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$L^p(\mathbb{R}^d)$ Boundedness for the Calderón Commutator with Rough Kernel

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Abstract

Let $k \in \mathbb{N}$, Ω be homogeneous of degree zero, integrable on S^{d-1} and have vanishing moment of order k, a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$, and $T_{\Omega,\,a;k}$ be the d-dimensional Calderón commutator defined by

$$T_{\Omega,\,a;k}f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+k}} \big(a(x)-a(y)\big)^k f(y)dy.$$

In this paper, the authors prove that if

$$\sup_{\zeta \in S^{d-1}} \int_{S^{d-1}} |\Omega(\theta)| \log^\beta \left(\frac{1}{|\theta \cdot \zeta|}\right) d\theta < \infty,$$

with $\beta \in (1,\ \infty)$, then for $\frac{2\beta}{2\beta-1} < p < 2\beta$, $T_{\Omega,\,a;\,k}$ is bounded on $L^p(\mathbb{R}^d)$.

Keywords Calderón commutator · Fourier transform · Littlewood–Paley theory · Calderón reproducing formula · Approximation

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1 Introduction

We will work on $\mathbb{R}^d$, $d \geq 2$. Let $k \in \mathbb{N}$, Ω be homogeneous of degree zero, integrable on S^{d-1}, the unit sphere in $\mathbb{R}^d$, and have vanishing moment of order k, that is, for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\int_{S^{d-1}} \Omega(\theta)\theta^\gamma d\theta = 0, \quad |\gamma| = k. \tag{1.1}$$

Let a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Define the d-dimensional Calderón commutator $T_{\Omega,a;\,k}$ by

$$T_{\Omega,a;k}f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+k}} \big(a(x) - a(y)\big)^k f(y)dy. \tag{1.2}$$

For simplicity, we denote $T_{\Omega,a;\,1}$ by $T_{\Omega,a}$. Commutators of this type were introduced by Calderón [1], who proved that if $\Omega \in L\log L(S^{d-1})$, then $T_{\Omega,a}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\ \infty)$. It should be pointed out that Calderón's result in [1] also holds for $T_{\Omega,a;\,k}$. Pan et al. [13] improved Calderón's result, and obtained the following conclusion.

Theorem 1.1 *Let Ω be homogeneous of degree zero, satisfy the vanishing moment (1.1) with $k = 1$, a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Suppose that $\Omega \in H^1(S^{d-1})$ (the Hardy space on S^{d-1}), then $T_{\Omega,a}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\ \infty)$.*

Chen et al. [4] showed that the converse of Theorem 1.1 is also true. Precisely, Chen at al. [4, p. 1501] established the following result.

Theorem 1.2 *Let Ω be homogeneous of degree zero, $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ for some $\alpha \in (0,\ 1]$, and satisfy the vanishing moment (1.1) with $k = 1$, $a \in L^1_{\text{loc}}(\mathbb{R}^d)$. If $T_{\Omega,\,a}$ is bounded on $L^p(\mathbb{R}^d)$ for some $p \in (1,\ \infty)$, then $\nabla a \in L^\infty(\mathbb{R}^d)$.*

Hofmann [10] considered the weighted L^p boundedness with A_p weights for $T_{\Omega,a;\,k}$, and proved that if $\Omega \in L^\infty(S^{d-1})$ and satisfies (1.1), then for $p \in (1,\ \infty)$ and $w \in A_p(\mathbb{R}^d)$, $T_{\Omega,a;k}$ is bounded on $L^p(\mathbb{R}^d,\ w)$, where and in the following, $A_p(\mathbb{R}^d)$ denotes the weight function class of Muckenhoupt, see [7, Chap. 9] for the definition and properties of $A_p(\mathbb{R}^d)$. Ding and Lai [5] considered the weak type endpoint estimate for $T_{\Omega,a}$, and proved that $\Omega \in L\log L(S^{d-1})$ is a sufficient condition such that $T_{\Omega,a}$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\,\infty}(\mathbb{R}^d)$.

For $\beta \in [1,\ \infty)$, we say that $\Omega \in GS_\beta(S^{d-1})$ if $\Omega \in L^1(S^{d-1})$ and

$$\sup_{\zeta \in S^{d-1}} \int_{S^{d-1}} |\Omega(\theta)| \log^\beta\left(\frac{1}{|\zeta \cdot \theta|}\right) d\theta < \infty. \tag{1.3}$$

The condition (1.3) was introduced by Grafakos and Stefanov [8] in order to study the $L^p(\mathbb{R}^d)$ boundedness for the homogeneous singular integral operator defined by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y)dy, \tag{1.4}$$

where Ω is homogeneous of degree zero and has mean value zero on S^{d-1}. Obviously, $L(\log L)^\beta(S^{d-1}) \subset GS_\beta(S^{d-1})$. On the other hand, as it was pointed out in [8], there exist integrable functions on S^{d-1} which are not in $H^1(S^{d-1})$ but satisfy (1.3) for all $\beta \in (1, \infty)$. Thus, it is of interest to consider the $L^p(\mathbb{R}^d)$ boundedness for operators such as T_Ω and $T_{\Omega,a;k}$ when $\Omega \in GS_\beta(S^{d-1})$. Grafakos and Stefanov [8] proved that if $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1, \infty]$, then T_Ω is bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/\beta < p < 1 + \beta$. Fan et al. [6] improved the result of [8], and proved the following result.

Theorem 1.3 *Let Ω be homogeneous of degree zero, integrable and have mean value zero on S^{d-1}. Suppose that $\Omega \in GS_\beta(S^{d-1})$ with $\beta \in (1, \infty)$, then for $\frac{2\beta}{2\beta-1} < p < 2\beta$, T_Ω is bounded on $L^p(\mathbb{R}^d)$.*

The purpose of this paper is to establish the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,a;k}$ when $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 1$. Our main result can be stated as follows.

Theorem 1.4 *Let $k \in \mathbb{N}$, Ω be homogeneous of degree zero, satisfy the vanishing moment (1.1), a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Suppose that $\Omega \in GS_\beta(S^{d-1})$ with $\beta \in (1, \infty)$, Then for $\frac{2\beta}{2\beta-1} < p < 2\beta$, $T_{\Omega,a;k}$ is bounded on $L^p(\mathbb{R}^d)$.*

Different from the operator T_Ω defined by (1.4), $T_{\Omega,a;k}$ is not a convolution operator, and the argument in [6, 8] does not apply to $T_{\Omega,a;k}$ directly. To prove Theorem 1.4, we will first prove the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,a;k}$ by employing the ideas used in [10], together with some new localizations and decompositions. The argument in the proof of $L^2(\mathbb{R}^d)$ boundedness is based on a refined decomposition appeared in (2.10). To prove the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,a;k}$, we will introduce a suitable approximation to $T_{\Omega,a;k}$ by a sequence of integral operators, whose kernels enjoy Hörmander's condition. We remark that the idea approximating rough convolution operators by smooth operators was originated by Watson [16].

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. Constant with subscript such as C_1, does not change in different occurrences. For any set $E \subset \mathbb{R}^d$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q. For a suitable function f, we denote $\widehat{f}$ the Fourier transform of f. For $p \in [1, \infty]$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$.

2 Proof of Theorem 1.4: $L^2(\mathbb{R}^d)$ Boundedness

This section is devoted to the proof of the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,a;\,k}$. For simplicity, we only consider the case $k = 1$. As it was pointed out in [10, Sect. 2], the argument in this section still works for all $k \in \mathbb{N}$, if we proceed by induction on the order k.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a radial function, $\operatorname{supp}\phi \subset B(0, 2)$, $\phi(x) = 1$ when $|x| \le 1$. Set $\varphi(x) = \phi(x) - \phi(2x)$. We then have that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \equiv 1, \ \ |x| > 0. \tag{2.1}$$

Let $\varphi_j(x) = \varphi(2^{-j}x)$ for $j \in \mathbb{Z}$.

