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Locally Strongly Convex Affine Hypersurfaces with Semi-parallel Cubic Form

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Abstract
In this paper, we investigate the locally strongly convex affine hypersurfaces with semi-parallel cubic form relative to the Levi-Civita connection of affine metric. We obtain two results on such hypersurfaces which admit at most one affine principal curvature of multiplicity one: (1) classify these being not affine hyperspheres; (2) classify these affine hyperspheres with constant scalar curvature. For the latter, by proving the parallelism of their cubic forms we translate the classification into that of affine hypersurfaces with parallel cubic form, which has been completed by Hu-Li-Vrancken (J Differ Geom 87:239–307, 2011).

Keywords Affine hypersurface · Semi-parallel cubic form · Levi-Civita connection · Affine principal curvature · Warped product

Mathematics Subject Classification Primary 53A15 · Secondary 53B25 · 53B30

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1 Introduction

The classical equiaffine differential geometry is mainly concerned with geometric properties of hypersurfaces in affine space, that are invariant under unimodular affine transformations. Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional real unimodular affine space. On a non-degenerate hypersurface immersion of $\mathbb{R}^{n+1}$, it is well known how to induce an affine connection ∇, an affine shape operator S whose eigenvalues are called affine principal curvatures, and a symmetric bilinear form h, called the affine metric. From a local point of view, there are two natural tensors, namely the difference tensor K which is defined as the difference between ∇ and the Levi-Civita connection $\hat{\nabla}$ of h, and the cubic form $C := \nabla h$. The classical Pick-Berwald theorem states that the cubic form or difference tensor vanishes, if and only if the hypersurface is a non-degenerate hyperquadric. In that sense, the cubic form or difference tensor plays the role as the second fundamental form for submanifolds of real space forms.

In the same style as the Pick-Berwald theorem, geometric conditions on the cubic form and difference tensor have been used to classify natural classes of affine hypersurfaces by many geometers in the past decades, see e.g. [5–7, 13, 17–20, 32, 35]. Among them, one of the most interesting developments may be the classification of locally strongly convex affine hypersurfaces with parallel cubic form relative to $\hat{\nabla}$. In this subject, F. Dillen, L. Vrancken, et al. obtain the classifications for lower dimensions in [11, 15, 21, 27], and finally Z. Hu, H. Li and L. Vrancken complete the classification for all dimensions as follows:

Theorem 1.1 (cf. [23]) *Let M be an n-dimensional ($n \geq 2$) locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{\nabla}C = 0$. Then, M is either a hyperquadric (i.e., $C = 0$) or a hyperbolic affine hypersphere with $C \neq 0$; in the latter case either*

- (i) *M is obtained as the Calabi product of a lower dimensional hyperbolic affine hypersphere with parallel cubic form and a point, or*
- (ii) *M is obtained as the Calabi product of two lower dimensional hyperbolic affine hyperspheres with parallel cubic form, or*
- (iii) *$n = \frac{1}{2}m(m+1) - 1,\ m \geq 3$, (M, h) is isometric to $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m)$, and M is affinely equivalent to the standard embedding $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m) \hookrightarrow \mathbb{R}^{n+1}$, or*
- (iv) *$n = m^2 - 1,\ m \geq 3$, (M, h) is isometric to $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m)$, and M is affinely equivalent to the standard embedding $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m) \hookrightarrow \mathbb{R}^{n+1}$, or*
- (v) *$n = 2m^2 - m - 1,\ m \geq 3$, (M, h) is isometric to $\mathbf{SU}^*\big(2m\big)/\mathbf{Sp}(m)$, and M is affinely equivalent to the standard embedding $\mathbf{SU}^*\big(2m\big)/\mathbf{Sp}(m) \hookrightarrow \mathbb{R}^{n+1}$, or*
- (vi) *$n = 26$, (M, h) is isometric to $\mathbf{E}_{6(-26)}/\mathbf{F}_4$, and M is affinely equivalent to the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$.*

As that did in [23, 24], we say that an affine hypersurface is of semi-parallel (resp. parallel) cubic form relative to the Levi-Civita connection of affine metric if $\hat{R} \cdot C = 0$ (resp. $\hat{\nabla}C = 0$), where $\hat{R}$ is the curvature tensor of affine metric, and the tensor $\hat{R} \cdot C$ is defined by

$$\hat{R}(X, Y) \cdot C = \hat{\nabla}_X \hat{\nabla}_Y C - \hat{\nabla}_Y \hat{\nabla}_X C - \hat{\nabla}_{[X,Y]} C \tag{1.1}$$

for tangent vector fields X, Y. Obviously, the parallelism of cubic form implies its semi-parallelism, the converse is not true, we refer to Remark 3.1 for the counter-examples.

In this paper, we investigate locally strongly convex affine hypersurfaces with semi-parallel cubic form relative to the Levi-Civita connection of affine metric. First, we prove that if all the affine principal curvatures of the hypersurface have multiplicity more than one, then the hypersurface is an affine hypersphere. If further assume that its affine metric is of constant scalar curvature, by proving the parallelism of the cubic form we translate the classification into that of Theorem 1.1. More precisely, let H, $\triangle$ and $\hat{Ric}$ be the affine mean curvature, Laplacian operator and Ricci curvature of affine metric h, respectively, we can state the first main result as follows.

Theorem 1.2 *Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R}\cdot C=0$ and $n\geq 2$. Assume that M^n does not admit any affine principal curvature of multiplicity one. Then M^n is either a hyperquadric (i.e., $C=0$) or a hyperbolic affine hypersphere with non-positive scalar curvature κ and $C\neq 0$. Moreover, there hold*

$$2\triangle\kappa=\|\hat{\nabla}C\|_h^2, \tag{1.2}$$

$$(n+1)\kappa H=\|\hat{R}\|_h^2+\|\hat{Ric}\|_h^2, \tag{1.3}$$

where $\|\cdot\|_h$ denotes the tensorial norm with respect to h. If additionally assume that κ is constant for $n\geq 3$, then $\hat{\nabla}C=0$, and M^n is affinely equivalent to one of the examples in Theorem 1.1.

Remark 1.1 For a locally strongly convex affine hypersurface M^n, Theorems 1.1 and 1.2 imply that:

(1) If $n\geq 3$, it is an affine hypersphere with $\hat{R}\cdot C=0$ and constant scalar curvature if and only if $\hat{\nabla}C=0$.
(2) If $n=2$, it is an affine sphere with $\hat{R}\cdot C=0$ if and only if $\hat{\nabla}C=0$.

We conjecture that any locally strongly convex affine hypersphere with $\hat{R}\cdot C=0$ must satisfy $\hat{\nabla}C=0$.

Second, if the hypersurface admits exactly one affine principal curvature of multiplicity one, then the number of its distinct affine principal curvatures is either three or two (i.e., the hypersurface is quasi-umbilical), which are further classified, respectively. These results are given precisely in the following theorems.

Theorem 1.3 *Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R}\cdot C=0$ and $n\geq 3$. If it admits exactly one affine principal curvature of multiplicity one, then the number of its distinct affine principal curvatures is either two or three.*

Theorem 1.4 *Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R}\cdot C=0$ and $n\geq 5$. Assume that there are exactly three distinct affine principal curvatures μ_1,μ_2,μ_3 of multiplicity $(1,n_2,n_3)$ with $n_2\geq 2$ and $n_3\geq 2$, respectively.*

Then (M^n, h) *is locally isometric to the warped product* $\mathbb{R}_+ \times M_2 \times_t M_3$*, and each* μ_i *is a function which depends only on* t *such that* $\mu_2\mu_3 \neq 0$*,*

$$t^2(\mu_1 - \mu_2)(\mu_2 - \mu_3)^2 = 2(\mu_1 - \mu_3)(\mu_2 + \mu_3).$$

Moreover, M^n *is affinely equivalent to*

$$F(t, p_2, p_3) = (\gamma_2(t)\phi_2(p_2), \gamma_3(t)\phi_3(p_3)),$$

where $\gamma_2,\ \gamma_3$ *are nonzero functions satisfying*

$$\gamma_2' = \tfrac{1}{2}(\mu_3 - \mu_2)t\gamma_2,\ \gamma_3' = \tfrac{2\mu_3}{(\mu_3-\mu_2)t}\gamma_3,$$

and $\phi_i : M_i \to \mathbb{R}^{n_i+1}$ *is a locally strongly convex proper affine hypersphere with the affine mean curvature* H_i *for* $i = 2, 3$*, which are nonzero constant defined by*

$$H_2 = \mu_2 - \tfrac{1}{4}(\mu_2 - \mu_3)^2 t^2,\ H_3 = \mu_3 t^2 + \tfrac{2}{\mu_2-\mu_3},$$

ϕ_2 *is an ellipsoid if* $H_2 > 0$ *and is of semi-parallel cubic form otherwise, whereas* ϕ_3 *is an ellipsoid if* $H_3 \geq 1$*.*

Theorem 1.5 *Let* M^n *be a locally strongly convex quasi-umbilical affine hypersurface in* $\mathbb{R}^{n+1}$ *with* $\hat{R} \cdot C = 0$ *and* $n \geq 3$*. Then* (M^n, h) *is locally isometric to the warped product* $\mathbb{R}_+ \times_f M_2$*, and* M^n *is affinely equivalent to one of the immersions explicitly described in Theorem 6.1, where the warped function* $f(t) = 1$ *or* t*.*

Remark 1.2 The examples in Theorems 1.4 and 1.5 are the generalized Calabi compositions of affine hyperspheres in some special forms. The construction method of such examples initially originates from E. Calabi [8], and now has been extended and characterized by F. Dillen, L. Vrancken, Z. Hu, H. Li, et al. in [1, 2, 12, 22].

This paper is organized as follows. In Sect. 2, we briefly review the local theory of equiaffine hypersurfaces, some results and concepts of warped product manifolds. In Sect. 3, we begin with the Tsinghua principle to study the properties of the hypersurfaces involving the affine principal curvatures, the difference tensor and the eigenvalue distributions of affine shape operator, and present the proof of Theorem 1.2. Based on these properties, in Sect. 4 we obtain Theorem 1.3 by showing the number of the affine principal curvatures being three or two. In either case, we prove the warped product structure, discuss all the possibilities of the immersion and complete the proofs of Theorems 1.4 and 1.5 in last two sections, respectively.

2 Preliminaries

In this section, we briefly recall the local theory of equiaffine hypersurfaces. For more details, we refer to the monographs [26, 29].

Let $\mathbb{R}^{n+1}$ denote the standard $(n+1)$-dimensional real unimodular affine space that is endowed with its usual flat connection D and a parallel volume form ω, given by the determinant. Let $F : M^n \to \mathbb{R}^{n+1}$ be an oriented non-degenerate hypersurface immersion. On such a hypersurface, up to a sign there exists a unique transversal vector field ξ, called the *affine normal*. A non-degenerate hypersurface equipped with the affine normal is called an *(equi)affine hypersurface*, or a *Blaschke hypersurface*. Denote by X, Y, Z, W the tangent vector fields on M^n from now on. By the affine normal we have

$$D_X F_* Y = F_* \nabla_X Y + h(X, Y)\xi, \qquad \text{(Gauss formula)} \qquad (2.1)$$

$$D_X \xi = -F_* S X, \qquad \text{(Weingarten formula)} \qquad (2.2)$$

which induce on M^n the *affine connection* ∇, a symmetric bilinear form h, called the *affine metric*, the *affine shape operator* S whose eigenvalues are called *affine principal curvatures*, and the *cubic form* $C := \nabla h$. An affine hypersurface is called *locally strongly convex* if h is definite, we always choose ξ, up to a sign, such that h is positive definite. We call a locally strongly convex affine hypersurface *quasi-umbilical* if it admits exactly two distinct affine principal curvatures, one of which is simple.

Let $\hat{\nabla}$ be the Levi-Civita connection of the affine metric h. The difference tensor K is defined by

$$K(X, Y) := \nabla_X Y - \hat{\nabla}_X Y. \qquad (2.3)$$

We also write $K_X Y$ and $K_X = \nabla_X - \hat{\nabla}_X$. Since both ∇ and $\hat{\nabla}$ have zero torsion, K is symmetric in X and Y. It is related to the totally symmetric cubic form C by

$$C(X, Y, Z) = -2h(K(X, Y), Z), \qquad (2.4)$$

which implies that the operator K_X is symmetric relative to h. Moreover, K satisfies the *apolarity condition*, namely, $\operatorname{tr} K_X = 0$ for all X.

