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# On Sprays of Scalar Curvature and Metrizability

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**Abstract**

Every Finsler metric naturally induces a spray but not so for the converse. The notion for sprays of scalar (resp. isotropic) curvature has been known as a generalization for Finsler metrics of scalar (resp. isotropic) flag curvature. In this paper, a new notion, sprays of constant curvature, is introduced and especially it shows that a spray of isotropic curvature is not necessarily of constant curvature even in dimension $n \geq 3$. Further, complete conditions are given for sprays of isotropic (resp. constant) curvature to be Finsler metrizable. Based on this result, the local structure is determined for locally projectively flat Berwald sprays of isotropic (resp. constant) curvature which are Finsler metrizable, and some more sprays of isotropic curvature are discussed for their metrizability. Besides, the metrizability problem is also investigated for sprays of scalar curvature under certain curvature conditions.

**Keywords** Finsler metric · Spray · Berwald spray · Metrizability · Scalar/isotropic/constant curvature · Projective flatness

**Mathematics Subject Classification** 53C60 · 53B40

## 1 Introduction

Spray geometry studies the properties of sprays on a manifold, and it is more general than Finsler geometry, because every Finsler metric induces a natural spray, but there are a lot of sprays which cannot be induced by any Finsler metric [8, 13, 20]. A spray **G** on a manifold $M$ is a family of compatible second-order ODEs which define a special vector filed on a conical region $\mathcal{C}$ of $TM \setminus \{0\}$ (an important case is $\mathcal{C} = TM \setminus \{0\}$). The integral curves of **G** projected onto $M$ are called geodesics of **G**. Many basic curvatures, such as Riemann curvature, Ricci curvature, Weyl curvature, Berwald curvature and Douglas curvature, appearing in Finsler geometry, are actually defined

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in spray geometry via the spray coefficients. Geodesics and these basic curvatures play an important role in the study of spray geometry.

The metrizability problem for a spray **G** seeks for a Finsler metric whose spray is just **G**, or whose geodesics coincide with that of **G**. A weaker problem is to consider the projective metrizability of a given spray **G**, which aims to look for a Finsler metric projectively related to **G**. So a natural question is to determine whether a given spray is (projectively) Finsler metrizable or not under certain curvature conditions.

In [14], Matsumoto proves that any two-dimensional spray is locally projectively Finsler metrizable. More generally, any spray of scalar curvature is locally projectively Finsler metrizable [4, 7]. In [15], Muzsnay gives some sprays which are not Finsler metrizable under some conditions satisfied by the holonomy distribution generated from the horizontal vector fields of a spray. In [20], the present author constructs a class of sprays whose metrizable and non-metrizable conditions are completely determined respectively. Inspired by the sprays constructed in [20], Elgendi and Muzsnay discuss a more general class of sprays and prove the non-metrizability of such sprays by computing the dimension of the holonomy distribution under certain conditions [8]. In [5], Bucataru and Muzsnay characterize metrizable sprays with non-zero Ricci constant in dimension greater than 2, and further they give necessary and sufficient conditions for sprays of scalar curvature and non-zero Ricci curvature to be metrizable in [6]. In [12], Li and Shen introduce the notion of sprays with isotropic curvature and give some non-metrizability conditions for locally projectively flat sprays, and shows that a locally projectively flat spray with vanishing Riemann curvature is metrizable (cf. [17]). In [13], Li, Mo, and Yu give a class of locally projectively flat Berwald sprays which are non-metrizable.

In this paper, we are going to study some special properties and the metrizability problem of some special classes of sprays: locally projectively flat Berwald sprays, sprays of scalar curvature (resp. isotropic curvature, constant curvature). A locally projectively flat spray, which is always of scalar curvature, means that its geodesics are locally straight lines. A Berwald spray means that its spray coefficients $G^i = G^i(x, y)$ are quadratic in $y$. A spray **G** is said to be of *scalar curvature* if its Riemann curvature ${R^i}_k$ satisfies

$$ {R^i}_k = R\delta^i_k - \tau_k y^i, \tag{1} $$

where $R = R(x, y)$ and $\tau_k = \tau_k(x, y)$ are some homogeneous functions [16]. In [5], the condition $R_{.i} = 2\tau_i$ is considered for a spray satisfying (1). If in (1) there holds $R_{.i} = 2\tau_i$, then **G** is said to be of *isotropic curvature* [12]. A spray **G** is said to be of *constant curvature*, a new notion we introduce in this paper, if **G** satisfies (1) with

$$ \tau_{i;k} = 0 \ ( \Leftrightarrow \ R = \tau_k = 0, \ or \ R_{;i} = 0). $$

In the above, we use $T_{i;j}$ and $T_{i.j}$ to denote respectively the horizontal and vertical covariant derivatives of the tensor $T$ with respect to Berwald connection of a given spray. Some basic properties for a spray of constant curvature are given in Theorem 5.2 below, in which, it especially shows that an $n(\geq 3)$-dimensional spray of isotropic curvature is not necessarily of constant curvature, which is different from the Finslerian case.

**Theorem 1.1** *Let* $\mathbf{G}$ *be an* $n$*-dimensional spray of isotropic curvature* $R^i{}_k = R\delta^i_k - \frac{1}{2}R_{.k}$.

(i) *If the spray manifold* $(\mathbf{G}, M)$ *is analytical with* $R = 0$*, then* $\mathbf{G}$ *is locally Finsler metrizable.*
(ii) *If* $R \neq 0$ *and* $R$ *is not a Finsler metric, then* $\mathbf{G}$ *is not Finsler metrizable.*
(iii) *If* $R$ *is a Finsler metric, then* $\mathbf{G}$ *is (locally) Finsler metrizable if and only if* $R_{;i} = R\omega_i$ *for some closed 1-form* $\omega = \omega_i(x)dx^i$*. In this case, we have*

(iiia) *if* $n \geq 3$*, then* $R_{;i} = 0$ *(or* $\omega = 0$*).*
(iiib) *if* $\omega = 0$*, then* $\mathbf{G}$ *is induced by the Finsler metric* $R$ *with the flag curvature* $\mathbf{K} = 1$.
(iiib) *if* $\omega \neq 0$*, then* $\mathbf{G}$ *is induced by the Finsler metric* $R/\lambda$ *with the flag curvature* $\mathbf{K} = \lambda$*, where* $\lambda \neq 0$ *is given by* $\omega_i = (\ln|\lambda|)_{;i}$*. This case happens only in dimension* $n = 2$.

**Theorem 1.2** *Let* $\mathbf{G}$ *be an* $n$*-dimensional spray of constant curvature and* $Ric$ *be the Ricci curvature of* $\mathbf{G}$*. Then* $\mathbf{G}$ *is (locally) Finsler metrizable iff.* $Ric = 0$ *or* $Ric$ *is a Finsler metric. In this case, the Finsler metric* $L$ *inducing* $\mathbf{G}$ *has vanishing flag curvature or* $L$ *is given by* $L = Ric/(n-1)$ *with the flag curvature* 1.

Note that Theorem 1.1 for $R \neq 0$ can be concluded from Theorem 3.1 in [6], and Theorem 1.1 (iiia) has been proved in [5]. In Theorem 1.1, if $\mathbf{G}$ is metrizable, the corresponding Finsler metric is easily obtained if $Ric \neq 0$. The metric in Theorem 1.1 can be multiplied by a suitable non-zero constant if we need the metric to be positive. Theorem 1.1(i) shows that a spray with vanishing Riemann curvature (not necessary to be locally projectively flat) is locally metrizable (cf. [12, 17]). As applications of Theorems 1.1 and 1.2, we generalize a result in [20] (Theorem 4.10 below), and make a check on some spays whether they are Finsler metrizable or not (see Sect. 7 below). On the other hand, we can use Theorem 1.1 to obtain the local structure for locally projectively flat Berwald sprays of isotropic curvature when they are Finsler metrizable.

**Theorem 1.3** *Let* $\mathbf{G}$ *be a projectively flat Berwald spray of isotropic curvature on an open set* $U \subset R^n$ *with* $Ric \neq 0$ *(on a conical region* $\mathcal{C}(U)$*). Then* $\mathbf{G}$ *is Finsler metrizable (on* $\mathcal{C}(U)$*) if and only if* $\mathbf{G}$ *can be expressed as*

$$G^i = Py^i, \quad P := -\frac{1}{2}\Big[\ln|\langle Ax, x\rangle + \langle B, x\rangle + C|\Big]_{x^k} y^k, \tag{2}$$

*where* $A \neq 0$ *is a constant symmetric matrix,* $B$ *is a constant vector and* $C$ *is a constant number satisfying certain condition such that the following function* $L$ *is a metric (defined on* $\mathcal{C}(U)$*),*

$$L := \frac{4(\langle Ax, x\rangle + \langle B, x\rangle + C)\langle Ay, y\rangle - (2\langle Ax, y\rangle + \langle B, y\rangle)^2}{4(\langle Ax, x\rangle + \langle B, x\rangle + C)^2}. \tag{3}$$

*In this case,* **G** *is induced by the metric* $L = Ric/(n-1)$ *of constant sectional curvature 1.*

In [17], there is a general description of the construction for locally projectively flat Finsler metrics with constant flag curvature 1. As a special case, putting $2A = (\delta_{ij})$, $B = 0$, $C = 1/2$ in Theorem 1.3, we obtain

$$L = \frac{(1+|x|^2)|y|^2 - \langle x, y\rangle^2}{(1+|x|^2)^2}. \tag{4}$$

Theorem 1.3 also implies that if we take in (2), $P = -\big[\ln\sqrt{|f(x)|}\big]_{x^k} y^k$ for an arbitrary non-constant function $f(x)$ which is not a polynomial of degree two, then the spray **G** in (2) is not Finsler metrizable. If **G** in Theorem 1.3 has zero Riemann curvature ($Ric = 0$), then **G** can be locally induced by a Minkowski metric (a trivial case, Remark 6.7 below).

