, \quad \forall X \in \mathrm{IR}^3_x, \quad \forall x \in \overline{\Omega},$$

where D is the connection of the metric $g = A^{-1}(x)$;

(iii) $\upsilon(x)$ has no critical point on $\overline{\Omega}$, namely,

$$\inf_{x \in \Omega} |\nabla_g \upsilon|_g \geq \beta > 0.$$

In the case of constant coefficients, conditions (i) and (ii) in (A.1) are due to the Neumann boundary conditions which are the physically correct boundary conditions of the hyperbolic problem and introduced in [27, Sect. 5]. We mention that in [28,

Appendix B], the authors have given some constructions of functions satisfying condition (i). Condition (iii) is needed for the validity of the pointwise Carleman estimate. Condition (ii) means that υ is an escape function which depends on the curvature of the metric $g = A^{-1}(x)$. For the case of constant coefficients, $\upsilon(x) = |x - x_0|^2$ is one of the choices, where x_0 is a fixed point outside $\overline{\Omega}$. For the general cases, there are some examples in [34, Chap. 2] to show how to find an escape function. We here given an example.

An example satisfying conditions (i) and (ii) in assumption (A.1). Similar to [25, Example 2.1], for a given

$$A(x) = \begin{pmatrix} \frac{1}{4}(1+|x|^2)^2 & 0 & 0 \\ 0 & \frac{1}{4}(1+|x|^2)^2 & 0 \\ 0 & 0 & \frac{1}{4}(1+|x|^2)^2 \end{pmatrix} \quad \text{for } x = (x_1, x_2, x_3) \in \mathrm{IR}^3,$$

the metric $g(x)$ is

$$g(x) = A^{-1}(x) = \begin{pmatrix} \frac{4}{(1+|x|^2)^2} & 0 & 0 \\ 0 & \frac{4}{(1+|x|^2)^2} & 0 \\ 0 & 0 & \frac{4}{(1+|x|^2)^2} \end{pmatrix}.$$

Let

$$\mathcal{M} = \{(x_1, x_2, x_3, x_4) \in \mathrm{IR}^4 : \; x_1^2 + x_2^2 + x_3^2 + (x_4 - 1)^2 = 1\},$$

a sphere of IR^4 with radius 1. Let $p = (0, 0, 0, 2)$. We define the stereographic projection P as

$$P : \mathcal{M}\backslash p \to (\mathrm{IR}^3, g), \; P(x) = \frac{1}{2 - x_4}(x_1, x_2, x_3) \;\text{ for }\; x = (x_1, x_2, x_3, x_4) \in \mathcal{M}\backslash p.$$

Then P is an isometry, which implies that the curvature of (IR^3, g) is 1.

Let $p_0 = (1, 0, 0, 1) \in \mathcal{M}$. Denote

$$\mathcal{C}(r, \theta_1, \theta_2) = (\cos r, \sin r \cos\theta_1, \sin r \sin\theta_1 \cos\theta_2, 1 + \sin r \sin\theta_1 \sin\theta_2) \quad \text{for} \quad 0 < r < \frac{\pi}{2}.$$

Then

$$B_{\mathcal{M}}(p_0, r_0) = \{\mathcal{C}(r, \theta_1, \theta_2) : \; 0 \le r \le r_0, 0 \le \theta_1 \le \pi, 0 \le \theta_2 < 2\pi\}$$

is a geodesic ball of $\mathcal{M}$ centered at p_0 with radius $r_0 \in \left(0, \frac{\pi}{2}\right)$. Let $x_0 = P(p_0) = (1, 0, 0) \in \mathrm{IR}^3$. Then, the geodesic ball of (IR^3, g) centered at x_0 with radius $0 < r_0 < \frac{\pi}{2}$ is given by

$$\begin{aligned} B_g(x_0, r_0) &= P(B_{\mathcal{M}}(p_0, r_0)) \\ &= \Big\{ \frac{1}{1 - \sin r \sin\theta_1 \sin\theta_2} (\cos r,\ \sin r\cos\theta_1,\ \sin r \sin\theta_1 \cos\theta_2) : \\ &\quad 0 \le r \le r_0, 0 \le \theta_1 \le \pi, 0 \le \theta_2 < 2\pi \Big\}. \end{aligned}$$

Let $\rho(x) = d_g(x, x_0)$ be the distance function subject to metric g from $x \in \mathrm{IR}^3$ to x_0. Let $H = \frac{1}{2} D\rho^2$. Then by [34, Theorem 2.5], condition (i) holds for the escape function $\upsilon(x) = \rho^2(x)$, provided that $\Omega \subset B_g(x_0, r_0)$.

Based on the above discussions, we give the following example satisfying condition (ii).

Example 1.1 Let $0 < r_0 < \frac{\pi}{2}$. Set

$$\Omega = (B_g(x_0, r_0) \cap \{(x_1, 0, x_3) \in \mathrm{IR}^3\}) \backslash \{x \in \mathrm{IR}^3 : \ |x| > 1\},$$

$$\begin{aligned} \Gamma_1 = &\Big\{ \frac{1}{1 - \sin r \sin\theta_2} (\cos r, 0, \sin r \cos\theta_2) : \\ &-r_0 < r < r_0, 0 \le \theta_2 < 2\pi \} \cap \{x \in \mathrm{IR}^3 : |x| \le 1 \Big\}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \Gamma_0 = &\Big\{ \frac{1}{1 - \sin r_0 \sin\theta_1 \sin\theta_2} (\cos r_0,\ \sin r_0 \cos\theta_1,\ \sin r_0 \sin\theta_1 \cos\theta_2) : \\ &0 \le \theta_1 \le \pi, 0 \le \theta_2 \le \pi \Big\}. \end{aligned} \tag{1.5}$$

Then conditions (i) and (ii) hold for the triple $\{(\Omega, g), \Gamma_0, \Gamma_1\}$.

It remains to show that the above example meets condition (i). Indeed, for a fixed $\theta_2 \in [0, 2\pi)$, it is easy to see that $\gamma(r) = (\cos r, \sin r\cos\theta_1, \sin r\cos\theta_2, 1 + \sin r \sin\theta_2)$ is a normal geodesic of $\mathcal{M}$ through $\gamma(0) = p_0 = (1, 0, 0, 1)$. Then

$$\beta(r) = P(\gamma(r)) = \frac{1}{1 - \sin r \sin\theta_2} (\cos r, \sin r\cos\theta_1, \sin r \cos\theta_2)$$

is a normal geodesic of (IR^3, g) through $\beta(0) = x_0 = (1, 0, 0)$. Notice that $\rho(\beta(r)) = r$. Then

$$\begin{aligned} &D\rho(\beta(r)) \\ &\quad = \frac{1}{(1 - \sin r \sin\theta_2)^2} (-\sin r + \sin\theta_1 \sin\theta_2,\ \cos r \cos\theta_1,\ \cos r \sin\theta_1 \cos\theta_2). \end{aligned}$$

Let

$$u_1(r,\theta_2) = \frac{\cos r}{1-\sin r \sin\theta_2}, \quad u_2(r,\theta_2) = 0, \quad u_3(r,\theta_2) = \frac{\sin r \cos\theta_2}{1-\sin r\sin\theta_2}.$$

Then

$$\nu|_{\Gamma_1} = \frac{(u_{1r}, 0, u_{3r}) \times (u_{1\theta_2}, 0, u_{3\theta_2})}{|(u_{1r}, 0, u_{3r}) \times (u_{1\theta_2}, 0, u_{3\theta_2})|} \sim (0,-1,0),$$

which implies that $\langle D\rho, \nu\rangle|_{\Gamma_1} = \langle D\rho, \nu\rangle|_{\theta_1=\frac{\pi}{2}} = 0$. Thus, condition (i) follows.

