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Homoclinic Solutions for Partial Difference Equations with Mixed Nonlinearities

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Abstract

In this paper, we consider a class of partial difference equations with sign-changing mixed nonlinearities and unbounded potentials. Some sufficient conditions for the existence and multiplicity of homoclinic solutions are obtained by using critical point theory. Even for ordinary difference equations, our results significantly improve some existing ones.

Keywords Homoclinic solution · Partial difference equation · Discrete nonlinear Schrödinger equation · Critical point theory

Mathematics Subject Classification Primary: 39A14

1 Introduction

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important nonlinear lattice systems, appearing in many areas of biology and physics such as the DNA double-strand [1], nonlinear optics [2], complex electronic materials [3], and Bose–Einstein condensates [4]. Some reviews on DNLS equations can be found in [5, 6]. Among them, the two-dimensional (2D) DNLS equation has a place. For example, the classic DNLS equation

$$i\dot{\psi}_{m,n} + C\Xi\psi_{m,n} + |\psi_{m,n}|^2\psi_{m,n} = 0, \quad (m,n) \in \mathbb{Z}^2, \tag{1}$$

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where Ξ is the discrete Laplacian operator defined by $\Xi\psi_{m,n} = \psi_{m,n+1} + \psi_{m+1,n} + \psi_{m,n-1} + \psi_{m-1,n} - 4\psi_{m,n}$, C is the coupling constant; note that the corresponding coupling length in the waveguide array, C^{-1}, is usually on the order of a few millimeters, in physical units. It could be used for research into a semi-infinite 2D array of optical waveguides with a horizontal edge, whose plane is parallel to the waveguides (which is a physically relevant representation of 2D lattices bounded by a flat surface). Indeed, the DNLS equation (1) describes, in the mean-field approximation, the dynamics of a Bose–Einstein condensate (BEC) trapped in a strong 2D optical lattice [7].

In 2006, Pankov [8, 9] studied periodic and decaying solutions by using the linking theorem and Nehari manifold approach. Since then, the existence of standing waves of DNLS equations has been studied extensively and deeply by many mathematicians and physicists [10–13]. As follows from the general theory of MacKay and Aubry [14], standing waves exist also in higher dimensions. Many fundamental features are expected to occur in higher dimensions, such as vortex lattice solitons, bright lattice solitons that carry angular momentum, and three-dimensional collisions between lattice solitons. Fleischer and his co-workers reported the experimental observation of 2D lattice solitons in [15]. Some theoretical and numerical simulation results are described in [16, 17], and the nonuniform dichotomy spectrum is introduced in [18].

When we look for standing waves of the more general 2D DNLS equation

$$i\dot{\psi}_{m,n} = -\Xi\psi_{m,n} + \epsilon_{m,n}\psi_{m,n} - f(m,n,\psi_{m,n}), \quad (m,n) \in \mathbb{Z}^2, \tag{2}$$

where $f(m,n,\cdot) \in C(\mathbb{R},\mathbb{R})$ for each $(m,n) \in \mathbb{Z}^2$, and the nonlinearity is gauge invariant, i.e.,

$$f(m,n,e^{i\theta}u) = e^{i\theta}f(m,n,u), \quad \theta \in \mathbb{R}.$$

Since solitons are spatially localized time-periodic solutions and decay to zero at infinity, we can make use of the standing wave ansata

$$\psi_{m,n} = u_{m,n}e^{-i\omega t}, \qquad \lim_{|m|+|n|\to\infty} \psi_{m,n} = 0,$$

where $\{u_{m,n}\}$ is a real number sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then we arrive at the partial difference equation

$$-\Delta_1^2 u(m-1,n) - \Delta_2^2 u(m,n-1) + \omega(m,n)u(m,n) = f(m,n,u(m,n)), \quad (m,n) \in \mathbb{Z}^2, \tag{3}$$

and

$$\lim_{|m|+|n|\to\infty} u(m,n) = 0, \tag{4}$$

where $\omega(m,n) = \epsilon_{m,n} - \omega$ is real number for each $(m,n) \in \mathbb{Z}^2$, $\Delta_1 u(m,n) = u(m+1,n) - u(m,n)$, $\Delta_2 u(m,n) = u(m,n+1) - u(m,n)$, $\Delta_1^2 u(m,n) = \Delta_1(\Delta_1 u(m,n))$.

We assume that $f(m,n,0)=0$ for each $(m,n)\in\mathbb{Z}^2$, then $\{u(m,n)\}=\{0\}$ is a solution of (3), which is called the trivial solution. As usual, we say that a solution $u=\{u(m,n)\}$ of (3) is homoclinic (to 0) if (4) holds. In addition, if $\{u(m,n)\}\neq\{0\}$, then u is called a nontrivial homoclinic solution. Therefore, the problem on standing waves of the 2D DNLS equation (2) has been reduced to that on homoclinic solutions of the partial difference equation (3).