For a spray of scalar curvature $R^i{}_k = R\delta^i_k - \tau_k y^i$, the quantities $R$ and $\tau_k$ are closely related (Proposition 3.1 below). Now we consider the following condition for $R$ and $\tau_k$,

$$R_{.i} - 2\tau_i = \omega_{i0}, \quad (\omega_{i0} := \omega_{ir} y^r), \tag{5}$$

where $\omega = \omega_{ij}(x)dx^i \wedge dx^j$ is a 2-form. For a spray of scalar curvature, the condition (5) is a special case of $\chi_i = \omega_{i0}$ (see [11]), where $\chi_i$ is called the $\chi$-curvature originally defined in [18]. For a spray satisfying (5), we have the following theorem.

**Theorem 1.4** *Let* **G** *be an* $n$*-dimensional spray of scalar curvature* $R^i{}_k = R\delta^i_k - \tau_k y^i$ *satisfying* (5)*. Suppose that* **G** *is induced by a Finsler metric* $L$*.*

(i) *For* $n \geq 3$*,* $L$ *is of constant flag curvature with* $\omega = 0$ [9].

(ii) *For* $n = 2$*, the flag curvature* $\lambda$ *of* $L$ *satisfies* $\lambda''(\theta) + \epsilon I(\theta)\lambda'(\theta) = 0$ *on each tangent space, where* $\epsilon = \pm 1$ *is the sign of* $L$*, and* $\theta$ *is the Landsberg angle.*

  (iia) *If* $L$ *is a Riemann metric, or regular Finsler metric, then* $L$ *is of isotropic flag curvature* ($\lambda = \lambda(x)$) *with* $\omega = 0$*.*

  (iib) *If* $L$ *has constant main scalar, then* (5) *is satisfied with* $\omega$ *not necessarily zero.*

Theorem 1.4(i) has essentially been proved in [9]. Starting from (5), we are also going to give a little different version of proof from that in [9]. If a spray **G** satisfies (5) with $\omega \neq 0$ and $n \geq 3$, then the spray **G** is not Finsler metrizable by Theorem 1.4(i). If a two-dimensional spray **G** satisfies (5) with $\omega \neq 0$, then **G** cannot be induced by a Riemann metric or a regular Finsler metric by Theorem 1.4(iia), but Theorem 1.4(iib) shows that there are such sprays which can be induced by a singular Finsler metric.

## 2 Preliminaries

Let $M$ be an $n$-dimensional manifold. A conical region $\mathcal{C} = \mathcal{C}(M)$ of $TM \setminus \{0\}$ means $\mathcal{C}_x := \mathcal{C} \cap T_xM \setminus \{0\}$ is conical region for $x \in M$ ($\lambda y \in \mathcal{C}_x$ if $\lambda > 0$, $y \in \mathcal{C}_x$). A *spray* on $M$ is a smooth vector field $\mathbf{G}$ on a conical region $\mathcal{C}$ of $TM \setminus \{0\}$ (an important case is $\mathcal{C} = TM \setminus \{0\}$) expressed in a local coordinate system $(x^i, y^i)$ in $TM$ as follows:

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ for any constant $\lambda > 0$. The integral curves of $\mathbf{G}$ projected onto $M$ are the geodesics of $\mathbf{G}$.

The Riemann curvature tensor $R^i{}_k$ of a given spray $G^i$ is defined by

$$R^i{}_k := 2\partial_k G^i - y^j(\partial_j \dot{\partial}_k G^i) + 2G^j(\dot{\partial}_j \dot{\partial}_k G^i) - (\dot{\partial}_j G^i)(\dot{\partial}_k G^j), \tag{6}$$

where we define $\partial_k := \partial/\partial x^k$, $\dot{\partial}_k := \partial/\partial y^k$. The trace of $R^i{}_k$ is called the Ricci curvature, $Ric := R^i{}_i$. A spray $\mathbf{G}$ is said to be *R-flat* if $R^i{}_k = 0$. In [18], Shen defines a non-Riemannian quantity called $\chi$-curvature $\chi = \chi_i dx^i$ expressed as follows:

$$\chi_i := 2R^m{}_{i.m} + R^m{}_{m.i}. \tag{7}$$

Plugging (1) into (7) yields

$$\chi_i = (n+1)(R_{.i} - 2\tau_i). \tag{8}$$

So a spray of scalar curvature is of isotropic curvature iff. it has vanishing $\chi$-curvature.

A spray $\mathbf{G}$ is called a Berwald spray if its Berwald curvature vanishes $G^i_{hjk} := \dot{\partial}_h \dot{\partial}_j \dot{\partial}_k G^i = 0$. A spray $\mathbf{G}$ is said to be locally projectively flat if locally $G^i$ can be expressed as $G^i = Py^i$, where $P$ is a positively homogeneous local function of degree one.

In the calculation of some geometric quantities of a spray, it is very convenient to use Berwald connection as a tool. For a spray manifold $(\mathbf{G}, M)$, Berwald connection is usually defined as a linear connection on the pull-back $\pi^*TM$ ($\pi : TM \to M$ the natural projection) over the base manifold $M$. The Berwald connection is defined by

$$D(\partial_i) = (G^k_{ir} dx^r)\partial_k, \quad (G^k_{ir} := \dot{\partial}_r \dot{\partial}_i G^k),$$

For a spray tensor $T = T_i dx^i$ as an example, the horizontal and vertical derivatives of $T$ with respect to Berwald connection are given by

$$T_{i;j} = \delta_j T_i - T_r G^r_{ij}, \quad T_{i.j} = \dot{\partial}_j T_i, \quad (\delta_i := \partial_i - G^r_i \dot{\partial}_r).$$

The $hh$-curvature tensor $H_j{}^i{}_{kl}$ of Berwald connection is defined by

$$H_j{}^i{}_{kl} := \frac{1}{3}\left\{R^i{}_{l.j.k} - (k/l)\right\}, \quad H_{ij} := H_i{}^m{}_{jm}, \quad H_i := \frac{1}{n-1}(nH_{0i} + H_{i0}),$$

where $T_{ij} - (i/j)$ means $T_{ij} - T_{ji}$, and $T_0$ is defined by $T_0 := T_r y^r$, as an example. For the Ricci identities and Bianchi identities of Berwald connection, readers can refer to [1].

In this paper, we define a Finsler metric $L(\neq 0)$ on a manifold $M$ as follows (cf. [16]): (i) for any $x \in M$, $L_x$ is defined on a conical region of $T_x M \setminus \{0\}$ and $L$ is $C^\infty$; (ii) $L$ is positively homogeneous of degree two; (iii) the fundamental metric tensor $g_{ij} := (\frac{1}{2}L)_{y^i y^j}$ is non-degenerate. A Finsler metric $L$ is said to be regular if additionally $L$ is defined on the whole $TM \setminus \{0\}$ and $(g_{ij})$ is positively definite. Otherwise, $L$ is called singular. In general case, we don't require that $L$ be regular. If a Finsler metric $L > 0$, we put $L = F^2$, and in this case, $F$ is called a Finsler norm function which is positively homogeneous of degree one.

Any Finsler metric $L$ induces a natural spray whose coefficients $G^i$ are given by

$$G^i := \frac{1}{4} g^{il} \big\{ L_{x^k y^l} y^k - L_{x^l} \big\},$$

where $(g^{ij})$ is the inverse of $(g_{ij})$. $L$ is said to be of *scalar flag curvature* $K = K(x, y)$ if

$$R^i{}_k = K(L\delta^i_k - y^i y_k),$$

where $y_k := (L/2)_{.k} = g_{km} y^m$. If $K_{.i} = 0$, then $L$ is said to be of *isotropic flag curvature*, and in this case, $K$ is a constant if the dimension $n \geq 3$.

A spray $\mathbf{G}$ is (globally) Finsler metrizable on $M$ (or on $\mathcal{C}(M)$) if there is a Finlser metric $L$ defined on a conical region $\mathcal{C}(M)$ and $L$ induces $\mathbf{G}$. A spray $\mathbf{G}$ is locally Finsler metrizable on $M$ if for each $x \in M$, there is a neighborhood $U$ of $x$ such that $\mathbf{G}$ is Finsler metrizable on $U$.

Let $(M, L)$ be a two-dimensional Finsler space with the Finsler metric $L$. We use $\epsilon(= \pm 1)$ to denote the sign of the determinant of the metric matrix. We have

$$\begin{aligned} L g_{ij} &= y_i y_j + \epsilon Y_i Y_j, \\ (Y^1, Y^2) &= \big(\sqrt{\epsilon g}\big)^{-1}(-y_2, y_1), \quad (g := det(g_{ij})), \\ L Y_{i.j} &= y_j Y_i - y_i Y_j + \epsilon I Y_i Y_j, \quad L^2 C_{ijk} = I Y_i Y_j Y_k, \end{aligned} \tag{9}$$

where $(y, Y)$ with $y = (y^1, y^2)$, $Y = (Y^1, Y^2)$ is called the Berwald frame, $C_{ijk}$ is the Cartan tensor and $I$ is the main scalar. The system $L_{.i} = 2y_i$, $L\theta_{.i} = Y_i$ is integrable. It defines the so-called Landsberg angle $\theta$, which is the arc-length parameter of the indicatrix $S_x M := \{y \in T_x M | F(x, y) = 1\}$ with respect to the Riemann metric $ds^2 = g_{ij} dy^i \otimes dy^j$ on the Minkowski plane $(M_x, F_x)$ if $L$ is positively definite. When $L = F^2$, the Berwald frame is denoted by $(\ell, m)$ with $\ell := y/F, m := Y/F$ (see [1]).

**Lemma 2.1** *For a positively homogeneous function $\lambda = \lambda(x, y)$ of degree zero, it satisfies on each tangent space,*

$$L\lambda_{.i} = \lambda'(\theta) Y_i.$$

## 3 Spays of Scalar Curvature

In this section, we will introduce some basic properties of sprays with scalar curvature, and the metrizability of such sprays under certain conditions.

### 3.1 Some Basic Formulas

For a spray of scalar curvature, $R$ and $\tau_k$ in (1) are related in the following formula (10).

**Proposition 3.1** *Let* $\mathbf{G}$ *be an* $n(\geq 3)$*-dimensional spray of scalar curvature* $R^i{}_k = R\delta^i_k - \tau_k y^i$. *Then there holds*

$$R_{.i;0} - 3R_{;i} + \tau_{i;0} = 0. \tag{10}$$

*In particular, if* $\mathbf{G}$ *is of isotropic curvature, then* (10) *becomes* $\tau_{i;0} = R_{;i}$, *or* $\tau_{i;0} = \tau_{0;i}$.