For given constants $0 < c < 1$ and $\;\; > 0$ small, we fix $T > 0$ satisfying

$$\left[\max_{x\in\Omega} \upsilon(x) - cT^2\right] < \log \min_{x\in\Omega} \mathrm{e}^{\;\;\upsilon(x)}. \tag{1.6}$$

Let $a(x) = (a_1(x), \ldots, a_9(x))\mathrm{T}$. We set

$$G(x) = \Big(a_{x_1x_1}(x), a_{x_1x_2}(x), a_{x_2x_3}(x), a_{x_2x_2}(x), a_{x_2x_3}(x), a_{x_3x_3}(x),$$
$$\times a_{x_1}(x), a_{x_2}(x), a_{x_3}(x)\Big)$$

for $x \in \Omega$. Note that $G(x)$ has 81 components, and is a 9×9 matrix of functions.

We further make the following assumption.

Assumption (A.2) Functions $a_1, \ldots, a_9$ are given such that

$$\det G(x) \neq 0 \quad \text{for} \quad x \in \Omega.$$

We mention that such an example has been given in [9].

Moreover, for a given positive constant C_0, we denote an admissible set of $A(x)$ as

$$\mathcal{U}(C_0) = \left\{A \in C^5(\overline{\Omega}, \mathrm{IR}^{3\times 3}) \mid A(x) = A_0(x)\; x \in \Gamma_0;\; ||A||_{C^5(\overline{\Omega})} \le C_0\right\}. \tag{1.7}$$

Our main results are the following.

Theorem 1.1 (Uniqueness of the inverse problem) *Let the assumption* (*A*.1) *of the metric* $g = A_1^{-1}(x)$ *and assumption* (*A*.2) *hold. Let* T *satisfy* (1.6). *Assume that* $A_1(x), A_2(x) \in \mathcal{U}(C_0)$, $\upsilon_0 \in H_0^2(\Gamma_0)$, *and* $a(x) \in H^4(\Omega)$, *such that*

$$||\partial_t^2 R||_{L^\infty(-T,T;W^{2,\infty}(\Omega))} + ||\partial_t^3 R||_{L^\infty(-T,T;W^{2,\infty}(\Omega))} \le M_0 < +\infty. \quad (1.8)$$

Then $\partial_t^2 U|_{\Sigma_0} = 0$ *implies that* $B(x) = 0$ *for* $x \in \Omega$.

Theorem 1.2 (Stability of the inverse problem) *Let all the assumptions in Theorem 1.1 hold. Let* $A_1(x),\ A_2(x) \in \mathcal{U}(C_0)$. *Let* $(a_j, 0, v_0, 0) \in \mathcal{F}$ *for* $1 \le j \le 9$, *where* $\mathcal{F}$ *is given by* (2.11). *Then there exists a positive constant* $C = C(T, \Omega, \Gamma_0, C_0, M_0, a, v_0)$ *such that*

$$||B(x)||^2_{H^1(\Omega)} \le C \left(||\partial_t^2 U_{tt}||^2_{L^2(\Sigma_0)} + ||\mathcal{A}_0^2 U_{tt}||^2_{L^2(\Sigma_0)} \right). \quad (1.9)$$

The PDE system (1.1) describing acoustic interactions has been known and studied for some time (e.g., see [6, 7]). Physical motivation for studying this kind of problem comes from a variety of engineering applications that arise, for example, in the context of controlling the pressure in a helicopter's cabin or reducing unwanted cabin noise generated by some exterior field. In the case where $A(x) = I_3$ the 3×3 identity matrix, many papers contributed to various topics: stability, controllability, regularity, and inverse problems [1–4, 23]. In [23], an inverse problem of

$$\begin{cases} z_{tt} - \Delta z = qz & \text{in } \Omega \times (0, T), \\ \frac{\partial z}{\partial \nu_{\mathcal{A}}} = 0 & \text{on } \Gamma_1 \times (0, T), \\ \frac{\partial z}{\partial \nu_{\mathcal{A}}} = v_t & \text{on } \Gamma_0 \times (0, T), \\ v_{tt} + \Delta_\Gamma^2 v + \Delta_\Gamma^2 v_t = -z_t & \text{on } \Gamma_0 \times (0, T), \\ v = \frac{\partial v}{\partial n_0} = 0 & \text{on } \partial\Gamma_0 \times (0, T), \\ (z(x, 0), z_t(x, 0)) = (z_0, z_1) & \text{in } \Omega, \\ (v(x, 0), v_t(x, 0)) = (v_0, v_1) & \text{on } \Gamma_0, \end{cases}$$

was studied, where only the stability about q was obtained.

In the case where $A(x)$ is not a constant matrix, not much literature (e.g., [25, 33]) is known for such a case. To the best knowledge of the authors, the present paper for the first time establishes the uniqueness and the Lipschitz stability (Theorems 1.1 and 1.2) in determining the important material coefficients matrix $A(x)$ of system (1.1) with the finitely many observation data v_{tt}. Moreover, the assumption (A.1) of the metric $g = A^{-1}(x)$ plays a key role to guarantee that the interior information of solutions to the system arrives at boundary Γ_0.

Inverse problems of PDEs have been the object of numerous studies not only at the theoretical level but also the practical. It is known that the Carleman estimates and microlocal analysis play an essential role in the inverse problems. We refer to [5, 8–17, 21, 26, 29–32, 35] and the references therein. Here, we shall adopt a differential geometrical approach [34] to study the inverse problems of system (1.1).

The rest of this paper is organized as follows: In Sect. 2, we give the Carleman estimates and observability inequalities for problem (1.1). Section 3 focuses on the proofs of Theorems 1.1 and 1.2. Some concluding remarks are given in the last Sect. 4.