For a function $\Omega \in L^1(S^{d-1})$, define the operator $W_{\Omega,j}$ by

$$W_{\Omega,j}h(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \varphi_j(x-y)h(y)dy. \tag{2.2}$$

Lemma 2.1 *Let Ω be homogeneous of degree zero, integrable on S^{d-1}, satisfy the vanishing moment (1.1) with $k = 1$ and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1, \infty)$, a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Then, for any $r \in (0, \infty)$, functions $\eta_1,\ \eta_2 \in C_0^\infty(\mathbb{R}^d)$ which are supported on balls of radius no larger than r,*

$$\Big|\int_{\mathbb{R}^d} \eta_2(x)T_{\Omega,\,a}\eta_1(x)dx\Big| \lesssim \|\Omega\|_{L^1(S^{d-1})} r^{-d} \prod_{j=1}^{2} \big(\|\eta_j\|_{L^\infty(\mathbb{R}^d)} + r\|\nabla\eta_j\|_{L^\infty(\mathbb{R}^d)}\big).$$

Recall that under the hypothesis of Lemma 2.1, the operator $T_{\Omega,m}$ defined by

$$T_{\Omega,m}f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)(x_m-y_m)}{|x-y|^{d+1}} f(y)dy, \ \ 1 \le m \le d \tag{2.3}$$

is bounded on $L^2(\mathbb{R}^d)$ (see [8]). Lemma 2.1 can be proved by repeating the proof of Lemma 2.5 in [10].

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial function, have integral zero and $\operatorname{supp}\psi \subset B(0, 1)$. Let Q_s be the operator defined by $Q_s f(x) = \psi_s * f(x)$, where $\psi_s(x) = s^{-d}\psi(s^{-1}x)$. We assume that

$$\int_0^\infty [\widehat{\psi}(s)]^4 \frac{ds}{s} = 1.$$

Then, the Calderón reproducing formula

$$\int_0^\infty Q_s^4 \frac{ds}{s} = I \tag{2.4}$$

holds true. In addition, the Littlewood–Paley theory tells us that

$$\Big\|\Big(\int_0^\infty |Q_s f|^2 \frac{ds}{s}\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

For each fixed $j \in \mathbb{Z}$, set

$$T^j_{\Omega,\,a} f(x) = \int_{\mathbb{R}^d} K_j(x,\ y) f(y)dy,$$

where

$$K_j(x,\ y) = \frac{\Omega(x-y)}{|x-y|^{d+1}}\big(a(x)-a(y)\big)\varphi_j(|x-y|).$$

Lemma 2.2 *Let Ω be homogeneous of degree zero, integrable on S^{d-1} and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1,\ \infty)$, then for $j \in \mathbb{Z}$ and $0 < s \le 2^j$,*

$$\|Q_s W_{\Omega,\,j} f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-j} \log^{-\beta}(2^j/s+1)\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof Let $K_{\Omega,j}(x) = \frac{\Omega(x)}{|x|^{d+1}}\varphi_j(|x|)$. By Plancherel's theorem, it suffices to prove that

$$|\widehat{\psi_s}(\xi)\widehat{K_{\Omega,j}}(\xi)| \lesssim 2^{-j} \log^{-\beta}(2^j/s+1). \tag{2.6}$$

As it was proved by Grafakos and Stefanov [8, p. 458], we know that

$$|\widehat{K_{\Omega,j}}(\xi)| \lesssim 2^{-j} \log^{-\beta}(|2^j\xi|+1).$$

On the other hand, it is easy to verify that

$$|\widehat{\psi_s}(\xi)| \lesssim \min\{1,\ |s\xi|\}.$$

Observe that (2.6) holds true when $|2^j\xi| \le 1$, since

$$|s\xi| \log^{-\beta}(2^j|\xi|+1) = \frac{s}{2^j}|2^j\xi| \log^{-\beta}(|2^j\xi|+1) \lesssim \frac{s}{2^j} \lesssim \log^{-\beta}(2^j/s+1).$$

If $|s\xi| \ge 1$, we certainly have that

$$|\widehat{\psi_s}(\xi)\widehat{K_{\Omega,j}}(\xi)| \lesssim 2^{-j} \log^{-\beta}(2^j|\xi|+1) \lesssim 2^{-j} \log^{-\beta}(2^j/s+1).$$

Now, we assume that $s|\xi| < 1$ and $|2^j\xi| > 1$, and

$$2^{-k}2^j < s \le 2^{-k+1}2^j,\ \ 2^{k_1-1} < |\xi| \le 2^{k_1}$$

for $k \in \mathbb{N}$ and $k_1 \in \mathbb{Z}$ respectively. Then $j + k_1 \in \mathbb{N}$, $j + k_1 \le k$ and

$$|s\xi| \log^{-\beta}(2^j|\xi| + 1) \lesssim 2^{j-k+k_1}(j + k_1)^{-\beta} \lesssim k^{-\beta} \lesssim \log^{-\beta}(2^j/s + 1).$$

This verifies (2.6). □

Lemma 2.3 *Let Ω be homogeneous of degree zero, satisfy the vanishing moment (1.1) with $k = 1$ and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1, \infty)$, a be a function on $\mathbb{R}^d$ with $\nabla a \in L^\infty(\mathbb{R}^d)$. Then*

(i) $T_{\Omega,a}1 \in \mathrm{BMO}(\mathbb{R}^d)$;
(ii) *for any $j \in \mathbb{Z}$ and $s \in (0, 2^j]$;*

$$\|Q_s T^j_{\Omega,a}1\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(S^{d-1})}2^{-j}s.$$

Conclusion (ii) is just Lemma 2.4 in [10], while (i) of Lemma 2.3 can be proved by mimicking the proof of Lemma 2.3 in [10], since for all $1 \le m \le d$, $T_{\Omega,m}$ defined by (2.3) is bounded on $L^2(\mathbb{R}^d)$ when $\Omega \in GS_\beta(S^{d-1})$ for $\beta > 1$. We omit the details for brevity.

Proof of Theorem 1.4 *$L^2(\mathbb{R}^d)$ boundedness.* By (2.4), it suffices to prove that for $f, g \in C_0^\infty(\mathbb{R}^d)$,

$$\Big|\int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q^4_s T_{\Omega,a} Q^4_t f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}, \tag{2.7}$$

and

$$\Big|\int_0^\infty \int_t^\infty \int_{\mathbb{R}^d} Q^4_s T_{\Omega,a} Q^4_t f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.8}$$

Observe that (2.8) can be deduced from (2.7) and a standard duality argument. Thus, we only need to prove (2.7).

We now prove (2.7). Without loss of generality, we assume that $\|\nabla a\|_{L^\infty(\mathbb{R}^d)} = 1$. Write

$$\begin{aligned}
&\int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q^4_s T_{\Omega,a} Q^4_t f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\\
&= \sum_{j\in\mathbb{Z}} \int_0^{2^j} \int_0^t \int_{\mathbb{R}^d} Q_s T^j_{\Omega,a} Q^4_t f(x)Q^3_s g(x)dx\frac{ds}{s}\frac{dt}{t}\\
&\quad + \sum_{j\in\mathbb{Z}} \int_{2^j}^\infty \int_0^{(2^j t^{\alpha-1})^{\frac{1}{\alpha}}} \int_{\mathbb{R}^d} Q_s T^j_{\Omega,a} Q^4_t f(x)Q^3_s g(x)dx\frac{ds}{s}\frac{dt}{t}\\
&\quad + \sum_{j\in\mathbb{Z}} \int_{2^j}^\infty \int_{(2^j t^{\alpha-1})^{\frac{1}{\alpha}}}^t \int_{\mathbb{R}^d} Q^4_s T^j_{\Omega,a} Q^4_t f(x)g(x)dx\frac{ds}{s}\frac{dt}{t} := \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3,
\end{aligned}$$

where $\alpha \in \left(\frac{d+1}{d+2},\ 1\right)$ is a constant.