***Proof*** By a Bianchy identity of Berwald connection

$$R^i{}_{jk;l} + R^i{}_{kl;j} + R^i{}_{lj;k} = 0,$$

we have

$$R^m{}_{k;m} + R^m{}_{km;0} - R^m{}_{m;k} = 0. \tag{11}$$

Since $R^i{}_k = R\delta^i_k - \tau_k y^i$, a direct computation gives

$$R^m{}_{k;m} = R_{;k} - \tau_{k;0}, \quad R^m{}_{km;0} = \frac{1}{3}\big[(n-1)(R_{.k;0} + \tau_{k;0}) + \tau_{k;0} - \tau_{m.k;0}y^m\big]. \tag{12}$$

By $\tau_0 = R$, we obtain

$$\tau_{m.k;0}y^m = R_{.k;0} - \tau_{k;0}. \tag{13}$$

Now plugging (12) and (13) into (11) we have

$$(n-2)(R_{.i;0} - 3R_{;i} + \tau_{i;0}) = 0,$$

which gives the proof. □

The isotropic case of Proposition 3.1 is given by Z. Shen in [19]. In Proposition 3.1, if $\mathbf{G}$ is Finsler metrizable induced by a Finsler metric $L$ with the flag curvature $\lambda$, we have $R = \lambda L$, $\tau_i = \lambda y_i$ and then putting them into (10) gives a known formula:

$$L\lambda_{.i;0} + 3\lambda_{;0}y_i - 3L\lambda_{;i} = 0. \tag{14}$$

If $L = F^2$ is of weakly isotropic flag curvature $\lambda = 3\theta/F + \sigma$, then (14) becomes

$$\theta_{;i} - \theta_{.i;0} + (F\sigma_{;r} + 2\theta_{;r})h^r_i = 0, \quad (h^i_j := \delta^i_j - \ell^i \ell_j).$$

For a two-dimensional spray **G** in Proposition 3.1, (10) generally does not hold. For example, if **G** is a two-dimensional spray induced by a Riemann metric $L$ of isotropic Gauss curvature $\lambda = \lambda(x)$, then (14) reduces to $\lambda_{;i} L = \lambda_{;0} y_i$, which is impossible if $\lambda$ is not constant.

In Proposition 3.1, if **G** is induced by a Finsler metric $L$ of isotropic flag curvature $\lambda$, then we have $\lambda = constant$ by (14), which is just the Schur's Theorem. But for a general spray, we cannot conclude from Proposition 3.1 that a spray of isotropic curvature in dimension $n \geq 3$ must be of constant curvature (see Examples 7.3 and 7.4 below).

The following proposition gives a useful formula on a spray manifold of scalar curvature.

**Proposition 3.2** *Let* **G** *be a spray of scalar curvature $R^i{}_k = R\delta^i_k - \tau_k y^i$, and $T$ be a homogeneous scalar function of degree $p$ satisfying $T_{;i} = 0$. Then we have*

$$RT_{.k} = pT\tau_k. \tag{15}$$

(i) *If $p = 0$ and $R \neq 0$, then $T$ is a constant.*
(ii) *If* **G** *is of isotropic curvature with $R \neq 0$, then there holds*

$$T = c|R|^{\frac{1}{2}p}, \quad (c = c(x)). \tag{16}$$

(iii) *If* **G** *is induced by a Finsler metric $L$ of non-zero flag curvature, then there holds*

$$T = c|L|^{\frac{1}{2}p}, \quad (c = constant). \tag{17}$$

***Proof*** By a Ricci identity and $R^i{}_k = R\delta^i_k - \tau_k y^i$, we have

$$0 = y^j(T_{;i;j} - T_{;j;i}) = T_{.r} R^r{}_i = RT_{.k} - y^r T_{.r}\tau_k.$$

Then we obtain (15) since $T$ is a homogeneous scalar function of degree $p$.

(i) If $p = 0$ and $R \neq 0$, then $T_{.i} = 0$ by (15). So $T$ is independent of $y$. Using $T_{;i} = 0$ again, we obtain $T = constant$.
(ii) Since **G** is of isotropic curvature, we have $\tau_k = \frac{1}{2}R_{.i}$. Putting it into (15) gives (16) since

$$\frac{T_{.i}}{T} = \frac{p}{2}\frac{R_{.i}}{R} \quad \left( \Leftrightarrow \left( \frac{T}{|R|^{\frac{1}{2}p}} \right)_{.i} = 0 \right).$$

(iii) The original version is given in [1]. Let $\lambda \neq 0$ be the flag curvature of $L$. Similarly as the proof in (i), plugging $R = \lambda L$ and $\tau_i = \frac{1}{2} L_{.i}$ into (15) gives

$$T = c|L|^{\frac{1}{2}p}, \quad c = c(x).$$

Then by $T_{;i} = 0$ and $L_{;i} = 0$ we obtain $c_{;i} = 0$, which means $c = constant$.

□

### 3.2 Some Basic Properties

Let $\mathbf{G}$ be a spray of scalar curvature satisfying (1). Then we have

$$H_h{}^i{}_{jk} = \frac{1}{3}\big[R_{.j.h}\delta^i_k - \tau_{k.j.h} y^i - \tau_{k.j}\delta^i_h - \tau_{k.h}\delta^i_j - (j/k)\big]. \tag{18}$$

By (18) we obtain

$$H_{ij} - H_{ji} = \frac{1}{3}(n+1)(\tau_{j.i} - \tau_{i.j}), \tag{19}$$

$$H_{0i} = \frac{1}{3}\big[(n-2)R_{.i} + (n+1)\tau_i\big], \quad H_{0i} = \frac{1}{3}\big[(2n-1)R_{.i} - (n+1)\tau_i\big] \tag{20}$$

$$H_{0i} - H_{i0} = -\frac{1}{3}(n+1)(R_{.i} - 2\tau_i), \tag{21}$$

$$H_i = \frac{1}{3}(n+1)(R_{.i} + \tau_i). \tag{22}$$

**Proposition 3.3** *Let $\mathbf{G}$ be a spray of scalar curvature satisfying (1).*

(i) *If $\mathbf{G}$ is of isotropic curvature, then $H_{ij} = H_{ji}$, $H_{i0} = H_{0i}$ and $H_i$ is proportional to $\tau_i$.*
(ii) *If $H_{ij} = H_{ji}$, or $R_{i0} = H_{0i}$, or $H_i$ is proportional to $\tau_i$ with $R \neq 0$, then $\mathbf{G}$ is of isotropic curvature.*

***Proof*** Firstly, we prove $R_{.i} = 2\tau_i$ is equivalent to $H_{ij} = H_{ji}$. If $R_{.i} = 2\tau_i$, then it is easy to see that $\tau_{i.j} = \tau_{j.i}$ and then by (19), we have $H_{ij} = H_{ji}$. Conversely, if $H_{ij} = H_{ji}$, then by (19) we have $\tau_{i.j} = \tau_{j.i}$. Differentiating $R = \tau_0$ by $y^i$ yields

$$R_{.i} = \tau_i + \tau_{m.i} y^m = \tau_i + \tau_{i.m} y^m = 2\tau_i.$$

Secondly, it is clear from (21) that $R_{.i} = 2\tau_i$ is equivalent to $H_{i0} = H_{0i}$.

Finally, if $R_{.i} = 2\tau_i$, it is an obvious result that $H_i$ is proportional to $\tau_i$ by (22). Conversely, if $H_i$ is proportional to $\tau_i$ and $R \neq 0$, then by (22) we get $R_{.i} = \lambda\tau_i$ for some scalar function $\lambda = \lambda(x, y)$. Contracting this by $y^i$ gives $2R = \lambda\tau_0 = \lambda R$. Since $R \neq 0$ by assumption, we have $\lambda = 2$, and thus, $R_{.i} = 2\tau_i$. □

**Proposition 3.4** *A spray* $\mathbf{G}$ *of scalar curvature is R-flat if and only if* $H_{ij} = 0$*, or* $H_{i0} = 0$*, or* $H_{0i} = 0$ *or* $H_i = 0$.

***Proof*** The Riemann curvature tensor of $\mathbf{G}$ satisfies (1). If $H_{ij} = 0$, then $H_i = 0$. So by (22) we have $R_{.i} + \tau_i = 0$. Contracting this by $y^i$ gives $3R = 0$ and so $R = 0$, $\tau_i = 0$. This shows that $\mathbf{G}$ is R-flat. If $H_{0i} = 0$ or $H_{i0} = 0$, we have $(n-2)R_{.i} + (n+1)\tau_i = 0$ or $(2n-1)R_{.i} - (n+1)\tau_i = 0$ by (20). Contracting either one by $y^i$ gives $R = 0$. Similarly we see that $\mathbf{G}$ is R-flat. □

### 3.3 Metrizability

Let $\mathbf{G}$ be a spray of scalar curvature $R^i{}_k = R\delta^i_k - \tau_k y^i$. If $\mathbf{G}$ is induced by a Finsler metric $L$, then $L$ is of scalar flag curvature with $R = \lambda L$, $\tau_k = \lambda y_k$ for some scalar function $\lambda = \lambda(x, y)$. Then if $R = 0$, then $\tau_k = 0$; if $R \neq 0$, then we can easily obtain

$$\left(\frac{\tau_i}{R}\right)_{;j} = 0, \quad \left(\frac{\tau_i}{R}\right)_{.j} = \left(\frac{\tau_j}{R}\right)_{.i}. \tag{23}$$

In [16], there are some two-dimensional sprays (of scalar curvature) satisfying the condition $R = 0$, $\tau_k \neq 0$ (also see Example 7.6 below). It is easy to check that a spray of constant curvature ($R \neq 0$) always satisfies (23). Example 7.4 below shows that the condition (23) is not sufficient for a spray of scalar curvature to be Finsler metrizable.

The condition (23) is almost sufficient for a spray of scalar curvature with $R \neq 0$ to be Finsler metrizable. In [6], Bucataru and Muzsnay show that (23) and the following non-degenerate condition:

$$\det\left(\left(\frac{\tau_i}{R}\right)_{.j} + \frac{2\tau_i\tau_j}{R^2}\right) \neq 0,$$

are necessary and sufficient for a spray of scalar curvature ($R \neq 0$) to be Finsler metrizable.