The curvature tensor $\hat{R}$ of affine metric h, S and K are related by the following Gauss and Codazzi equations:

$$\begin{aligned} \hat{R}(X, Y)Z = {} & \tfrac{1}{2}[h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y] \\ & -[K_X, K_Y]Z, \end{aligned} \qquad (2.5)$$

$$\begin{aligned} & (\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) \\ & \quad = \tfrac{1}{2}[h(Y, Z)SX - h(X, Z)SY - h(SY, Z)X + h(SX, Z)Y], \end{aligned} \qquad (2.6)$$

$$(\hat{\nabla}_X S)Y - (\hat{\nabla}_Y S)X = K(SX, Y) - K(SY, X), \qquad (2.7)$$

where, by definitions, $[K_X, K_Y]Z = K_X K_Y Z - K_Y K_X Z$, and

$$\hat{R}(X,Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X,Y]}Z,$$
$$(\hat{\nabla}_X K)(Y,Z) = \hat{\nabla}_X(K(Y,Z)) - K(\hat{\nabla}_X Y, Z) - K(Y, \hat{\nabla}_X Z),$$
$$(\hat{\nabla}_X S)Y = \hat{\nabla}_X(SY) - S\hat{\nabla}_X Y.$$

Contracting Gauss equation (2.5) we obtain

$$\chi = H + J, \tag{2.8}$$

where $J = \frac{1}{n(n-1)}h(K,K)$, $H = \frac{1}{n}\mathrm{tr}\, S$, $\chi = \frac{\kappa}{n(n-1)}$ and κ are the *Pick invariant*, *affine mean curvature*, *normalized scalar curvature* and scalar curvature of h, respectively. Recall the second covariant differentiation of K, defined by

$$\hat{\nabla}^2_{X,Y} K = \hat{\nabla}_X \hat{\nabla}_Y K - \hat{\nabla}_{\hat{\nabla}_X Y} K,$$

and the following Ricci identity:

$$(\hat{\nabla}^2_{X,Y} K)(Z,W) - (\hat{\nabla}^2_{Y,X} K)(Z,W) = (\hat{R}(X,Y)\cdot K)(Z,W)$$
$$= \hat{R}(X,Y)K(Z,W) - K(\hat{R}(X,Y)Z,W) - K(Z,\hat{R}(X,Y)W).$$

The affine hypersurface M^n is called an affine hypersphere if $S = H\,id$. Then it follows from (2.7) that H is constant if $n \geq 2$. M^n is said to be a proper (resp. improper) affine hypersphere if H is nonzero (resp. zero). Moreover, a locally strongly convex affine hypersphere is called parabolic, elliptic or hyperbolic according to $H = 0$, $H > 0$ or $H < 0$, respectively. For affine hyperspheres, the Gauss and Codazzi equations reduce to

$$\hat{R}(X,Y)Z = H[h(Y,Z)X - h(X,Z)Y] - [K_X, K_Y]Z, \tag{2.9}$$
$$(\hat{\nabla}_X K)(Y,Z) = (\hat{\nabla}_Y K)(X,Z). \tag{2.10}$$

We collect the following two results for later use.

Theorem 2.1 (cf. Theorem 6.2 of [24]) *A locally strongly convex affine surface M^2 in $\mathbb{R}^3$ satisfies $\hat{R}\cdot C = 0$ if and only if either M^2 is locally a quadric or (M^2, h) is flat.*

Theorem 2.2 (cf. Theorem 1 of [1]) *Let M^{m+1}, $m \geq 2$, be a locally strongly convex affine hypersurface of the affine space $\mathbb{R}^{m+2}$ such that its tangent bundle is an orthogonal sum, with respect to the affine metric h, of two distributions: a one-dimensional distribution $\mathcal{D}_1$ spanned by a unit vector field T and an m-dimensional distribution $\mathcal{D}_2$, such that*

$$K(T,T) = \lambda_1 T, \quad K(T,X) = \lambda_2 X,$$
$$ST = \mu_1 T, \quad SX = \mu_2 X, \quad \forall\, X \in \mathcal{D}_2.$$

Then either M^{m+1} is an affine hypersphere such that $K_T = 0$ or is affinely congruent to one of the following immersions:

(1) $f(t, x_1, \ldots, x_m) = (\gamma_1(t), \gamma_2(t)g_2(x_1, \ldots, x_m))$*, for* γ_1, γ_2 *such that*

$$\epsilon\gamma_1'\gamma_2(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') < 0;$$

(2) $f(t, x_1, \ldots, x_m) = \gamma_1(t)C(x_1, \ldots, x_m) + \gamma_2(t)e_{m+1}$*, for* γ_1, γ_2 *such that*

$$\operatorname{sgn}\Big(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'\Big) = \operatorname{sgn}(\gamma_1'\gamma_1) \neq 0;$$

(3) $f(t, x_1, \ldots, x_m) = C(x_1, \ldots, x_m) + \gamma_2(t)e_{m+1} + \gamma_1(t)e_{m+2}$*, for* γ_1, γ_2 *such that*

$$\operatorname{sgn}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sgn}(\gamma_1') \neq 0.$$

Here $g_2 : \mathbb{R}^m \to \mathbb{R}^{m+1}$ is a proper affine hypersphere centered at the origin with affine mean curvature ϵ, and $C : \mathbb{R}^m \to \mathbb{R}^{m+2}$ is an improper affine hypersphere, given by $C(x_1, \ldots, x_m) = (x_1, \ldots, x_m, p(x_1, \ldots, x_m), 1)$, with the affine normal e_{m+1}.

Finally, we review some notions of warped product manifolds and subbundles. For Riemannian manifolds (B, g_B), (M_1, g_1), ..., (M_k, g_k) and positive functions $f_1, \ldots, f_k : B \to \mathbb{R}$, the manifold $M := B \times M_1 \times \cdots \times M_k$ equipped with the metric $h = g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k$ is a warped product manifold with warped functions f_i, denoted by $B \times_{f_1} M_1 \times \cdots \times_{f_k} M_k$. Let $\hat{\nabla}$ be the Levi-Civita connection of a Riemannian manifold (M, h). A subbundle $E \subset TM$ is called auto-parallel if $\hat{\nabla}_X Y \in E$ holds for all $X,\ Y \in E$. Whereas a subbundle E is called totally umbilical if there exists a vector field $V \in E^{\perp}$ such that $h(\hat{\nabla}_X Y, Z) = h(X, Y)h(V, Z)$ for all $X,\ Y \in E$ and $Z \in E^{\perp}$, here we call V the mean curvature vector of E. If, moreover, $h(\hat{\nabla}_X V, Z) = 0$ holds, we say that E is spherical. We conclude this section by the decomposition theorem of Riemannian manifolds.

Theorem 2.3 (cf. Theorem 4 of [28]) *Let M be a Riemannian manifold, and let $TM = \bigoplus_{i=0}^k E_i$ be an orthogonal decomposition into nontrivial vector subbundles such that E_i is spherical and $E_i^{\perp}$ is autoparallel for $i = 1, \ldots, k$. Then, for every point $p \in M$ there is an isometry ψ of a warped product $M_0 \times_{f_1} M_1 \times \cdots \times_{f_k} M_k$ onto a neighbourhood of p in M such that $\psi(\{p_0\} \times \cdots \times \{p_{i-1}\} \times M_i \times \{p_{i+1}\} \times \cdots \times \{p_k\})$ is an integral manifold of E_i for $i = 0, \ldots, k$ and all $p_0 \in M_0, \ldots, p_k \in M_k$.*

3 Properties of Affine Hypersurfaces with $\hat{R} \cdot C = 0$

From this section on, when we say that an affine hypersurface has *semi-parallel cubic form*, it always means that $\hat{R} \cdot C = 0$, equivalently $\hat{R} \cdot K = 0$. Then, by the Ricci identity of K we have

$$\hat{R}(X, Y)K(Z, W) = K(\hat{R}(X, Y)Z, W) + K(Z, \hat{R}(X, Y)W). \tag{3.1}$$

In fact, by (2.4) and the Ricci identities of C and K, the equivalence above follows from the following formula:

$$\begin{aligned}(\hat{R}\cdot C)(U,V,X,Y,Z) &= (\hat{R}(U,V)\cdot C)(X,Y,Z)\\ &= -C(X,Y,\hat{R}(U,V)Z) - C(X,\hat{R}(U,V)Y,Z) - C(\hat{R}(U,V)X,Y,Z)\\ &= 2[h(K_XY,\hat{R}(U,V)Z) + h(K_X\hat{R}(U,V)Y,Z) + h(K_Y\hat{R}(U,V)X,Z)]\\ &= -2h(\hat{R}(U,V)K_XY - K_X\hat{R}(U,V)Y - K_Y\hat{R}(U,V)X,Z)\\ &= -2h((\hat{R}(U,V)\cdot K)(X,Y),Z).\end{aligned}\tag{3.2}$$

Remark 3.1 Besides examples in Theorem 1.1, we see from (3.1) that all flat affine hypersurfaces satisfy $\hat{R}\cdot C = 0$. Therefore, to see the examples whose cubic forms are semi-parallel but not parallel, we refer to Remark 6.2 in [24] for such flat surfaces, and Theorem 4.1 in [3] for the flat and quasi-umbilical affine hypersurfaces.

In what follows, if no other stated, we always assume that M^n is a locally strongly convex affine hypersurface with semi-parallel cubic form. First, by using the Codazzi equations for both the shape operator and the difference tensor, we obtain some linear equations involving the components of the difference tensor and affine principal curvatures as follows.

Lemma 3.1 *Let M^n $(n \geq 2)$ be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R}\cdot C = 0$. Denote by $\{e_1, \dots, e_n\}$ the orthonormal frame of M^n, where e_i are the eigenvector fields of the shape operator S with corresponding eigenvalues μ_i, $i = 1, \dots, n$. Then, for any i, j, k, ℓ, there holds*

$$\begin{aligned}&(\mu_k-\mu_i)[\delta_{j\ell}K(e_k,e_i) + h(K(e_k,e_i),e_\ell)e_j]\\ &\quad + (\mu_i-\mu_j)[\delta_{k\ell}K(e_i,e_j) + h(K(e_i,e_j),e_\ell)e_k]\\ &\quad + (\mu_j-\mu_k)[\delta_{i\ell}K(e_j,e_k) + h(K(e_j,e_k),e_\ell)e_i] = 0.\end{aligned}\tag{3.3}$$

Proof By the second covariant differentiation of K and (3.2) it holds that

$$(\hat{\nabla}^2_{W,X}K)(Y,Z) - (\hat{\nabla}^2_{X,W}K)(Y,Z) = (\hat{R}(W,X)\cdot K)(Y,Z) = 0. \tag{3.4}$$

On the other hand, direct calculations show that

$$\begin{aligned}&\sigma_{W,X,Y}\{(\hat{\nabla}^2_{W,X}K)(Y,Z) - (\hat{\nabla}^2_{X,W}K)(Y,Z)\}\\ &\quad = \sigma_{W,X,Y}\{(\hat{\nabla}^2_{W,X}K)(Y,Z) - (\hat{\nabla}^2_{W,Y}K)(X,Z)\},\end{aligned}\tag{3.5}$$

where $\sigma_{W,X,Y}$ denotes the cyclic summation over W, X, Y. Moreover, by the second covariant differentiation of K we have (see also (3.3) in [3])

$$\begin{aligned}&(\hat{\nabla}^2_{W,X}K)(Y,Z) - (\hat{\nabla}^2_{W,Y}K)(X,Z)\\ &\quad = (\hat{\nabla}_W\hat{\nabla}K)(X,Y,Z) - (\hat{\nabla}_W\hat{\nabla}K)(Y,X,Z)\\ &\quad = \tfrac{1}{2}\{h(Y,Z)(\hat{\nabla}_WS)X - h(X,Z)(\hat{\nabla}_WS)Y\\ &\qquad - h((\hat{\nabla}_WS)Y,Z)X + h((\hat{\nabla}_WS)X,Z)Y\},\end{aligned}\tag{3.6}$$

where the last equality follows from the covariant differentiation of (2.6) along W. Together with (2.7), by (3.4) and (3.5) we see that

$$\begin{aligned}0 &= 2\sigma_{W,X,Y}\{(\hat{\nabla}^2_{W,X}K)(Y,Z) - (\hat{\nabla}^2_{W,Y}K)(X,Z)\}\\ &= h(Y,Z)(K(SW,X)-K(SX,W))+h(W,Z)(K(SX,Y)-K(SY,X))\\ &\quad + h(X,Z)(K(SY,W)-K(SW,Y))+h(K(SY,W)-K(SW,Y),Z)X\\ &\quad + h(K(SW,X)-K(SX,W),Z)Y+h(K(SX,Y)-K(SY,X),Z)W.\end{aligned} \tag{3.7}$$

Finally, by taking $X = e_i$, $Y = e_j$, $W = e_k$, $Z = e_\ell$ in (3.7) we have (3.3). □

Remark 3.2 The technique used in Lemma 3.1, is based on the Tsinghua principle due to H. Li, L. Vrancken and X. Wang [3]. For some tensor, it allows one to take the cyclic permutation of the covariant derivative of its Codazzi equation, use the Ricci identity in an indirect way and express the tensor in a conveniently chosen frame, see [4, 9, 10, 14, 25] for its applications in various purposes.

By the notations of Lemma 3.1, we always denote by $\mathfrak{D}(\mu_i)$ the eigenvalue distribution of S corresponding to the eigenvalue μ_i and by n_i its dimension. Note that the conclusion of Lemma 3.1 is the same as Lemma 3.1 of [3], although the assumptions are different. Therefore, following the proof of Lemma 3.2 in [3], by (3.3) we obtain the same results as below.

Lemma 3.2 *The difference tensor K satisfies:*

(i) If $\mu_i \neq \mu_j$ and $n_i, n_j \geq 2$, then $K(e_i, e_j) = 0$.
(ii) If $n_j = 1$ and $n_i \geq 2$, then there exist functions $\lambda_i^j := h(K_{e_i}e_i, e_j)$ depending on the choice of μ_i, μ_j such that $K(e_j, e_i) = \lambda_i^j e_i$.
(iii) If there are at least two different eigenvalues $\mu_i \neq \mu_k$ such that $n_i, n_k \geq 2$ and $n_j = 1$, then there exists a differentiable function $\bar{\lambda}_j$ such that it holds that $(\mu_j - \mu_i)\lambda_i^j = (\mu_j - \mu_k)\lambda_k^j = \bar{\lambda}_j$.

Furthermore, by Codazzi equation (2.7) we have

$$\begin{aligned}&e_i(\mu_j)e_j - e_j(\mu_i)e_i + \mu_j\hat{\nabla}_{e_i}e_j - \mu_i\hat{\nabla}_{e_j}e_i\\ &\quad = S\hat{\nabla}_{e_i}e_j - S\hat{\nabla}_{e_j}e_i + (\mu_i - \mu_j)K(e_i, e_j).\end{aligned} \tag{3.8}$$

By multiplying this with the eigenvector e_k, we get the following lemma.