We first consider term D_2. For each fixed $j \in \mathbb{Z}$, let $\{I_{j,l}\}_l$ be a sequence of cubes having disjoint interiors and side length 2^j, such that $\mathbb{R}^d = \cup_l I_{j,l}$. For each fixed j, l, let $\omega_{j,l} \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\omega_{j,l} \subset 48dI_{j,l}$, $0 \le \omega_{j,l} \le 1$ and $\omega_{j,l}(x) \equiv 1$ when $x \in 32dI_{j,l}$. Let $I^*_{j,l} = 64dI_{j,l}$ and $x_{j,l}$ be the center of $I_{j,l}$. For each l, set $a_{j,l}(y) = (a(y) - a(x_{j,l}))\omega_{j,l}(y)$, and $h_{s,j,l}(y) = Q_s^2 g(y)\chi_{I_{j,l}}(y)$. It is obvious that for all l,

$$\|a_{j,l}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^j,\ \ \|\nabla a_{j,l}\|_{L^\infty(\mathbb{R}^d)} \lesssim 1,$$

and for $s \in (0,\ 2^j]$ and $x \in \operatorname{supp} Q_s h_{s,j,l}$,

$$T^j_{\Omega,\,a}h(x) = a_{j,l}(x)W_{\Omega,j}h(x) - W_{\Omega,j}(a_{j,l}h)(x).$$

For each fixed j and l, let

$$\begin{aligned}
\mathrm{D}_{j,l,1}(s,t) &= -\int_{\mathbb{R}^d}[a_{j,l},\ Q_s]W_{\Omega,j}Q_t^4 f(x)Q_s h_{s,j,l}(x)dx,\\
\mathrm{D}_{j,l,2}(s,t) &= \int_{\mathbb{R}^d} a_{j,l}(x)Q_s W_{\Omega,j}Q_t^4 f(x)Q_s h_{s,j,l}(x)dx,\\
\mathrm{D}_{j,l,3}(s,t) &= \int_{\mathbb{R}^d} Q_s W_{\Omega,\,j}[a_{j,l},\ Q_s]Q_t^4 f(x)h_{s,j,l}(x)dx,
\end{aligned}$$

and

$$\mathrm{D}_{j,l,4}(s,\ t) = -\int_{\mathbb{R}^d} Q_s W_{\Omega,\,j}(a_{j,l}Q_s Q_t^4 f)(x)h_{s,j,l}(x)dx,$$

where and in the following, for a locally integrable function b and an operator U, $[b,\ U]$ denotes the commutator of U with symbol b, namely,

$$[b,\ U]h(x) = b(x)Uh(x) - U(bh)(x). \tag{2.9}$$

Observe that both of Q_s and $W_{\Omega,j}$ are convolution operators and $Q_s W_{\Omega,j} = W_{\Omega,j}Q_s$. For $j \in \mathbb{Z}$ and $s \in (0,\ 2^j]$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q_s^4 T^j_{\Omega,\,a}Q_t^4 f(x)g(x)dx\\
&\quad = \sum_l \int_{\mathbb{R}^d} Q_s T^j_{\Omega,\,a}Q_t^4 f(x)Q_s h_{s,j,l}(x)dx\\
&\quad = \sum_{n=1}^4 \sum_l \mathrm{D}_{j,l,n}(s,t).
\end{aligned} \tag{2.10}$$

It now follows from Hölder's inequality that

$$\begin{aligned}
&\Big|\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}\mathrm{D}_{j,l,1}(s,t)\frac{ds}{s}\frac{dt}{t}\Big|\\
&\le\Big\|\Big(\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}|\chi_{I_{j,l}^*}Q_t^4f|^2 2^{-j}s\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\\
&\quad\times\Big\|\Big(\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}|W_{\Omega,j}[a_{j,l},\,Q_s]Q_sh_{s,j,l}|^2\frac{1}{2^{-j}s}\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Invoking the fact that $\sum_l\chi_{I_{j,l}^*}\lesssim 1$, we deduce that

$$\begin{aligned}
&\Big\|\Big(\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}|\chi_{I_{j,l}^*}Q_t^4f|^2 2^{-j}s\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\\
&\lesssim\Big\|\Big(\int_0^{\infty}|Q_t^4f|^2\int_0^t\sum_{j:\,2^j\ge s^\alpha t^{1-\alpha}}2^{-j}s\frac{ds}{s}\frac{dt}{t}\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)}\lesssim\|f\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Let M_Ω be the operator defined by

$$M_\Omega f(x)=\sup_{r>0}r^{-d}\int_{|x-y|<r}|\Omega(x-y)||f(y)|dy.$$

The method of rotation of Calderón and Zygmund states that

$$\|M_\Omega f\|_{L^p(\mathbb{R}^d)}\lesssim\|\Omega\|_{L^1(S^{d-1})}\|f\|_{L^p(\mathbb{R}^d)},\ \ p\in(1,\,\infty). \tag{2.11}$$

Let M be the Hardy–Littlewood maximal operator. Observe that when $s\in(0,\,2^j]$,

$$\big|[a_{j,l},\,Q_s]h(x)\big|\le\int_{\mathbb{R}^d}|\psi_s(x-y)||a_{j,l}(x)-a_{j,l}(y)||h(y)|dy\lesssim sMh(x).$$

This, together with (2.11), yields

$$\begin{aligned}
&\Big\|\Big(\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}|W_{\Omega,j}[a_{j,l},\,Q_s]Q_sh_{s,j,l}|^2(2^{-j}s)^{-1}\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}^2\\
&\lesssim\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}\|M_\Omega MQ_sh_{s,j,l}\|_{L^2(\mathbb{R}^d)}^2 2^{-j}s\frac{ds}{s}\frac{dt}{t}\\
&\lesssim\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}\|h_{s,j,l}\|_{L^2(\mathbb{R}^d)}^2 2^{-j}s\frac{ds}{s}\frac{dt}{t}\lesssim\|g\|_{L^2(\mathbb{R}^d)}^2,
\end{aligned}$$

where the last inequality follows from the fact that

$$\int_s^\infty \sum_{j:2^j\geq s^\alpha t^{1-\alpha}} 2^{-j} s\frac{dt}{t} \lesssim 1.$$

Therefore,

$$\begin{aligned}&\Big|\sum_j\sum_l\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} \mathrm{D}_{j,l,1}(s,t)\frac{ds}{s}\frac{dt}{t}\Big|\\&\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{2.12}$$

Similar to the estimate (2.12), we have that

$$\Big|\sum_j\sum_l\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} \mathrm{D}_{j,l,3}(s,t)\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\tag{2.13}$$

To estimate the term $\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{1/\alpha}} \mathrm{D}_{j,l,2}(s,t)\frac{ds}{s}\frac{dt}{t}$, we write

$$\begin{aligned}\mathrm{D}_{j,l,2}(s,t) &= \int_{\mathbb{R}^d} Q_sW_{\Omega,j}Q_t^4f(x)[a_{j,l},Q_s]h_{s,j,l}(x)dx\\&\quad+\int_{\mathbb{R}^d} Q_sW_{\Omega,j}Q_t^4f(x)Q_s(a_{j,l}h_{s,j,l})(x)dx\\&=\mathrm{D}^1_{j,l,2}(s,t)+\mathrm{D}^2_{j,l,2}(s,t).\end{aligned}$$

Repeating the estimate for $\mathrm{D}_{j,l,1}$, we have that

$$\Big|\sum_j\sum_l\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} \mathrm{D}^1_{j,l,2}(s,t)\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\tag{2.14}$$

Write

$$\begin{aligned}&\Big|\sum_j\sum_l\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} \mathrm{D}^2_{j,l,2}(s,t)\frac{ds}{s}\frac{dt}{t}\Big|\\&\leq\Big\|\Big(\sum_j\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} |Q_s^2(2^jW_{\Omega,j})Q_t^3f|^2\log^\sigma(2^j/s+1)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\\&\quad\times\Big\|\Big(\sum_j\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}} \big|2^{-j}Q_t\big(\sum_l a_{j,l}h_{s,j,l}\big)\big|^2\log^{-\sigma}\Big(\frac{2^j}{s}+1\Big)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\\&:=\mathrm{I}_1\mathrm{I}_2,\end{aligned}$$

where $\sigma > 1$ is a constant such that $2\beta - \sigma > 1$. Invoking the estimate (2.5), we obtain that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \Big(\sum_j \int_0^{2^j} \big\|2^{-j}\sum_l a_{j,l}h_{s,j,l}\big\|^2_{L^2(\mathbb{R}^d)} \log^{-\sigma}(2^j/s+1)\frac{ds}{s}\Big)^{\frac12}\\
&\lesssim \Big(\sum_j \int_0^{2^j} \big\|\sum_l |h_{s,j,l}|\big\|^2_{L^2(\mathbb{R}^d)} \log^{-\sigma}(2^j/s+1)\frac{ds}{s}\Big)^{\frac12}\\
&= \Big(\int_0^\infty \|Q_s^2 g\|^2_{L^2(\mathbb{R}^d)} \sum_{j:2^j\ge s} \log^{-\sigma}(2^j/s+1)\frac{ds}{s}\Big)^{\frac12} \lesssim \|g\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Note that $Q_s^2(2^jW_{\Omega,j}) = Q_s(2^jW_{\Omega,j})Q_s$. It follows from Lemma 2.2 and (2.5) that

$$\begin{aligned}
\mathrm{I}_1 &= \Big(\sum_j \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{\frac1\alpha}} \|Q_s^2(2^jW_{\Omega,j})Q_t^3 f\|^2_{L^2(\mathbb{R}^d)} \log^{\sigma}\Big(\frac{2^j}{s}+1\Big)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\\
&\lesssim \Big(\sum_j \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{\frac1\alpha}} \|Q_sQ_t^3 f\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma}(2^j/s+1)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\\
&\lesssim \Big\|\Big(\int_0^\infty\int_0^\infty |Q_sQ_t^3 f|^2\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|^2_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