Partial difference equations predated partial differential equations, but unfortunately, they were not as popular as the latter, and their development did not continue until the 20th century [19]. At present, there are few kinds of research on the qualitative theory of partial difference equations, mainly involving oscillation, stability, chaos, and other problems [20–24], and even fewer discussions on its homoclinic solutions [25, 26]. The main reason is that there is no effective tool for studying partial difference equations. Since critical point theory was introduced into difference equations by Guo and Yu [27], it has been developed as a powerful tool for studying homoclinic solutions of difference equations. In 2006, Ma and Guo [28] studied homoclinic solutions of a class of second order self-adjoint difference equations by using critical point theory. In recent years, more and more scholars have made use of the relevant tools and methods of critical point theory to study some nonlinear discrete systems, and a lot of meaningful research results have been obtained [29–33]. In particular, Lin and Yu [31] studied homoclinic solutions of periodic discrete systems with sign-changing mixed nonlinearities by new arguments including weak*-compactness.

We note that many practical partial difference equation models have a variational structure in which the solutions of these equations can be transformed into the critical points of corresponding variational functional in a suitable space, this makes it possible to study partial difference equations by means of the variational method. Motivated by the interesting studies above and the references therein, we shall attempt to investigate the existence and multiplicity of homoclinic solutions for the partial difference equation (3).

It is worth pointing out that in the search for infinitely many homoclinic solutions of discrete nonlinear systems, most of the existing literature considers only the case where the nonlinear terms are superlinear, whereas the mixed nonlinear condition we adopt is more applicable and weaker. Not only that, but we can circumvent an important global condition by some technical means, and the nonlinearities are allowed to be sign-changing. Details can be found in the remarks.

Assume the following condition on $\{\omega(m,n)\}$ holds.

(Ω) $\omega(m,n)\to+\infty$ as $|m|+|n|\to\infty$, $\omega_*=\min\{\omega(m,n):(m,n)\in\mathbb{Z}^2\}>0$.

Let S be the vector space of all real sequences of the form

$$\begin{aligned}u&=\{u(m,n)\}_{(m,n)\in\mathbb{Z}^2}\\&=(u(0,0),u(1,0),u(0,1),u(-1,0),u(0,-1),u(2,0),u(1,1),\\&\quad u(0,2),\cdots,u(m,n),\cdots),\end{aligned}$$

namely

$$S = \left\{ u = \{u(m,n)\} \;\middle|\; u(m,n) \in \mathbb{R}, (m,n) \in \mathbb{Z}^2 \right\}.$$

Define the space

$$E = \left\{ u \in S \;\middle|\; \sum_{(m,n)\in\mathbb{Z}^2} \omega(m,n)|u(m,n)|^2 < \infty \right\},$$

and the norm

$$\|u\| = \left(\sum_{(m,n)\in\mathbb{Z}^2} \omega(m,n)|u(m,n)|^2 \right)^{\frac{1}{2}} \quad \text{for } u \in E.$$

We assume that nonlinearities $f(m,n,u)$ and $F(m,n,u) = \int_0^u f(m,n,s)ds$ satisfy the following conditions:

- (F_1) $\limsup_{u\to 0} \frac{f(m,n,u)}{u} = a(m,n)$ and $\liminf_{u\to 0} \frac{f(m,n,u)}{u} = b(m,n)$ uniformly for $(m,n) \in \mathbb{Z}^2$, where $\sup_{(m,n)\in\mathbb{Z}^2} a(m,n) < \omega_*$ and $\inf_{(m,n)\in\mathbb{Z}^2} b(m,n) > -\omega_*$;
- (F_2) $\liminf_{|u|\to\infty} \frac{F(m,n,u)}{u^2} = c(m,n) \le \infty$ for $(m,n) \in \mathbb{Z}^2$;
- (F_3) there exists constant $\theta > 0$ such that $\theta\mathcal{F}(m,n,u) \ge \mathcal{F}(m,n,tu)$ for $(m,n) \in \mathbb{Z}^2$, $u \in \mathbb{R}$, and $t \in [0,1]$, where $\mathcal{F}(m,n,u) = f(m,n,u)u - 2F(m,n,u)$;
- (F_4) $f(m,n,u)u - 2F(m,n,u) \to +\infty$ as $|u| \to \infty$ for $(m,n) \in \mathbb{Z}^2$.

Now, we give the main results of this paper:

Theorem 1.1 *Assume that* (Ω) *holds, and* $f(m,n,u)$ *satisfies* $(F_1) - (F_4)$. *If there exists a constant* c_* *such that* $c(m,n) \ge c_* > \omega_*/2 + 2$ *for* $(m,n) \in \mathbb{Z}^2$, *then (3) has at least one nontrivial homoclinic solution in* E.

Theorem 1.2 *Assume that* (Ω) *holds,* $f(m,n,u)$ *is odd in* u *for each* $(m,n) \in \mathbb{Z}^2$ *and satisfies* $(F_1) - (F_4)$. *If* $c(m,n) > \omega(m,n)/2 + 4$ *for* $(m,n) \in \mathbb{Z}^2$, *then (3) has infinitely many high energy homoclinic solutions in* E.

Remark 1.1 If $c(m,n) \equiv \infty$, then the condition (F_4) in Theorems 1.1 and 1.2 can be removed.