Lemma 3.3 *It holds that*

(i) $e_i(\mu_j) = (\mu_j - \mu_i)h(\hat{\nabla}_{e_j}e_j - K_{e_j}e_j, e_i)$ for $k = j \neq i$;
(ii) $e_i(\mu_j)\delta_{jk} - e_j(\mu_i)\delta_{ik} + (\mu_j - \mu_k)h(\hat{\nabla}_{e_i}e_j, e_k)$ $= (\mu_i - \mu_k)h(\hat{\nabla}_{e_j}e_i, e_k) + (\mu_i - \mu_j)h(K_{e_i}e_j, e_k)$ for any i, j, k.

By taking $e_i, e_j \in \mathfrak{D}(\mu_i)$ and $e_k \in \mathfrak{D}(\mu_i)^\perp$ in Lemma 3.3 (ii), we see that each eigenvalue distribution $\mathfrak{D}(\mu_i)$ forms an integrable subbundle. Similarly, taking $e_i, e_j \in$

$\mathfrak{D}(\mu_i)$ in Lemma 3.3 (i) for $i \neq j$, we get that each eigenvalue of multiplicities more than one is constant on its integral submanifolds.

Next, for more information we also denote by $\tilde{\mu}_1, \ldots, \tilde{\mu}_r$ the eigenvalue functions of affine shape operator S with the multiplicity one, and by $u_1, \ldots, u_r$ the corresponding unit eigenvector fields. Let $\mu_1, \ldots, \mu_s$ be the eigenvalues of higher multiplicities, and $v_1^i, \ldots, v_{n_i}^i$ be the orthonormal eigenvector fields of μ_i, which span the distribution $\mathfrak{D}(\mu_i)$ for $i = 1, \ldots, s$. Note from Lemma 3.2 that $K(v_j^i, v_k^i) \in \bigoplus_{j=1}^r \operatorname{Span}(u_j) \bigoplus \mathfrak{D}(\mu_i)$. Define tensors $L^i : \mathfrak{D}(\mu_i) \times \mathfrak{D}(\mu_i) \to \mathfrak{D}(\mu_i)$ given by

$$L^i(X, X') = K(X, X') - \sum_{j=1}^r \lambda_i^j h(X, X') u_j, \ i = 1, \ldots, s, \tag{3.9}$$

which are the projection of K onto the distribution $\mathfrak{D}(\mu_i)$, then we are ready to prove the next two results.

Lemma 3.4 *Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R} \cdot C = 0$. If the multiplicity of affine principal curvature μ_i is more than one, then*

(i) The eigenvalue distribution $\mathfrak{D}(\mu_i)$ is integrable, on which μ_i is constant.
(ii) L^i is totally symmetric and satisfies the apolarity condition.
(iii) For any $X, X', X'' \in \mathfrak{D}(\mu_i), W \in \mathfrak{D}(\mu_i)^{\perp}$, there hold $\hat{R}(X, X') \cdot L^i = 0$, and

$$\begin{aligned} \hat{R}(X, X')X'' &= (\mu_i - \sum_{j=1}^r (\lambda_i^j)^2)[h(X', X'')X - h(X, X'')X'] \\ &\quad - [L_X^i, L_{X'}^i]X'', \\ \hat{R}(X, X')W &= 0. \end{aligned} \tag{3.10}$$

(iv) Assume that $\mathfrak{D}(\mu_i)$ is spherical. Denote by $\hat{R}^{\perp}$ the curvature tensor of the connection $\hat{\nabla}^{\perp}$ on the integral manifold of $\mathfrak{D}(\mu_i)$ induced from (M^n, h), and by $\rho_i T$ the mean curvature vector with the unit vector $T \in \mathfrak{D}(\mu_i)^{\perp}$. Then

$$\begin{aligned} &(\hat{R}^{\perp}(X, X') \cdot L^i)(X'', \tilde{X}) \\ &= \rho_i^2 \{h(X, X'')L^i(X', \tilde{X}) - h(X', X'')L^i(X, \tilde{X}) \\ &\quad + h(X, \tilde{X})L^i(X', X'') - h(X', \tilde{X})L^i(X, X'') \\ &\quad + h(X', L^i(X'', \tilde{X}))X - h(X, L^i(X'', \tilde{X}))X'\}, \end{aligned} \tag{3.11}$$

where $\hat{R}^{\perp}(X, X')X'' = \hat{R}(X, X')X'' + \rho_i^2(h(X', X'')X - h(X, X'')X')$. In particular, if $\mathfrak{D}(\mu_i)$ is auto-parallel, i.e., $\rho_i = 0$, then $\hat{R}^{\perp} \cdot L^i = 0$.

Proof The previous analysis after Lemma 3.3 gives the proof of the part (i). Note that $h(L^i(v_j^i, v_k^i), v_\ell^i) = h(K(v_j^i, v_k^i), v_\ell^i)$ is totally symmetric. The apolarity condition yields $\sum_{j=1}^r K(u_j, u_j) + \sum_{\ell=1}^s \sum_{p=1}^{n_\ell} K(v_p^\ell, v_p^\ell) = 0$. Then, for arbitrary $v_q^i \in$

$\mathfrak{D}(\mu_i)$, by (3.9) and Lemma 3.2 there holds

$$\begin{aligned}\sum_{p=1}^{n_i} h(L^i(v_p^i, v_p^i), v_q^i) &= \sum_{p=1}^{n_i} h(K(v_p^i, v_p^i), v_q^i)\\ &= -\sum_{\ell\neq i}\sum_{p=1}^{n_\ell} h(K(v_p^\ell, v_p^\ell), v_q^i) - \sum_{j=1}^{r} h(K(u_j, u_j), v_q^i) = 0.\end{aligned} \tag{3.12}$$

Therefore, $\sum_{p=1}^{n_i} L^i(v_p^i, v_p^i) = 0$, i.e., the tensor L^i satisfies the apolarity condition. We have proved part (ii).

Denote also by $L_X^i X' = L^i(X, X')$. By the total symmetry of L^i, Lemma 3.2 and (3.9) we have

$$\begin{aligned}[L_X^i, L_{X'}^i]X'' &= L^i(X, L^i(X', X'')) - L^i(X', L^i(X, X''))\\ &= K(X, L^i(X', X'')) - K(X', L^i(X, X''))\\ &= K(X, K(X', X'')) - K(X', K(X, X''))\\ &\quad - \sum_j \lambda_i^j [h(X', X'')K(u_j, X) - h(X, X'')K(u_j, X')]\\ &= [K_X, K_{X'}]X'' - \sum_j (\lambda_i^j)^2 [h(X', X'')X - h(X, X'')X'].\end{aligned}$$

Together with Gauss equation (2.5), by Lemma 3.2 we further have (3.10). Combining this with (3.1) we deduce that

$$\begin{aligned}\hat{R}(X, X')L^i(X'', \tilde{X}) &= \hat{R}(X, X')K(X'', \tilde{X}) - \sum_j \lambda_i^j h(X'', \tilde{X})\hat{R}(X, X')u_j\\ &= K(\hat{R}(X, X')X'', \tilde{X}) + K(X'', \hat{R}(X, X')\tilde{X})\\ &= L^i(\hat{R}(X, X')X'', \tilde{X}) + L^i(X'', \hat{R}(X, X')\tilde{X})\\ &\quad + \sum_j \lambda_i^j [h(\hat{R}(X, X')X'', \tilde{X}) + h(X'', \hat{R}(X, X')\tilde{X})]u_j\\ &= L^i(\hat{R}(X, X')X'', \tilde{X}) + L^i(X'', \hat{R}(X, X')\tilde{X}).\end{aligned}$$

This shows the part (iii).

For part (iv), as $\mathfrak{D}(\mu_i)$ is spherical, we have

$$\hat{\nabla}_X X' = \hat{\nabla}_X^{\perp} X' + \rho_i h(X, X')T,\ \ \hat{\nabla}_X T = -\rho_i X,\ \ X(\rho_i) = 0.$$

It is well known that the projection $\hat{\nabla}_{X'}^{\perp} X$ of $\hat{\nabla}_{X'} X$ on $\mathfrak{D}(\mu_i)$ defines a connection that turns out to be the Levi-Civita connection of the induced metric on the integral manifold of $\mathfrak{D}(\mu_i)$ from (M^n, h). Then, by the definition of curvature tensor and $\hat{R}(X, X') \cdot L^i = 0$, direct computations give (3.11). □

Proposition 3.1 *Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R} \cdot C = 0$. If the multiplicity of affine principal curvature μ_i is more than one, then, by the notations as above, it holds that either $L^i = 0$, or $L^i \neq 0$ and*

$$\mu_i - \sum_{j=1}^{r} (\lambda_i^j)^2 < 0. \tag{3.13}$$

In particular, if M^n is an affine hypersphere, then M^n is either a hyperquadric (i.e., $C = 0$), or a hyperbolic affine hypersphere with $C \neq 0$.

Proof Set $\lambda = \mu_i - \sum_{j=1}^{r} (\lambda_i^j)^2$. It follows from Lemma 3.4 that L^i is totally symmetric and satisfies the apolarity condition. Moreover,

$$\begin{aligned} &\hat{R}(X, X')L^i(X'', \tilde{X}) = L^i(\hat{R}(X, X')X'', \tilde{X}) + L^i(X'', \hat{R}(X, X')\tilde{X}), \\ &\hat{R}(X, X')X'' = \lambda[h(X', X'')X - h(X, X'')X'] - [L^i_X, L^i_{X'}]X'' \end{aligned} \tag{3.14}$$

for any vector fields $X, X', X'' \in \mathfrak{D}(\mu_i)$.

Assume that $L^i \neq 0$. Fix a point $p \in M^n$, we now choose an orthonormal basis of $\mathfrak{D}(\mu_i)(p)$ with respect to the affine metric h in the following way. Let $U_p\mathfrak{D}(\mu_i) := \{u \in \mathfrak{D}(\mu_i)(p) | h(u, u) = 1\}$. Since h is positive definite, $U_p\mathfrak{D}(\mu_i)$ is compact. We define a function $f(u) = h(L^i_u u, u)$ on $U_p\mathfrak{D}(\mu_i)$. Let e_1 be an element of $U_p\mathfrak{D}(\mu_i)$ at which f attains an absolute maximum. Since $L^i \neq 0$, we have $f(e_1) > 0$.

Let $u \in U_p\mathfrak{D}(\mu_i)$ such that $h(e_1, u) = 0$, and define another function $g(t) = f(e_1 \cos t + u \sin t)$. Then we have $g'(0) = 3h(L^i_{e_1} e_1, u)$, $g''(0) = 6h(L^i_{e_1} u, u) - 3f(e_1)$. Since g attains an absolute maximum at $t = 0$, we have $g'(0) = 0$, thus $h(L^i_{e_1} e_1, u) = 0$. Then e_1 is an eigenvector of $L^i_{e_1}$ with eigenvalue $\nu_1 = f(e_1) > 0$. Let $e_2, \dots, e_{n_i}$ be orthonormal vectors of $\mathfrak{D}(\mu_i)(p)$, orthogonal to e_1, which are the remaining eigenvectors of $L^i_{e_1}$ corresponding to the eigenvalues $\nu_2, \dots, \nu_{n_i}$, respectively.

Since e_1 is an absolute maximum point of f, we know that $g''(0) \leq 0$. This implies that for every $j \geq 2$, we have $\nu_1 - 2\nu_j \geq 0$. From the apolarity condition of $L^i_{e_1}$ we have

$$\nu_1 + \nu_2 + \cdots + \nu_{n_i} = 0. \tag{3.15}$$

By applying (3.14) we have

$$\begin{aligned} &\nu_1 \hat{R}(e_1, e_j)e_1 = \hat{R}(e_1, e_j)L^i(e_1, e_1) = 2L^i(\hat{R}(e_1, e_j)e_1, e_1), \\ &\hat{R}(e_1, e_j)e_1 = -\lambda e_j - [L^i_{e_1}, L^i_{e_j}]e_1 = (-\lambda - \nu_j^2 + \nu_1 \nu_j)e_j, \end{aligned}$$

which imply that

$$(\nu_1 - 2\nu_j)(-\lambda - \nu_j^2 + \nu_1 \nu_j) = 0. \tag{3.16}$$

If $\nu_1 = 2\nu_j$ for all $j \geq 2$, then (3.15) implies that $\nu_1 = 0$, this is a contradiction to $\nu_1 = f(e_1) > 0$. Hence, there exists an integer $k \in \{1, \ldots, n_i - 1\}$ such that, after rearranging the ordering,

$$\nu_2 = \cdots = \nu_k = \tfrac{1}{2}\nu_1,\ \nu_{k+1} < \tfrac{1}{2}\nu_1, \ldots, \nu_{n_i} < \tfrac{1}{2}\nu_1. \tag{3.17}$$

Moreover, if $j > k$, we see from (3.16) that

$$-\lambda - \nu_j^2 + \nu_1\nu_j = 0. \tag{3.18}$$

Subtracting this for $j, \ell > k$, we have

$$(\nu_j - \nu_\ell)(\nu_1 - \nu_j - \nu_\ell) = 0.$$

Note from (3.17) that $\nu_1 - \nu_j - \nu_\ell > 0$. Thus $\nu_{k+1} = \cdots = \nu_{n_i} := \nu_0$. Then, it follows from (3.15) and (3.18) that

$$\nu_0 = -\tfrac{k+1}{2(n_i-k)}\nu_1, \quad -\lambda = \tfrac{(k+1)(2n_i-k+1)}{4(n_i-k)^2}\nu_1^2 > 0.$$

Hence, (3.13) follows. The first part has been proved.