***Proof of Theorem 1.4*** Let $\mathbf{G}$ be induced by the Finsler metric $L$. Here we provide a version of proof of Theorem 1.4(ia) for $L = F^2$. Put $R = \lambda L$, $\tau_i = \lambda y_i = \lambda F \ell_i$, where $\lambda$ is the flag curvature of $F$. By (5) we have $(R_{.i} - 2\tau_i)_{.k} + (i/k) = 0$, which just is

$$F\lambda_{.i.k} + \lambda_{.k}\ell_i + \lambda_{.i}\ell_k = 0, \quad (or\ L\lambda_{.i.k} + \lambda_{.k}y_i + \lambda_{.i}y_k = 0). \tag{24}$$

Differentiating (24) by $y^j$ gives

$$F\lambda_{.i.j.k} + \lambda_{.i.k}\ell_j + \lambda_{.j.k}\ell_i + F^{-1}\lambda_{.k}h_{ij} + \lambda_{.i.j}\ell_k + F^{-1}\lambda_{.i}h_{jk} = 0, \tag{25}$$

where $h_{ij} := g_{ij} - \ell_i\ell_j$. Interchanging $i, j$ in (25) and making a subtraction, we obtain $h_{jk}\lambda_{.i} - h_{ik}\lambda_{.j} = 0$. Contracting this by $g^{jk}$ gives $(n-2)\lambda_{.i} = 0$.

(i) If $n > 2$, then we have $\lambda_{.i} = 0$. So $\lambda$ is constant by Schur's Theorem.

(ii) If $n = 2$, let $(y, Y)$ be the Berwald frame and $\theta$ be the Landsberg angle. By Lemma 2.1, we put

$$L\lambda_{.i} = \eta Y_i, \quad \eta(\theta) = \lambda'(\theta). \tag{26}$$

Differentiating (26) by $y^j$ and using (9) we obtain

$$2y_j\lambda_{.i} + L\lambda_{.i.j} = \eta_{.j}Y_i - \eta L^{-1}(y_jY_i - y_iY_j + \epsilon IY_iY_j),$$

which, by (26) again, is reduced to

$$y_i\lambda_{.j} + y_j\lambda_{.i} + L\lambda_{.i.j} = \eta_{.j}Y_i + \epsilon\eta L^{-1}IY_iY_j. \tag{27}$$

Then by (24) we have

$$L\eta_{.j} + \epsilon\eta IY_j = 0, \quad \text{or} \quad \eta' + \epsilon\eta I = 0, \quad \text{or} \quad \lambda'' + \epsilon\lambda' I = 0.$$

(iia) If $L$ is regular, then $\epsilon = 1$ and $T_xM$ is compact. There is a $\theta_0$ such that $\lambda'(\theta_0) = 0$. So we get $\eta(\theta) = \lambda'(\theta) \equiv 0$. Thus, we have $\lambda_{.i} = 0$ from (26).

(iib) It is known that all Finsler metrics with isotropic main scalar $I = I(x)$ on a two-dimensional manifold are divided into the following three classes ([1]):

$$L = c\beta^{2s}\gamma^{2(1-s)}, \quad (s = s(x) \neq 0, \quad s(x) \neq 1), \tag{28}$$

$$L = c\beta^2 e^{\frac{2\gamma}{\beta}}, \tag{29}$$

$$L = c(\beta^2 + \gamma^2)e^{2r\cdot arctan(\frac{\beta}{\gamma})}, \quad r = r(x), \tag{30}$$

where $\beta = p_i(x)y^i$ and $\gamma = q_i(x)y^i$ are two independent 1-forms, and $c \neq 0$ is constant. The main scalar $I = I(x)$ is given respectively by

$$\epsilon I^2 = \frac{(2s-1)^2}{s(s-1)}, \quad I^2 = 4, \quad I^2 = \frac{4r^2}{1+r^2}.$$

In the following, we assume that the Finsler metric $L$ is of constant main scalar, that is, $s$ in (28) and $r$ in (30) is constant. For the convenience of computation, we may put $\beta = py^1$ and $\gamma = qy^2$ under certain local coordinate system, where $p = p(x^1, x^2)$ and $q = q(x^1, x^2)$ are scalar functions. By a direct computation, we can obtain the flag curvature $\lambda$ of the Finsler metric given by (28)–(30) respectively taking $c = 1$ there for convenience). Then we can verify (5) from $R = \lambda L$ and $\tau_k = \lambda y_k$. If $L$ is given by (28), then (5) holds with

$$\omega_{12} = \frac{1-2s}{s(1-s)} \frac{(p^2qq_{12} - pq^2p_{12} + q^2p_1p_2 - p^2q_1q_2)s - p^2qq_{12} + p^2q_1q_2}{p^2q^2}.$$

If $L$ is given by (29), then (5) holds with

$$\omega_{12} = -\frac{2(p^2 q p_{22} + p q^2 p_{12} - p^2 q q_{12} + p^2 q_1 q_2 - q^2 p_1 p_2 - p^2 p_2 q_2)}{p^2 q^2}.$$

If $L$ is given by (30), then (5) holds with

$$\omega_{12} = \frac{2r}{1+r^2} \frac{(p^2 q q_{12} - p q^2 p_{12} - p^2 q_1 q_2 + q^2 p_1 p_2) r + p^2 q p_{22} + p q^2 q_{11} - q^2 p_1 q_1 - p^2 p_2 q_2}{p^2 q^2}.$$

So generally, $\omega_{12}$ in the above is not zero.

□

## 4 Sprays of Isotropic Curvature

In this section, we are going to prove Theorem 1.1 and give a metrizibility result as an application of Theorem 1.1. For this, we first introduce a simple necessary and sufficient condition for a general spray to be metrizable (see Lemma 4.3).

In the following lemmas, the horizontal and vertical covariant derivative is taken with respect to the given spray $\mathbf{G}$.

**Lemma 4.1** *Let* $\mathbf{G}$ *be a spray and* $L$ *be a Finsler function. Then we have*

$$L_{.i;0} - L_{;i} = 0 \Longleftrightarrow G^i = \frac{1}{4} g^{il} \{ L_{x^k y^l} y^k - L_{x^l} \}.$$

***Proof*** It follows from $L_{.i;0} - L_{;i} = -2G^r L_{.r.i} + L_{x^r y^i} y^r - L_{x^i}$. □

**Lemma 4.2** *Let* $\mathbf{G}$ *be a spray and* $L$ *be a Finsler function. Then we have*

$$L_{.0} = 2L, \quad L_{;i} = 0 \Longleftrightarrow L_{.0} = 2L, \quad L_{.i;0} - L_{;i} = 0.$$

***Proof*** We only need to prove "$\Longleftarrow$". By $L_{.i;0} = L_{;i}$ we have $L_{;0.i} - L_{;i} = L_{;i}$, or $L_{;0.i} = 2L_{;i}$. Further, $L_{.i;0} = L_{;i}$ implies $L_{.0;0} = L_{;0}$. So by $L_{.0} = 2L$ we obtain $L_{;0} = 0$. Thus, it follows from $L_{;0.i} = 2L_{;i}$ that $L_{;i} = 0$. □

**Lemma 4.3** *A spray* $\mathbf{G}$ *is Finsler metrizable if and only if there is a Finsler function* $L$ *satisfies* $L_{;i} = 0$. *In this case,* $\mathbf{G}$ *is induced by* $L$.

### 4.1 Formal Integrability

To prove Theorem 1.1(i), we need a theory on Spencer's technique of formal integrability for linear partial differential systems. Here we only give some basic notions for this theory and more details are refereed to [3, 10].

Let $B$ be a vector bundle over an $n$-dimensional manifold $M$, and denote by $J_k B$ the bundles of $k$th-order jets of the sections of $B$. For two vector bundles $B_1$, $B_2$ over $M$, consider $P : Sec(B_1) \to Sec(B_2)$, which is a linear partial differential operator of order $k$. $P$ can be identified with a map $p_0(P) : J_k B_1 \to B_2$, a morphism of vector bundles over $M$. We also denote by $p_l(P) : J_{k+l} B_1 \to J_l B_2$ the morphisms of vector bundles over $M$, which is called the $l$th-order jet prolongation of $P$. Let $R_{k+l,x}(P) := Ker p_l(P)_x$ be the space of $(k+l)$th-order formal solutions of $P$ at a point $x \in M$. The operator $P$ is said to be formally integrable at $x \in M$, if $R_{k+l}(P)$ is a vector bundle for all $l \geq 0$ and the projection $\pi_{k+l,x}(P) : R_{k+l,x}(P) \to R_{k+l-1,x}(P)$ is onto for all $l \geq 1$.

Let $\sigma_k(P) : S^k(T^*M) \otimes B_1 \to B_2$ be the symbol of $P$, which is defined by the highest order term of $P$, and let $\sigma_{k+l}(P) : S^{k+l}(T^*M) \otimes B_1 \to S^l(T^*M) \otimes B_2$ be the symbol of the $l$th-order prolongation of $P$. Define

$$\begin{aligned} g_{k,x}(P) : \quad &= Ker\ \sigma_{k,x}(P), \\ g_{k,x}(P)_{e_1 \dots e_j} : \quad &= \{A \in g_{k,x}(P) | i_{e_1} A = \dots = i_{e_j} A = 0\}, \quad 1 \leq j \leq n, \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is a basis of $T_x M$. Such a basis is said to be quasi-regular if it satisfies

$$dim\ g_{k+1,x}(P) = dim\ g_{k,x}(P) + \sum_{j=1}^{n} dim\ g_{k,x}(P)_{e_1 \cdots e_j}.$$

The symbol $\sigma_k(P)$ is said to be involutive at $x \in M$ if there exists a quasi-regular basis of $T_x M$. For the proof of Theorem 1.1(i), we need the following theorem and lemma ([10]).