The estimates for I_1 and I_2 show that

$$\Big|\sum_j\sum_l \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{1/\alpha}} \mathrm{D}^2_{j,l,2}(s,t)\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

This, together with (2.14), gives us that

$$\Big|\sum_j\sum_l \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{1/\alpha}} \mathrm{D}_{j,l,2}(s,t)\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.15}$$

We now estimate term corresponding to $\int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{1/\alpha}} \mathrm{D}_{j,l,4}(s,t)\frac{ds}{s}\frac{dt}{t}$. Write

$$\begin{aligned}
&\Big|\sum_j\sum_l \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{1/\alpha}} \mathrm{D}_{j,l,4}(s,t)\frac{ds}{s}\frac{dt}{t}\Big|\\
&\le \Big\|\Big(\sum_j \int_{2^j}^\infty \int_0^{(2^jt^{\alpha-1})^{1/\alpha}} |Q_sQ_t^3 f|^2 \log^{-\sigma}(2^j/s+1)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

$$\times \Big\| \Big(\sum_j \int_{2^j}^\infty \int_0^{(2^j t^{\alpha-1})^{1/\alpha}} \Big| Q_t \Big(\sum_l a_{j,l} W_{\Omega,j} Q_s h_{s,j,l} \Big) \Big|^2 \log^\sigma \Big(\frac{2^j}{s} + 1 \Big) \frac{ds}{s} \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)}$$

$$:= \mathrm{I}_3 \mathrm{I}_4.$$

Obviously,

$$\mathrm{I}_3 \lesssim \Big\| \Big(\int_0^\infty \int_0^\infty |Q_s Q_t^3 f|^2 \frac{ds}{s} \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, it follows from Littlewood–Paley theory and Lemma 2.2 that

$$\begin{aligned}
\mathrm{I}_4 &\lesssim \Big(\sum_j \int_0^{2^j} \Big\| \sum_l a_{j,l} W_{\Omega,\, j} Q_s h_{s,j,l} \Big\|^2_{L^2(\mathbb{R}^d)} \log^\sigma \Big(\frac{2^j}{s} + 1 \Big) \frac{ds}{s} \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_j \int_0^{2^j} 2^{2j} \sum_l \|W_{\Omega,j} Q_s h_{s,j,l}\|^2_{L^2(\mathbb{R}^d)} \log^\sigma \Big(\frac{2^j}{s} + 1 \Big) \frac{ds}{s} \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_j \int_0^{2^j} \sum_l \|h_{s,j,l}\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma} \Big(\frac{2^j}{s} + 1 \Big) \frac{ds}{s} \Big)^{\frac{1}{2}} \lesssim \|g\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

since $\|a_{j,l}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^j$, and the supports of functions $\{a_{j,l} W_{\Omega,\, j} Q_s h_{s,j,l}\}$ have bounded overlaps. The estimate for I_4, together with the estimate for I_3, gives us that

$$\Big| \sum_j \sum_l \int_{2^j}^\infty \int_0^{(2^j t^{\alpha-1})^{1/\alpha}} \mathrm{D}_{j,l,4}(s,t) \frac{ds}{s} \frac{dt}{t} \Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.16}$$

Combining inequalities (2.12), (2.13), (2.15) and (2.16) leads to that

$$|\mathrm{D}_2| \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

The estimate for D_1 is fairly similar to the estimate D_2. For example, since

$$\int_0^t \sum_{j:\, 2^j \ge t} 2^{-j} s \frac{ds}{s} \lesssim 1, \quad \int_s^\infty \sum_{j: 2^j \ge t} 2^{-j} s \frac{dt}{t} \lesssim 1,$$

we have that

$$\begin{aligned}
&\Big| \sum_j \sum_l \int_0^{2^j} \int_0^t \mathrm{D}_{j,l,1}(s,t) \frac{ds}{s} \frac{dt}{t} \Big| \\
&\le \Big\| \Big(\sum_j \sum_l \int_0^{2^j} \int_0^t |\chi_{I^*_{j,l}} Q_t^4 f|^2 2^{-j} s \frac{ds}{s} \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

$$\times\Big\|\Big(\sum_j\sum_l\int_0^{2^j}\int_0^{\infty}|W_{\Omega,j}[a_{j,l},Q_s]Q_sh_{s,j,l}|^2(2^{-j}s)^{-1}\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

The estimates for terms $\sum_j\sum_l\int_0^{2^j}\int_0^t \mathrm{D}_{j,l,i}(s,t)\frac{ds}{s}\frac{dt}{t}$ $(i=2,3,4)$ are parallel to the estimates for $\sum_j\sum_l\int_{2^j}^{\infty}\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}\mathrm{D}_{j,l,i}(s,t)\frac{ds}{s}\frac{dt}{t}$. Altogether, we have that

$$|\mathrm{D}_1|\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

It remains to consider D_3. This was essentially proved in [10, pp. 1281–1283]. For the sake of self-contained, we present the details here. Set

$$h(x,y)=\int\int\psi_s(x-z)\sum_{j:2^j\le s^\alpha t^{1-\alpha}}K_j(z,u)[\psi_t(u-y)-\psi_t(x-y)]dudz.$$

Let H be the operator with integral kernel h. It then follows that

$$\begin{aligned}|\mathrm{D}_3|&\lesssim\Big|\int_0^{\infty}\int_0^t\int_{\mathbb{R}^d}HQ_t^3f(x)Q_s^3g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|\\&\quad+\Big|\sum_{j\in\mathbb{Z}}\int_{2^j}^{\infty}\int_{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}}^{t}\int_{\mathbb{R}^d}(Q_sT^j_{\Omega,a}1)(x)Q_t^4f(x)Q_s^3g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|\\&=|\mathrm{D}_{31}|+|\mathrm{D}_{32}|.\end{aligned}$$

As in [10, p. 1282], we obtain by Lemma 2.1 and the mean value theorem that

$$|h(x,y)|\lesssim\Big(\frac{s}{t}\Big)^{\varrho}t^{-d}\chi_{\{(x,y):|x-y|\le Ct\}}(x,y),$$

where $\varrho=(d+2)\alpha-d-1\in(0,\ 1)$. Then we have

$$|HQ_t^3f(x)|\lesssim\Big(\frac{s}{t}\Big)^{\varrho}M(Q_t^3f)(x),$$

and

$$\begin{aligned}|\mathrm{D}_{31}|&\lesssim\int_0^{\infty}\int_0^t\int_{\mathbb{R}^d}|M(Q_t^3f)(x)||Q_s^3g(x)|dx\Big(\frac{s}{t}\Big)^{\varrho}\frac{ds}{s}\frac{dt}{t}\\&\lesssim\Big\|\Big(\int_0^{\infty}\int_0^t|M(Q_t^3f)|^2\Big(\frac{s}{t}\Big)^{\varrho}\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\\&\quad\times\Big\|\Big(\int_0^{\infty}\int_s^{\infty}|Q_s^3g|^2\Big(\frac{s}{t}\Big)^{\varrho}\frac{dt}{t}\frac{ds}{s}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

$$\lesssim \Big\|\Big(\int_0^\infty |M(Q_t^3 f)|^2 \frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)} \Big\|\Big(\int_0^\infty |Q_s^3 g|^2 \frac{ds}{s}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