If M^n is an affine hypersphere, from Lemma 3.4 we see that $\mathfrak{D}(\mu_i)(p)$ is the tangent space T_pM^n with $n_i = n$, the projection tensor L^i is nothing but K, $\mu_i = H$, and $r = 0$ (i.e., $\sum_{j=1}^{r}(\lambda_i^j)^2 = 0$). Following the same process as above, we see that either $K = 0$ or $K \neq 0$ and $H < 0$. The conclusion follows. □

Remark 3.3 For the affine hypersphere, Proposition 3.1 extends the result of Proposition 2.1 in [15] from $\hat{\nabla}C$ to $\hat{R} \cdot C = 0$. The technique, which is employed to construct a typical orthonormal basis on $\mathfrak{D}(\mu_i)(p)$, was introduced by Ejiri [16] and has been extended and widely applied for various purposes, see e.g. [5, 7, 9–11, 15, 21, 23, 30, 31, 33, 34].

Finally, we conclude this section by proving Theorem 1.2.

Completion of Theorem 1.2's Proof Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R} \cdot C = 0$ and $n \geq 2$, whose affine principal curvatures are all of the multiplicity at least two. Then, for $n = 2, 3$, M^n is an affine hypersphere.

For $n \geq 4$, we assume that there exist at least two different affine principal curvatures, namely μ_i and μ_j. Then, M^n is not an affine hypersphere, and thus $K \neq 0$. By Gauss equation (2.5) and Lemma 3.2 (i) we see that, for any unit vector fields $X, X' \in \mathfrak{D}(\mu_i)$, $Y \in \mathfrak{D}(\mu_j)$,

$$\begin{aligned} &K(X, Y) = 0,\ K(X, X') \in \mathfrak{D}(\mu_i),\\ &\hat{R}(X, Y)Y = \tfrac{1}{2}(\mu_i + \mu_j)X,\ \hat{R}(X, Y)X = -\tfrac{1}{2}(\mu_i + \mu_j)Y. \end{aligned}$$

By (3.1) we further obtain that

$$
\begin{aligned}
0 &= \hat{R}(X,Y)K(Y,X) = K(\hat{R}(X,Y)Y,X) + K(Y,\hat{R}(X,Y)X) \\
&= \tfrac{1}{2}(\mu_i + \mu_j)(K_X X - K_Y Y).
\end{aligned}
\tag{3.19}
$$

As $K \neq 0$, there must exist a unit vector $X_0 \in \mathfrak{D}(\mu_{i_0})$ for some μ_{i_0} such that $h(K_{X_0}X_0, X_0) \neq 0$. Let $X = X_0$ in (3.19), by multiplying this with X_0, we have

$$
\mu_{i_0} + \mu_j = 0 \tag{3.20}
$$

for any $\mu_j \neq \mu_{i_0}$. This means that there are exactly two different affine principal curvatures, namely μ_1 and μ_2, and $\mu_1 = -\mu_2$.

For any unit vector fields $X, X' \in \mathfrak{D}(\mu_1)$ and $Y, Y' \in \mathfrak{D}(\mu_2)$, by Lemma 3.3 (i) we have $X(\mu_1) = -X(\mu_2) = 0$. Similarly, $Y(\mu_2) = -Y(\mu_1) = 0$. Hence, μ_1 and μ_2 are constant. Then, taking $e_i = X, e_j = Y, e_k = X'$ and $e_i = Y, e_j = X, e_k = Y'$ respectively in Lemma 3.3 (ii), by Lemma 3.2 (i) we have

$$
h(\hat{\nabla}_X Y, X') = 0, \quad h(\hat{\nabla}_Y X, Y') = 0, \tag{3.21}
$$

which imply that $\hat{\nabla}_Y X, \hat{\nabla}_X X' \in \mathfrak{D}(\mu_1)$ and $\hat{\nabla}_X Y, \hat{\nabla}_Y Y' \in \mathfrak{D}(\mu_2)$. Together with the Codazzi equation (2.6) we see that

$$
0 = h((\hat{\nabla}_X K)(Y,X) - (\hat{\nabla}_Y K)(X,X), Y) = \tfrac{1}{2}(\mu_1 - \mu_2) = \mu_1, \tag{3.22}
$$

which implies that $\mu_2 = -\mu_1 = 0$. This is a contradiction to $\mu_1 \neq \mu_2$. Therefore, M^n has a single affine principal curvature, i.e., it is an affine hypersphere.

In summary, M^n is an affine hypersphere for $n \geq 2$. Then, Proposition 3.1 implies that M^n is either a hyperquadric or a hyperbolic affine hypersphere with $C \neq 0$.

For such affine hyperspheres, denote by $\{e_1, \ldots, e_n\}$ an orthonormal frame relative to h, set $A_{ijk} = h(K_{e_i}e_j, e_k)$, we see from (2.10) that the components of first covariant differentiation $A_{ijk,\ell}$ are totally symmetric. It follows from $\hat{R}\cdot K = 0$ and the apolarity condition that the components of second covariant differentiation $A_{ijk,\ell s}$ are symmetric and trace-free in any two indices. Then, by $n(n-1)J = h(K,K) = \sum (A_{ijk})^2$ we have

$$
\tfrac{1}{2}n(n-1)\triangle J = \sum (A_{ijk,\ell})^2 + \sum A_{ijk}A_{ijk,\ell\ell} = \sum (A_{ijk,\ell})^2, \tag{3.23}
$$

where, by (2.4) there holds (cf. (7) of [6])

$$
A_{ijk,\ell} = h((\hat{\nabla}_{e_\ell} K)(e_i,e_j), e_k) = -\tfrac{1}{2}(\hat{\nabla}_{e_\ell} C)(e_i,e_j,e_k).
$$

Together with $n(n-1)J = \kappa - n(n-1)H$, we have (1.2). Recall the formula (3.32) in [26] for the Laplacian of J on affine hyperspheres:

$$
\tfrac{1}{2}n(n-1)\triangle J = \sum (A_{ijk,\ell})^2 + \sum (\hat{R}_{ij})^2 + \sum (\hat{R}_{ijk\ell})^2 - (n+1)\kappa H. \tag{3.24}
$$

Combining with (3.23) we obtain (1.3), which implies that $\kappa H \geq 0$, and thus $\kappa \leq 0$ on the hyperbolic affine hypersphere.

Furthermore, for $n = 2$, it follows from Theorem 2.1 that M^2 is affinely equivalent to either a quadric or a flat affine sphere, and thus κ is constant. Together with the assumption that κ is constant for $n \geq 3$, we see from (1.2) that $\hat{\nabla} C = 0$ for $n \geq 2$. Then, M^n is affinely equivalent to one of the examples in Theorem 1.1. □

4 Proof of Theorem 1.3

Let M^n be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R} \cdot C = 0$ and $n \geq 3$. Assume that $\mu_1, \ldots, \mu_m$ are the m distinct affine principal curvatures of multiplicity $(1, n_2, \ldots, n_m)$ with $m \geq 2$ and $n_i \geq 2$, respectively. Then, M^n is not an affine hypersphere, and thus $K \neq 0$.

Assume that $m \geq 3$, and thus $n \geq 5$. For our purpose, it is sufficient to prove $m = 3$. In Lemmas 3.2-3.4, as $r = 1$ we always omit the upper index $j = 1$ of λ_i^j for simplicity. Denote by T the unit eigenvector field of the affine principal curvature μ_1, by Lemma 3.2 we have

$$\begin{aligned} &ST = \mu_1 T,\ SX = \mu_i X,\ SY = \mu_j Y, \\ &K_T T = \lambda_1 T,\ K_T X = \lambda_i X,\ K_T Y = \lambda_j Y, \\ &K(X, Y) = 0,\ \forall\ X \in \mathfrak{D}(\mu_i),\ \forall\ Y \in \mathfrak{D}(\mu_j), \end{aligned} \tag{4.1}$$

where

$$(\mu_1 - \mu_i)\lambda_i = (\mu_1 - \mu_j)\lambda_j = \bar{\lambda}_1,\ \ i \neq j \geq 2. \tag{4.2}$$

In the following, if no other stated, we always assume the unit vector fields

$$X, X' \in \mathfrak{D}(\mu_i),\ \ Y, Y' \in \mathfrak{D}(\mu_j),\ \ i \neq j \geq 2.$$

From the Gauss equation (2.5), by (4.1) we have

$$\begin{aligned} &\hat{R}(T, X)X = (\lambda_i^2 - \lambda_1\lambda_i + \tfrac{1}{2}(\mu_1 + \mu_i))T, \\ &\hat{R}(T, X)T = -(\lambda_i^2 - \lambda_1\lambda_i + \tfrac{1}{2}(\mu_1 + \mu_i))X, \\ &\hat{R}(Y, X)Y = (\lambda_i\lambda_j - \tfrac{1}{2}(\mu_i + \mu_j))X, \\ &\hat{R}(Y, X)X = (\tfrac{1}{2}(\mu_i + \mu_j) - \lambda_i\lambda_j)Y, \\ &\hat{R}(T, X)Y = 0. \end{aligned} \tag{4.3}$$

Notice that $K_Y Y - \lambda_j T \in \mathfrak{D}(\mu_j)$, by (3.1) and (4.3) we can compute

$$\begin{aligned}
&\hat{R}(T,X)K(X,T) = K(\hat{R}(T,X)X,T) + K(X,\hat{R}(T,X)T),\\
&\hat{R}(Y,X)K(X,Y) = K(\hat{R}(Y,X)X,Y) + K(X,\hat{R}(Y,X)Y),\\
&\hat{R}(T,X)K(Y,Y) = 2K(\hat{R}(T,X)Y,Y)
\end{aligned} \tag{4.4}$$

to obtain respectively that

$$\begin{aligned}
&(\lambda_i^2 - \lambda_1\lambda_i + \tfrac{1}{2}(\mu_1+\mu_i))(K_X X + (\lambda_i - \lambda_1)T) = 0,\\
&(\lambda_i\lambda_j - \tfrac{1}{2}(\mu_i+\mu_j))(K_X X - K_Y Y) = 0,\\
&(\lambda_i^2 - \lambda_1\lambda_i + \tfrac{1}{2}(\mu_1+\mu_i))\lambda_j X = 0.
\end{aligned} \tag{4.5}$$

Remark 4.1 $\bar{\lambda}_1 \neq 0$, and it follows from (4.2) that $\lambda_2, \ldots, \lambda_m$ are all nonzero and distinct. Otherwise, if $\bar{\lambda}_1 = 0$, by (4.2) and the apolarity condition we see that $\lambda_i = 0$ for all $i \geq 1$, thus $K_T = 0$. As $K \neq 0$, there must exist a unit vector $X_0 \in \mathfrak{D}(\mu_{i_0})$ for some eigenvalue $\mu_{i_0} \neq \mu_1$ such that $h(K_{X_0}X_0, X_0) \neq 0$. Taking the inner product with X_0 of the first two equations in (4.5) for $X = X_0$, we have

$$\mu_1 + \mu_{i_0} = \mu_j + \mu_{i_0} = 0,$$

thus $\mu_j = \mu_1 = -\mu_{i_0}$, i.e., $m = 2$. This is a contradiction to $m \geq 3$.

By multiplying the last two equations in (4.5) respectively with T and X, by Remark 4.1 we have

$$\begin{aligned}
&\mu_i + \mu_j = 2\lambda_i\lambda_j \neq 0,\ i \neq j > 1,\\
&\mu_1 + \mu_i = 2\lambda_i(\lambda_1 - \lambda_i),\ i > 1.
\end{aligned} \tag{4.6}$$

By subtracting these equations, we further obtain that

$$\begin{aligned}
&\mu_1 - \mu_j = 2\lambda_i(\lambda_1 - \lambda_i - \lambda_j) \neq 0,\\
&\mu_i - \mu_j = 2(\lambda_i - \lambda_j)(\lambda_1 - \lambda_i - \lambda_j) \neq 0.
\end{aligned} \tag{4.7}$$

Remark 4.2 $m \leq 4$. If there exist three different affine principal curvatures μ_i, μ_j, μ_k of multiplicity more than one, then from (4.2) and (4.6) we have

$$\frac{\lambda_j}{\lambda_k} = \frac{\mu_1 - \mu_k}{\mu_1 - \mu_j} = \frac{\mu_i + \mu_j}{\mu_i + \mu_k}, \tag{4.8}$$

which further implies that

$$\mu_1 = \mu_i + \mu_j + \mu_k. \tag{4.9}$$

Then μ_k is uniquely determined by (4.9) for fixed μ_i and μ_j. Therefore, $m \leq 4$.

By taking special vector fields in Lemma 3.3, we can obtain that

$$
\begin{aligned}
&X(\mu_1) = (\mu_1 - \mu_i)h(\hat{\nabla}_T T, X),\ \ X(\mu_i) = 0,\\
&Y(\mu_i)h(X, X') = (\mu_j - \mu_i)h(\hat{\nabla}_X Y, X'),\\
&(\mu_i - \mu_j)h(\hat{\nabla}_T X, Y) = (\mu_1 - \mu_j)h(\hat{\nabla}_X T, Y),\\
&(\mu_j - \mu_i)h(\hat{\nabla}_T Y, X) = (\mu_1 - \mu_i)h(\hat{\nabla}_Y T, X),\\
&T(\mu_i)h(X, X') = (\mu_1 - \mu_i)h(\hat{\nabla}_X T + \lambda_i X, X').
\end{aligned}
\tag{4.10}
$$

By the Codazzi equation (2.6) and (4.1), taking the inner product of

$$
(\hat{\nabla}_X K)(T, T) = (\hat{\nabla}_T K)(X, T) + \tfrac{1}{2}(\mu_i - \mu_1)X \tag{4.11}
$$

with T, X' and Y, respectively, we see that

$$
\begin{aligned}
&X(\lambda_1) = (\lambda_1 - 2\lambda_i)h(\hat{\nabla}_T T, X),\\
&(\lambda_1 - 2\lambda_i)h(\hat{\nabla}_X T, X') = h((T(\lambda_i) + \tfrac{1}{2}(\mu_i - \mu_1))X - K_X\hat{\nabla}_T T, X'),\\
&(\lambda_i - \lambda_j)h(\hat{\nabla}_T X, Y) = (\lambda_1 - 2\lambda_j)h(\hat{\nabla}_X T, Y).
\end{aligned}
\tag{4.12}
$$

By changing the role of X, Y in the last equation of (4.12), we also get

$$
(\lambda_j - \lambda_i)h(\hat{\nabla}_T Y, X) = (\lambda_1 - 2\lambda_i)h(\hat{\nabla}_Y T, X). \tag{4.13}
$$

Then we are ready to prove the following results.