**Theorem 4.4** (Cartan–Kahler) *Let $P$ be a regular linear partial differential operator of order $k$. If $\pi_{k+1,x}(P) : R_{k+1,x}(P) \to R_{k,x}(P)$ is onto and the symbol $\sigma_k(P)$ is involutive, then $P$ is formally integrable.*

**Lemma 4.5** *For two vector bundles $B_1$, $B_2$ over $M$, let $P : Sec(B_1) \to Sec(B_2)$ be a regular linear partial differential operator of order $k$. Then $\pi_{k+1,x}(P) : R_{k+1,x}(P) \to R_{k,x}(P)$ is onto iff.*

$$P(s)_x = 0 \Longrightarrow (DP(s))_x = \sigma_{k+1}(P)(A) :$$

*for some $A \in S^{k+1}(T_x^*M) \otimes B_1$, where $D$ is an arbitrary linear connection of the bundle $B_2$ over the base manifold $M$.*

### 4.2 Proof of Theorem 1.1(i)

Let $T_v^*TM$ denote the subbundle of $T^*TM$, in which, if $\omega \in T_v^*TM$, then $\omega$ can be written locally as $\omega = \omega_i(x, y)dx^i$, and $\wedge^2 T_v^*TM$ the subbundle of $T^*TM$, every element $\omega$ of which is locally in the form $\omega = \omega_{ij}(x, y)dx^i \wedge dx^j$ with $\omega_{ij} = -\omega_{ji}$.

Besides, we designate $S^k(T^*TM)$ as the bundle of symmetric $k$-forms over $TM$. For an $n$-dimensional manifold $M$, let $B_1$, $B_2$ be two vector bundles over $TM$ with

$$B_1 := T_v^*TM, \quad B_2 := T_v^*TM \oplus \wedge^2 T_v^*TM \oplus (T_v^*TM \otimes T_v^*TM).$$

We define a linear partial differential operator $P : Sec(B_1) \to Sec(B_2)$ in component form as follows

$$P(\theta_i) = (\theta_{i.0}, \theta_{i.j} - \theta_{j.i}, \theta_{i;j}). \tag{31}$$

**Lemma 4.6** *For the operator $P$ in (31), the symbol $\sigma_1(P)$ is involutive.*

***Proof*** We are going to prove that $\{\dot{\partial}_1, \ldots, \dot{\partial}_n, \delta_1, \ldots, \delta_n\}$ is a quasi-regular basis of $P$.

By definition, for $A = (A_{ji}, A_{\underline{j}i}) (= A_{ji} dx^j \otimes dx^i + A_{\underline{j}i} \delta y^j \otimes dx^i) \in T^*TM \otimes B_1$, we have

$$\sigma_1(P)A = (A_{\underline{0}i}, A_{\underline{j}i} - A_{\underline{i}j}, A_{ji}) \in B_2.$$

Assume $\sigma_1(P)(A) = 0$. Then for the computation of $dim(g_1(P))$, we see that $A_{\underline{0}i} = 0$ and $A_{\underline{i}j} = A_{\underline{j}i}$ together contribute the number $(n-1)n/2$, and $A_{ij} = 0$ gives 0. Therefore, we obtain

$$dim(g_1(P)) = \frac{(n-1)n}{2}. \tag{32}$$

Now with respect to the basis $\{dx^i, \delta y^i\}$, an element $B \in S^2(T^*TM) \otimes B_1$ can be expressed as follows:

$$B = (B_{ijk},\ B_{\underline{i}jk},\ B_{i\underline{j}k},\ B_{\underline{i}\underline{j}k})$$
$$\big(B_{ijk} = B_{jik}, B_{\underline{i}jk} = B_{j\underline{i}k}, B_{\underline{i}\underline{j}k} = B_{\underline{j}\underline{i}k}\big).$$

By definition, we have

$$\sigma_2(P)B = (B_{i\underline{0}k}, B_{\underline{i}\underline{0}k}; B_{i\underline{j}k} - B_{i\underline{k}j}, B_{\underline{i}\underline{j}k} - B_{\underline{i}\underline{k}j}; B_{ijk}, B_{\underline{i}jk}).$$

Assume $\sigma_2(P)(B) = 0$. Then $B_{ijk}$, $B_{i\underline{j}k}$ and $B_{\underline{i}jk}$ gives 0 to $dim(g_2(P))$. By the fact that $B_{\underline{i}\underline{j}k}$ is symmetric in $i, j, k$ satisfying additional condition: $B_{\underline{i}\underline{0}k} = 0$, we obtain the number

$$\sum_{k=0}^{n-2} \frac{(n-k-1)(n-k)}{2}.$$

to $dim(g_2(P))$. So altogether, we have

$$dim(g_2(P)) = \sum_{k=0}^{n-2} \frac{(n-k-1)(n-k)}{2}. \tag{33}$$

Next we verify under the basis $\{\dot{\partial}_1, \dots, \dot{\partial}_n, \delta_1, \dots, \delta_n\}$ at a point $(x, y) \in TM$,

$$dim(g_2(P)) = dim\ g_1(P) + \sum_{j=1}^{n} dim\ g_1(P)_{\dot{\partial}_1 \cdots \dot{\partial}_j} + \sum_{j=1}^{n} dim\ g_1(P)_{\dot{\partial}_1 \cdots \dot{\partial}_n \delta_1 \cdots \delta_j}. \tag{34}$$

By a direct computation, we see

$$dim\ g_1(P)_{\dot{\partial}_1 \dots \dot{\partial}_j} = \frac{(n-j-1)(n-j)}{2}, \quad dim\ g_1(P)_{\dot{\partial}_1 \dots \dot{\partial}_n \delta_1 \dots \delta_j} = 0, \quad (1 \le j \le n).$$

Plugging them into (34) and using (32), (33), we see that both sides of (34) are equal. Therefore, $\sigma_1(P)$ is involutive. □

**Lemma 4.7** *For the operator $P$ in (31), a first-order solution of $P(\theta) = 0$ can be lifted into a second-order solution iff.*

$$\theta_r H_i{}^r{}_{jk} + \theta_{i.r} R^r{}_{jk} = (\theta_r R^r{}_{jk})_{.i} = 0. \tag{35}$$

***Proof*** If $P(\theta) = 0$, then we have $\theta_{i;j} = \theta_{j;i}$. Then by a Ricci identity and $\theta_{i.j} = \theta_{j.i}$, it gives (35):

$$0 = \theta_{i;j;k} - \theta_{i;k;j} = -\theta_r H_i{}^r{}_{jk} - \theta_{i.r} R^r_{jk} = -(\theta_r R^r{}_{jk})_{.i}.$$

Conversely, suppose that (35) holds for a $\theta$ satisfying $P(\theta) = 0$ with $\theta_0 \neq 0$. Let $D$ be the Berwald connection of the bundle $\pi^* TM$ over the base manifold $TTM$. Then $D$ can be naturally extended to the bundle $B_2$. We use Lemma 4.5. In component form, we have

$$DP(\theta_i) = (\theta_{i.0;j},\ \theta_{i.0.j},\ \ \theta_{i.j;k} - \theta_{j.i;k},\ \theta_{i.j.k} - \theta_{j.i.k},\ \ \theta_{i;j;k},\ \theta_{i;j.k}), \tag{36}$$

where $\theta$ ($\theta_0 \neq 0$) satisfies $P(\theta) = 0$ at a point $w = (x, y) \in TM$, that is, at the point $w$ there holds

$$\theta_{i.0} = 0, \quad \theta_{i.j} = \theta_{j.i}, \quad \theta_{i;j} = 0. \tag{37}$$

Next we are going to prove that there is an $A_{\alpha\beta i} \in S^2(T_w^* TM) \otimes B_1$ satisfying

$$DP(\theta_i) = \sigma_2(A_{\alpha\beta i}) = (A_{\alpha \underline{r} i} y^r,\ \ A_{\alpha \underline{j} i} - A_{\alpha \underline{i} j},\ \ A_{\alpha j i}). \tag{38}$$

By (36), we see that (38) is equivalent to

① $A_{j\underline{r}i}y^r = \theta_{i.r;j}y^r$, ② $A_{\underline{j}\underline{r}i}y^r = \theta_{i.0.j} = \theta_{i.r.j}y^r + \theta_{i.j}$,

③ $A_{k\underline{j}i} - A_{k\underline{i}j} = \theta_{i.j;k} - \theta_{j.i;k}$, ④ $A_{\underline{k}\underline{j}i} - A_{\underline{k}\underline{i}j} = \theta_{i.j.k} - \theta_{j.i.k}$,

⑤ $A_{kji} = \theta_{i;j;k}$, ⑥ $A_{\underline{k}ji} = \theta_{i;j.k}$.

In the following, we will construct $A_{\alpha\beta i}$ which satisfies the above six relations. Put

$$\begin{aligned} A_{\underline{j}\underline{k}i} &= \theta_{i.k.j} + \theta_0^{-1}\big[\theta_{i.j}\theta_k + (i, j, k)\big], \\ A_{j\underline{k}i} &= \theta_{i.k;j} - \theta_r G^r_{ijk}, \qquad A_{\underline{k}ji} = \theta_{i;j.k}, \\ A_{kji} &= \theta_{i;j;k}. \end{aligned}$$

Firstly, $A_{\underline{j}\underline{k}i}$ is symmetric in $j, k$ by $\theta_{i.j} = \theta_{j.i}$ in (37) and it also satisfies ②④ by $\theta_{i.j} = \theta_{j.i}, \theta_{i.0} = 0$ in (37). Next, by a Ricci identity of Berwald connection, we see that

$$A_{\underline{k}ji} - A_{j\underline{k}i} = \theta_{i;j.k} - \theta_{i.k;j} + \theta_r G^r_{ijk} = 0,$$

which gives $A_{\underline{k}ji} = A_{j\underline{k}i}$. It is also clear that $A_{\underline{k}ji}$ and $A_{j\underline{k}i}$ satisfy ①③⑥. Finally, $A_{kji}$ is symmetric in $j, k$, and satisfies ⑤ by a Ricci identity of Berwald connection and (35). So (38) holds. This finishes the proof of the lemma. □

Now in Theorem 1.1(i), we have $R = 0$. So (35) automatically holds. It follows from Lemmas 4.6, 4.7 and then Theorem 4.4, the operator $P$ is formally integrable, that is, for each point $u_0 := (x^i_0, y^i_0)$, there exist a neighborhood $U$ of $u_0$ and a analytic $\theta$ defined on $U$ such that $P(\theta) = 0$.