2 Some Key Lemmas and Theorems

We introduce an abstract operator-theoretic formulation associated with (1.1) as in [23]. To achieve this, we consider an operator on $L^2(\Omega)$ as follows.

$$\mathcal{A}u = \operatorname{div} A(x)\nabla u \quad \text{with} \quad D(\mathcal{A}) = \left\{ z \in H^2(\Omega) : \frac{\partial z}{\partial \nu_{\mathcal{A}}}|_{\Gamma} = 0 \right\}. \tag{2.1}$$

It is easy to check that $-\mathcal{A}$ is a nonnegative, self-adjoint operator. We define the Neumann map $z = Np : L^2(\Gamma_0) \to L^2(\Omega)$ by:

$$\begin{cases} \operatorname{div} A(x)\nabla(Np) = 0 & \text{in } \Omega, \\ \frac{\partial Np}{\partial \nu_{\mathcal{A}}} = 0 & \text{on } \Gamma_1, \\ \frac{\partial Np}{\partial \nu_{\mathcal{A}}} = p & \text{on } \Gamma_0. \end{cases} \tag{2.2}$$

It is well known that $N \in \mathcal{L}(L^2(\Gamma_0), H^{3/2}(\Omega))$ by the elliptic theory (see [22]). Then, by the Green's formula and [18, 23], the operator $-N^*\mathcal{A}$ have the following property

$$-N^*\mathcal{A}\zeta = \begin{cases} \zeta & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1 \end{cases} \quad \text{for} \quad \zeta \in D(\mathcal{A}). \tag{2.3}$$

Extending $\zeta \in D(\mathcal{A})$ by continuity to $\zeta \in H^1(\Omega)$. We set

$$\mathcal{B} = -\mathcal{A}N : \ L^2(\Gamma_0) \to [H^1(\Omega)]', \tag{2.4}$$

where $[H^1(\Omega)]'$ is the dual of $H^1(\Omega)$ related to $L^2(\Omega)$. Then $\mathcal{B}^* = -N^*\mathcal{A}$. Therefore, by (2.3), $\mathcal{B}^*$ is the restriction of the trace map from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma_0)$.

Let $\mathcal{A}_0 v = \operatorname{div}_{\Gamma_0} A_0(x)\nabla v$ be given in (1.2). Define

$$\mathcal{C} = \mathcal{A}_0^2 : \ D(\mathcal{C}) \to L^2(\Gamma_0), \quad D(\mathcal{C}) = \left\{ v \in H_0^2(\Gamma_0) : \mathcal{A}_0^2 v \in L^2(\Gamma_0) \right\}, \tag{2.5}$$

where

$$H_0^2(\Gamma_0) = \left\{ v \in H^2(\Gamma_0) : v\Big|_{\partial\Gamma_0} = \frac{\partial v}{\partial n_0}\Big|_{\partial\Gamma_0} = 0 \right\}.$$

It is easy to check that $\mathcal{C}$ is self-adjoint, positive definite and

$$D\left(\mathcal{C}^{\frac{1}{2}}\right) = H_0^2(\Gamma_0).$$

Based on the original system and the above setting, we set

$$\Lambda = \begin{pmatrix} 0 & I & 0 & 0 \\ \mathcal{A} & 0 & 0 & \mathcal{B} \\ 0 & 0 & 0 & I \\ 0 & -\mathcal{B}^* & -\mathcal{C} & 0 \end{pmatrix} : \ D(\Lambda) \subset \Xi \to \Xi, \tag{2.6}$$

where

$$\Xi = H^1(\Omega) \times L^2(\Omega) \times H_0^2(\Gamma_0) \times L^2(\Gamma_0). \tag{2.7}$$

The domain of Λ is given by

$$\begin{aligned} D(\Lambda) = \Big\{ &(z_0, z_1, v_0, v_1)^{\mathrm{T}} : z_0 \in H^2(\Omega), z_1 \in H^1(\Omega), v_1 \in H_0^2(\Gamma_0), \\ &\frac{\partial z_0}{\partial \nu_{\mathcal{A}}} = v_1 \text{ on } \Gamma_0, v_0 \in D(\mathcal{C}) \Big\}. \end{aligned} \tag{2.8}$$

Then the original system can be re-written as the following abstract evolution equation

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \Lambda \mathcal{E}, \quad \mathcal{E}(x, 0) = \mathcal{E}_0, \tag{2.9}$$

where $\mathcal{E} = (z, z_t, v, v_t)^{\mathrm{T}}$ and $\mathcal{E}_0 = (z_0, z_1, v_0, v_1)^{\mathrm{T}}$. Therefore, the semigroup theory (e.g., see [1, 2]) yields that Λ is the generator of a C_0-semigroup on Ξ, and

$$\begin{aligned} &\mathcal{E}_0 \in \Xi \quad \text{implies } \{z, z_t, v, v_t\} \in C(0, T; \Xi), \\ &\mathcal{E}_0 \in D(\Lambda) \quad \text{implies } \{z, z_t, v, v_t\} \in C(0, T; D(\Lambda)). \end{aligned} \tag{2.10}$$

Moreover, the time interval of (2.10) can be evenly extended to $[-T, T]$. For our inverse problem, we let

$$\begin{aligned} \mathcal{F} = D(\Lambda^6) \cap \Big\{ &[z_0, z_1, v_0, v_1]^{\mathrm{T}} : z_0 \in H^7(\Omega), z_1 \in H^6(\Omega), \\ &v_0 \in H_0^2(\Gamma_0) \cap H^7(\Gamma_0), v_1 \in H_0^2(\Gamma_0) \cap H^6(\Gamma_0) \}. \end{aligned} \tag{2.11}$$

Similarly, we have the abstract equation for the linearized system (1.2):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} W^{\mathrm{T}} \\ W_t^{\mathrm{T}} \\ U^{\mathrm{T}} \\ U_t^{\mathrm{T}} \end{pmatrix} = \Lambda \begin{pmatrix} W^{\mathrm{T}} \\ W_t^{\mathrm{T}} \\ U^{\mathrm{T}} \\ U_t^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} O_{1\times 9} \\ F^{\mathrm{T}} \\ O_{1\times 9} \\ O_{1\times 9} \end{pmatrix}, \tag{2.12}$$

where $F(x,t) = \operatorname{div} B(x)\nabla R(x,t)$.

2.1 A Carleman Estimate

There are many papers on the Carleman estimates for the wave equation, for example, [10, 28] and the references therein. Among them, there is a compact form of such Carleman estimates from [10].

Suppose that $\upsilon : \overline{\Omega} \to (0, +\infty)$ is a strictly convex function that satisfies the assumption $(A.1)$ in the metric $g = A^{-1}(x)$. Let

$$\psi(x,t) = \upsilon(x) - ct^2 \quad \text{for} \quad x \in \Omega, \quad t \in [-T, T], \tag{2.13}$$

$$\varphi(x,t) = \mathrm{e}^{\gamma\psi}, \quad (x,t) \in Q, \tag{2.14}$$

where $\gamma > 0$ is a constant. Set

$$m = \min_{x\in\overline{\Omega}} \upsilon(x), \quad d = \min_{x\in\overline{\Omega}} \varphi(x,0) \geq \mathrm{e}^{\gamma m}.$$

Suitably choose $0 < c < 1$ and $T > 0$, such that

$$\gamma \max_{x\in\overline{\Omega}} \upsilon(x) < \log d + c\gamma T^2. \tag{2.15}$$

Then φ satisfies

$$\varphi(x,0) \geq d, \quad \varphi(x,T) = \varphi(x,-T) < d \quad \text{uniformly on } \overline{\Omega}. \tag{2.16}$$

Therefore, for given $\varepsilon > 0$ small, we choose $\delta > 0$ such that

$$\varphi(x,t) \geq d - \varepsilon \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [-\delta, \delta], \tag{2.17}$$

$$\varphi(x,t) \leq d - 2\varepsilon \quad \text{for} \quad (x,t) \in \overline{\Omega} \times ([-T, -T+2\delta] \cup [T-2\delta, T]). \tag{2.18}$$