Remark 1.2 The conditions (F_1) and (F_2) allow for the non-existence of limits of $f(m,n,u)/u$ for all $(m,n) \in \mathbb{Z}^2$ both at the origin and at infinity, which of course means that our conditions encompass cases of superlinear, asymptotically linear and a mixture of them. In contrast to existing results (see [8, 11, 12, 28]), we do not need f to be only superlinear or asymptotically linear at the origin or at infinity.

Remark 1.3 In comparison with the conditions in [11], we remove the following condition (F_1'): there exist $a > 0$ and $p > 2$ such that $|f(u)| \le a(1 + |u|^{p-1})$ for all $u \in \mathbb{R}$.

Remark 1.4 Our nonlinearity can be sign-changing, which is more general than the non-negativity case ($F(u) \geq 0$ for all $u \in \mathbb{R}$) in most related papers [10–12].

Next we give two typical examples to illuminate our results.

Example 1.1 Let

$$F(m,n,u) = \alpha(m,n)(u^4 - u^2), \quad (m,n,u) \in \mathbb{Z}^2 \times \mathbb{R},$$

where $0 < \inf\{\alpha(m,n) : (m,n) \in \mathbb{Z}^2\} \leq \sup\{\alpha(m,n) : (m,n) \in \mathbb{Z}^2\} < +\infty$. Then we know

$$\mathcal{F}(m,n,u) = f(m,n,u)u - 2F(m,n,u) = \alpha(m,n)2u^4.$$

Let's say $\alpha(m,n) \equiv 1$. For the sake of visualization, we can draw them as follows

Clearly, F is sign-changing and satisfies our conditions $(F_1) - (F_4)$ for $\omega_* = 3$ (Fig. 1).

Example 1.2 Let

$$f(m,n,u) = \frac{(\omega(m,n)+4)u^3 - 0.2\omega_* u}{\tau(m,n)u^2 + 1}, \quad (m,n,u) \in \mathbb{Z}^2 \times \mathbb{R},$$

where $\{\tau(m,n)\}$ is a sequence with

$$\tau(m,n) = \begin{cases} 0, & (m,n) = (0,0), \\ 1, & (m,n) \neq (0,0). \end{cases}$$

Obviously, f satisfies conditions $(F_1) - (F_4)$ and is neither superlinear nor asymptotically linear at infinity, so our results extend and improve those in the existing literature [11].

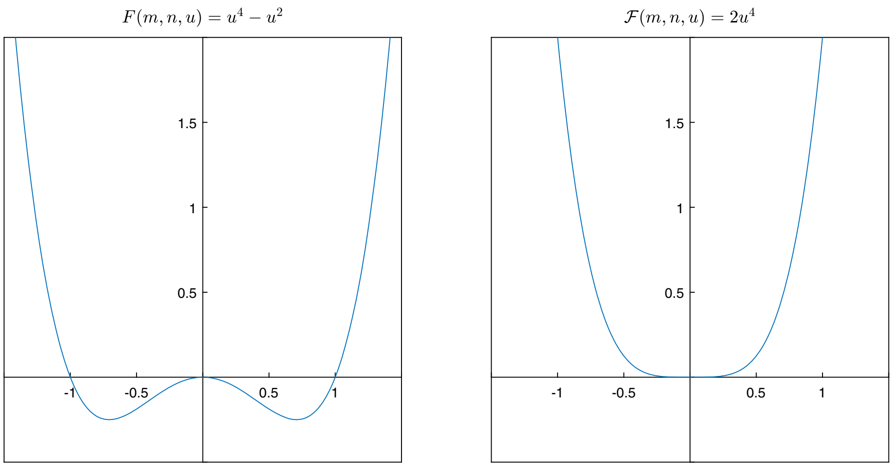

Fig. 1 The images of $F(m,n,u)$ and $\mathcal{F}(m,n,u)$

Of course, we can also solve this problem with the classic AR condition. It is easy to see that the AR condition is a special case of our conditions, i.e., $a(m,n) = b(m,n) = 0$, $c(m,n) = \infty$ for all $(m,n) \in \mathbb{Z}^2$. Here, we write them down as corollaries.

We assume that the nonlinearity $f(m,n,u)$ satisfies the following conditions:

(G_1) $\lim_{u\to 0} \frac{f(m,n,u)}{u} = 0$ uniformly for $(m,n) \in \mathbb{Z}^2$;

(G_2) there exists constant $\beta > 2$ such that $f(m,n,u)u \geq \beta \int_0^u f(m,n,s)ds > 0$ for all $(m,n) \in \mathbb{Z}^2$, $u \in \mathbb{R}\backslash\{0\}$.

Corollary 1.1 *Assume that* (Ω) *holds, and* $f(m,n,u)$ *satisfies* (G_1), (G_2). *Then (3) has at least one nontrivial homoclinic solution in* E.