Under the basis $(\delta_i, \dot{\partial}_i)$, the local coordinate of $J_1B_1$ is expressed as $(x^i, y^i, \theta_i, \theta_{i\underline{j}}, \theta_{ij})$. An initial data $(x^i_0, y^i_0, \theta^0_i, \theta^0_{i\underline{j}}, \theta^0_{ij})$ satisfied by the operator $P$ means $\theta^0_{i\underline{0}} = 0, \theta^0_{i\underline{j}} = \theta^0_{j\underline{i}}, \theta^0_{ij} - \theta^0_r G^r_i = 0$. Further, we let the initial data satisfy $\theta^0_i y^i_0 > 0$ and $Rank(\theta^0_{i\underline{j}}) = n - 1$.

Now for the above analytic solution $\theta$ of $P$ which is defined on a neighborhood $U$ of $u_0$ and satisfies the above initial data, we obtain a local metric $F := \theta_0$ defined on $U$. To prove that $F$ is a Finsler metric, we need the following lemma which can be proved by an elementary discussion in linear algebra.

**Lemma 4.8** *Let $F$ be positively homogeneous of degree one with $F(y) \neq 0$ at a point $y$. Then $g_{ij} := \frac{1}{2}(F^2)_{.i.j}$ is non-degenerate at $y$ iff. $Rank(F_{.i.j}) = n - 1$ at $y$.*

Now for $F = \theta_0$, we have $F_{.i.j} = \theta_{i.j}$. Then at $u_0$, we have $Rank(F_{.i.j}) = Rank(\theta^0_{i\underline{j}}) = n - 1$. So by the above lemma, $g_{ij}$ is non-degenerate at $u_0$. By continuity, $g_{ij}$ is non-degenerate in $U$ (when it is small enough). Thus, we obtain a Finsler metric $F$ with each $F_x$ defined on the conical region formed by $y = 0$ and $\{y|(x, y) \in U\}$. Further, $F$ satisfies $F_{;i} = 0$, which means that the spray $\mathbf{G}$ in Theorem 1.1 is induced by $F$ by Lemma 4.3. This completes the proof of Theorem 1.1(i). □

***Remark 4.9*** The idea of the proof of Theorem 1.1(i) can be referred to that in [4] for the formal integrability of the operator $P_1(\theta) := (\theta_{i.0}, \theta_{i.j} - \theta_{j.i}, \delta_i\theta_j - \delta_j\theta_i)$. On the other hand, a suitable change of the proof of Theorem 1.1(i) can give the proof for the formal integrability of the operator $P_1$ in [4] (where actually we can redefine $P_1$ as $\bar{P}_1 := (\theta_{i.0}, \theta_{i.j} - \theta_{j.i}, \theta_{j;i} - \theta_{i;j})$). Besides, we may also consider the system $F_{.0} = F,\ F_{;i} = 0$ for a possible proof (cf. [15] for a more general discussion).

### 4.3 Proof of Theorem 1.1(ii) and (iii)

(ii) Assume that $R \neq 0$ is not a Finsler metric. If $\mathbf{G}$ is Finsler metrizable induced by a Finsler metric $F$, then $F$ is of isotropic curvature $\lambda \neq 0$. By $R = \lambda F^2$, we see that $R$ is a Finsler metric, which gives a contradiction.

(iii) Assume that $R$ is a Finsler metric. If $\mathbf{G}$ is Finsler metrizable induced by a Finsler metric $L$, then $L$ is of isotropic curvature $\lambda = \lambda(x) \neq 0$. By $R = \lambda L$, we have $R_{;i} = \lambda_{;i} L$. Thus, we obtain

$$R_{;i} = \frac{\lambda_{;i}}{\lambda} R = R(\ln|\lambda|)_{;i}.$$

Let $\omega_i := (\ln|\lambda|)_{;i}$. Then $\omega$ is closed and $R_{;i} = R\omega_i$. Conversely, if $R_{;i} = R\omega_i$ for some closed 1-form $\omega = \omega_i(x)dx^i$, then locally there is a scalar function $\lambda = \lambda(x) \neq 0$ such that $\omega_i = (\ln|\lambda|)_{;i}$. It is easy to check that $R_{;i} = R(\ln|\lambda|)_{;i}$ is equivalent to $(R/\lambda)_{;i} = 0$. Therefore, $\mathbf{G}$ is Finsler metrizable induced by the Finsler metric $L := R/\lambda$ by Lemma 4.3. By $R = \lambda L$, we see that $L$ is of isotropic flag curvature $\lambda$. If $\omega = 0$, we may choose $\lambda = 1$, and then the Finsler metric $L = R$ is of constant flag curvature $\lambda = 1$. If $n \geq 3$, then the Finsler metric $L$ is of constant flag curvature $\lambda$ by Schur's theorem, which gives $R_{;i} = 0$. □

### 4.4 A Metrizability Result

As an application of Theorem 1.1, we show the following theorem.

**Theorem 4.10** *Let $G^i$ be the spray of a Finsler metric $F$ of constant flag curvature $\lambda$ and $\bar{G}^i$ be a spray defined by $\bar{G}^i = G^i + cFy^i$ for a constant $c$. Then $\bar{\mathbf{G}}$ is (locally) Finsler metrizable iff. $\lambda = -c^2$ or $c = 0$. When $\lambda = -c^2$, $\bar{G}^i$ is locally induced by a Finsler metric of zero flag curvature.*

***Proof*** The Riemann curvature ${R^i}_k$ of $\mathbf{G}$ is given by

$${R^i}_k = \lambda(F^2\delta^i_k - FF_{.k}y^i).$$

Then by a direct computation, the Riemann curvature ${\bar{R}^i}_k$ of $\bar{\mathbf{G}}$ is given by

$${\bar{R}^i}_k = \bar{R}\delta^i_k - \bar{\tau}_k y^i, \quad \big(\bar{R} := (\lambda + c^2)F^2, \quad \bar{\tau}_k := (\lambda + c^2)FF_{.k}\big).$$

So $\bar{\boldsymbol{G}}$ is of isotropic curvature since $\bar{R}_{.i} = 2\bar{\tau}_i$. Further, we have

$$\bar{R}_{\bar{;}i} = \bar{R}_{;i} + \bar{R}_{.r}(cF_{.i}y^r + cF\delta_i^r) = 4c(\lambda + c^2)F^2F_{.i}.$$

Assume that $\bar{\boldsymbol{G}}$ is Finsler metrizable. If $c(\lambda + c^2) \neq 0$, then $\bar{R}$ is a Finsler metric. By Theorem 1.1(iii), we have $\bar{R}_{\bar{;}i} = \bar{R}\omega_i$ for some closed 1-form $\omega = \omega_i(x)dx^i$. But clearly this does not hold. Therefore, we have $c = 0$ or $\lambda = -c^2$. Conversely, if $c = 0$, then $\bar{\boldsymbol{G}}$ is induced by $F$. If $\lambda + c^2 = 0$, then $\bar{\boldsymbol{G}}$ has zero Riemann curvature. So $\bar{\boldsymbol{G}}$ is (locally) Finsler metrizable by Theorem 1.1(i). □

Theorem 4.10 is a generalization of a result in [20], where we have an additional condition that $F$ is projectively flat.

## 5 Sprays of Constant Curvature

In this part, we introduce a new notion: a spray of *constant curvature*, which is a generalization of a Finsler metric of constant flag curvature. For this new notion, some basic properties for Finsler metrics still remain unchanged for sprays (see Theorem 5.2 below).

**Definition 5.1** A spray $\mathbf{G}$ of scalar curvature ${R^i}_k = R\delta^i_k - \tau_k y^i$ is said to be of *constant curvature* if $\tau_{i;j} = 0$.

The following theorem gives some basic properties for sprays of constant curvature.

**Theorem 5.2** *A spray has the following properties on constant curvature:*

(i) *A spray of scalar curvature is of constant curvature iff. its Riemann curvature is zero or its Ricci curvature $Ric$ satisfies $Ric_{;i} = 0 (Ric \neq 0)$.*
(ii) *A spray of constant curvature must be of isotropic curvature.*
(iii) *A Finsler metric is of constant flag curvature iff. its spray is of constant curvature.*
(iv) *An $n$-dimensional spray of isotropic curvature is not necessarily of constant curvature even for $n \geq 3$.*

A Finsler metric has the same conclusions as shown for sprays in Theorem 5.2(i)(ii). Meanwhile, Theorem 5.2(iv) shows a different property of sprays from that of Finsler metrics.

To prove Theorem 5.2, we first show the following lemma.

**Lemma 5.3** *Let $\mathbf{G}$ be a spray of scalar curvature ${R^i}_k = R\delta^i_k - \tau_k y^i$. Then we have*

$$R_{;i} = 0 \ (R \neq 0) \implies R_{.i} = 2\tau_i, \tag{39}$$

$$\tau_{i;k} = 0 \implies R = \tau_k = 0 \ \text{ or } \ R_{;i} = 0 \ (R \neq 0), \tag{40}$$

$$R_{;i} = 0 \ (R \neq 0) \implies \tau_{i;k} = 0. \tag{41}$$

***Proof*** Assume $R_{;i} = 0 (R \neq 0)$. We have $RR_{.i} = 2R\tau_i$ from (15), where we have put $T = R$ with $p = 2$ in (15). This gives $R_{.i} = 2\tau_i$ since $R \neq 0$, which gives the proof of (39).

Assume $\tau_{i;k} = 0$. We have $R_{;i} = 0$ since $\tau_0 = R$ and then $R_{;i} = \tau_{0;i} = \tau_{m;i} y^m = 0$. Now we prove that if $R = 0$, then $\tau_i = 0$. By a Ricci identity we obtain

$$0 = y^j(\tau_{i;j;k} - \tau_{i;k;j}) = y^j(-\tau_r H_i{}^r{}_{jk} - \tau_{i.r} R^r{}_{jk}) = -\tau_r H_i{}^r{}_{0k} - \tau_{i.r} R^r{}_k. \quad (42)$$

Now by $R^i{}_k = R\delta^i_k - \tau_k y^i$ we have

$$H_i{}^r{}_{jk} = \frac{1}{3}\big[R_{.j.i}\delta^r_k - \tau_{k.j.i} y^r - \tau_{k.j}\delta^r_i - \tau_{k.i}\delta^r_j - (j/k)\big]. \quad (43)$$

Plugging (43), $R^i{}_k = R\delta^i_k - \tau_k y^i$, $\tau_{i;j} = 0$ and $R = 0$ into (42) we obtain $\tau_i \tau_k = 0$, which gives $\tau_i = 0$. This gives the proof of (40).