Lemma 4.1 *It holds that*

$$
\begin{aligned}
&h(\hat{\nabla}_T X, Y) = h(\hat{\nabla}_X T, Y) = h(\hat{\nabla}_Y T, X) = 0,\\
&\hat{\nabla}_X T = -\rho_i X,\ \ \rho_i := \tfrac{T(\mu_i)}{\mu_i - \mu_1} + \lambda_i,
\end{aligned}
\tag{4.14}
$$

where $X \in \mathfrak{D}(\mu_i)$, $Y \in \mathfrak{D}(\mu_j)$, $i \neq j > 1$.

Proof From (4.10), (4.12) and (4.13) we have

$$
\begin{aligned}
&(\mu_i - \mu_j)h(\hat{\nabla}_T X, Y) = (\mu_1 - \mu_j)h(\hat{\nabla}_X T, Y) = (\mu_1 - \mu_i)h(\hat{\nabla}_Y T, X),\\
&(\lambda_i - \lambda_j)h(\hat{\nabla}_T X, Y) = (\lambda_1 - 2\lambda_j)h(\hat{\nabla}_X T, Y) = (\lambda_1 - 2\lambda_i)h(\hat{\nabla}_Y T, X).
\end{aligned}
\tag{4.15}
$$

Assume on the contrary that $h(\hat{\nabla}_T X, Y) \neq 0$, then (4.15) imply that $h(\hat{\nabla}_X T, Y)$ and $h(\hat{\nabla}_Y T, X)$ are nonzero, too. And we further see from (4.15) that

$$
\begin{aligned}
&(\mu_1 - \mu_j)(\lambda_i - \lambda_j) = (\mu_i - \mu_j)(\lambda_1 - 2\lambda_j),\\
&(\mu_1 - \mu_i)(\lambda_i - \lambda_j) = (\mu_i - \mu_j)(\lambda_1 - 2\lambda_i).
\end{aligned}
$$

By subtracting these equations, we get $(\mu_i - \mu_j)(\lambda_i - \lambda_j) = 0$. This is a contraction to $\lambda_i \neq \lambda_j$. Therefore, $h(\hat{\nabla}_T X, Y) = 0$. Then, the conclusions follow from the first line equation in (4.15), and the last equation of (4.10). □

By the Codazzi equation (2.6) and (4.1), taking the inner product of

$$(\hat{\nabla}_T K)(X, Y) - (\hat{\nabla}_X K)(T, Y) = 0 \tag{4.16}$$

with X' and Y, respectively, by Lemma 4.1 we have

$$\begin{aligned} &(\lambda_i - \lambda_j)h(\hat{\nabla}_X Y, X') = h(K_X X', \hat{\nabla}_T Y) = -\lambda_i h(X, X')h(\hat{\nabla}_T T, Y), \\ &X(\lambda_j) = h(K_Y Y, \hat{\nabla}_X T - \hat{\nabla}_T X) = \lambda_j h(\hat{\nabla}_T T, X). \end{aligned} \tag{4.17}$$

Lemma 4.2 *It holds that*

$$\begin{aligned} &\hat{\nabla}_T T = 0, \ h(\hat{\nabla}_X Y, X') = 0, \\ &T(\lambda_i) = (2\lambda_i - \lambda_1)\rho_i + \tfrac{1}{2}(\mu_1 - \mu_i), \\ &X(\lambda_1) = X(\lambda_i) = X(\mu_1) = X(\mu_i) = 0, \\ &\rho_i \lambda_j - \rho_j \lambda_i + \tfrac{1}{2}(\mu_i - \mu_j) = 0, \end{aligned} \tag{4.18}$$

where $X, X' \in \mathfrak{D}(\mu_i)$, $Y \in \mathfrak{D}(\mu_j)$, $i \neq j > 1$.

Proof Since $h(\hat{\nabla}_T T, T) = 0$, there exist unit vector fields $V_0^i \in \mathfrak{D}(\mu_i)$ such that

$$\hat{\nabla}_T T = a_2 V_0^2 + \cdots + a_m V_0^m$$

for some differential functions a_i. Then, we see from the first equation of (4.12) and (4.17) that

$$V_0^i(\lambda_1) = a_i(\lambda_1 - 2\lambda_i), \ V_0^i(\lambda_j) = a_i \lambda_j, \ j \neq i > 1. \tag{4.19}$$

Recall from the apolarity condition that $\lambda_1 + n_2\lambda_2 + \cdots + n_m\lambda_m = 0$, then

$$V_0^i(\lambda_i) = (1 + 2/n_i)a_i\lambda_i. \tag{4.20}$$

Let $\{V_0^i, \ldots, V_{n_i-1}^i\}$ be an orthonormal frame of $\mathfrak{D}(\mu_i)$ for $i > 1$. Taking $X = V_j^i$, $X' = L^i(V_0^i, V_j^i)$ in the last equation of (4.10), by Lemma 3.4 we obtain

$$\begin{aligned} &(\mu_1 - \mu_i)h(\hat{\nabla}_{V_j^i} T, L^i(V_0^i, V_j^i)) = (T(\mu_i) + (\mu_i - \mu_1)\lambda_i)h(V_j^i, L^i(V_0^i, V_j^i)), \\ &(\mu_1 - \mu_i)\sum_{j=0}^{n_i-1} h(\hat{\nabla}_{V_j^i} T, L^i(V_0^i, V_j^i)) \\ &\quad = (T(\mu_i) + (\mu_i - \mu_1)\lambda_i)\sum_{j=0}^{n_i-1} h(V_0^i, L^i(V_j^i, V_j^i)) = 0. \end{aligned} \tag{4.21}$$

Considering the Codazzi equation (2.6) of the following form

$$h((\hat{\nabla}_{V_j^i}K)(V_0^i,T),V_j^i)=h((\hat{\nabla}_{V_0^i}K)(V_j^i,T),V_j^i),\ \ j\neq 0 \tag{4.22}$$

we get

$$V_0^i(\lambda_i)=h(K_{V_j^i}V_j^i,\hat{\nabla}_{V_0^i}T)-h(K_{V_0^i}V_j^i,\hat{\nabla}_{V_j^i}T),\ \ j=1,\ldots,n_i-1,$$

which together with (4.21) further shows that

$$\begin{aligned}(n_i-1)V_0^i(\lambda_i)&=\sum_{j=1}^{n_i-1}h(K_{V_j^i}V_j^i,\hat{\nabla}_{V_0^i}T)-\sum_{j=1}^{n_i-1}h(K_{V_0^i}V_j^i,\hat{\nabla}_{V_j^i}T)\\&=-h(K_{V_0^i}V_0^i,\hat{\nabla}_{V_0^i}T)-\sum_{j=1}^{n_i-1}h(K_{V_0^i}V_j^i,\hat{\nabla}_{V_j^i}T)\\&=-\sum_{j=0}^{n_i-1}h(\hat{\nabla}_{V_j^i}T,L^i(V_0^i,V_j^i))=0.\end{aligned} \tag{4.23}$$

It follows from (4.20) that $a_i\lambda_i=0$, thus $a_i=0$. Then $\hat{\nabla}_T T=0$. Together with (4.17), (4.10) and (4.12), by (4.14) we have (4.18) except the last equation.

Finally, we consider the Codazzi equation (2.6) of the following form

$$h((\hat{\nabla}_Y K)(X,X),Y)=h((\hat{\nabla}_X K)(Y,X),Y)+\tfrac{1}{2}(\mu_j-\mu_i). \tag{4.24}$$

Since $h(\hat{\nabla}_X X,Y)=0$, and similarly $h(\hat{\nabla}_Y Y,X)=0$, then by (4.1) direct computations from (4.24) show the last equation of (4.18). □

Lemma 4.3 *If the number m of distinct affine principal curvatures is at least three, then $m=3$.*

Proof By Remark 4.2 it is sufficient to prove $m\neq 4$. On the contrary, assume $m=4$, let μ_2,μ_3,μ_4 be the three different affine principal curvatures of multiplicity more than one. For any $X\in\mathfrak{D}(\mu_2)$, $Y\in\mathfrak{D}(\mu_3)$, $Z\in\mathfrak{D}(\mu_4)$, by (4.1) we consider the Codazzi equation (2.6) of the following form

$$h((\hat{\nabla}_Y K)(X,Z),T)=h((\hat{\nabla}_X K)(Y,Z),T)$$

to obtain that

$$(\lambda_3-\lambda_4)h(\hat{\nabla}_X Y,Z)=(\lambda_2-\lambda_4)h(\hat{\nabla}_Y X,Z). \tag{4.25}$$

It follows from Lemma 3.3 (ii) that

$$(\mu_3-\mu_4)h(\hat{\nabla}_X Y,Z)=(\mu_2-\mu_4)h(\hat{\nabla}_Y X,Z)=(\mu_2-\mu_3)h(\hat{\nabla}_Z X,Y). \tag{4.26}$$

First, we claim that

$$h(\hat{\nabla}_X Y, Z) = h(\hat{\nabla}_Y X, Z) = h(\hat{\nabla}_Z X, Y) = 0. \tag{4.27}$$

On the contrary, assume that $h(\hat{\nabla}_X Y, Z) \neq 0$, then the linear homogeneous system of equations (4.25) and (4.26) has nonzero solutions, thus its determinant vanishes:

$$(\mu_2 - \mu_4)(\lambda_3 - \lambda_4) = (\mu_3 - \mu_4)(\lambda_2 - \lambda_4). \tag{4.28}$$

By the first equation of (4.6) we have

$$\mu_2 - \mu_4 = 2\lambda_3(\lambda_2 - \lambda_4), \ \mu_3 - \mu_4 = 2\lambda_2(\lambda_3 - \lambda_4),$$

which together with (4.28) imply that $\lambda_2 = \lambda_3$, a contradiction to Remark 4.1. Therefore, $h(\hat{\nabla}_X Y, Z) = 0$. Together with (4.26) the claim (4.27) follows.

Next, we consider the Gauss equations for unit vector fields $X \in \mathfrak{D}(\mu_i)$, $Y \in \mathfrak{D}(\mu_j)$. From (4.27), Lemmas 4.1 and 4.2 we see that

$$\hat{\nabla}_Y Y - \rho_j T \in \mathfrak{D}(\mu_j), \ \hat{\nabla}_X Y \in \mathfrak{D}(\mu_j), \ \hat{\nabla}_Y X, \hat{\nabla}_T X \in \mathfrak{D}(\mu_i),$$

which imply that $h(\hat{\nabla}_X \hat{\nabla}_Y Y, X) = -h(\hat{\nabla}_Y Y, \hat{\nabla}_X X) = -\rho_i \rho_j$. Then, by straightforward computation we obtain

$$\begin{aligned} h(\hat{R}(X, Y)Y, X) &= h(\hat{\nabla}_X \hat{\nabla}_Y Y - \hat{\nabla}_Y \hat{\nabla}_X Y, X) - h(\hat{\nabla}_{\hat{\nabla}_X Y} Y - \hat{\nabla}_{\hat{\nabla}_Y X} Y, X) \\ &= -\rho_i \rho_j + h(\hat{\nabla}_X Y, \hat{\nabla}_Y X) - 0 - h(Y, \hat{\nabla}_{\hat{\nabla}_Y X} X) \\ &= -\rho_i \rho_j. \end{aligned} \tag{4.29}$$

On the other hand, it follows from (4.3) and (4.6) that $h(\hat{R}(X, Y)Y, X) = 0$, thus $\rho_i \rho_j = 0$, which means that $\rho_2\rho_3 = \rho_3\rho_4 = \rho_2\rho_4 = 0$. Then at least two of ρ_2, ρ_3, ρ_4 are zero locally. Without loss of generality, we assume that $\rho_2 = \rho_3 = 0$. From the last equation of (4.18) for $i = 2$, $j = 3$ we have $\mu_2 = \mu_3$, a contradiction to $\mu_2 \neq \mu_3$. □

By Lemma 4.3 we finish the proof of Theorem 1.3.

5 Proof of Theorem 1.4

In this section, we continue the analysis of Sect. 4 for $m = 3$ to complete the proof of Theorem 1.4. Let $F : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersurface with $\hat{R} \cdot C = 0$ and $n \geq 5$. Assume that there are exactly three distinct affine principal curvatures μ_1, μ_2, μ_3 of multiplicity $(1, n_2, n_3)$ with $n_2 \geq 2$ and $n_3 \geq 2$, respectively.

First, we will prove the warped product structure of (M^n, h). By the apolarity condition we have

$$\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = 0, \tag{5.1}$$

which together with the equations of the third line in (4.18) gives that

$$X(\lambda_i) = Y(\lambda_i) = X(\mu_i) = Y(\mu_i) = 0,\ i = 1, 2, 3 \tag{5.2}$$

for any $X \in \mathfrak{D}(\mu_2)$, $Y \in \mathfrak{D}(\mu_3)$. Then, we can show the following lemma.