Let $Pv = \partial_t^2 v - \operatorname{div} A(x)\nabla v$ and

$$\mathcal{H} = \Big\{ v \in H^1(-T,T;L^2(\Omega)) \cap L^2(-T,T;H^1(\Omega)) : \\ \partial_t^j v(x,l) = 0, l = \pm T, j = 0,1 \Big\}.$$

Theorem 2.1 ([10]) *Under assumption* $(A.1)$ *of the metric* $g = A^{-1}(x)$, *there exist constants* $C > 0$ *and* $\gamma_* > 0$ *such that for any* $\gamma > \gamma_*$, *there exists* $s_0 = s(\gamma)$ *such that for all* $s > s_0 > 1$, *the following Carleman estimate hold*:

$$\int_Q \left[\sigma(|\nabla_g v|_g^2 + v_t^2) + \sigma^3 v^2 \right] \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \le C \left(\int_Q |Pv|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + \int_\Sigma \mathrm{BT}|_\Sigma \mathrm{d}\Sigma \right), \tag{2.19}$$

whenever $v \in \mathcal{H}$ *and the right-hand side of* (2.19) *is finite, with* $\sigma = s\gamma\varphi$. *In addition, the boundary terms in* $\mathrm{BT}|_\Sigma$ *are given explicitly by*

$$\begin{aligned} \mathrm{BT}|_\Sigma = {} & \sigma z_t^2 \frac{\partial \psi}{\partial \nu_\mathcal{A}} - 2\sigma \frac{\partial z}{\partial \nu_\mathcal{A}} \left(z_t \psi_t - \langle \nabla_g z, \nabla_g \psi \rangle_g \right) \\ & - \sigma \frac{\partial \psi}{\partial \nu_\mathcal{A}} |\nabla_g z|_g^2 + \frac{\gamma^2}{2} \sigma \frac{\partial \psi}{\partial \nu_\mathcal{A}} z^2 \left(\psi_t^2 - |\nabla_g \psi|_g^2 \right) \\ & - \gamma\sigma \frac{\partial z}{\partial \nu_\mathcal{A}} z \left(\psi_t^2 - |\nabla_g \psi|_g^2 \right) - \sigma^3 \frac{\partial \psi}{\partial \nu_\mathcal{A}} \left(\psi_t^2 - |\nabla_g \psi|_g^2 \right) z^2 \\ & + \sigma \frac{\partial z}{\partial \nu_\mathcal{A}} z (\psi_{tt} - \mathrm{div} A(x) \nabla \psi), \end{aligned} \tag{2.20}$$

where $z = \mathrm{e}^{s\varphi} v$.

We mention that in [10] the boundary terms are given by

$$\int_\Sigma \mathrm{BT}|_\Sigma \mathrm{d}\Sigma - \frac{\gamma}{2} \int_\Sigma \sigma z^2 \frac{\partial |\nabla_g \psi|_g^2}{\partial \nu_\mathcal{A}} \mathrm{d}\Sigma := \int_\Sigma \mathrm{BT}|_\Sigma \mathrm{d}\Sigma - I_1. \tag{2.21}$$

Thanks to the inner estimate, the second term I_1 on the right-hand side of (2.21) can be absorbed. In fact, I_1 arises from the following term (see [10, (40)])

$$\gamma \int_Q \sigma \left\langle \nabla_g z, \nabla_g |\nabla_g \psi|_g^2 \right\rangle_g z \mathrm{d}x\mathrm{d}t := I_2. \tag{2.22}$$

Since there exists a positive constant $C = C(\psi)$, such that

$$I_2 \le C\gamma \int_Q |\nabla_g z|_g^2 \mathrm{d}x\mathrm{d}t + \gamma \int_Q \sigma^2 z^2 \mathrm{d}x\mathrm{d}t, \tag{2.23}$$

the term I_1 is absorbed by the left-hand side of (2.19) when $s > 0$ is sufficiently large.

2.2 Observability Inequalities

We consider the observability inequalities for the following system with a non-homogeneous term:

$$\begin{cases} v_{tt} - \operatorname{div} A(x)\nabla v = h & \text{for} \quad (x,t) \in Q, \\ \frac{\partial v}{\partial \nu_{\mathcal{A}}} = 0 & \text{for} \quad (x,t) \in \Sigma_1, \\ \frac{\partial v}{\partial \nu_{\mathcal{A}}} = f & \text{for} \quad (x,t) \in \Sigma_0, \\ (v(x,0), v_t(x,0)) = (v_0, v_1) & \text{for} \quad x \in \Omega. \end{cases} \tag{2.24}$$

Let $\mathcal{V} = L^2(Q) \times L^2(\Sigma_0)$. We have the following.

Theorem 2.2 *Assume that the assumption (A.1) holds. Let T satisfy (2.15). Let $(h, f) \in \mathcal{V}$ and $(v_0, v_1) \in H^1(\Omega) \times L^2(\Omega)$. Then there exists a constant $C = C(T, C_0) > 0$ such that for all $t \in (-T, T)$,*

$$\begin{aligned} \int_\Omega \left(v^2 + |\nabla_g v|_g^2 + v_t^2\right) \mathrm{d}x \le C\mathrm{e}^{-2s(d-\varepsilon)} \int_Q h^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + C \int_Q h^2 \mathrm{d}x\mathrm{d}t \\ + C\mathrm{e}^{2sM} \int_{\Sigma_0} [\sigma^3 v^2 + \sigma(v_t^2 + |\nabla_g v|_g^2)] \mathrm{d}\Sigma \end{aligned} \tag{2.25}$$

for $s > 0$ large, where $M = \sup_{(x,t)\in Q} \varphi(x,t)$.

Proof Since $(h, f) \in \mathcal{V}$ and $(v_0, v_1) \in H^1(\Omega) \times L^2(\Omega)$, the regularity of hyperbolic problems implies that (2.24) admits a unique solution v such that

$$v \in H^1(-T, T; L^2(\Omega)) \cap L^2(-T, T; H^1(\Omega)).$$

Let $\mathrm{BT}|_\Sigma$ be given by (2.20) and let $z = \mathrm{e}^{s\varphi} v$. By condition (i) in assumption (A.1) and the boundary condition in (2.24), we have

$$\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} = 0, \quad \frac{\partial z}{\partial \nu_{\mathcal{A}}} = \sigma z \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} + \mathrm{e}^{s\varphi} \frac{\partial v}{\partial \nu_{\mathcal{A}}} = 0 \quad \text{for} \quad (x,t) \in \Sigma_1.$$

It then follows from (2.20) that

$$\int_\Sigma \mathrm{BT}|_\Sigma \mathrm{d}\Sigma \le C\mathrm{e}^{2sM} \Gamma([-T, T], v), \tag{2.26}$$

where

$$\Gamma([-T, T], v) = \int_{-T}^T \int_{\Gamma_0} \left[\sigma^3 v^2 + \sigma \left(v_t^2 + |\nabla_g v|_g^2\right)\right] \mathrm{d}\Gamma \mathrm{d}t.$$

Let

$$E(t) = \int_{\Omega} \left(v_t^2 + |\nabla_g v|_g^2 \right) \mathrm{d}x + \int_{\Gamma_0} v^2 \mathrm{d}\Gamma.$$