As for D_{32}, we split it into three parts as follows:

$$\mathrm{D}_{32} = \sum_{j\in\mathbb{Z}} \int_0^\infty \int_0^t - \sum_{j\in\mathbb{Z}} \int_0^{2^j} \int_0^t - \sum_{j\in\mathbb{Z}} \int_{2^j}^\infty \int_0^{(2^j t^{\alpha-1})^{\frac{1}{\alpha}}} = \mathrm{D}_{321} - \mathrm{D}_{322} - \mathrm{D}_{323}.$$

Let

$$\zeta(x) = \int_1^\infty \psi_t * \psi_t * \psi_t * \psi_t(x) \frac{dt}{t},\;\; P_s = \int_s^\infty Q_t^4 \frac{dt}{t}.$$

Han and Sawyer [9] proved that ζ is a radial function which is supported on a ball having radius C and has mean value zero. Observe that $P_s f(x) = \zeta_s * f(x)$ with $\zeta_s(x) = s^{-d}\zeta(s^{-1}x)$. The Littlewood–Paley theory tells us that

$$\Big\|\Big(\int_0^\infty |P_s f|^2 \frac{ds}{s}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

(i) of Lemma 2.3 states that $T_{\Omega,a}1 \in \mathrm{BMO}(\mathbb{R}^d)$. Recall that $\mathrm{supp}\,\psi \subset B(0,\,1)$ and ψ has integral zero. Thus for $x \in \mathbb{R}^d$,

$$|Q_s(T_{\Omega,a}1)(x)| \le s^{-d} \int_{|x-y|\le s} |\psi(s^{-1}(x-y))||T_{\Omega,a}1(y) - \langle T_{\Omega,a}1\rangle_{B(x,s)}|dy \lesssim 1,$$

where $\langle T_{\Omega,a}1\rangle_{B(x,s)}$ denotes the mean value of $T_{\Omega,a}1$ on the ball centered at x and having radius s. Therefore,

$$|\mathrm{D}_{321}| = \Big|\int_{\mathbb{R}^d}\int_0^\infty Q_s T_{\Omega,a}1(x) P_s f(x) Q_s^3 g(x) \frac{ds}{s} dx\Big|$$
$$\lesssim \Big\|\Big(\int_0^\infty |P_s f|^2 \frac{ds}{s}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)} \Big\|\Big(\int_0^\infty |Q_s^3 g|^2 \frac{ds}{s}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

From (ii) of Lemma 2.3 and Hölder's inequality, we obtain that

$$|\mathrm{D}_{322}| \lesssim \Big\|\Big(\sum_j \int_0^{2^j}\int_0^t 2^{-j} s |Q_t^4 f|^2 \frac{ds}{s}\frac{dt}{t}\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\times \Big\|\Big(\sum_j \int_0^{2^j}\int_0^t 2^{-j} s |Q_s^3 g|^2 \frac{ds}{s}\frac{dt}{t}\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

The same result holds true for D_{323}. Combining the estimates for terms D_{321}, D_{322} and D_{323} give us that

$$|D_3| \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

This leads to (2.7) and then establishes the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,a}$. □

3 Proof of Theorem 1.4: L^p Boundedness

We begin with some lemmas.

Lemma 3.1 *Let $\varpi \in C_0^\infty(\mathbb{R}^d)$ be a radial function such that* $\operatorname{supp} \varpi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \varpi^3(2^{-l}\xi) = 1, \quad |\xi| > 0,$$

and S_l be the multiplier operator defined by

$$\widehat{S_l f}(\xi) = \varpi(2^{-l}\xi)\widehat{f}(\xi).$$

Let $k \in \mathbb{Z}_+$, a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Then

$$\Big\| \Big(\sum_{l\in\mathbb{Z}} \big|2^{kl}[a, S_l]^k f\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}, \tag{3.1}$$

and

$$\Big\| \sum_{l\in\mathbb{Z}} 2^{kl}[a, S_l]^k f_l \Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big\| \Big(\sum_{l} |f_l|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^d)}, \tag{3.2}$$

where and in the following, for a locally integrable function a and an operator U, $[a, U]^0 f = Uf$, while for $k \in \mathbb{N}$ $[a, U]^k$ denotes the commutator of $[a, U]^{k-1}$ and a, defined as (2.9).

Note that (3.2) follows from (3.1) and a duality argument. For the case of $k = 0$, (3.1) follows from Littlewood–Paley theory. Inequality (3.1) with $k = 1$ was proved in [3, Lemma 2.3], while for the case of $k \ge 2$, the proof of (3.1) is similar to the proof of [3, Lemma 2.3].

Lemma 3.2 *Let $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$ with $n \le k$, D, E be positive constants and $E \le 1$, m be a multiplier such that $m \in L^1(\mathbb{R}^d)$, and*

$$\|m\|_{L^\infty(\mathbb{R}^d)} \le D^{-k} E$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \le D^{|\gamma|-k}.$$

Let a *be a function on* $\mathbb{R}^d$ *with* $\nabla a \in L^\infty(\mathbb{R}^d)$*, and* T_m *be the multiplier operator defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then for any $\varepsilon \in (0,\ 1)$,

$$\|[a,\ T_m]^n f\|_{L^2(\mathbb{R}^d)} \lesssim D^{n-k} E^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}.$$

Proof Our argument here is a generalization of the proof of Lemma 2 in [11], together with some more refined estimates, see also [12, Lemma 2.3] for the original version. We only consider the case $1 \le n \le k$, since

$$\|[a,\ T_m]^0 f\|_{L^2(\mathbb{R}^d)} \lesssim D^{-k} E^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}$$

holds obviously.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be the same as in (2.1). Recall that $\operatorname{supp}\varphi \subset \{1/4 \le |x| \le 4\}$, and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \equiv 1,\ \ |x| > 0.$$

Let $\varphi_{l,D}(x) = \varphi(2^{-l}D^{-1}x)$ for $l \in \mathbb{Z}$. Set

$$W_l(x) = K(x)\varphi_{l,D}(x),\ \ l \in \mathbb{Z},$$

where K is the inverse Fourier transform of m. Observing that for all multi-indices $\gamma \in \mathbb{Z}_+^d$, $\partial^\gamma\varphi(0) = 0$, we thus have that

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\xi^\gamma d\xi = 0.$$

This, in turn, implies that for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\widehat{W_l}(\xi)| &= \Big| \int_{\mathbb{R}^d} \Big(m(\xi - \frac{\eta}{2^l D}) - \sum_{|\gamma|\le N} \frac{1}{\gamma!}\partial^\gamma m(\xi)\Big(\frac{\eta}{2^l D}\Big)^\gamma \Big)\widehat{\varphi}(\eta)d\eta \Big| \\ &\lesssim 2^{-l(N+1)} D^{-(N+1)} \sum_{|\gamma|=N+1} \|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\eta|^{N+1}|\widehat{\varphi}(\eta)|d\eta \\ &\lesssim 2^{-l(N+1)} D^{-k}. \end{aligned} \tag{3.3}$$

On the other hand, a trivial computation gives that for $l \in \mathbb{Z}$,

$$\|\widehat{W_l}\|_{L^\infty(\mathbb{R}^d)} \le \|m\|_{L^\infty(\mathbb{R}^d)} \|\widehat{\varphi_{l,D}}\|_{L^1(\mathbb{R}^d)} \lesssim D^{-k} E. \tag{3.4}$$

Combining the inequalities (3.3) and (3.4) shows that for any $l \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\varepsilon \in (0,\ 1)$,

$$\|\widehat{W_l}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-l(N+1)(1-\varepsilon)} D^{-k} E^\varepsilon. \tag{3.5}$$

Let $T_{m,l}$ be the convolution operator with kernel W_l. Inequality (3.5), via Plancherel's theorem, tells us that for $l \in \mathbb{Z}$ and $N \in \mathbb{N}$,

$$\|T_{m,l} f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-l(N+1)(1-\varepsilon)} D^{-k} E^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.6}$$

We claim that for all $l \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\varepsilon \in (0,\ 1)$,

$$\|[a,\ T_{m,l}]^n f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-l(N+1)(1-\varepsilon)+ln} D^{n-k} E^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.7}$$

Observe that supp $W_l \subset \{x : |x| \le D2^{l+2}\}$. If I is a cube having side length $2^l D$, and $f \in L^2(\mathbb{R}^d)$ with supp $f \subset I$, then $T_{m,l} f \subset 100dI$. Therefore, to prove (3.7), we may assume that supp $f \subset I$ with I a cube having side length $2^l D$. Let $x_0 \in I$ and $a_I(y) = (a(y) - a(x_0))\chi_{100dI}(y)$. Then

$$\|a_I\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^l D.$$

Write

$$[a,\ T_{m,l}]^n f(x) = \sum_{i=0}^{n} (a_I(x))^i C_n^i T_{m,l}\big((-a_I)^{k-i} f\big)(x).$$

It then follows from (3.6) that

$$\begin{aligned}\|[a,\ T_{m,\,l}]^n f\|_{L^2(\mathbb{R}^d)} &\lesssim \sum_{i=0}^{n} 2^{il} D^i \|T_{m,l}\big((-a_I)^{n-i} f\big)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim 2^{nl-l(N+1)(1-\varepsilon)} D^{n-k} E^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

This yields (3.7).