Corollary 1.2 *Assume that* (Ω) *holds,* $f(m,n,u)$ *is odd in* u *for* $(m,n) \in \mathbb{Z}^2$ *and satisfies* (G_1), (G_2). *Then there exists an unbounded sequence in* E *of homoclinic solutions of (3).*

The rest of this paper is organized as follows. In Sect. 2, we establish the variational framework associated with (3) and cite the Mountain Pass Lemma and the Symmetric Mountain Pass Lemma. Then we give some lemmas which will be of fundamental importance in proving our main results in Sect. 3. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2.

2 The Variational Structure

In this section, we first establish the variational framework associated with (3) and state some basic notations. We denote by l^2 the set of all functions $u : \mathbb{Z}^2 \to \mathbb{R}$ such that

$$\|u\|_2 = \left(\sum_{(m,n)\in\mathbb{Z}^2} |u(m,n)|^2 \right)^{\frac{1}{2}} < \infty.$$

Moreover, we denote by l^∞ the set of all functions $u : \mathbb{Z}^2 \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{(m,n)\in\mathbb{Z}^2} |u(m,n)| < \infty.$$

On the Hilbert space E, we consider the functional

$$J(u) = \sum_{(m,n)\in\mathbb{Z}^2} \left[\frac{1}{2} (\Delta_1 u(m-1,n))^2 + \frac{1}{2}(\Delta_2 u(m,n-1))^2 + \frac{1}{2}\omega(m,n)u^2(m,n) - F(m,n,u(m,n)) \right].$$

Standard arguments show that the functional J is well-defined C^1 functional on E and satisfies

$$\begin{aligned}\langle J'(u), v\rangle &= \sum_{(m,n)\in\mathbb{Z}^2} [\Delta_1 u(m-1,n)\Delta_1 v(m-1,n) + \Delta_2 u(m,n-1)\Delta_2 v(m,n-1) \\ &\quad + \omega(m,n)u(m,n)v(m,n) - f(m,n,u(m,n))v(m,n)] \\ &= \sum_{(m,n)\in\mathbb{Z}^2} [\Delta_1 u(m-1,n)v(m,n) - \Delta_1 u(m,n)v(m,n) + \Delta_2 u(m,n-1)v(m,n) \\ &\quad - \Delta_2 u(m,n)v(m,n) + \omega(m,n)u(m,n)v(m,n) - f(m,n,u(m,n))v(m,n)] \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \left[-\Delta_1^2 u(m-1,n) - \Delta_2^2 u(m,n-1) + \omega(m,n)u(m,n) \right. \\ &\quad \left. - f(m,n,u(m,n))\right] v(m,n), \; u, v \in E.\end{aligned} \tag{5}$$

It follows from the above equation that $\langle J'(u), v\rangle = 0$ for all $v \in E$ if and only if

$$-\Delta_1^2 u(m-1,n) - \Delta_2^2 u(m,n-1) + \omega(m,n)u(m,n) - f(m,n,u(m,n)) = 0.$$

Therefore, we have reduced the problem of finding a nontrivial homoclinic solution of (3) to that of seeking a nonzero critical point of the functional J.

Let B_r denote the open ball of radius r about 0, and let ∂B_r denote its boundary.

Definition 2.1 For $J \in C^1(E, \mathbb{R})$, we say J satisfies the Palais–Smale condition if any sequence $\{x_j\} \subset E$ for which $J(x_j)$ is bounded and $J'(x_j) \to 0$ as $j \to \infty$ possesses a convergent subsequence.

Definition 2.2 Let $J \in C^1(E, \mathbb{R})$. A sequence $\{x_j\} \subset E$ is called a Cerami sequence for J if $J(x_j) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|x_j\|)J'(x_j) \to 0$ as $j \to \infty$. We say J satisfies the Cerami condition if any Cerami sequence for J possesses a convergent subsequence.

Lemma 2.1 (Mountain Pass Lemma [34]) *Suppose $J \in C^1(E, \mathbb{R})$, satisfies the Palais–Smale condition, $J(0) = 0$,*

(i) *there exist constants $\rho, a > 0$ such that $J|_{\partial B_\rho} \geq a$, and*
(ii) *there is an $e \in E \backslash \bar{B}_\rho$ such that $J(e) \leq 0$.*
Then J possesses a critical value $c \geq a$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)),$$

where $\Gamma = \{h \in C([0,1], E) \mid h(0) = 0, h(1) = e\}$.

Lemma 2.2 (Symmetric Mountain Pass Lemma [35]) *Let $J \in C^1(E, \mathbb{R})$ be even. Suppose J satisfies the Palais–Smale condition, $J(0) = 0$,*

(i) *there exist constants $\rho, a > 0$ such that $J|_{\partial B_\rho} \geq a$, and*
(ii) *for each finite-dimensional subspace $\tilde{E} \subset E$, there is $\gamma = \gamma(\tilde{E})$ such that $J \leq 0$ on $\tilde{E} \backslash B_\gamma$.*
Then J possesses an unbounded sequence of critical values.

Remark 2.1 A deformation lemma can show that conclusion of the above lemmas remains true if the Palais–Smale condition is replaced with the Cerami condition [36].