Lemma 5.1 *There hold that*

$$\begin{aligned} &X(\rho_i) = Y(\rho_i) = 0,\ \forall\, X \in \mathfrak{D}(\mu_2),\ \forall\, Y \in \mathfrak{D}(\mu_3),\\ &T(\lambda_i) = (2\lambda_i - \lambda_1)\rho_i + \tfrac{1}{2}(\mu_1 - \mu_i),\\ &T(\mu_i) = (\mu_i - \mu_1)(\rho_i - \lambda_i),\quad i = 2, 3,\\ &\rho_2\lambda_3 - \rho_3\lambda_2 + \tfrac{1}{2}(\mu_2 - \mu_3) = 0,\\ &\rho_2\rho_3 = 0,\ T(\rho_2) = \rho_2^2,\ T(\rho_3) = \rho_3^2. \end{aligned} \tag{5.3}$$

Proof Let $\{X_1, \ldots, X_{n_2}\}$ (resp. $\{Y_1, \ldots, Y_{n_3}\}$) be an orthonormal frame of $\mathfrak{D}(\mu_2)$ (resp. $\mathfrak{D}(\mu_3)$). Lemmas 4.1 and 4.2 imply that

$$\begin{aligned} &\hat{\nabla}_X Y = \sum b_j Y_j,\quad \hat{\nabla}_Y X = \sum a_i X_i,\\ &[X, X'] = \hat{\nabla}_X X' - \hat{\nabla}_{X'} X \in \mathfrak{D}(\mu_2), \end{aligned} \tag{5.4}$$

which simplify the Gauss equation as

$$\begin{aligned} 0 = \hat{R}(X, X')T &= \hat{\nabla}_X(-\rho_2 X') - \hat{\nabla}_{X'}(-\rho_2 X) + \rho_2[X, X']\\ &= -X(\rho_2)X' + X'(\rho_2)X, \end{aligned}$$

so we have $X(\rho_2) = 0$. Similarly, we get $Y(\rho_3) = 0$.

Analogously, using (5.4), from

$$\begin{aligned} 0 = \hat{R}(X, Y)T &= -\hat{\nabla}_X(\rho_3 Y) + \hat{\nabla}_Y(\rho_2 X) - \sum_j b_j \hat{\nabla}_{Y_j} T + \sum_i a_i \hat{\nabla}_{X_i} T\\ &= -X(\rho_3)Y + Y(\rho_2)X, \end{aligned}$$

we get $X(\rho_3) = Y(\rho_2) = 0$. Together with Lemmas 4.1 and 4.2 we have proved (5.3) except the equations of last line in (5.3).

By the same computations as that did in (4.29), we have $\rho_2\rho_3 = 0$. Analogously, it follows from (4.3) and (4.6) that $\hat{R}(X, T)T = 0$. On the other hand, by Lemmas 4.1 and 4.2 we also have

$$\begin{aligned} \hat{R}(X, T)T &= \hat{\nabla}_T(\rho_2 X) - \hat{\nabla}_{\hat{\nabla}_X T} T + \hat{\nabla}_{\hat{\nabla}_T X} T\\ &= T(\rho_2)X + \rho_2\hat{\nabla}_T X - \rho_2^2 X - \rho_2\hat{\nabla}_T X\\ &= (T(\rho_2) - \rho_2^2)X. \end{aligned}$$

Thus, $T(\rho_2) = \rho_2^2$. Similarly, we get $T(\rho_3) = \rho_3^2$. □

Now, it follows from Lemma 4.1 that

$$\hat{\nabla}_X T = -\rho_2 X, \quad \hat{\nabla}_Y T = -\rho_3 Y.$$

Together with previous lemmas in Sect. 4 we see that $\mathfrak{D}(\mu_i)$ $(i = 1, 2, 3)$ are integrable, and both $\mathfrak{D}(\mu_1) \oplus \mathfrak{D}(\mu_3)$ and $\mathfrak{D}(\mu_1) \oplus \mathfrak{D}(\mu_2)$ are auto-parallel. Moreover, one can show that $\mathfrak{D}(\mu_2)$ (resp. $\mathfrak{D}(\mu_3)$) is spherical with the mean curvature vector $\rho_2 T$ (resp. $\rho_3 T$). Therefore, by Theorem 2.3 we conclude that M^n is locally a warped product $\mathbb{R} \times_{f_2} M_2 \times_{f_3} M_3$, where $\mathbb{R}$, M_2 and M_3 are, respectively, integral manifolds of the distributions $\mathfrak{D}(\mu_1)$, $\mathfrak{D}(\mu_2)$ and $\mathfrak{D}(\mu_3)$. The warping functions f_2 and f_3 are determined by

$$\rho_i = -T(\ln f_i), \; i = 2, \; 3.$$

By the warped product structure, we always take the local coordinates $\{t, x_i, y_j\}$ on M^n such that $\frac{\partial}{\partial t} = T$, $\text{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n_2}}\} = \mathfrak{D}(\mu_2)$ and $\text{span}\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n_3}}\} = \mathfrak{D}(\mu_3)$, and also let $X, X' \in \mathfrak{D}(\mu_2)$ and $Y, Y' \in \mathfrak{D}(\mu_3)$ for convention. Then we see from (5.2) and (5.3) that all the functions μ_i, λ_i, ρ_j and f_j depend only on t. Denote by $\partial_t() = (\cdot)'$, they are related by (4.2), (4.6) and (5.3):

$$\begin{aligned} &(\mu_1 - \mu_2)\lambda_2 = (\mu_1 - \mu_3)\lambda_3 \neq 0, \; \mu_2 + \mu_3 = 2\lambda_2\lambda_3 \neq 0, \\ &\mu_1 + \mu_i = 2\lambda_i(\lambda_1 - \lambda_i), \; \rho_i' = \rho_i^2, \; i = 2, 3, \\ &\rho_2\rho_3 = 0, \; \rho_2\lambda_3 - \rho_3\lambda_2 + \tfrac{1}{2}(\mu_2 - \mu_3) = 0. \end{aligned} \tag{5.5}$$

By the equations of last line in (5.5), without loss of generality, from now on we assume $\rho_2 = 0$ locally, thus $\rho_3 \neq 0$. Then, we can solve from the equations above for the warping function f_3 and ρ_3 to get that, up to a translation and a direction of the parametric t,

$$f_2 = 1, \; f_3 = t, \; \rho_3 = -\tfrac{1}{t}, \tag{5.6}$$

where locally we take $t > 0$. Together with (5.5) we further see that

$$\lambda_2 = \tfrac{1}{2}(\mu_2 - \mu_3)\rho_3^{-1}, \; \lambda_3 = \tfrac{\mu_2+\mu_3}{\mu_2-\mu_3}\rho_3, \; \lambda_1 - \lambda_2 = \tfrac{\mu_1+\mu_2}{\mu_2-\mu_3}\rho_3. \tag{5.7}$$

Second, we will show some properties for the functions as above in next two lemmas. Recall from Remark 4.1 that λ_2, λ_3 are nonzero and distinct, we can prove the similar results for μ_2 and μ_3 as follows.

Lemma 5.2 *Locally, both μ_2 and μ_3 are nonzero and distinct.*

Proof As μ_1, μ_2, μ_3 are distinct, by $\rho_2 = 0$ we see from the equations of third line in (5.3) that $T(\mu_2) \neq 0$, thus μ_2 cannot vanish identically, locally let $\mu_2 \neq 0$.

Assume that $\mu_3 = 0$. Then $\mu_2\mu_1 \neq 0$. From Lemma 5.1 and (5.5) we see that

$$\begin{aligned} &\lambda_3 = \rho_3, \ (\mu_1 - \mu_2)\lambda_2 = \mu_1\lambda_3, \\ &2\lambda_3\lambda_2 = \mu_2, \ \lambda_1 - \lambda_2 = \tfrac{\mu_1+\mu_2}{2\lambda_2}, \end{aligned} \tag{5.8}$$

which together with (5.1) imply that

$$\begin{aligned} &\lambda_2 = \tfrac{1}{2}\mu_2\rho_3^{-1}, \qquad \lambda_1 = \tfrac{1}{2}\mu_2\rho_3^{-1} + (1+\mu_1/\mu_2)\rho_3, \\ &\mu_1 - \mu_2 = 2\rho_3^2\mu_1/\mu_2, \quad \lambda_1 = -\tfrac{1}{2}n_2\mu_2\rho_3^{-1} - n_3\rho_3. \end{aligned} \tag{5.9}$$

Therefore, it holds that

$$\begin{aligned} &\tfrac{1}{2}(n_2+1)\mu_2\rho_3^{-1} + (n_3+1)\rho_3 + \tfrac{\mu_1}{\mu_2}\rho_3 = 0, \\ &\tfrac{1}{2}(\mu_2-\mu_1)\rho_3^{-1} + \tfrac{\mu_1}{\mu_2}\rho_3 = 0. \end{aligned}$$

By subtracting these equations we get

$$\mu_1 + n_2\mu_2 = -2(n_3+1)\rho_3^2 \neq 0, \tag{5.10}$$

which together with the third equation of (5.9) shows that

$$\frac{\mu_1 + n_2\mu_2}{\mu_1 - \mu_2} = -\frac{n_3+1}{\mu_1/\mu_2}. \tag{5.11}$$

Then we see that $\kappa_0 := \mu_1/\mu_2$ is the solution of the quadric equation

$$\kappa_0^2 + n\kappa_0 - n_3 - 1 = 0.$$

It follows from this and (5.10) that κ_0 is a constant, $\kappa_0 \notin \{0, 1, 2, -n_2\}$, and

$$\mu_2 = -\frac{2(n_3+1)}{n_2+\kappa_0}\rho_3^2, \ \ \mu_1 = -\frac{2(n_3+1)\kappa_0}{n_2+\kappa_0}\rho_3^2. \tag{5.12}$$

By taking the derivative on both sides of $\mu_2 = 2\lambda_2\rho_3$ in (5.8), we see from (5.3) and (5.9) that

$$\mu_2' = 2\rho_3^2\lambda_2 + \rho_3(\mu_1 - \mu_2) = \mu_1\rho_3.$$

On the other hand, by (5.12) we have $\mu_2' = 2\rho_3\mu_2 = \frac{2}{\kappa_0}\mu_1\rho_3$. Combining this with the equation above, as $\kappa_0 \neq 2$, we get $\mu_1\rho_3 = 0$, a contradiction to $\mu_1\rho_3 \neq 0$. Therefore, $\mu_3 \neq 0$. □

Furthermore, we see from (5.5) that

$$\rho_3\lambda_2 = \tfrac{1}{2}(\mu_2 - \mu_3) = \mu_2 - \lambda_2\lambda_3 = \lambda_2\lambda_3 - \mu_3, \tag{5.13}$$

which implies that

$$\mu_2 = \lambda_2(\lambda_3 + \rho_3), \quad \mu_3 = \lambda_2(\lambda_3 - \rho_3). \tag{5.14}$$

Then, it holds that

$$\mu_2(\rho_3 - \lambda_3) + \mu_3\lambda_2 = \lambda_2(\rho_3 - \lambda_3)(\rho_3 + \lambda_3 - \lambda_2) = \mu_3(\lambda_2 - \rho_3 - \lambda_3). \tag{5.15}$$

Now, we are ready to prove the following lemma.

Lemma 5.3 *Set $H_2 = \mu_2 - \lambda_2^2$, $H_3 = 1 + (\mu_3 - \lambda_3^2)/\rho_3^2$. Then H_2 and H_3 are nonzero constant. Moreover,*

$$\begin{aligned} &4\mu_3 H_2 + (\mu_2 - \mu_3)^2 H_3 = 0, \\ &\mu_2(\rho_3 - \lambda_3) + \mu_3\lambda_2 = (\rho_3 - \lambda_3)H_2 \neq 0. \end{aligned} \tag{5.16}$$

Proof By (5.3) and the equations of second line in (5.5) we can check that

$$(\mu_2 - \lambda_2^2)' = 0, \ (\tfrac{\mu_3 - \lambda_3^2}{\rho_3^2})' = 0.$$

Therefore, H_2 and H_3 are all constant. By Lemma 5.2 and (5.14) we have

$$\rho_3 - \lambda_3 \neq 0, \ H_2 = \lambda_2(\rho_3 + \lambda_3 - \lambda_2), \ H_3 = (\rho_3 - \lambda_3)(\rho_3 + \lambda_3 - \lambda_2)\rho_3^{-2}. \tag{5.17}$$

Assume that $H_2 = 0$. Then we see from (5.17) and $\lambda_2 \neq 0$ that

$$\begin{aligned} &\rho_3 = \lambda_2 - \lambda_3 \neq 0, \ \mu_2 = \lambda_2^2, \\ &\mu_3 - \lambda_3^2 + \rho_3^2 = \rho_3^2 H_3 = 0. \end{aligned} \tag{5.18}$$

Taking the derivative on both sides of $\rho_3 = \lambda_2 - \lambda_3$, by (5.3) and (5.14) we have

$$\rho_3^2 = (\lambda_1 - \lambda_2 - 2\lambda_3)\rho_3,$$

which together with (5.18) implies that

$$\lambda_1 = 2\lambda_2 + \lambda_3. \tag{5.19}$$

Combining with (5.1) and (5.18) we obtain

$$\lambda_3 = -\tfrac{n_2+2}{n_3+1}\lambda_2, \ (\lambda_3 - \lambda_2)^2 = (\tfrac{n+2}{n_3+1})^2\lambda_2^2 = \rho_3^2,$$

which further show that

$$\mu_2 = \lambda_2^2 = (\tfrac{n_3+1}{n+2})^2\rho_3^2, \ \mu_3 = \lambda_3^2 - \rho_3^2 = -\tfrac{(n_3+1)(n+n_2+4)}{(n+2)^2}\rho_3^2.$$

By combining this with (5.13) we have

$$\lambda_2 = \tfrac{1}{2}(\mu_2 - \mu_3)\rho_3^{-1} = \tfrac{n_3+1}{n+2}\rho_3,\ \lambda_3 = -\tfrac{n_2+2}{n+2}\rho_3, \tag{5.20}$$

which imply that

$$\lambda_2' = \tfrac{n_3+1}{n+2}\rho_3^2. \tag{5.21}$$

On the other hand, by (5.19) and (5.20) we see from the second equation of (5.3) and the first equation of (4.7) that

$$\lambda_2' = \tfrac{1}{2}(\mu_1 - \mu_2) = \lambda_3\lambda_2 = -\tfrac{(n_2+2)(n_3+1)}{(n+2)^2}\rho_3^2.$$

Together with (5.21) we have $\rho_3 = 0$. This contradiction shows that $H_2 \neq 0$.