We now conclude the proof of Lemma 3.2. Recall that $E \in (0,\ 1]$. It suffices to prove Lemma 3.2 for the case of $\varepsilon \in (2/3,\ 1)$. For fixed $\varepsilon \in (2/3,\ 1)$, we choose $N_1 \in \mathbb{N}$ such that $(N_1 + 1)(1 - \varepsilon) > n$, $N_2 \in \mathbb{N}$ such that $(N_2 + 1)(1 - \varepsilon) < n$. It follows from (3.7) that

$$\|[a,\ T_m]^n f\|_{L^2(\mathbb{R}^d)} \le \sum_{l \le 0} \|[a,\ T_{m,\,l}]^n f\|_{L^2(\mathbb{R}^d)} + \sum_{l \in \mathbb{N}} \|[a,\ T_{m,l}]^n f\|_{L^2(\mathbb{R}^d)}$$

$$\begin{aligned} &\lesssim D^{n-k} E^{\varepsilon} \sum_{l\in\mathbb{N}} 2^{-l(N_1+1)(1-\varepsilon)+ln} \|f\|_{L^2(\mathbb{R}^d)} \\ &\quad + D^{n-k} E^{\varepsilon} \sum_{l\le 0} 2^{-l(N_2+1)(1-\varepsilon)+ln} \|f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim D^{n-k} E^{\varepsilon} \|f\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3 *Let $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$ with $n \le k$, D, A and B be positive constants with $A,\ B < 1$, m be a multiplier such that $m \in L^1(\mathbb{R}^d)$, and*

$$\|m\|_{L^\infty(\mathbb{R}^d)} \le D^{-k}(AB)^{k+1},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \le D^{|\gamma|-k} B^{-|\gamma|}.$$

Let T_m be the multiplier operator defined by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Let a be a function on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$. Then for any $\sigma \in (0,\ 1)$,

$$\left\|[a,\ T_m]^n f\right\|_{L^2(\mathbb{R}^d)} \lesssim D^{n-k} A^\sigma B^{k-n+\sigma} \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.8}$$

Proof Let $T_{m,l}$ be the same as in the proof of Lemma 3.2. As in the proof of Lemma 3.2, we know that for all $l \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\varepsilon \in (0,\ 1)$,

$$\begin{aligned} \left\|[a,\ T_{m,l}]^n f\right\|_{L^2(\mathbb{R}^d)} &\lesssim 2^{-l(N+1)(1-\varepsilon)+nl} D^{n-k} \\ &\qquad \times B^{-(N+1)(1-\varepsilon)+(k+1)\varepsilon} A^{(k+1)\varepsilon} \|f\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.9}$$

For each fixed $\sigma \in (0,\ 1)$, we choose $\varepsilon \in (0,\ 1)$ such that

$$(k+1)\varepsilon - k - \sigma > 1 - \varepsilon,$$

and choose $N_1 \in \mathbb{N}$ such that

$$(N_1+1)(1-\varepsilon) > n,\ \ -(N_1+1)(1-\varepsilon) + (k+1)\varepsilon > k - n + \sigma.$$

Also, we choose $N_2 \in \mathbb{N}$ such that $(N_2+1)(1-\varepsilon) < n$. Note that such a N_2 satisfies

$$-(N_2+1)(1-\varepsilon) + (k+1)\varepsilon > k - n + \sigma.$$

Recalling that $B < 1$, we have that

$$B^{-(N_1+1)(1-\varepsilon)+(k+1)\varepsilon} \leq B^{k-n+\sigma}, \quad B^{-(N_2+1)(1-\varepsilon)+(k+1)\varepsilon} \leq B^{k-n+\sigma}.$$

Our desired estimate (3.8) now follows (3.9) by

$$\begin{aligned}\|[a, T_m]^n f\|_{L^2(\mathbb{R}^d)} &\lesssim D^{n-k} A^\sigma B^{k-n+\sigma} \sum_{l\in\mathbb{N}} 2^{-l(N_1+1)(1-\varepsilon)+ln} \|f\|_{L^2(\mathbb{R}^d)} \\ &\quad + D^{n-k} A^\sigma B^{k-n+\sigma} \sum_{l\leq 0} 2^{-l(N_2+1)(1-\varepsilon)+ln} \|f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim D^{n-k} B^{k-n+\sigma} A^\sigma \|f\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

since $(k+1)\varepsilon > \sigma$ and $A < 1$. This completes the proof of Lemma 3.3. □

The following conclusion is a variant of Theorem 1 in [11], and will be useful in the proof of Theorem 1.4.

Theorem 3.4 *Let $k \in \mathbb{N}$, $A \in (0, 1/2)$ be a constant, $\{\mu_j\}_{j\in\mathbb{Z}}$ be a sequence of functions on $\mathbb{R}^d\backslash\{0\}$. Suppose that for some $\beta \in (1, \infty)$,*

$$\|\mu_j\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-jk}, \ |\widehat{\mu_j}(\xi)| \lesssim 2^{-jk} \min\{|A2^j\xi|^{k+1}, \log^{-\beta}(2+|2^j\xi|)\},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma \widehat{\mu_j}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{j(|\gamma|-k)}.$$

Let $K(x) = \sum_{j\in\mathbb{Z}} \mu_j(x)$ and T be the convolution operator with kernel K. Then for any $\varepsilon \in (0, 1)$, function a on $\mathbb{R}^d$ with $\nabla a \in L^\infty(\mathbb{R}^d)$,

$$\|[a, T]^k f\|_{L^2(\mathbb{R}^d)} \lesssim \log^{-\varepsilon\beta+1}\Big(\frac{1}{A}\Big)\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof At first, we claim that for $k_1 \in \mathbb{Z}$ with $0 \leq k_1 \leq k$,

$$\|Tf\|_{L^2_{k_1-k}(\mathbb{R}^d)} \lesssim \|f\|_{L^2_{k_1}(\mathbb{R}^d)}, \tag{3.10}$$

where $\|f\|_{L^2_{k_w}(\mathbb{R}^d)}$ for $k_2 \in \mathbb{Z}$ is the Sobolev norm defined as

$$\|f\|^2_{L^2_{k_2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2k_2} |\widehat{f}(\xi)|^2 d\xi.$$

In fact, by the Fourier transfrom estimate of μ_j, we have that for each fixed $\xi \in \mathbb{R}^d\backslash\{0\}$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\mu_j}(\xi)| \lesssim \sum_{j:\,2^j\geq|\xi|^{-1}} 2^{-jk} + |\xi|^{k+1} \sum_{j:\,2^j\leq|\xi|^{-1}} 2^j \lesssim |\xi|^k.$$

This, together with Plancherel's theorem, gives (3.10).