By Poincaré's inequality, we have

$$\int_{\Omega} \left(v^2 + v_t^2 + |\nabla_g v|_g^2 \right) \mathrm{d}x \leq C E(t). \tag{2.27}$$

For given $\varepsilon > 0$ small, we fixed $\delta > 0$ small such that (2.17) and (2.18) hold. Taking a cut-off function $\chi(t) \in C_0^2([-T, T])$ satisfying

$$\chi(t) = \begin{cases} 1, & t \in [-T + 2\delta, T - 2\delta], \\ 0, & t \in [-T, -T + \delta] \cup [T - \delta, T]. \end{cases} \tag{2.28}$$

Then $\chi v \in \mathcal{H}$ and

$$P(\chi v) = \chi P(v) + \chi'' v + 2\chi' v_t = \chi h + \chi'' v + 2\chi' v_t. \tag{2.29}$$

Applying the Carleman estimate (2.19) to (2.29), for $T \geq 3\delta$, we obtain, by (2.26), the following:

$$\begin{aligned} &\mathrm{e}^{2s(d-\varepsilon)} \int_{-\delta}^{\delta} \int_{\Omega} (v^2 + v_t^2 + |\nabla_g v|_g)^2 \mathrm{d}x \mathrm{d}t \\ &\leq \int_Q \left(|\nabla_g (\chi v)|_g^2 + |(\chi v)_t|^2 + |\chi v|^2 \right) \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\ &\leq C \int_Q (|\chi h|^2 + |\chi'' v|^2 + |\chi' v_t|^2) \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t + C\mathrm{e}^{2sM} \Gamma([-T, T], \chi v) \\ &\leq C \int_Q |h|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t + C \left(||v||_{L^2(Q)}^2 + ||v_t||_{L^2(Q)}^2 \right) \mathrm{e}^{2s(d-2\varepsilon)} \\ &\quad + C\mathrm{e}^{2sM} \Gamma([-T + \delta, T - \delta], v), \end{aligned} \tag{2.30}$$

where we have used the inequality $\varphi \leq d - 2\varepsilon$ only in the case where $\chi' \neq 0$. By the mean value theorem, there exists a $t_1 \in (-\delta, \delta)$ such that

$$\begin{aligned} \int_{\Omega} \left(|\nabla_g v|_g^2 + v_t^2 + v^2 \right) \mathrm{d}x \Big|_{t=t_1} \leq\ & C\mathrm{e}^{-2s(d-\varepsilon)} \int_Q |h|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\ & + C\mathrm{e}^{2s(M-d+\varepsilon)} \Gamma([-T + \delta, T - \delta], f) \\ & + C\mathrm{e}^{-2s\varepsilon} \left(||v||_{L^2(Q)}^2 + ||v_t||_{L^2(Q)}^2 \right). \end{aligned} \tag{2.31}$$

It then follows from (2.27) and the standard energy integration that

$$\int_{\Omega}\left(v^2+v_t^2+|\nabla_g v|_g^2\right)\mathrm{d}x \le C\int_{\Omega}\left(v^2+v_t^2+|\nabla_g v|_g^2\right)\mathrm{d}x\Big|_{t=t_1} \\ +C\int_{\Sigma_0}\left(v_t^2+f^2+v^2\right)\mathrm{d}\Sigma+C\int_{Q}|h|^2\mathrm{d}x\mathrm{d}t. \tag{2.32}$$

Then

$$\int_{Q}\left(v^2+v_t^2+|\nabla_g v|_g^2\right)\mathrm{d}x\mathrm{d}t \le CT\int_{\Omega}\left(v^2+v_t^2+|\nabla_g v|_g^2\right)\mathrm{d}x\Big|_{t=t_1} \\ +CT\int_{\Sigma_0}\left(v_t^2+f^2+v^2\right)\mathrm{d}\Sigma \\ +CT\int_{Q}|h|^2\mathrm{d}x\mathrm{d}t. \tag{2.33}$$

Taking s large enough. By (2.31) and (2.33), the term $C\mathrm{e}^{-2s\varepsilon}\left(||v||^2_{L^2(Q)}+||v_t||^2_{L^2(Q)}\right)$ on the right-hand side of (2.31) is absorbed. Thus, it follows (2.32) that

$$\int_{\Omega}(|\nabla_g v|_g^2+v_t^2+v^2)\mathrm{d}x \\ \le C\mathrm{e}^{-2s(d-\varepsilon)}\int_{Q}|h|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ +C\int_{Q}h^2\mathrm{d}x\mathrm{d}t+C\mathrm{e}^{2s(M-d+\varepsilon)}\Gamma([-T+\delta,T-\delta],v), \tag{2.34}$$

and hence (2.25) follows. □

The following lemma is quoted from [19], from which the tangential derivative $\nabla_{\Gamma_g}v$ on Σ_0 can be removed from the right-hand side of (2.25) in the case where $h=0$.

Lemma 2.1 ([19]) *Let v solve problem (2.24) with $h=0$. Then, for given small δ : $0<\delta<T$, there exists a positive constant $C=C(T,\delta)$ such that*

$$\int_{-T+\delta}^{T-\delta}\int_{\Gamma_0}\left|\nabla_{\Gamma_g}v\right|^2\mathrm{d}\Gamma\mathrm{d}t \le C\int_{\Sigma_0}\left(v_t^2+\left|\frac{\partial v}{\partial\nu_{\mathcal{A}}}\right|^2\right)\mathrm{d}\Sigma+L(v), \tag{2.35}$$

where $L(v)$ denotes the lower-order terms of v with respect to the norm of $C(-T,T;H^1(\Omega))$.

Using (2.35) in (2.34) and by a compactness–uniqueness argument, we have the following.

Corollary 2.1 *Let v solve problem (2.24) with $h = 0$. Then for $s > 0$ large,*

$$\int_{\Omega} \left(v^2 + |\nabla_g v|_g^2 + v_t^2\right) \mathrm{d}x \leq C \mathrm{e}^{2sM} \int_{\Sigma_0} \left(v^2 + v_t^2 + f^2\right) \mathrm{d}\Sigma. \tag{2.36}$$

3 Proofs of the Main Theorems

A similar argument as in the proof of [9, Lemma 3.1] yields the following lemma.