Now, by $H_2 \neq 0$, (5.13)-(5.15) and (5.17) we obtain (5.16), which together with Lemma 5.2 implies $H_3 \neq 0$. □

Finally, based on previous lemmas, we can prove Theorem 1.4.

***Completion of Theorem* 1.4's Proof** Define a vector field by

$$g_3 = M(\lambda_2\xi + \mu_2 T), \tag{5.22}$$

where $M(t)$ is a nonzero solution of the equation $M' + M(\lambda_1 - \lambda_2) = 0$. Then direct computations give that

$$\begin{aligned} D_T g_3 &= (M' + M(\lambda_1 - \lambda_2))(\lambda_2\xi + \mu_2 T) = 0,\\ D_X g_3 &= M(-\mu_2\lambda_2 X + \mu_2(\hat{\nabla}_X T + K(X,T))) = 0,\\ D_Y g_3 &= g_{3*}Y = -M(\mu_2(\rho_3 - \lambda_3) + \mu_3\lambda_2)Y,\\ D_{Y'}D_Y g_3 &= -M(\mu_2(\rho_3 - \lambda_3) + \mu_3\lambda_2)\\ &\quad \cdot [\hat{\nabla}^{\perp}_{Y'}Y + L^3(Y,Y') + h(Y,Y')(\xi + (\rho_3 + \lambda_3)T)], \end{aligned} \tag{5.23}$$

where $\hat{\nabla}^{\perp}_{Y'}Y = \hat{\nabla}_{Y'}Y - \rho_3 h(Y,Y')T$ is the projection of $\hat{\nabla}_{Y'}Y$ on $\mathfrak{D}(\mu_3)$, and L^3 is the projection tensor of K on $\mathfrak{D}(\mu_3)$ defined by (3.9).

Similarly, define another vector field

$$g_2 = N((\lambda_3 - \rho_3)\xi + \mu_3 T), \tag{5.24}$$

where $N(t)$ is a nonzero solution of $N' + N(\rho_3 + \lambda_1 - \lambda_3) = 0$. It holds that

$$\begin{aligned} D_T g_2 &= (N' + N(\rho_3 + \lambda_1 - \lambda_3))((\lambda_3 - \rho_3)\xi + \mu_3 T) = 0,\\ D_Y g_2 &= N(-\mu_3(\lambda_3 - \rho_3)Y + \mu_3(\hat{\nabla}_Y T + K(Y,T))) = 0,\\ D_X g_2 &= g_{2*}X = N(\mu_3\lambda_2 + \mu_2(\rho_3 - \lambda_3))X,\\ D_{X'}D_X g_2 &= N(\mu_3\lambda_2 + \mu_2(\rho_3 - \lambda_3))\\ &\quad \cdot [\hat{\nabla}^{\perp}_{X'}X + L^2(X,X') + h(X,X')(\xi + \lambda_2 T)], \end{aligned} \tag{5.25}$$

where $\hat{\nabla}^{\perp}_{X'}X$ is the projection of $\hat{\nabla}_{X'}X$ on $\mathfrak{D}(\mu_2)$, and L^2 is the projection tensor of K on $\mathfrak{D}(\mu_2)$ defined by (3.9).

From Lemma 5.3 we have $\mu_2(\rho_3-\lambda_3)+\mu_3\lambda_2\neq 0$. Then, by (5.23) (resp. (5.25)) we see that g_3 (resp. g_2) is an immersion from the integral manifold M_3 (resp. M_2) of $\mathfrak{D}(\mu_3)$ (resp. $\mathfrak{D}(\mu_2)$) into the affine space. Moreover, by (5.14), (5.22) and (5.23) there holds

$$D_{Y'}D_Y g_3 = g_{3*}(\hat{\nabla}^{\perp}_{Y'}Y + L^3(Y,Y')) - h(Y,Y')(\mu_3-\lambda_3^2+\rho_3^2)g_3 \in \mathfrak{D}(\mu_3)+\mathrm{span}(g_3),$$

where $\mu_3-\lambda_3^2+\rho_3^2 = H_3\rho_3^2\neq 0$. It follows from Lemma 3.4 that L^3 satisfies apolarity condition, and $\hat{R}^{\perp}(Y,Y')\cdot L^3\neq 0$ in general, thus g_3 is a proper affine hypersphere with affine metric $\rho_3^2 h = f_3^{-2}h$ (cf. (5.6)), affine mean curvature H_3, and difference tensor L^3. It follows from Proposition 3.1 that g_3 is an ellipsoid if $\mu_3-\lambda_3^2=\rho_3^2(H_3-1)\geq 0$, i.e., $H_3\geq 1$.

Similarly, we have

$$D_{X'}D_X g_2 = g_{2*}(\hat{\nabla}^{\perp}_{X'}X + L^2(X,X')) - h(X,X')(\mu_2-\lambda_2^2)g_2 \in \mathfrak{D}(\mu_2)+\mathrm{span}(g_2),$$

where $\mu_2-\lambda_2^2 = H_2\neq 0$. Then, we see from $\rho_2=0$ and Lemma 3.4 that L^2 satisfies apolarity condition and $\hat{R}^{\perp}\cdot L^2=0$. Therefore, g_2 is a proper affine hypersphere with affine metric h, affine mean curvature H_2, and difference tensor L^2. Hence, g_2 has semi-parallel cubic form. It follows from Proposition 3.1 that g_2 is an ellipsoid if $H_2>0$.

Let $\beta_1(t)$ and $\beta_2(t)$ be functions such that

$$\beta_1' = -\beta_2,\quad \beta_2' = 1+\beta_1\mu_1-\beta_2\lambda_1.$$

Denote by $\delta_1 = 1+\mu_2\beta_1-\lambda_2\beta_2$ and $\delta_2 = 1+\mu_3\beta_1+(\rho_3-\lambda_3)\beta_2$. It follows from Lemma 5.3 that $\mu_2(\rho_3-\lambda_3)+\mu_3\lambda_2\neq 0$. Then, by choosing the initial conditions for β_1 and β_2 appropriately we can let $\delta_1(0)=\delta_2(0)=0$. Moreover, from (4.6) and (5.3) we see that

$$\delta_1' = -\lambda_2\delta_1,\ \ \delta_2' = (\rho_3-\lambda_3)\delta_2.$$

Therefore, by the initial conditions we have $\delta_1=\delta_2=0$ identically.

Now, straight computations from above show that

$$\begin{aligned} &D_X(\beta_1\xi+\beta_2T) = X,\ \ \forall\, X\in\mathfrak{D}(\mu_2),\\ &D_Y(\beta_1\xi+\beta_2T) = Y,\ \ \forall\, Y\in\mathfrak{D}(\mu_3),\\ &D_T(\beta_1\xi+\beta_2T) = T. \end{aligned}$$

Then, up to a translation constant, we can write $F: M^n\to\mathbb{R}^{n+1}$ as

$$F = \beta_1\xi+\beta_2T.$$

From (5.22) and (5.24), as $\mu_2(\rho_3 - \lambda_3) + \mu_3\lambda_2 \neq 0$, by (5.15) and (5.17) we can uniquely express ξ and T to obtain

$$F(t, x, y) = \gamma_2(t)g_2(x) + \gamma_3(t)g_3(y), \tag{5.26}$$

where $x = (x_1, \ldots, x_{n_2})$, $y = (y_1, \ldots, y_{n_3})$,

$$\gamma_2(t) = \tfrac{N^{-1}}{\mu_2(\rho_3-\lambda_3)+\mu_3\lambda_2} = \tfrac{1}{H_3 N\lambda_2\rho_3^2},$$
$$\gamma_3(t) = \tfrac{-M^{-1}}{\mu_2(\rho_3-\lambda_3)+\mu_3\lambda_2} = -\tfrac{1}{H_3 M\lambda_2\rho_3^2}.$$

By (5.3), (5.5) and (5.14), direct computations show that

$$\gamma_2'(t) = \lambda_2\gamma_2(t),\ \ \gamma_3'(t) = (\lambda_3 - \rho_3)\gamma_3(t),$$

where, it follows from (5.6) and (5.7) that

$$\lambda_2 = \tfrac{1}{2}(\mu_3 - \mu_2)t,\ \ \lambda_3 - \rho_3 = \tfrac{2\mu_3}{(\mu_3-\mu_2)t}.$$

Furthermore, we put $\rho_3 = -1/t$ and (5.7) into the first equation of (5.5) to get

$$t^2(\mu_1 - \mu_2)(\mu_2 - \mu_3)^2 = 2(\mu_1 - \mu_3)(\mu_2 + \mu_3).$$

By (5.7) and Lemma 5.3 we can rewrite the nonzero constants H_2 and H_3 by

$$H_2 = \mu_2 - \tfrac{1}{4}(\mu_2 - \mu_3)^2t^2,\ \ H_3 = \mu_3 t^2 + \tfrac{2}{\mu_2-\mu_3}.$$

Summing above, we have completed the proof of Theorem 1.4. $\square$

6 Proof of Theorem 1.5

Let $F : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex quasi-umbilical affine hypersurface with $\hat{R}\cdot C = 0$ and $n \geq 3$. Denote by μ_1, μ_2 the two distinct affine principal curvatures of multiplicity $(1, n-1)$, respectively. Then, M^n is not an affine hypersphere. Let T be the unit eigenvector field of the affine principal curvature μ_1. As before, omit the upper index $j = 1$ for λ_i^j in Lemma 3.2, we have

$$\begin{aligned} &ST = \mu_1 T, \qquad SX = \mu_2 X,\\ &K_T T = \lambda_1 T, \quad K_T X = \lambda_2 X,\ \forall\ X \in \mathfrak{D}(\mu_2), \end{aligned} \tag{6.1}$$

where by apolarity condition it holds that

$$\lambda_1 + (n-1)\lambda_2 = 0. \tag{6.2}$$

Remark 6.1 λ_1, λ_2 are distinct and nonzero, M^n is affinely equivalent to one of the three classes of immersions in Theorem 2.2 by taking $m = n - 1$. In fact, it follows from (6.1) that M^n satisfies the conditions of Theorem 2.2. In the proof of Theorem 2.2 in [1], it was shown in Lemma 3 that if $\lambda_2 = 0$, then $K_T = 0$ and M^n is an affine hypersphere. In our situation, by M^n being not an affine hypersphere we can exclude this possibility in Theorem 2.2, and obtain the conclusions.

Next, for more information we will show the warped product structure and discuss all the possibilities of the immersion. By (6.1) we see from (2.5) that

$$\hat{R}(X, T)T = (\lambda_2^2 - \lambda_1\lambda_2 + \tfrac{1}{2}(\mu_1 + \mu_2))X$$

for any unit vector field $X \in \mathfrak{D}(\mu_2)$. As $\lambda_2 \neq 0$, it follows from (6.2) that $\lambda_1 - 2\lambda_2 \neq 0$. Then, by (3.1) we can compute

$$h(\hat{R}(X, T)K(T, T), X) = 2h(K(\hat{R}(X, T)T, T), X)$$

to obtain that

$$\lambda_2^2 - \lambda_1\lambda_2 + \tfrac{1}{2}(\mu_1 + \mu_2) = 0, \tag{6.3}$$

which together with (6.2) implies that

$$\mu_1 + \mu_2 = -2n\lambda_2^2 < 0. \tag{6.4}$$

In the proof of Theorem 2.2 in [1], together with (6.3) it was shown that

$$\begin{aligned}
&\hat{\nabla}_T T = 0,\ \hat{\nabla}_X T = -\alpha X,\ T(\alpha) = \alpha^2,\\
&X(\alpha) = X(\mu_1) = X(\mu_2) = X(\lambda_2) = 0,\ \forall\, X \in \mathfrak{D}(\mu_2),\\
&T(\lambda_2) = (n+1)\lambda_2\alpha + \tfrac{1}{2}(\mu_1 - \mu_2),\\
&T(\mu_2) = (\mu_2 - \mu_1)(\alpha - \lambda_2).
\end{aligned} \tag{6.5}$$

Therefore, $\mathfrak{D}(\mu_1)$ is auto-parallel and the distribution $\mathfrak{D}(\mu_2)$ is spherical with the mean curvature vector αT. It follows from Theorem 2.3 that M^n is locally a warped product $\mathbb{R} \times_f M_2$, where $\mathbb{R}$ and M_2 are, respectively, integral manifolds of the distributions $\mathfrak{D}(\mu_1)$ and $\mathfrak{D}(\mu_2)$. The warping function f is determined by $\alpha = -T(\ln f)$. As before, we take the local coordinate $\{t, x_1, \ldots, x_{n-1}\}$ on M^n such that $\frac{\partial}{\partial t} = T$, $\mathrm{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\} = \mathfrak{D}(\mu_2)$. Hence all functions μ_i, λ_i, α and f depend only on t.