Let U_j be the convolution operator with kernel μ_j, and $\varpi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varpi \le 1$, $\operatorname{supp}\varpi \subset \{1/4 \le |\xi| \le 4\}$ and

$$\sum_{l\in\mathbb{Z}} \varpi^3(2^{-l}\xi) = 1, \ \ |\xi| > 0.$$

Set $m_j(x) = \widehat{\mu_j}(\xi)$, and $m_j^l(\xi) = m_j(\xi)\varpi(2^{j-l}\xi)$. Define the operator U_j^l by

$$\widehat{U_j^l f}(\xi) = m_j^l(\xi)\varpi(2^{j-l}\xi)\widehat{f}(\xi).$$

Now let S_l be the multiplier operator defined as in Lemma 3.1. Let $f \in C_0^\infty(\mathbb{R}^d)$, $B = B(0,\ R)$ be a ball such that $\operatorname{supp} f \subset B$, and let $x_0 \in B$. We can write

$$\begin{aligned}
[a,\ T]^k f &= \sum_{n=0}^{k} C_k^n (a - a(x_0))^{k-n} T\big((a(x_0) - a)^n f\big)(x) \\
&= \sum_{n=0}^{k} C_k^n (a - a(x_0))^{k-n} \sum_l \sum_j (S_{l-j} U_j^l S_{l-j})\big((a(x_0) - a)^n f\big) \\
&= \sum_l \sum_j [a,\ S_{l-j} U_j^l S_{l-j}]^k f.
\end{aligned} \tag{3.11}$$

We now estimate $\big\|[a,\ S_{l-j}U_j^l S_{l-j}]^k f\big\|_{L^2(\mathbb{R}^d)}$. At first, we have that $m_j^l \in L^1(\mathbb{R}^d)$ and

$$|m_j^l(\xi)| \lesssim 2^{-jk}\min\{A^{k+1}2^{l(k+1)},\ \log^{-\beta}(2+2^l)\}.$$

Furthermore, by the fact that

$$|\partial^\gamma \phi(2^{j-l}\xi)| \lesssim 2^{(j-l)|\gamma|},\ \ |\partial^\gamma m_j(\xi)| \lesssim 2^{j(|\gamma|-k)},$$

it then follows that for all $\gamma \in \mathbb{Z}_+^d$,

$$|\partial^\gamma m_j^l(\xi)| \lesssim \begin{cases} 2^{j(|\gamma|-k)} & \text{if } l \in \mathbb{N} \\ 2^{j(|\gamma|-k)} 2^{-|\gamma| l}, & \text{if } l \le 0. \end{cases}$$

An application of Lemma 3.2 (with $D = 2^j$, $E = \min\{(A2^l)^{k+1},\ l^{-\beta}\}$) yields

$$\|[a,\ U_j^l]^n f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{j(n-k)}\min\{(A2^l)^{k+1},\ l^{-\beta}\}^\varepsilon \|f\|_{L^2(\mathbb{R}^d)},\ \ l \in \mathbb{N}. \tag{3.12}$$

On the other hand, we deduce from Lemma 3.3 (with $D = 2^j$ and $B = 2^l$) that for some $\sigma \in (0,\ 1)$,

$$\|[a,\ U_j^l]^n f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{j(n-k)} 2^{l(k-n)} A^\sigma 2^{\sigma l} \|f\|_{L^2(\mathbb{R}^d)},\ \ l \le 0. \tag{3.13}$$

Write

$$[a,\ S_{l-j}U_j^l S_{l-j}]^k = \sum_{n_1=0}^{k} C_k^{n_1}[a,\ S_{l-j}]^{n_1} \sum_{n_2=0}^{k-n_1} C_{k-n_1}^{n_2}[a,\ U_j^l]^{n_2}[a,\ S_{l-j}]^{k-n_1-n_2}.$$

For fixed $n_1,\ n_2,\ n_3 \in \mathbb{Z}_+$ with $n_1+n_2+n_3=k$, a standard computation involving Lemma 3.1, estimates (3.12) and (3.13) leads to that for $l \in \mathbb{N}$,

$$\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} [a,\ S_{l-j}]^{n_1}[a,\ U_j^l]^{n_2}[a,\ S_{l-j}]^{n_3} f\Big\|^2_{L^2(\mathbb{R}^d)}\\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{2(j-l)n_1}\|[a,\ U_j^l]^{n_2}[a,\ S_{l-j}]^{n_3} f\|^2_{L^2(\mathbb{R}^d)}\\
&\lesssim \min\{(A2^l)^{k+1},\ l^{-\beta}\}^{2\varepsilon}\|f\|^2_{L^2(\mathbb{R}^d)};
\end{aligned}$$

and for $l \in \mathbb{Z}_-$,

$$\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} [a,\ S_{l-j}]^{n_1}[a,\ U_j^l]^{n_2}[a,\ S_{l-j}]^{n_3} f\Big\|^2_{L^2(\mathbb{R}^d)}\\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{2(j-l)n_1}\|[a,\ U_j^l]^{n_2}[a,\ S_{l-j}]^{n_3} f\|^2_{L^2(\mathbb{R}^d)}\\
&\lesssim A^{2\sigma}2^{2\sigma l}\|f\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_l \|[a,\ S_{l-j}U_j^l S_{l-j}]^k f\|_{L^2(\mathbb{R}^d)} &= \sum_{l:\, l>\log(\frac{1}{\sqrt{A}})} \|[a,\ S_{l-j}U_j^l S_{l-j}]^k f\|_{L^2(\mathbb{R}^d)}\\
&\quad + \sum_{l:\, 0\le l\le \log(\frac{1}{\sqrt{A}})} \|[a,\ S_{l-j}U_j^l S_{l-j}]^k f\|_{L^2(\mathbb{R}^d)}\\
&\quad + \sum_{l:\, l<0} \|[a,\ S_{l-j}U_j^l S_{l-j}]^k f\|_{L^2(\mathbb{R}^d)}\\
&\lesssim \Big(\sum_{l:\, l>\log(\frac{1}{\sqrt{A}})} l^{-\varepsilon\beta} + A^{\sigma}\sum_{l:\, l<0} 2^{\sigma l}\Big)\|f\|_{L^2(\mathbb{R}^d)}\\
&\quad + A^{(k+1)\varepsilon} \sum_{l:\, 0\le l\le \log(\frac{1}{\sqrt{A}})} 2^{(k+1)l\varepsilon}\|f\|_{L^2(\mathbb{R}^d)}\\
&\lesssim \log^{-\varepsilon\beta+1}(\frac{1}{A})\|f\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

This, via (3.11), leads to our desired conclusion. □

Proof of Theorem 1.4 *$L^p(\mathbb{R}^d)$ boundedness*. By duality, it suffices to prove that $T_{\Omega,a;\,k}$ is bounded on $L^p(\mathbb{R}^d)$ for $2 < p < 2\beta$.

For $j \in \mathbb{Z}$, let $K_j(x) = \frac{\Omega(x)}{|x|^{d+k}} \chi_{\{2^{j-1} \le |x| < 2^j\}}(x)$. Let $\omega \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative radial function such that

$$\operatorname{supp} \omega \subset \{x : |x| \le 1/4\}, \quad \int_{\mathbb{R}^d} \omega(x)dx = 1,$$

and

$$\int_{\mathbb{R}^d} x^\gamma \omega(x)dx = 0, \quad 1 \le |\gamma| \le k.$$

For $j \in \mathbb{Z}$, set $\omega_j(x) = 2^{-dj}\omega(2^{-j}x)$. For a positive integer l, define

$$H_l(x) = \sum_{j \in \mathbb{Z}} K_j * \omega_{j-l}(x).$$

Let R_l be the convolution operator with kernel H_l. For a function a on $\mathbb{R}^d$ such that $\nabla a \in L^\infty(\mathbb{R}^d)$, recall that $[a,\ R_l]^k$ denotes the k-th commutator of R_l with symbol a.