Lemma 3.1 *Let assumption $(A.2)$ hold. Then there is a $C > 0$ such that*

$$\int_{\Omega} (|B(x)|^2 + |\nabla B(x)|^2) \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x \leq C \int_{\Omega} (|J|^2 + |\nabla J|^2) \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x \tag{3.1}$$

for $s > 0$ large, where $B(x) = (b_{ij}(x))_{1 \leq i,j \leq 3}$ and

$$J(x) = (\mathrm{div} B(x) \nabla a_1(x), \ldots, \mathrm{div} B(x) \nabla a_9(x))^{\mathrm{T}}.$$

Let (W, U) solve problem (1.2). Set

$$\overline{W} = W_t.$$

By (1.2), $(\overline{W}, U)$ satisfies problem

$$\begin{cases} \overline{W}_{tt} - \mathrm{div}\, A_1(x) \nabla \overline{W} = \mathrm{div} B \nabla R_t & \text{in} \quad \Omega \times (0, T), \\ \frac{\partial \overline{W}}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{on} \quad \Gamma_1 \times (0, T), \\ \frac{\partial \overline{W}}{\partial \nu_{\mathcal{A}_1}} = U_{tt} & \text{on} \quad \Gamma_0 \times (0, T), \\ U_{tt} + \mathcal{A}_0^2 U = -\overline{W} & \text{on} \quad \Gamma_0 \times (0, T), \\ U = \frac{\partial U}{\partial n_0} = 0 & \text{on} \quad \partial \Gamma_0 \times (0, T), \\ \overline{W}(x, 0) = 0, \ \overline{W}_t(x, 0) = J(x) & \text{in} \quad \Omega, \\ U(x, 0) = U_t(x, 0) = 0 & \text{on} \quad \Gamma_0. \end{cases} \tag{3.2}$$

Proof of Theorem 1.1 Let $(\overline{W}, U)$ solve problem (3.2). Let $U_{tt}(x, t) = 0$ on Σ_0. We proceed to prove that

$$B(x) = 0 \quad \text{for} \quad x \in \Omega$$

holds as follows.

The assumptions $U_{tt}(t, x) = 0$ for $(t, x) \in \Sigma_0$ and $U(x, 0) = U_t(x, 0) = 0$ for $x \in \Gamma_0$ imply that

$$U(t, x) = 0 \quad \text{for} \quad (t, x) \in \Sigma_0.$$

Let

$$\widetilde{W} = \overline{W}_t, \quad \widetilde{\widetilde{W}} = \overline{W}_{tt} \quad \text{for} \quad (t, x) \in Q.$$

It is easy to check from (3.2) that $\widetilde{W}$ satisfies the following:

$$\begin{cases} \widetilde{W}_{tt} - \operatorname{div} A_1(x)\nabla \widetilde{W} = \operatorname{div} B\nabla R_{tt} & \text{in} \quad \Omega \times (0,T), \\ \frac{\partial \widetilde{W}}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{on} \quad \Gamma_1 \times (0,T), \\ \frac{\partial \widetilde{W}}{\partial \nu_{\mathcal{A}_1}} = \widetilde{W} = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ \widetilde{W}(x,0) = J(x), \ \widetilde{W}_t(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{3.3}$$

and $\widetilde{\widetilde{W}}$ solves the following:

$$\begin{cases} \widetilde{\widetilde{W}}_{tt} - \operatorname{div} A_1(x)\nabla \widetilde{\widetilde{W}} = \operatorname{div} B\nabla R_{ttt} & \text{in} \quad \Omega \times (0,T), \\ \frac{\partial \widetilde{\widetilde{W}}}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{on} \quad \Gamma_1 \times (0,T), \\ \frac{\partial \widetilde{\widetilde{W}}}{\partial \nu_{\mathcal{A}_1}} = \widetilde{\widetilde{W}} = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ \widetilde{\widetilde{W}}(x,0) = 0, \ \widetilde{\widetilde{W}}_t(x,0) = \hat{F}(x) & \text{in} \quad \Omega, \end{cases} \tag{3.4}$$

where

$$\hat{F}(x) = \operatorname{div} A_1(x)\nabla J(x) + \operatorname{div} B(x)\nabla R_{tt}(x,0) \in L^2(\Omega),$$

since $A_1(x)$ and $A_2(x)$ are in $\mathcal{U}(C_0)$.

Let the cut-off function $\chi(t)$ given by (2.28). We apply the Carleman estimate in Theorem 2.1 with

$$P(\chi\widetilde{W}) = \partial_t^2(\chi\widetilde{W}) - \operatorname{div} A_1(x)\nabla(\chi\widetilde{W}) = \chi''\widetilde{W} + 2\chi'\widetilde{W}_t + \chi \operatorname{div} B\nabla R_{tt}$$

to (3.3), to obtain

$$\begin{aligned} &\int_Q \left[\sigma\left(|(\chi\widetilde{W})_t|^2 + |\chi\nabla_g\widetilde{W}|_g^2\right) + \sigma^3|\chi\widetilde{W}|^2\right] e^{2s\varphi} dx dt \\ &\quad \le C\int_Q |\operatorname{div} B\nabla R_{tt}|^2 e^{2s\varphi} dx dt + Ce^{2s(d-2\varepsilon)}\int_Q \left(|\widetilde{W}_t|^2 + |\widetilde{W}|^2\right) dx dt, \end{aligned} \tag{3.5}$$

where $\sigma = s\ \varphi$. Similarly, applying Theorem 2.1 to (3.3) yields

$$
\begin{aligned}
&\int_Q \left[\sigma\left(|(\chi \widetilde{\widetilde{W}})_t|^2 + |\chi \nabla_g \widetilde{\widetilde{W}}|_g^2\right) + \sigma^3 |\chi \widetilde{\widetilde{W}}|^2\right] \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\
&\quad \le C \int_Q |\operatorname{div} B \nabla R_{ttt}|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t + C \mathrm{e}^{2s(d-2\varepsilon)} \int_Q (|\widetilde{\widetilde{W}}_t|^2 + |\widetilde{\widetilde{W}}|^2) \mathrm{d}x \mathrm{d}t.
\end{aligned} \tag{3.6}
$$

Next, since $\widetilde{W}(x,0) = J(x)$, by (3.5) we have

$$
\begin{aligned}
\int_\Omega |J(x)|^2 \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x &= \int_{-T}^0 \frac{\partial}{\partial t} \int_\Omega |\chi(t) \widetilde{W}(x,t)|^2 \mathrm{e}^{2s\varphi(x,t)} \mathrm{d}x \mathrm{d}t \\
&\le C \int_Q (|\chi'||\widetilde{W}|^2 + \sigma |\chi \widetilde{W}|^2 + |(\chi \widetilde{W})_t|^2) \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\
&\le C \int_Q |\operatorname{div} B \nabla R_{tt}|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\
&\quad + C \mathrm{e}^{2s(d-2\varepsilon)} \int_Q (|\widetilde{W}_t|^2 + |\widetilde{W}|^2) \mathrm{d}x \mathrm{d}t.
\end{aligned} \tag{3.7}
$$

Moreover, since

$$
\varsigma |\nabla J(x)| \le |\nabla_g J(x)| \le C |\nabla J(x)| \quad \text{for} \quad x \in \Omega
$$

for some $\varsigma > 0$ small, it follows from (3.5) and (3.6) that

$$
\begin{aligned}
&\int_\Omega |\nabla J(x)|^2 \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x \\
&\quad \le C \int_\Omega |\nabla_g J(x)|_g^2 \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x \\
&\quad = C \int_{-T}^0 \frac{\partial}{\partial t} \int_\Omega |\chi \nabla_g \widetilde{W}|_g^2 \mathrm{e}^{2s\varphi(x,t)} \mathrm{d}x \mathrm{d}t \\
&\quad \le C \int_Q \left(|\chi'||\nabla_g \widetilde{W}|_g^2 + \sigma |\chi \nabla_g \widetilde{W}|_g^2 + |\chi \nabla_g \widetilde{\widetilde{W}}|_g^2\right) \mathrm{e}^{2s\varphi(x,t)} \mathrm{d}x \mathrm{d}t \\
&\quad \le C \int_Q (|\operatorname{div} B \nabla R_{tt}|^2 + |\operatorname{div} B \nabla R_{ttt}|^2) \mathrm{e}^{2s\varphi} \mathrm{d}x \mathrm{d}t \\
&\qquad + C \mathrm{e}^{2s(d-2\varepsilon)} \int_Q (|\nabla_g \widetilde{W}|_g^2 + |\widetilde{W}_t|^2 + |\widetilde{W}|^2 + |\widetilde{\widetilde{W}}_t|^2 + |\widetilde{\widetilde{W}}|^2) \mathrm{d}x \mathrm{d}t.
\end{aligned} \tag{3.8}
$$