3 Some Lemmas

Similar to the proof of [28], we generalize and obtain the following lemma, which gives a discrete version of compact embedding theorem and plays a crucial role in the subsequent proof.

Lemma 3.1 *Under the assumption* (Ω), *the embedding map from* E *into* l^2 *is compact.*

Proof Let $\{u_k\} \subset E$ be a bounded sequence, i.e., there exists $M_0 > 0$ such that $\|u_k\|^2 < M_0$ for all $k \in \mathbb{N}$. Up to a subsequence if necessary, we have

$$u_k \rightharpoonup u \quad \text{in } E.$$

We may assume $u = 0$, in particular $u_k(m, n) \to 0$ as $k \to \infty$ for all $(m, n) \in \mathbb{Z}^2$. For any $\varepsilon_0 > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\omega(m, n) > \frac{1 + M_0}{\varepsilon_0} \quad \text{for all } |m| + |n| > N_0.$$

By continuity of the finite sum, there exists $K_0 \in \mathbb{N}$ such that

$$\sum_{|m|+|n| \le N_0} |u_k(m, n)|^2 < \frac{\varepsilon_0}{1 + M_0} \quad \text{for all } k > K_0.$$

So for $k > K_0$, we have

$$\begin{aligned} \sum_{(m,n)\in\mathbb{Z}^2} |u_k(m, n)|^2 &\le \frac{\varepsilon_0}{1 + M_0} + \frac{\varepsilon_0}{1 + M_0} \sum_{|m|+|n|>N_0} \omega(m, n)|u_k(m, n)|^2 \\ &\le \frac{\varepsilon_0}{1 + M_0}(1 + \|u_k\|^2) \\ &< \varepsilon_0. \end{aligned}$$

Thus, $u_k \to 0$ in l^2. □

Lemma 3.2 *Assume that the conditions of Theorem 1.1 hold. Then the functional* J *satisfies the Cerami condition for any given* $c \in \mathbb{R}$.

Proof Let $\{u_k\} \subset E$ be a Cerami sequence of J, that is

$$J(u_k) \to c, \quad (1 + \|u_k\|)\|J'(u_k)\| \to 0 \quad \text{as } k \to \infty. \tag{6}$$

First, we prove that $\{u_k\}$ is bounded in E. In fact, if not, we may assume that $\|u_k\| \to \infty$ as $k \to \infty$. Set $\xi_k = u_k/\|u_k\|$. Up to a subsequence if necessary, we have

$$\xi_k \rightharpoonup \xi \quad \text{in } E,$$

$$\xi_k \to \xi \quad \text{in } l^2. \tag{7}$$

Case 1: $\xi \neq 0$. Let $\Lambda = \{(m,n) \in \mathbb{Z}^2 : \xi(m,n) \neq 0\}$. Then it follows from (7) that

$$u_k(m_0,n_0) = \xi_k(m_0,n_0)\|u_k\| \to \infty \quad \text{as } k \to \infty, \quad \text{for } (m_0,n_0) \in \Lambda,$$

and by (F_4), we have

$$f(m_0,n_0,u_k(m_0,n_0))u_k(m_0,n_0) - 2F(m_0,n_0,u_k(m_0,n_0)) \to +\infty \quad \text{as } k \to \infty. \tag{8}$$

By (6), there is a constant $c_0 > 0$ such that $|J(u_k)| \le c_0$, then we have

$$\begin{aligned}
&\sum_{(m,n)\in\mathbb{Z}^2} (f(m,n,u_k(m,n))u_k(m,n) - 2F(m,n,u_k(m,n)))\\
&= 2J(u_k) - \langle J'(u_k), u_k\rangle\\
&\le 2|J(u_k)| + \|u_k\|\|J'(u_k)\|\\
&\le 2c_0 + (1 + \|u_k\|)\|J'(u_k)\|\\
&< +\infty.
\end{aligned}$$

This contradicts (8).

In particular, as noted in Remark 1.1, if $c(m,n) \equiv \infty$, the above proof can be obtained without (F_4). In fact, by (6), there exists a constant C such that $J(u_k) \ge C$. Thus, we have

$$C \le J(u_k) \le \frac{4}{\omega_*}\|u_k\|^2 + \frac{1}{2}\|u_k\|^2 - \sum_{(m,n)\in\mathbb{Z}^2} F(m,n,u_k(m,n)). \tag{9}$$

We divide both sides of (9) by $\|u_k\|^2$ and get

$$\sum_{(m,n)\in\mathbb{Z}^2} \frac{F(m,n,u_k(m,n))}{\|u_k\|^2} \le \frac{4}{\omega_*} + \frac{1}{2} - \frac{C}{\|u_k\|^2} < +\infty. \tag{10}$$

In view of (F_2), we have

$$\frac{F(m_0,n_0,u_k(m_0,n_0))}{\|u_k\|^2} = \frac{F(m_0,n_0,u_k(m_0,n_0))}{u_k^2(m_0,n_0)}\xi_k^2(m_0,n_0) \to +\infty \quad \text{as } k \to \infty.$$

This contradicts (10).