Assume $R_{;i} = 0 (R \neq 0)$. By (39) we have $R_{.i} = 2\tau_i$. So $\tau_{i;k} = \frac{1}{2}R_{.i;k} = \frac{1}{2}R_{;k.i} = 0$. This gives the proof of (41). □

***Proof of Theorem 5.2*** (i) Assume that **G** is of constant curvature. Then we have $R^i{}_k = R\delta^i_k - \tau_k y^i$ with $\tau_{i;k} = 0$ by definition. By (40) we immediately obtain the desired conclusion since $Ric = (n-1)R$. Conversely, let the Riemann curvature be zero or the Ricci curvature $Ric$ satisfy $Ric_{;i} = 0 (Ric \neq 0)$. If $R^i{}_k = 0$, then $\tau_i = 0$ and so $\tau_{i;k} = 0$. If $Ric_{;i} = 0 (Ric \neq 0)$, then we have $R_{;i} = 0$ $(R \neq 0)$. So by (39) we have $R_{.i} = 2\tau_i$. Thus, we obtain $\tau_{i;k} = \frac{1}{2}R_{.i;k} = \frac{1}{2}R_{;k.i} = 0$. By definition, **G** is of constant curvature.

(ii) It follows directly from (40) and (39), and the definitions for a spray of isotropic curvature and constant curvature.

(iii) If a Finsler metric $L$ is of constant flag curvature $\lambda$, then its spray has the Riemann curvature $R^i{}_k = \lambda(L\delta^i_k - y_k y^i)$. So its spray is of scalar curvature $R^i{}_k = R\delta^i_k - \tau_k y^i$ with $R = \lambda L$ and $\tau_k = \lambda y_k$. Thus, we have $\tau_{i;k} = 0$, which implies that the spray is of constant curvature. Conversely, if the spray **G** of a Finsler metric $L$ is of constant curvature, then the Riemann curvature of **G** has the form $R^i{}_k = R\delta^i_k - \tau_k y^i$ with $\tau_{i;k} = 0$. So $L$ is of scalar flag curvature (put the flag curvature as $\lambda$). We have $R = \lambda L$ and $\tau_k = \lambda y_k$. Thus, we get $\lambda_{;k} = 0$ by $\tau_{i;k} = 0$. If $\lambda = 0$, then $L$ has constant flag curvature 0. If $\lambda \neq 0$, then since $\lambda$ is a homogeneous function of degree 0 satisfying $\lambda_{;k} = 0$, we immediately obtain $\lambda = constant$ by putting $T = \lambda$, $p = 0$ in (17) of Proposition 3.2.

(iv) See Examples 7.3 and 7.4 below.

□

***Proof of Theorem 1.2*** The spray **G** has the Riemann curvature $R^i{}_k = R\delta^i_k - \tau_k y^i$ with $\tau_{i;k} = 0$ (note that $Ric = (n-1)R$). By Theorem 5.2(ii), the spray **G** is of isotropic curvature since **G** is of constant curvature.

If **G** is (locally) Finsler metrizable, then by Theorem 1.1, it is clear that there has $Ric = 0$ or $Ric$ is a Finsler metric when $Ric \neq 0$.

Conversely, if $Ric = 0$, then by Theorem 1.1(i), $\mathbf{G}$ is (locally) Finsler metrizable. If $Ric \neq 0$, then by assumption $Ric$ is a Finsler metric. It follows from Theorem 5.2(i) that $Ric_{;i} = 0$. By Theorem 1.1(iii), $\mathbf{G}$ is (locally) Finsler metrizable. □

# 6 Locally Projectively Flat Sprays

In this section, we consider the properties and metrizability of locally projectively flat sprays on a manifold. A locally projectively flat spray is always of scalar curvature. Locally, we let $\mathbf{G}$ be a projectively flat spray with $G^i = Py^i$ defined on an open set of $R^n$. Then by (6), the Riemann curvature tensor $R^i{}_k$ is in the form $R^i{}_k = R\delta^i_k - \tau_k y^i$ with

$$R = P^2 - P_{x^r} y^r, \quad \tau_k = P P_{y^k} + P_{x^r y^k} y^r - 2P_{x^k}. \tag{44}$$

## 6.1 Some Basic Results

**Lemma 6.1** *Let $G^i = Py^i$ be a spray defined on an open set of $R^n$. Then (44) becomes*

$$R = -P^2 - P_{;0}, \quad \tau_k = P_{k;0} - 2P_{;k} - PP_k, \quad (P_k := P_{.k}). \tag{45}$$

**Lemma 6.2** *In (45), $R = 0$ and $\tau_k = 0 \Longleftrightarrow P_{;k} + PP_k = 0$. So the spray $G^i = Py^i$ has vanishing Riemann curvature iff. $P_{;k} + PP_k = 0$.*

***Proof*** If $R = 0$ and $\tau_k = 0$, then $P_{;0} = -P^2$ implies $P_{;k} + P_{k;0} = -2PP_k$. Further by $P_{k;0} = 2P_{;k} + PP_k$ we obtain $P_{;k} + PP_k = 0$. Conversely, let $P_{;k} + PP_k = 0$. Then we have $P_{;0} = -P^2$ $(R = 0)$. Now again $P_{;0} = -P^2$ implies $P_{;k} + P_{k;0} = -2PP_k$, which implies $\tau_k = 0$ by $P_{;k} + PP_k = 0$. □

By Theorem 1.1(i) and Lemma 6.2, a spray $G^i = Py^i$ satisfying $P_{;k} + PP_k = 0$ is locally Finsler metrizable (also see [12, 17]). By (45) we can easily obtain the following lemma.

**Lemma 6.3** *Let $G^i = Py^i$ be a spray with $R^i{}_k = R\delta^i_k - \tau_k y^i$. Then we have*

$$R_{.i} - 2\tau_i = 3(P_{;i} - P_{i;0}). \tag{46}$$

*So $\mathbf{G}$ is of isotropic curvature iff. $P_{;i} = P_{i;0}$.*

By Lemma 6.3, we can easily obtain the following Proposition 6.4.

**Proposition 6.4** *Let $G^i = Py^i$ be a Berwald spray. Then $\mathbf{G}$ is of isotropic curvature iff. $P$ is a (local) exact 1-form given by $P = \sigma_{x^i} y^i$ for a scalar function $\sigma = \sigma(x)$.*

The following Proposition 6.5 directly follows from Lemma 6.3 and Theorem 1.1.

**Proposition 6.5** *Let $G^i = Py^i$ $(P \neq 0)$ be a spray defined on an open set of $R^n$ with $P_{;i} = 0$. Then $\mathbf{G}$ is of isotropic curvature. Further $\mathbf{G}$ is Finsler metrizable iff. $P$ is a Finsler metric, and in this case, $P^2$ is a Finsler metric of constant flag curvature $-1$.*

By Theorem 1.4(i) and (46) we have the following Proposition 6.6 (cf. [13]).

**Proposition 6.6** *For an $n(\geq 3)$-dimensional Berwald spray* $\mathbf{G}$ *with* $G^i = Py^i$*, if* $\mathbf{G}$ *is not of isotropic curvature (or equivalently,* $P$ *is not a closed 1-form), then* $\mathbf{G}$ *is not Finsler metrizable.*

### 6.2 Proof of Theorem 1.3

Now we give the proof of Theorem 1.3 as follows.

If the spray $\mathbf{G}$ is given by (2) (defined on $\mathcal{C}(U)$), then $\mathbf{G}$ is Finsler metrizable (on $\mathcal{C}(U)$) since it is easy to check that the metric $L$ given by (3) induces the spray $\mathbf{G}$. Further, a direct computation shows that the Ricci curvature $Ric$ of $\mathbf{G}$ is equal to $(n-1)L^*$ (on $\mathcal{C}(U)$) since $\mathbf{G}$ is given by (2). So $L := Ric/(n-1)$ is a Riemann metric and $L$ induces $\mathbf{G}$.

Conversely, suppose that the spray $\mathbf{G}$ is Finsler metrizable (on $\mathcal{C}(U)$). Since $\mathbf{G}$ is a projectively flat Berwald spray of isotropic curvature, it follows from Proposition 6.4 that $\mathbf{G}$ has the form $G^i = \sigma_0 y^i$, where $\sigma_i = \sigma_{x^i}$ is the differential of some scalar function $\sigma = \sigma(x)$. Since $Ric \neq 0$ and $\mathbf{G}$ is Finsler metrizable, $Ric$ is a Finsler metric and $Ric_{;i} = 0$ by theorem 1.1 and the fact that a locally projectively flat Finsler metric of isotropic flag curvature is of constant flag curvature ([2]). Since $Ric = (n-1)[-(\sigma_0)^2 - \sigma_{0;0}]$ (see (45)) and $Ric_{;i} = 0$, we have

$$\left[(\sigma_0)^2 + \sigma_{0;0}\right]_{;i} = 0. \tag{47}$$

Now we only need to solve the scalar function $\sigma$ from (47). It is easy to see that (47) is equivalent to

$$(\sigma_j\sigma_k + \sigma_{j;k})_{;i} = 0. \tag{48}$$

Now by $G^i = \sigma_0 y^i$ we have

$$G^i_j = \sigma_j y^i + \sigma_0\delta^i_j, \quad G^i_{jk} = \sigma_j\delta^i_k + \sigma_k\delta^i_j.$$

For convenience, we put $\sigma_{ij} := \sigma_{x^ix^j}, \sigma_{ijk} := \sigma_{x^ix^jx^k}$. By a direct computation we obtain

$$\begin{aligned} \sigma_{i;j} &= \sigma_{ij} - 2\sigma_i\sigma_j, \quad \sigma_i\sigma_j + \sigma_{i;j} = \sigma_{ij} - \sigma_i\sigma_j, \\ (\sigma_j\sigma_k + \sigma_{j;k})_{;i} &= \sigma_{ijk} - 2(\sigma_i\sigma_{jk} + \sigma_j\sigma_{ik} + \sigma_k\sigma_{ij}) + 4\sigma_i\sigma_j\sigma_k. \end{aligned} \tag{49}$$