On the other hand, assumption (1.8) implies

$$
|\operatorname{div} B \nabla R_{tt}|^2 + |\operatorname{div} B \nabla R_{ttt}|^2 \le C(|J(x)|^2 + |\nabla J(x)|^2) \quad \text{for} \quad (t,x) \in Q.
$$

From (3.7) and (3.8), we obtain

$$\begin{aligned}
&\int_{\Omega}(|J(x)|^2+|\nabla J(x)|^2)\mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x\\
&\quad\leq C\int_{\Omega}(|J(x)|^2+|\nabla J(x)|^2)\mathrm{e}^{2s\varphi(x,0)}\left|\int_{-T}^{\mathrm{T}}\mathrm{e}^{2s[\varphi(x,t)-\varphi(x,0)]}\mathrm{d}t\right|\mathrm{d}x\\
&\qquad+C\mathrm{e}^{2s(d-2\varepsilon)}\int_{Q}(|\nabla_g\widetilde{W}|_g^2+|\widetilde{W}_t|^2+|\widetilde{W}|^2+|\widetilde{\widetilde{W}}_t|^2+|\widetilde{\widetilde{W}}|^2)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.9}$$

We assume that there is a small number $c>0$ such that

$$\mathrm{e}^{-c\,t^2}-1\leq -c\varsigma\,t^2\quad\text{for}\quad t\in(-T,T)$$

for some $\varsigma>0$ small. It is easy to check that

$$\sup_{x\in\Omega}\left|\int_{-T}^{T}\mathrm{e}^{2s[\varphi(x,t)-\varphi(x,0)]}\mathrm{d}t\right|\to 0\quad\text{at}\quad s\to\infty.$$

Thus, the first term on the right-hand side of (3.9) can be absorbed by the left-hand side of (3.9). By (3.1) and (3.9), we obtain

$$\begin{aligned}
\int_{\Omega}(|B(x)|^2+|\nabla B(x)|^2)\mathrm{e}^{2sd}\mathrm{d}x&\leq\int_{\Omega}(|B(x)|^2+|\nabla B(x)|^2)\mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x\\
&\leq C\mathrm{e}^{2s(d-2\varepsilon)}\int_{Q}(|\nabla_g\widetilde{W}|_g^2+|\widetilde{W}_t|^2+|\widetilde{W}|^2\\
&\quad+|\widetilde{\widetilde{W}}_t|^2+|\widetilde{\widetilde{W}}|^2)\mathrm{d}x\mathrm{d}t
\end{aligned}\tag{3.10}$$

for $s>0$ large. From (3.10), the proof of Theorem 1.1 is complete by taking $s\to\infty$. □

Proof of Theorem 1.2 Let $(\overline{W},U)$ solve problem (3.2). Set $\widetilde{W}=\overline{W}_t$. By (3.2) $(\widetilde{W},U)$ solves problem

$$\begin{cases}
\widetilde{W}_{tt}-\operatorname{div}A_1\nabla\widetilde{W}=\operatorname{div}B\nabla R_{tt} & \text{in}\quad \Omega\times(0,T),\\
\frac{\partial\widetilde{W}}{\partial\nu_{\mathcal{A}_1}}=0 & \text{on}\quad \Gamma_1\times(0,T),\\
\frac{\partial\widetilde{W}}{\partial\nu_{\mathcal{A}_1}}=\partial_t^3U & \text{on}\quad \Gamma_0\times(0,T),\\
\partial_t^3U+\mathcal{A}_0^2U_t=-\widetilde{W} & \text{on}\quad \Gamma_0\times(0,T),\\
U=\frac{\partial U}{\partial n_0}=0 & \text{on}\quad \partial\Gamma_0\times(0,T),\\
\widetilde{W}(x,0)=J,\ \ \widetilde{W}_t(x,0)=0 & \text{in}\quad \Omega,\\
U(x,0)=U_t(x,0)=U_{tt}(x,0)=0 & \text{on}\quad \Gamma_0.
\end{cases}\tag{3.11}$$

Noting that the assumption

$$A_1(x) = A_2(x) = A_0(x) \quad \text{for} \quad x \in \Gamma_0,$$

we have

$$J(x) = 0 \quad \text{for} \quad x \in \Gamma_0.$$

Then

$$\partial_t^3 U(x,0) = -J(x) - \mathcal{A}_0^2 U_t(x,0) = 0 \quad \text{for} \quad x \in \Gamma_0.$$

Thus

$$\|\partial_t^3 U\|_{L^2(\Sigma_0)} \le C\|\partial_t^4 U\|_{L^2(\Sigma_0)}, \quad \|\mathcal{A}_0^2 U_t\|_{L^2(\Sigma_0)} \le C\|\mathcal{A}_0^2 U_{tt}\|_{L^2(\Sigma_0)}.$$

For given U by (3.11), we define Φ as the unique solution to the following problem:

$$\begin{cases} \Phi_{tt} - \operatorname{div} A_1 \nabla \Phi = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial \Phi}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{in } \Gamma_1 \times (0,T), \\ \frac{\partial \Phi}{\partial \nu_{\mathcal{A}_1}} = \partial_t^3 U & \text{on } \Gamma_0 \times (0,T), \\ \Phi(x,0) = J(x),\ \Phi_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.12}$$

We apply Theorem 2.2 to problem (3.12) with $v = \Phi$, and recall B in (3.1), to have

$$\begin{aligned} &\|B\|^2_{H^1(\Omega)} \\ &\le C\left(||\partial_t^3 U||^2_{L^2(\Sigma_0)} + ||\Phi||^2_{L^2(\Sigma_0)} + ||\Phi_t||^2_{L^2(\Sigma_0)}\right) \\ &\le C\left(||\partial_t^3 U||^2_{L^2(\Sigma_0)} + ||Y||^2_{L^2(\Sigma_0)} + ||Y_t||^2_{L^2(\Sigma_0)} + \|\widetilde{W}\|^2_{L^2(\Sigma_0)} + \|\widetilde{W}_t\|^2_{L^2(\Sigma_0)}\right) \\ &\le C\left(||\partial_t^4 U||^2_{L^2(\Sigma_0)} + \|\mathcal{A}_0^2 U_{tt}\|^2_{L^2(\Sigma_0)} + ||Y||^2_{L^2(\Sigma_0)} + ||Y_t||^2_{L^2(\Sigma_0)}\right), \end{aligned} \tag{3.13}$$

where

$$Y = \widetilde{W} - \Phi \quad \text{for} \quad (t,x) \in Q,$$

that solves problem

$$\begin{cases} Y_{tt} - \operatorname{div} A_1 \nabla Y = \operatorname{div} B \nabla R_{tt} & \text{in } \Omega \times (0,T), \\ \frac{\partial Y}{\partial \nu_{\mathcal{A}_1}} = 0 & \text{on } \Gamma \times (0,T), \\ Y(x,0) = 0, \quad Y_t(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.14}$$

by (3.11) and (3.12).