Case 2: $\xi = 0$. Set

$$J(t_k u_k) = \max_{t \in [0,1]} J(t u_k).$$

For any given $M > \max\{2\theta c, 1\}$, let k be large enough such that $\|u_k\| \geq M$ and $\overline{\xi}_k = M^{1/2}\xi_k$. For any $\varepsilon > 0$, set

$$p(m,n) = \max\{|a(m,n)+\varepsilon|, |b(m,n)-\varepsilon|\} \quad \text{for } (m,n) \in \mathbb{Z}^2,$$

and

$$p^* = \sup_{(m,n)\in\mathbb{Z}^2} p(m,n).$$

By (F_1) and (7), it is easy to see that

$$\sum_{(m,n)\in\mathbb{Z}^2} F(m,n,\overline{\xi}_k(m,n)) \leq \frac{p^*}{2}\|\overline{\xi}_k\|_2^2 \to 0 \quad \text{as } k \to \infty.$$

Thus, for k large enough, we have

$$\begin{aligned} J(t_k u_k) &\geq J(\overline{\xi}_k) \\ &\geq \frac{1}{2}\|\overline{\xi}_k\|^2 - \sum_{(m,n)\in\mathbb{Z}^2} F(m,n,\overline{\xi}_k(m,n)) \\ &\geq \frac{1}{2}M - \sum_{(m,n)\in\mathbb{Z}^2} F(m,n,\overline{\xi}_k(m,n)). \end{aligned}$$

This implies that

$$\liminf_{k\to\infty} J(t_k u_k) \geq \frac{1}{2}M > \theta c. \tag{11}$$

Noting that $J(0) = 0$ and $J(u_k) \to c$, as $k \to \infty$, $J(tu_k)$ attains its maximum at $t_k \in (0,1)$ when k is big enough. Thus, $\langle J'(t_k u_k), t_k u_k\rangle = 0$. It follows from (F_3) that

$$
\begin{aligned}
J(t_k u_k) &= J(t_k u_k) - \frac{1}{2}\langle J'(t_k u_k), t_k u_k\rangle \\
&= \sum_{(m,n)\in\mathbb{Z}^2}\left(\frac{1}{2}f(m,n,t_k u_k(m,n))t_k u_k(m,n) - F(m,n,t_k u_k(m,n))\right) \\
&\leq \theta \sum_{(m,n)\in\mathbb{Z}^2}\left(\frac{1}{2}f(m,n,u_k(m,n))u_k(m,n) - F(m,n,u_k(m,n))\right) \\
&\leq \theta\left(J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle\right),
\end{aligned}
$$

which implies that

$$
\limsup_{k\to\infty} J(t_k u_k) \leq \theta c.
$$

This contradicts (11). Hence, $\{u_k\}$ is bounded in E.

Second, we show that there exists a convergent subsequence of $\{u_k\}$. Actually, there is a subsequence, still denoted by the same notation, such that $\{u_k\}$ weakly converges to some $u \in E$. By Lemma 3.1, we can see that

$$
u_k \to u \quad \text{in } l^2. \tag{12}
$$

Then by (5), we have

$$
\begin{aligned}
&\sum_{(m,n)\in\mathbb{Z}^2} \omega(m,n)(u_k(m,n) - u(m,n))^2 \\
&\leq \left\langle J'(u_k) - J'(u), u_k - u\right\rangle \\
&\quad + \sum_{(m,n)\in\mathbb{Z}^2} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)).
\end{aligned}
$$

Due to the weak convergence and (6), we see that

$$
\left\langle J'(u_k) - J'(u), u_k - u\right\rangle \to 0 \quad \text{as } k \to \infty.
$$

By (F_1), there exists $\zeta \leq \varepsilon$ such that

$$
|f(m,n,u)| \leq p(m,n)|u| \quad \text{for all } (m,n) \in \mathbb{Z}^2 \text{ and } |u| \leq \zeta.
$$

We know $u(m,n) \to 0$ as $|m| + |n| \to \infty$, then there exists $N \in \mathbb{N}$ such that

$$
|u(m,n)| \leq \frac{\zeta}{2} \quad \text{for all } |m| + |n| > N.
$$

By (12), there exists $K \in \mathbb{N}$ such that

$$\|u_k - u\|_2 < \frac{\zeta}{2} \quad \text{for all } k > K.$$

Then for all $k > K$, $|m| + |n| > N$, we have

$$|u_k(m,n) - u(m,n)| \le \frac{\zeta}{2},$$

then

$$|u_k(m,n)| \le |u(m,n)| + \frac{\zeta}{2} \le \zeta,$$

and

$$|f(m,n,u_k(m,n))| \le p(m,n)|u_k(m,n)|, \quad |f(m,n,u(m,n))| \le p(m,n)|u(m,n)|.$$

We know that

$$\begin{aligned} &\sum_{(m,n)\in\mathbb{Z}^2} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)) \\ &\quad = \sum_{|m|+|n|\le N} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)) \\ &\qquad + \sum_{|m|+|n|>N} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)). \end{aligned} \tag{13}$$