Putting $u = e^{-2\sigma}$, it follows from (49) that (48) is equivalent to $u_{ijk} = 0$. So $u$ is a polynomial in $(x^i)$ of degree two. Thus, the scalar function $\sigma$ satisfying (48) is given by

$$\sigma(x) = -\frac{1}{2}\ln|\langle Ax, x\rangle + \langle B, x\rangle + C|, \tag{50}$$

where $A$ is a constant symmetric matrix, $B$ is a constant vector and $C$ is a constant number. By $G^i = \sigma_0 y^i$ and (50), we see that $\mathbf{G}$ is locally given by (2). Meanwhile, by (50), we have

$$-(\sigma_0)^2 - \sigma_{0;0} = \frac{4(\langle Ax, x\rangle + \langle B, x\rangle + C)\langle Ay, y\rangle - (2\langle Ax, y\rangle + \langle B, y\rangle)^2}{4(\langle Ax, x\rangle + \langle B, x\rangle + C)^2}. \quad (51)$$

Since $R(= -(\sigma_0)^2 - \sigma_{0;0}) \neq 0$ by assumption, by Theorem 1.1, $R$ is a metric, which means that $A, B, C$ should satisfy certain condition such that $L$ given by (3) is a metric.

A simple computation shows that if $det(A) \neq 0$, then $L$ given by (3) is a metric iff. the constant quantities $A, B, C$ satisfy $4C \neq B' A^{-1} B$. □

***Remark 6.7*** If $-(\sigma_0)^2 - \sigma_{0;0} = 0$, we can obtain $\sigma = -\ln|\langle B, x\rangle + C|$. In this case, $\mathbf{G}$ has vanishing Riemann curvature, and it can be locally induced by a Finsler metric $L$. It is clear that $L$ is a Minkowskin metric.

The above proof and Remark 6.7 have actually given the proof of the following theorem for the local structure of a locally projectively flat Berwald spray with constant curvature.

**Theorem 6.8** *Let* $\mathbf{G}$ *be a locally projectively flat Berwald spray on a manifold* $M$. *Then* $\mathbf{G}$ *is of constant curvature if and only if* $\mathbf{G}$ *can be locally expressed as* (2).

## 7 Examples

As an application of Theorem 1.1, Theorem 1.3 gives the necessary and sufficient condition for the Berwald spray $G^i = (\sigma_{x^r} y^r) y^i$ to be Finsler metrizable. In this section, we are going to give more examples to support some of the basic theories we introduce in the above sections.

***Example 7.1*** Let $\mathbf{G}$ be a two-dimensional Berwald spray given by

$$G^1 = f(x^1)[(y^1)^2 - (y^2)^2], \qquad G^2 = 2f(x^1)y^1y^2,$$

where $f(x^1)$ is a non-constant function. It can be directly verified that $\mathbf{G}$ is of isotropic curvature $R^i{}_k = R\delta^i_k - \frac{1}{2} R_{.k} y^i$ satisfying

$$R = -2f'(x^1)[(y^1)^2 + (y^2)^2], \qquad R_{;i} = R\omega_i$$

$$\omega_1 := \frac{f''(x^1)}{f'(x^1)} - 4f(x^1) = \left[\ln|f'(x^1)e^{-4\int f(x^1)dx^1}|\right]_{x^1} = [\ln|\lambda|]_{x^1}, \qquad \omega_2 := 0.$$

Then by Theorem 1.1, $\mathbf{G}$ is Finsler metrizable induced by the Riemann metric

$$L := R/\lambda = -2e^{4\int f(x^1)dx^1}[(y^1)^2 + (y^2)^2]$$

of isotropic sectional curvature

$$\lambda = f'(x^1)e^{-4\int f(x^1)dx^1}.$$

***Example 7.2*** Let $\mathbf{G}$ be a two-dimensional spray given by

$$G^1 = \frac{1}{2r}y^2\sqrt{(y^1)^2+(y^2)^2}, \quad G^2 = -\frac{1}{2r}y^1\sqrt{(y^1)^2+(y^2)^2},$$

where $r$ is a constant. This spray appears in Example 4.1.3 of [16]. From the completeness of geodesics of $\mathbf{G}$ and Hopf-Rinow theorem, it is concluded that $\mathbf{G}$ is not globally Finsler metrizable. But actually, it is even not locally Finsler metrizable by Theorem 1.1 (Example 12.4.1 in [16] shows it is locally projectively Finsler metrizable).

By a direct computation, $\mathbf{G}$ is of isotropic curvature $R^i{}_k = R\delta^i_k - \frac{1}{2}R_{.k}y^i$ satisfying

$$R = r^{-2}\big[(y^1)^2+(y^2)^2\big], \quad R_{;1} = r^{-3}y^2\sqrt{(y^1)^2+(y^2)^2},$$
$$R_{;2} = -r^{-3}y^1\sqrt{(y^1)^2+(y^2)^2}.$$

Therefore, by Theorem 1.1, $\mathbf{G}$ is not locally Finsler metrizable anywhere since there exists no closed 1-form $\omega = \omega_i(x)dx^i$ such that $R_{;i} = R\omega_i$.

***Example 7.3*** Let $\mathbf{G}_c$ be a Berwald spray given by

$$G^i_c = \frac{-|y|^2x^i + c\langle x, y\rangle y^i}{1-|x|^2},$$

where $c$ is a constant. The spray $\mathbf{G}_2$ is induced by the Riemann metric $L = 4|y|^2/(1-|x|^2)^2$ of constant sectional curvature $-1$, and by computation, we can see that $\mathbf{G}_1$ is just the spray given by Example 4.1.4 in [16]. $\mathbf{G}_c$ actually is projectively flat although at present coordinate it is not in the form $Py^i$.

By a direct computation, $\mathbf{G}_c$ is of isotropic curvature $R^i{}_k = R\delta^i_k - \frac{1}{2}R_{.k}y^i$ with

$$R = -\frac{\big[(c-2)|x|^2+c+2\big]|y|^2 - c(c-2)\langle x, y\rangle^2}{(1-|x|^2)^2}.$$

Further, we have

$$R_{;i} = \frac{2(c-2)}{(1-|x|^2)^3}\Big\{\big[(c-1)|x|^2+c+1\big](|y|^2x^i + 2\langle x, y\rangle y^i) - 2c(c-1)\langle x, y\rangle^2x^i\Big\}.$$

(i) By Theorem 1.1, it is easy to see that $\mathbf{G}_c$ is Finsler metrizable iff. $c = 2$. When $c = 2$, we obtain the Riemann metric $L := R = -4|y|^2/(1-|x|^2)^2$ of constant curvature 1.
(ii) $\mathbf{G}_c$ is of constant curvature iff. $c = 2$.

(iii) When $c \neq 2$, $\mathbf{G}_c$ is of isotropic curvature but not of constant curvature in any dimension. This fact is different from the Finslerian case.

***Example 7.4*** Let $\mathbf{G}$ be a spray given by $G^i = \sigma_0 y^i$ with $\sigma_i := \sigma_{x^i}, \sigma = \sigma(x)$.

(i) $\mathbf{G}$ is of isotropic curvature for any $\sigma$ but not of constant curvature by Theorem 6.8 if

$$\sigma(x) \neq -\frac{1}{2} \ln |\langle Ax, x\rangle + \langle B, x\rangle + C|.$$

(ii) Taking $A = (\delta_{ij})$, $B = 0$, $C = 0$ in (2) and (3), we have

$$G^i = -\frac{\langle x, y\rangle}{|x|^2} y^i, \quad L = \frac{|x|^2|y|^2 - \langle x, y\rangle^2}{|x|^4}. \tag{52}$$

In (52), $L$ is not a metric since $(L_{.i.j})$ is degenerate. By Theorem 1.3, or Theorem 1.1(ii) ($R = L$ is not a Finsler metric), the spray $\mathbf{G}$ given by (52) is not Finsler metrizable, although this spray $\mathbf{G}$ satisfies (23).

***Example 7.5*** Let $\mathbf{G}$ be a spray on $R^3$ given by ([16], Example 4.1.2)

$$G^1 = 0, \quad G^2 = x^1(y^1)^2 + x^3(y^3)^2, \quad G^3 = 0.$$

$\mathbf{G}$ has zero Riemann curvature. Then by Theorem 1.1(i), $\mathbf{G}$ is locally Finsler metrizable. Actually, if the following function is a Finsler metric

$$L(x, y) = \left[ f\left(\frac{y^3}{y^1}, (x^1)^2 + \frac{y^2}{y^1} + (x^3)^2\frac{y^3}{y^1}\right) y^1 \right]^2,$$

then $L$ induces $\mathbf{G}$ on some open set of $R^3$, and $F$ should be a locally Minkowski metric since $\mathbf{G}$ is a Berwald spray (with zero Riemann curvature). Consider a special case with $f(r, s) = \sqrt{r^2 + s}$ and we obtain

$$L(x, y) = (y^3)^2 + (x^1)^2(y^1)^2 + y^1y^2 + (x^3)^2y^1y^3,$$

which is a singular Riemann metric globally defined on $R^3$. So the spray $\mathbf{G}$ is globally Finsler metrizable.

***Example 7.6*** Let $\mathbf{G}$ be a two-dimensional spray given by

$$G^1 = -f(x^1)g'(t)(y^1)^2, \quad G^2 = f(x^1)\big[g(t)y^1 - g'(t)y^2\big]y^1, \quad (t := y^2/y^1).$$

where $f, g$ are two smooth functions. A direct computation gives $R^i{}_k = R\delta^i_k - \tau_k y^i$ with

$$R = 0, \quad \tau_1 = Ay^2, \quad \tau_2 = -Ay^1, \quad A := 2f^2(x^1)g(t)g'''(t) - f'(x^1)g''(t).$$

We can choose the two functions $f, g$ satisfying $A \neq 0$. So in this case, $\mathbf{G}$ is not Finsler metrizable.

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