Next, we remove the term $||Y||^2_{L^2(\Sigma_0)} + ||Y_t||^2_{L^2(\Sigma_0)}$ from the right-hand side of (3.13) by a compactness–uniqueness argument below as in [23] (see also [24]).

For simplicity, denote

$$||U||^2_S = ||\partial_t^4 U||^2_{L^2(\Sigma_0)} + ||\mathcal{A}_0^2 U_{tt}||^2_{L^2(\Sigma_0)}.$$

We define a map $\mathcal{K} : H^1(\Omega) \to L^2(\Sigma_0) \times L^2(\Sigma_0)$ by

$$\mathcal{K}B = (Y, Y_t),$$

where Y solves problem (3.14) for given $B(x)$.

Since the initial data $(a_k, 0, v_0, 0) \in \mathcal{F}$, where $\mathcal{F}$ is given by (2.11), the semigroup theory gives that

$$(R, R_t, V_2, V_{2t}) \in C((-T, T); D(\Lambda^6)).$$

Therefore, we deduce that

$$\partial_t^6 R \in C((-T, T); H^1(\Omega)), \quad \partial_t^7 R \in C((-T, T); L^2(\Omega)).$$

Since $\mathcal{A}(\partial_t^5 R) = (\partial_t^5 R)_{tt} \in C((-T, T); L^2(\Omega))$, elliptic theory yields

$$\partial_t^5 R \in C((-T, T); H^2(\Omega)).$$

By the Sobolev embedding theorems for dimension $n = 3$, we obtain

$$\partial_t^2 R \in C((-T, T); H^5(\Omega)) \to L^\infty((-T, T); W^{2,\infty}(\Omega)), \text{ continuously;}$$
$$\partial_t^3 R \in C((-T, T); H^4(\Omega)) \to L^\infty((-T, T); W^{2,\infty}(\Omega)), \text{ continuously.}$$

It is easy to check from (3.14) that

$$||R_{tt}||_{L^\infty(-T,T;W^{2,\infty}(\Omega))} + ||R_{ttt}||_{L^\infty(-T,T;W^{2,\infty}(\Omega))} < +\infty,$$

which implies that

$$\operatorname{div} B(x)\nabla R_{tt} \in L^2(Q), \quad \partial_t \operatorname{div} B(x)\nabla R_{tt} \in L^2(Q).$$

As a consequence, operator $\mathcal{K} : H^1(\Omega) \to L^2(\Sigma_0) \times L^2(\Sigma_0)$ is compact.

We proceed to complete the proof by contradiction. By assumption (1.7), suppose that there exists a sequence $\{B_n\}_{n\geq 1} \in H^1(\Omega)$ such that

$$||B_n||_{H^1(\Omega)} = 1, \quad n \geq 1, \tag{3.15}$$

and

$$||Y_n||^2_{L^2(\Sigma_0)} + ||Y_{nt}||^2_{L^2(\Sigma_0)} \geq n||U_n||^2_S, \tag{3.16}$$

where Y_n and U_n are given by (3.14) and (3.11), respectively, with $B = B_n$. Then we have

$$\lim_{n\to+\infty} ||U_n||_S = 0. \tag{3.17}$$

By (3.15), there exists a subsequence, still denoted by $\{B_n\}_{n\geq 1}$, such that

$$B_n \rightharpoonup B_0 \in H^1(\Omega) \text{ weakly in, } H^1(\Omega), \tag{3.18}$$

for some $B_0 \in H^1(\Omega)$. Let $(\widetilde{W}_n, U_n)$ and $(\widetilde{W}_0, U_0)$ be given by (3.11) with $B = B_n$ and $B = B_0$, respectively. It follows from (3.13) that

$$B_n \to B_0 \in H^1(\Omega) \text{ strongly in } H^1(\Omega),$$

and $\|B_0\|_{H^1(\Omega)} = 1$.

By the trace theorem and an a priori estimate of (3.11), we obtain

$$\|\widetilde{W}_n\|_{L^2(\Sigma_0)} \leq C\|\widetilde{W}_n\|_{H^{1/2}(Q)} \leq C\|\widetilde{W}_n\|_{H^1(Q)} \leq C\left(||U_n||_S + ||B_n||_{H^1(\Omega)}\right),$$

yielding

$$\widetilde{W}_n \to \widetilde{W}_0 \text{ strongly in } L^2(\Sigma_0). \tag{3.19}$$

Thus

$$\|\widetilde{W}_0\|_{L^2(\Sigma_0)} = \lim_{n\to\infty} ||\widetilde{W}_n||_{L^2(\Sigma_0)} \leq C \lim_{n\to\infty} ||U_n||_S = 0,$$

that is,

$$\partial_t^3 U_0 + \mathcal{A}_0^2 U_0 = 0.$$

By (3.11), $U_0 = \Psi_t$ solves the following:

$$\begin{cases} \Psi_{tt} + \mathcal{A}_0^2 \Psi = 0 & \text{in} \quad \Sigma_0, \\ \Psi = \frac{\partial \Psi}{\partial n_0} = 0 & \text{on} \quad \partial \Gamma_0, \\ \Psi(x, 0) = \Psi_t(x, 0) = 0 & \text{in} \quad \Gamma_0. \end{cases} \tag{3.20}$$

The uniqueness of problem (3.20) implies

$$U_0 = 0 \quad \text{on} \quad \Sigma_0.$$

By Theorem 1.1, $B_0 = 0$, which contradicts with the fact that $||B_0||_{H^1(\Omega)} = 1$. □

4 Concluding Remarks

The main prominent feature of the structural acoustic system (1.1) lies in the presence of a variable coefficient matrix $A(x)$, which arises naturally from the non-homogeneous material properties. We may further study the inverse problems for the structural model with a curved wall whose middle surface is a part of a surface in IR^3. For the modeling of the structural acoustic systems with variable coefficients and curved walls, we refer to Appendix in [33]. The above two characters not only make the structural acoustic system much more realistic, but also gain additional complexities to the mathematical analysis.

We mention that all the results obtained in this paper are also valid for the case where the dimension $n = 2$. That is, the plate Γ_0 reduces to the beam. It is also pointed out that the observability inequality (2.36) obtained by the Carleman estimate can also be proved by the well-known multiplier technique only. See for example [34, Chap. 2].

Assumption (A.2) means that we need to repeat observations 9 times for the determination of 6 unknown coefficients $(a_{ij}(x))_{1\le i,j\le 3}$. An interesting question is: Can we suitably choose 6 or less groups of inputs (observations) for determining $(a_{ij}(x))_{1\le i,j\le 3}$? However, we do not know how to achieve this. Anyways, this needs further considerations, and some estimates (e.g., Lemma 3.1) should be refined.

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