By the uniformly continuity of $f(m,n,u)$ in u and $u_k \to u$ in l^2, the first term on the righthand side of (13) approaches 0 as $k \to \infty$. It remains to show the second term also tends to 0 as $k \to \infty$. From Hölder's inequality, there exists a constant $\sigma > 0$, such that

$$\begin{aligned} &\sum_{|m|+|n|>N} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)) \\ &\le \left(\sum_{|m|+|n|>N} |f(m,n,u_k(m,n)) - f(m,n,u(m,n))|^2\right)^{\frac{1}{2}} \left(\sum_{|m|+|n|>N} |u_k(m,n) - u(m,n)|^2\right)^{\frac{1}{2}} \\ &\le p^* \left(\sum_{|m|+|n|>N} (|u_k(m,n)| + |u(m,n)|)^2\right)^{\frac{1}{2}} \|u_k - u\|_2 \\ &\le p^* \left[\left(\sum_{|m|+|n|>N} |u_k(m,n)|^2\right)^{\frac{1}{2}} + \left(\sum_{|m|+|n|>N} |u(m,n)|^2\right)^{\frac{1}{2}}\right] \varepsilon \\ &\le p^* \sigma \varepsilon. \end{aligned}$$

So we have

$$\sum_{(m,n)\in\mathbb{Z}^2} (f(m,n,u_k(m,n)) - f(m,n,u(m,n)))(u_k(m,n) - u(m,n)) \to 0 \quad \text{as } k \to \infty. \tag{14}$$

Therefore, combining (12) and the boundedness of $\{u_k\}$, it follows that

$$\lim_{k\to\infty} \|u_k - u\| = 0.$$

The proof is completed. □

4 Proofs of Main Results

Proof of Theorem 1.1. Let $\varepsilon = \min\left\{\omega_* - \sup_{(m,n)\in\mathbb{Z}^2} a(m,n), \omega_* + \inf_{(m,n)\in\mathbb{Z}^2} b(m,n)\right\}/2$, we have $\omega_* = p^* + \varepsilon$, then by (F_1), there is $\delta > 0$ such that

$$|F(m,n,u)| \leq \frac{p^*}{2}u^2 \quad \text{for all } (m,n) \in \mathbb{Z}^2 \text{ and } |u| \leq \delta.$$

Let $\|u\| = \rho = \sqrt{\omega_*}\delta$. We have $\|u\|_\infty \leq (\sqrt{\omega_*})^{-1}\|u\| = \delta$, then

$$\begin{aligned}
J(u) &\geq \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \omega(m,n)u(m,n)^2 - \sum_{(m,n)\in\mathbb{Z}^2} F(m,n,u(m,n)) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{p^*}{2} \sum_{(m,n)\in\mathbb{Z}^2} u^2(m,n) \\
&\geq \frac{1}{2}\left(1 - \frac{p^*}{\omega_*}\right)\|u\|^2 \\
&= \frac{\varepsilon}{2\omega_*}\delta^2 = a > 0.
\end{aligned}$$

□

Since $\omega_* = \min\{\omega(m,n) : (m,n) \in \mathbb{Z}^2\}$, there is $(m_*, n_*) \in \mathbb{Z}^2$ such that $\omega(m_*, n_*) = \omega_*$. Define $e = \{e(m,n)\}$ by

$$e(m,n) = \begin{cases} 1, & (m,n) = (m_*, n_*), \\ 0, & (m,n) \neq (m_*, n_*). \end{cases}$$

As $c_* > \omega_*/2 + 2$, then there exists $\epsilon_0 > 0$, such that

$$c_* > \frac{\omega_*}{2} + 2 + \epsilon_0.$$

By (F_2), there exists $\eta > 0$ such that

$$F(m_*, n_*, u) \geq (c_* - \epsilon_0)u^2 \quad \text{for } |u| \geq \eta. \tag{15}$$

Taking $|t|$ large enough, such that $|t| > \eta$. Then combining (15), we have

$$\begin{aligned} J(te) &= \sum_{(m,n)\in\mathbb{Z}^2} \left[\frac{1}{2} t^2 (\Delta_1 e(m-1, n))^2 \right. \\ &\quad \left. + \frac{1}{2} t^2 (\Delta_2 e(m, n-1))^2 + \frac{1}{2} t^2 \omega(m, n) e^2(m, n) - F(m, n, te(m, n)) \right] \\ &\leq 2t^2 + \frac{1}{2}\omega_* t^2 - (c_* - \epsilon_0)t^2 \\ &\leq (2 + \frac{\omega_*}{2} + \epsilon_0 - c_*)t^2. \end{aligned}$$

Letting $|t| \to \infty$ gives us $J(te) \to -\infty$. Then there exists a real number t_0 such that

$$\|t_0 e\| > \rho \quad \text{and} \quad J(t_0 e) < 0.$$

Since we have verified all assumptions of Lemma 2.1, it follows that J possesses a Cerami sequence $\{u_j\} \subset E$ for the mountain pass level $c \geq a$ with

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)),$$

where

$$\Gamma = \{h \in C([0, 1], E) : h(0) = 0, h(1) = t_0 e\}.$$

A nontrivial critical point u of J as the corresponding critical value $c \geq a > 0$. Hence, (3) has at least one nontrivial solution in E.