We claim that for each fixed $\varepsilon \in (0,\ 1)$, $l \in \mathbb{N}$,

$$\|T_{\Omega,a;\,k} f - [a,\ R_l]^k f\|_{L^2(\mathbb{R}^d)} \lesssim l^{-\varepsilon\beta+1} \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.14}$$

To prove this, write

$$H_l(x) - \sum_{j \in \mathbb{Z}} K_j(x) = \sum_{j \in \mathbb{Z}} \big(K_j(x) - K_j * \omega_{j-l}(x)\big) =: \sum_{j \in \mathbb{Z}} \mu_{j,l}(x).$$

By the vanishing moment of ω, we know that for all multi-indices $\gamma \in \mathbb{Z}_+^d$ with $1 \le |\gamma| \le k$, $\partial^\gamma \widehat{\omega}(0) = 0$. By Taylor series expansion and the fact that $\widehat{\omega}(0) = 1$, we deduce that

$$|\widehat{\omega}(2^{j-l}\xi) - 1| \lesssim \min\{1,\ |2^{j-l}\xi|^{k+1}\}.$$

When $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1,\ \infty)$, it was proved in [8, p. 458] that

$$|\widehat{K_j}(\xi)| \lesssim 2^{-jk} \min\{1,\ \log^{-\beta}(2 + |2^j\xi|)\}.$$

Thus, the Fourier transform estimate

$$|\widehat{\mu_{j,l}}(\xi)| = |\widehat{K_j}(\xi)||\widehat{\omega}(2^{j-l}\xi) - 1| \lesssim 2^{-jk} \min\{\log^{-\beta}(2 + |2^j\xi|),\ |2^{j-l}\xi|^{k+1}\} \tag{3.15}$$

holds true. On the other hand, a trivial computation shows that for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma \widehat{K_j}\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(S^{d-1})} 2^{(|\gamma|-k)j},$$

and so for all $\xi \in \mathbb{R}^d$,

$$|\partial^\gamma \widehat{\mu_{j,l}}(\xi)| \lesssim \sum_{\gamma_1+\gamma_2=\gamma} |\partial^{\gamma_1} \widehat{K_j}(\xi)||\partial^{\gamma_2} \widehat{\omega}(2^{j-l}\xi)| \lesssim \|\Omega\|_{L^1(S^{d-1})} 2^{j(|\gamma|-k)}. \tag{3.16}$$

The Fourier transforms (3.15) and (3.16), via Theorem 3.4 with $A = 2^{-l}$, lead to (3.14) immediately.

Let $\varepsilon \in (0,\ 1)$ be a constant which will be chosen later. An application of (3.14) gives us that

$$\left\|[a,\ R_{2^l}]^k f - [a,\ R_{2^{l+1}}]^k f\right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{(-\varepsilon\beta+1)l} \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.17}$$

Therefore, the series

$$T_{\Omega,a;\,k} = [a,\ R_2]^k + \sum_{l=1}^{\infty} ([a,\ R_{2^{l+1}}]^k - [a,\ R_{2^l}]^k) \tag{3.18}$$

converges in $L^2(\mathbb{R}^d)$ operator norm.

For $l \in \mathbb{N}$, let $L_l(x,\ y) = H_l(x-y)(a(x)-a(y))^k$. We claim that for any $y,\ y' \in \mathbb{R}^d$,

$$\int_{|x-y|\geq 2|y-y'|} |L_l(x,y) - L_l(x,y')|dx + \int_{|x-y|\geq 2|y-y'|} |L_l(y,x) - L_l(y',x)|dx \lesssim l. \tag{3.19}$$

To prove this, let $|y-y'| = r$. A trivial computation yields

$$\begin{aligned}\int_{|x-y|\geq 2r} \left|H_l(x-y)(a(y)-a(y'))^k\right|dx &\lesssim r\sum_j \int_{|x|\geq 2r} |K_j * \omega_{j-l}(x)|dx \\ &\lesssim r^k \sum_{j:\,2^{j-2}\geq r} \|K_j\|_{L^1(\mathbb{R}^d)} \|\omega_{j-l}\|_{L^1(\mathbb{R}^d)} \lesssim 1,\end{aligned}$$

since $\|K_j\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-j}$. For each fixed $j \in \mathbb{Z}$, observe that

$$\|\omega_{j-l}(\cdot - y) - \omega_{j-l}(\cdot - y')\|_{L^1(\mathbb{R}^d)} \lesssim \min\{1,\ 2^{l-j}|y-y'|\}.$$

It then follows from Young's inequality that

$$\begin{aligned}
&\int_{|x-y\geq 2r} |H_l(x-y)-H_l(x-y')||a(x)-a(y)|^k dx \\
&= \sum_{n=1}^{\infty}\int_{2^n r\leq|x-y\leq 2^{n+1}r} |H_l(x-y)-H_l(x-y')||a(x)-a(y)|^k dx \\
&\lesssim \sum_{n=1}^{\infty}(2^n r)^k \sum_{j:\,2^j\approx 2^n r} \|K_j\|_{L^1(\mathbb{R}^d)}\|\omega_{j-l}(\cdot-y)-\omega_{j-l}(\cdot-y')\|_{L^1(\mathbb{R}^d)} \\
&\lesssim \sum_{k=1}^{\infty}\min\{1,\ 2^{-k}2^l\}\lesssim l.
\end{aligned}$$

Combining the estimates above gives us that

$$\begin{aligned}
&\int_{|x-y|\geq 2|y-y'|} |L_l(x,y)-L_l(x,y')|dx \\
&\quad\leq \int_{|x-y|\geq 2r}\left|H_l(x-y)(a(y)-a(y'))^k\right|dx \\
&\qquad+\int_{|x-y\geq 2r} |H_l(x-y)-H_l(x-y')||a(x)-a(y)|^k dx\lesssim l.
\end{aligned}$$

Similarly, we can verify that

$$\int_{|x-y|\geq 2|y-y'|} |L_l(y,x)-L_l(y',x)|dx\lesssim l.$$

This establishes (3.19).

Recall that $T_{\Omega,\,a;k}$ is bounded on $L^2(\mathbb{R}^d)$. It follows from (3.14) that $[a,\ R_l]^k$ is also bounded on $L^2(\mathbb{R}^d)$ with bound independent of l. This, along with (3.19) and Calderón-Zygmud theory, tells us that

$$\left\|[a,\ R_l]^k f-[a,\ R_{l+1}]^k f\right\|_{L^p(\mathbb{R}^d)}\lesssim l\|f\|_{L^p(\mathbb{R}^d)},\ \ p\in(1,\ \infty),$$

and so

$$\left\|[a,\ R_{2^l}]^k f-[a,\ R_{2^{l+1}}]^k f\right\|_{L^p(\mathbb{R}^d)}\lesssim 2^l\|f\|_{L^p(\mathbb{R}^d)},\ \ p\in(1,\ \infty). \tag{3.20}$$

Interpolating inequalities (3.17) and (3.20) shows that for any $\varrho\in(0,\ 1)$ and $p\in(2,\ \infty)$,

$$\left\|[a,\ R_{2^l}]^k f-[a,\ R_{2^{l+1}}]^k f\right\|_{L^p(\mathbb{R}^d)}\lesssim 2^{(-2\varepsilon\beta/p+1+\varrho)l}\|f\|_{L^p(\mathbb{R}^d)}.$$

For each p with $2 < p < 2\beta$, we can choose $\varepsilon > 0$ close to 1 sufficiently, and $\varrho > 0$ close to 0 sufficiently, such that $2\varepsilon\beta/p - 1 - \varrho > 0$. This, in turn, shows that

$$\sum_{l=1}^{\infty} \left\| [a,\ R_{2^l}]^k f - [a,\ R_{2^{l+1}}]^k f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

and the series (3.18) converges in the $L^p(\mathbb{R}^d)$ operator norm. Therefore, $T_{\Omega,\,a;k}$ is bounded on $L^p(\mathbb{R}^d)$ for $2 < p < 2\beta$. This finishes the proof of Theorem 1.4. □

Remark 3.5 Let Ω be homogeneous of degree zero, integrable and have mean value zero on S^{d-1}, T_Ω be the homogeneous singular integral operator defined by (1.4). For $b \in \mathrm{BMO}(\mathbb{R}^d)$, define the commutator of T_Ω and b by

$$[b,\ T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x).$$

When $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ with $\alpha \in (0,\ 1]$, Uchiyama [15] proved that $[b,\ T_\Omega]$ is a compact operator on $L^p(\mathbb{R}^d)$ ($p \in (1,\ \infty)$) if and only if $b \in \mathrm{CMO}(\mathbb{R}^d)$, where $\mathrm{CMO}(\mathbb{R}^d)$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in the $\mathrm{BMO}(\mathbb{R}^d)$ topology, which coincide with the space of functions of vanishing mean oscillation. When $\Omega \in GS_\beta(S^{d-1})$ for $\beta \in (2,\ \infty)$, Chen and Hu [2] considered the compactness of $[b,\ T_\Omega]$ on $L^p(\mathbb{R}^d)$ with $\beta/(\beta - 1) < p < \beta$. For other work about the compactness of $[b,\ T_\Omega]$, see [14] and the references therein. It is of interest to characterize the compactness of Calderón commutator $T_{\Omega,\,a;\,k}$ on $L^p(\mathbb{R}^d)$ ($p \in (1,\ \infty)$). We will consider this in a forthcoming paper.

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