Denote by $\partial_t() = (\cdot)'$, we have $\alpha = -f'/f$. By solving from the equations above for f and α, we get that, up to a translation and a direction of the parametric t,

$$f = 1,\ \alpha = 0;\ \text{or}\ f = t,\ \alpha = -\tfrac{1}{t}, \tag{6.6}$$

where locally we take $t > 0$. From (6.4)-(6.6) we can check that $(\mu_2 - \lambda_2^2)' = 0$ if $f = 1$, and $((\mu_2 - \lambda_2^2)/\alpha^2)' = 0$ if $f = t$. Therefore, by (6.4) and (6.6) we have a constant:

$$H_0 = \begin{cases} \mu_2 - \lambda_2^2 = \frac{\mu_1 + (2n+1)\mu_2}{2n}, & \text{if } f = 1, \\ 1 + (\mu_2 - \lambda_2^2)/\alpha^2 = 1 + \frac{\mu_1 + (2n+1)\mu_2}{2n} t^2, & \text{if } f = t. \end{cases} \tag{6.7}$$

Finally, based on the proof of Theorem 2.2 and (6.6) we will follow the computations in [3] for three kinds of immersion in Theorem 2.2 to prove the following theorem, which give the explicit expressions of the immersions in Theorem 1.5.

Theorem 6.1 *Let M^n be a locally strongly convex quasi-umbilical affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{R} \cdot C = 0$ and $n \geq 3$. Denote by μ_1, μ_2 the two distinct affine principal curvatures of multiplicity $(1, n-1)$, respectively. Then, (M^n, h) is locally isometric to the warped product $\mathbb{R}_+ \times_f M_2$, where $f(t) = 1$ or t. Moreover, H_0 defined by (6.7) is a constant, and M^n is affinely equivalent to one of the following hypersurfaces:*

(1) The immersion $(\gamma_1(t), \gamma_2(t) g_2(x_1, \ldots, x_{n-1}))$ if $H_0 \mu_2 \neq 0$ and $f(t) = 1$, where $\gamma_1' \gamma_2^n = 1$, γ_2 is explicitly given in (6.10), g_2 is a hyperbolic affine hypersphere with semi-parallel cubic form if $H_0 < 0$, or an ellipsoid if $H_0 > 0$.

(2) The immersion $(\gamma_1(t), \gamma_2(t) g_2(x_1, \ldots, x_{n-1}))$ if $H_0 \mu_2 \neq 0$ and $f(t) = t$, where $\gamma_1' \gamma_2^n = t^{n+1}$, γ_2 is a positive solution to the differential equation

$$\gamma_2 = k(t)^{1/(n+1)}, \ t^2 k''(t) - (n+1) t k'(t) + (n+1) H_0 k(t) = 0,$$

g_2 is a locally strongly convex proper affine hypersphere with affine mean curvature H_0, and it is an ellipsoid if $H_0 \geq 1$.

(3) The immersion $(\gamma_1(t) x, \frac{1}{2} \gamma_1(t) \sum_{i=1}^{n-1} x_i^2 + \gamma_2(t), \gamma_1(t))$ if $\mu_2 \neq 0$, $H_0 = 0$ and $f(t) = 1$, where

$$\gamma_1 = ((n+1)t)^{\frac{1}{n+1}}, \gamma_2 = \frac{t((n+1)t)^{(n+2)/(n+1)}}{4n+6}, x = (x_1, \ldots, x_{n-1}).$$

(4) The immersion $(\gamma_1(t) x, \gamma_1(t) g(x) + \gamma_2(t), \gamma_1(t))$ if $\mu_2 \neq 0$, $H_0 = 0$ and $f(t) = t$, where

$$\gamma_1 = (\tfrac{n+1}{n+2} t^{n+2} + c_1)^{\frac{1}{n+1}}, \gamma_2' = \tfrac{n+1}{n+2} \gamma_1' \ln t - \tfrac{\gamma_1}{(n+2)t}, x = (x_1, \ldots, x_{n-1}),$$

c_1 is a constant, and $g(x)$ is a convex function whose graph immersion is a parabolic affine hypersphere.

(5) The immersion $(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}) - \frac{1}{n+2} \ln t, \frac{1}{n+2} t^{n+2})$ if $\mu_2 = 0$, where the warped function $f(t) = t$ and g is a convex function whose graph immersion is a parabolic affine hypersphere.

Proof We continue the analysis as above. First, we remark that $\mu_2 = 0$ if and only if $\alpha = \lambda_2$. In fact, it follows from the last equation of (6.5) that $\alpha = \lambda_2$ if $\mu_2 = 0$. If $\alpha = \lambda_2$, by taking its derivative on both sides, we see from (6.4) and (6.5) that $\mu_2 = 0$.

Therefore, $\mu_2 \neq 0$ if and only if $\alpha \neq \lambda_2$. Then, by H_0 defined by (6.7) we divide our discussions into three cases:

Case I: $H_0\mu_2 \neq 0$; **Case** II: $\mu_2 \neq 0$, $H_0 = 0$; **Case** III: $\mu_2 = 0$.

Case I. In this case, by (6.6) and (6.7) we have

$$\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0, \ \mu_2 - \lambda_2^2 + \alpha^2 \neq 0. \tag{6.8}$$

Then it was shown in [1] that M^n is locally given by

$$F(t, x_1, \ldots, x_{n-1}) = (\gamma_1(t), \gamma_2(t) g_2(x_1, \ldots, x_{n-1})),$$

where g_2 is the proper affine hypersphere. The same proof in [1] implies that the projection tensor L^2 of the difference tensor on $\mathfrak{D}(\mu_2)$ (cf. (3.9)) is the difference tensor of g_2, and g_2 has the affine metric $f^{-2}h$, affine mean curvature $(\mu_2 - \lambda_2^2 + \alpha^2) f^2 = H_0 \neq 0$ (cf. (6.6)-(6.8)) and the affine normal $-H_0 g_2$. Then, by the computations of this immersion on page 292-294 in [3] we take $\lambda = -H_0$ in (4.3) of [3], and deduce that

$$\begin{aligned} &\gamma_1' \gamma_2^n = f^{n+1}, \ \gamma_2 = k(t)^{1/(n+1)}, \\ &f^2 k''(t) - (n+1) f f' k'(t) + (n+1) H_0 k(t) = 0, \end{aligned} \tag{6.9}$$

where $k(t)$ and γ_2 are positive functions.

If $f = 1$, then $\alpha = 0$. By Lemma 3.4 (iv) the integral manifold M_2 of $\mathfrak{D}(\mu_2)$ is totally geodesic and $\hat{R}^{\perp} \cdot L^2 = 0$, i.e., g_2 has semi-parallel cubic form. Proposition 3.1 and (6.7) further imply that g_2 is an ellipsoid if $H_0 = \mu_2 - \lambda_2^2 > 0$. Moreover, (6.9) reduces to $\gamma_1' \gamma_2^n = 1$ and $\gamma_2 = k(t)^{1/(n+1)}$, where $k''(t) + (n+1) H_0 k(t) = 0$. Solving this equation we obtain that

$$\gamma_2 = \begin{cases} (c_1 e^{\sqrt{-(n+1)H_0 t}} + c_2 e^{-\sqrt{-(n+1)H_0 t}})^{\frac{1}{n+1}}, & \text{if } H_0 < 0, \\ (c_1 \cos(\sqrt{(n+1)H_0} t) + c_2 \sin(\sqrt{(n+1)H_0} t))^{\frac{1}{n+1}}, & \text{if } H_0 > 0, \end{cases} \tag{6.10}$$

where the constants c_1, c_2 are chosen such that $\gamma_2 > 0$. This is the immersion (1).

If $f = t$, then $\alpha = -1/t$. We see from (6.7) and Proposition 3.1 that g_2 is an ellipsoid if $\mu_2 - \lambda_2^2 = \alpha^2 (H_0 - 1) \geq 0$, i.e., $H_0 \geq 1$. Then, (6.9) reduces to $\gamma_1' \gamma_2^n = t^{n+1}$, $\gamma_2 = k(t)^{1/(n+1)}$ and

$$t^2 k''(t) - (n+1) t k'(t) + (n+1) H_0 k(t) = 0. \tag{6.11}$$

In particular, if $k(t)$ is a power function of t, we deduce that

$$\gamma_2(t) = \begin{cases} c_1 t^{\frac{n+2}{2(n+1)}}, & \text{if } H_0 = \frac{(n+2)^2}{4(n+1)}, \\ (c_2 t^{\tau_1} + c_3 t^{\tau_2})^{\frac{1}{n+1}}, & \text{if } H_0 < \frac{(n+2)^2}{4(n+1)}, \\ 0, & \text{if } H_0 > \frac{(n+2)^2}{4(n+1)}, \end{cases} \tag{6.12}$$

where c_1 is a positive constant, c_2, c_3 are chosen such that $\gamma_2 > 0$, and τ_1, τ_2 are the solutions of the quadric equation $\tau^2 - (n+2)\tau + (n+1)H_0 = 0$. This is the immersion (2).

Case II. In this case, by (6.6) and (6.7) we have

$$\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0, \;\; \mu_2 - \lambda_2^2 + \alpha^2 = 0.$$

It was shown in [1] that M^n is locally given by

$$F(t, x) = (\gamma_1(t)x, \gamma_1(t)g(x) + \gamma_2(t), \gamma_1(t)),$$

where $x = (x_1, \ldots, x_{n-1})$, and $g(x)$ is a convex function whose graph immersion is a parabolic affine hypersphere. As before, the same proof in [1] implies that the projection tensor L^2 of the difference tensor on $\mathfrak{D}(\mu_2)$ (cf. (3.9)) is the difference tensor of g. It follows from the computations of such hypersurfaces on page 294 of [3] that γ_1, γ_2 satisfy

$$(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2')f^2 = \gamma_1\gamma_1', \;\; f = |\gamma_1^n\gamma_1'|^{1/(n+1)}. \tag{6.13}$$

If $f = 1$, then $\alpha = 0$, and thus $\mu_2 - \lambda_2^2 = 0$, it follows from Proposition 3.1 that $L^2 = 0$, which together with (3.10) of Lemma 3.4 (iii) implies that M^n is a flat and quasi-umbilical affine hypersurface. We see from Theorem 4.1 of [3] that this is the immersion (3).

If $f = t$, by (6.13) we have $(\gamma_1^{n+1})' = \varepsilon(n+1)t^{n+1}$, $\varepsilon \in \{-1, 1\}$, which gives that $\gamma_1^{n+1} = \frac{n+1}{n+2}\varepsilon t^{n+2} + c_1$. By applying an affine reflection we may assume $\gamma_1 > 0$, then put $\varepsilon = 1$ and $\gamma_1 = (\frac{n+1}{n+2}t^{n+2} + c_1)^{1/(n+1)}$. By (6.13) we get $(\gamma_2'/\gamma_1')' = \frac{n+1}{n+2}t^{-1} + c_1 t^{-n-3}$, which yields that

$$\gamma_2'/\gamma_1' = \tfrac{n+1}{n+2}\ln t - \tfrac{c_1}{(n+2)t^{n+2}} + c_2.$$

Then, since $\gamma_1^n\gamma_1' = t^{n+1}$ and $c_1 = \gamma_1^{n+1} - \frac{n+1}{n+2}t^{n+2}$, we have

$$\gamma_2' = \tfrac{n+1}{n+2}\gamma_1'\ln t - \tfrac{\gamma_1}{(n+2)t} + (\tfrac{n+1}{(n+2)^2} + c_2)\gamma_1'$$

and

$$\gamma_2 = \int \big(\tfrac{n+1}{n+2}\gamma_1'\ln t - \tfrac{\gamma_1}{(n+2)t}\big)dt + (\tfrac{n+1}{(n+2)^2} + c_2)\gamma_1 + c_3.$$

Here, by applying equiaffine transformations we may put $c_2 = -(n+1)/(n+2)^2$ and $c_3 = 0$. We have the immersion (4).

Case III. By $\mu_2 = 0$ we have $\lambda_2 = \alpha$. It follows from $\lambda_2 \neq 0$ and (6.4)-(6.6) that

$$\lambda_2 = \alpha = -\tfrac{1}{t}, \;\; \mu_2 = 0, \;\; f = t, \;\; H_0 = 0, \;\; \mu_1 = -2n\alpha^2. \tag{6.14}$$

Moreover, it was shown in [1] that M^n is locally given by

$$F(t,x) = (x, g(x) + \gamma_1(t), \gamma_2(t)),$$

where $x = (x_1, \ldots, x_{n-1})$, $g(x)$ is a convex function whose graph immersion is a parabolic affine hypersphere. It follows from the computations of such hypersurfaces on page 295 of [3] that γ_1, γ_2 satisfy

$$\gamma_2'^3 = (\gamma_1''\gamma_2' - \gamma_1'\gamma_2'')f^{2(n+2)}, \;\; f = \mid \gamma_2' \mid^{1/(n+1)}. \tag{6.15}$$

Then, as $f = t$ in (6.14), we deduce that

$$\gamma_2' = \epsilon t^{n+1}, \;\; t^2\gamma_1'' - (n+1)t\gamma_1' - 1 = 0, \tag{6.16}$$

where $\epsilon \in \{-1, 1\}$. Then, we can directly solve these equations to obtain

$$\gamma_2(t) = \tfrac{\epsilon}{n+2}t^{n+2} + c_1, \;\; \gamma_1(t) = -\tfrac{\ln t}{n+2} + c_2 t^{n+2} + c_3.$$

By applying a translation and a reflection in $\mathbb{R}^{n+1}$ we may assume that $c_1 = c_3 = 0$, and $\gamma_2 > 0$, i.e., $\epsilon = 1$. Also, by possibly applying an equiaffine transformation we may put $c_2 = 0$. Hence, we obtain the immersion (5). $\square$

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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