Proof of Theorem 1.2 Using an argument similar to the one given in the proof of Theorem 1.1, one can prove that J satisfies the Cerami condition as well as establishes part (i) of Lemma 2.2. Let us establish part (ii) of the Symmetric Mountain Pass Lemma.

Let $\tilde{E} \subset E$ be a finite-dimensional subspace. To prove our conclusion, we only need to prove

$$J(u) \to -\infty \quad \text{when } \|u\| \to \infty.$$

Assume, by contradiction, that there exist a sequence $\{u_k\} \subset \tilde{E}$ with $\|u_k\| \to \infty$ as $k \to \infty$ and a constant C_0 such that $J(u_k) \geq C_0$ for all $k \in \mathbb{N}$. Set $v_k = u_k/\|u_k\|$, then $\|v_k\| = 1$. Since $\tilde{E}$ is finite dimensional, up to a subsequence if necessary, we can assume that $v_k \to v$ in $\tilde{E}$, and $v_k(m,n) \to v(m,n)$ for all $(m,n) \in \mathbb{Z}^2$, thus $\|v\| = 1$. Then, we have

$$C_0 \leq J(u_k) \leq 4\|u_k\|_2^2 + \frac{1}{2}\|u_k\|^2 - \sum_{(m,n)\in\mathbb{Z}^2} F(m,n,u_k(m,n)). \tag{16}$$

We divide both sides of (16) by $\|u_k\|^2$ and get

$$\sum_{(m,n)\in\mathbb{Z}^2} \frac{F(m,n,u_k(m,n))}{\|u_k\|^2} \leq 4\|v_k\|_2^2 + \frac{1}{2} - \frac{C_0}{\|u_k\|^2},$$

which implies that

$$\limsup_{k\to\infty} \sum_{(m,n)\in\mathbb{Z}^2} \frac{F(m,n,u_k(m,n))}{\|u_k\|^2} \leq 4\|v\|_2^2 + \frac{1}{2}. \tag{17}$$

On the other hand, let $\Omega = \{(m,n) \in \mathbb{Z}^2 : v(m,n) \neq 0\}$. We know that, for all $(m,n) \in \Omega$,

$$u_k(m,n) = v_k(m,n)\|u_k\| \to \infty \quad \text{as } k \to \infty.$$

By (F_2) and Fatou's Lemma, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \sum_{(m,n)\in\mathbb{Z}^2} \frac{F(m,n,u_k(m,n))}{\|u_k\|^2} \\
&\geq \sum_{(m,n)\in\mathbb{Z}^2} \liminf_{k\to\infty} \frac{F(m,n,u_k(m,n))}{\|u_k\|^2} \\
&\geq \sum_{(m,n)\in\Omega} \liminf_{k\to\infty} \frac{F(m,n,u_k(m,n))}{u_k^2(m,n)} v_k^2(m,n) \\
&> \sum_{(m,n)\in\Omega} \left(\frac{\omega(m,n)}{2} + 4\right) v^2(m,n) \\
&= \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{\omega(m,n)}{2} + 4\right) v^2(m,n) \\
&= 4\|v\|_2^2 + \frac{1}{2}.
\end{aligned}$$

This contradicts (17), and part (ii) of the Symmetric Mountain Pass Lemma follows.

It follows from Lemma 2.2 that J has a sequence of critical points $\{u_k\} \subset E$, such that $J(u_k) \to \infty$. Therefore, (3) has infinitely many solutions $\{u_k\}$ in E satisfying

$$\sum_{(m,n)\in\mathbb{Z}^2} \left[\frac{1}{2}(\Delta_1 u_k(m-1,n))^2 + \frac{1}{2}(\Delta_2 u_k(m,n-1))^2 + \frac{1}{2}\omega(m,n)u_k^2(m,n) - F(m,n,u_k(m,n))\right] \to \infty \quad \text{as } k \to \infty.$$

The proof of Theorem 1.2 is finished. □

5 Conclusions

In this work, by using critical point theory, we obtain sufficient conditions for the existence and multiplicity of homoclinic solutions for a class of partial difference equations with unbounded potentials. Specifically, under weak conditions, the existence of nontrivial homoclinic solutions for the partial difference equation (3) is obtained by using the Mountain Pass Lemma. Moreover, when the nonlinear term is odd, the existence of infinitely many nontrivial homoclinic solutions for the partial difference equation (3) is obtained by the Symmetric Mountain Pass Lemma.

Here, our conditions allow for nonlinear term to be superlinear, asymptotically linear and a mixture of them at the origin and at infinity, and even allow the limit of f/u to be non-existent. In many similar results for ordinary difference equations, f is required to be either superlinear or asymptotically linear at the origin or at infinity. We also allow the nonlinear term to change sign, which is required to satisfy non-negativity in most of the known results. In summary, even for ordinary difference equations, our results significantly improve the existing ones. We conclude by giving two examples to verify our conclusions.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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