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Short-Time Existence for Harmonic Map Heat Flow with Time-Dependent Metrics

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Abstract
In this work, we obtain a short-time existence result for harmonic map heat flow coupled with a smooth family of complete metrics in the domain manifold. Our results generalize short-time existence results for harmonic map heat flow by Li-Tam (Invent Math 105(1):1–46, 1991) and Chen-Zhu (J Differ Geom 74:119–154, 2006). In particular, we prove the short-time existence of harmonic map heat flow along a complete Ricci flow $g(t)$ on M into a complete manifold with curvature bounded from above, under the assumption: (i) $|\text{Rm}(g(t))| \le a/t$; (ii) $g(t)$ is uniformly equivalent to $g(0)$; and (iii) the initial map is smooth and with uniformly bounded energy density.

Keywords Harmonic map heat flow · Short-time existence · Unbounded curvature

Mathematics Subject Classification Primary 53E20; Secondary 35K58

1 Introduction

In this work, we want to extend some previous short-time existence results of harmonic map heat flow. Harmonic map heat flow was first introduced by Eells and Sampson

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We would like to dedicate this paper to Professor Peter Li on the occasion of his seventieth birthday.

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[4] to obtain harmonic map between two Riemannian manifolds. As a first step, they proved the short-time existence for harmonic map heat flows between compact manifolds. Later in [16], Peter Li and the second author proved the short-time existence for harmonic map heat flow from a complete noncompact Riemannian manifold (M^m, g) with Ricci curvature bounded from below to another complete Riemannian manifold (N^n, h) so that the initial map f has bounded energy density and $f(M)$ is *bounded*. Under an additional condition that the curvature $\mathrm{Rm}(h)$ of h is nonpositive, one can remove the assumption that $f(M)$ is bounded. From the point of view of PDE, one would like to understand whether this is still true under a weaker assumption that $\mathrm{Rm}(h) \leq \kappa$ for some $\kappa \geq 0$.

On the other hand, Hamilton [7] used harmonic map heat flow on compact manifolds along a Ricci flow of the domain manifold to obtain uniqueness result of Ricci flow. Later, Chen and Zhu studied the uniqueness of Ricci flow on complete noncompact manifolds following Hamilton's strategy. In [2], Chen and Zhu proved that if a Ricci flow $g(t), 0 \leq t \leq T$, which is complete on a noncompact manifold M, has *uniformly bounded curvature*, then one can obtain short-time solution for harmonic map heat flow along the Ricci flow from $(M, g(t))$ to $(M, g(T))$ with identity map as initial data. From this together with some careful estimates, they obtained uniqueness result on Ricci flow with uniformly bounded curvature on noncompact manifold.

The uniqueness result was generalized to Ricci flow which may have unbounded curvature. In [10], Kotschwar introduced an energy method to obtain a more general uniqueness result. The method has been developed further by Lee [12] and Ma–Lee [18]. In [18], Ma and Lee proved that if two complete solutions of the Ricci flows with the same initial metric on a noncompact manifolds with curvature bounded by a/t for some $a > 0$ so that the deformed metrics are uniformly equivalent to the initial metric, then they are the same.

One may wonder if one can use harmonic map heat flow to obtain similar results. Short-time existence results on harmonic map heat flow in [16] and the above uniqueness results on Ricci flow motivate the study in this work.

Our main result can be described as follows. Let M^m be a noncompact manifold with dimension m and let $g(t)$ be a smooth family of complete metrics defined on $M \times [0, T]$ so that

$$\frac{\partial}{\partial t} g(x,t) = H(x,t). \tag{1.1}$$

Let (N^n, h) be another complete Riemannian manifold with dimension n. We want to study the initial value problem for the harmonic map heat flow:

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \tau(F)(x,t) \\ F(x,0) = f(x) \end{cases} \tag{1.2}$$

where $f(x)$ is a smooth map from M to N and $\tau(F)(x,t)$ is the tension field of the map $F(\cdot,t) : M \to N$ with respect to $g(t)$ and h. For more details of the definitions of harmonic map heat flow and related quantities, see Sect. 4.1.

Consider the following assumptions:

(a1) $2\mathrm{Ric}(g(t)) + H(t) \geq -K(t)g(t)$ in $M \times [0, T]$ where $K(t) \geq 0$ and

$$K_0 =: \int_0^T K(t)\mathrm{d}t < \infty.$$

(a2) $|H| \leq at^{-1}$ and $|\nabla H| \leq at^{-\frac{3}{2}}$ for some $a > 0$. Here, the norm and the covariant derivative are with respect to $g(t)$.

(a3) The curvature of h is bounded from above: $\mathrm{Rm}(h) \leq \kappa$ for some $\kappa \geq 0$.

We obtain the following short-time existence result.

Theorem 1.1 *Let $(M^m, g(t))$ and (N, h) be as above satisfying assumptions* **(a1–a3)**. *Suppose there exists a smooth exhaustion function γ on M and $C_0 > 0$ such that*

$$C_0^{-1}(d_T(p, x) + 1) \leq \gamma(x) \leq C_0(d_T(p, x) + 1)$$

and

$$|\nabla_T^k \gamma| \leq C_0$$

for $1 \leq k \leq 2$, where d_T is the distance function and ∇_T is the covariant derivative with respect to $g(T)$. Given any smooth map $f : M \to N$ such that

$$\sup_M e(f; g(t)) \leq e_0$$

for all $t \in [0, T]$ for some constant e_0 where $e(f; g(t))$ is the energy density of the map f from $(M, g(t))$ to (N, h), the harmonic heat flow (1.2) has a short-time smooth solution F with initial map f defined on $M \times [0, T_0]$ such that

$$\sup_{M\times[0,T_0]} e(F) \leq C; \quad \sup_{M\times[0,T_0]} |\tau(F)|_{g(t)} \leq Ct^{-\frac{1}{2}}$$

for some $C > 0$ depending only on $m, n, K_0, \kappa, a, e_0$ and

$$T_0 = \min\{T, \frac{1}{2}(2\kappa e_0 \exp(K_0))^{-1}\}.$$

In particular, if $\kappa = 0$, then the harmonic map heat flow exists on $M \times [0, T]$.

In the theorem, the assumption on the existence of γ is satisfied if $g(T)$ has bounded curvature, see [21]. The condition that $e(f; g(t))$ is uniformly bounded is satisfied if (i) $e(f; g(0))$ is uniformly bounded and $g(t)$ is uniformly equivalent to $g(0)$; or more generally, if (ii) $e(f; g(t_0))$ is uniformly bounded for some t_0 and $g(t) \geq Cg(t_0)$ for some $C > 0$. We should also remark that the bounds in conclusion of the theorem do not depend on C_0.

Suppose $g(t) = g(0)$ is fixed, then $H = 0$. In this case, **(a1)** is satisfied if $\mathrm{Ric}(g) \geq -Kg$ for some $K \geq 0$. If g has bounded curvature, then the short-time existence result in [16] is still true without assuming the initial map has bounded image, provided (N, h) has curvature bounded from above. See Corollary 4.1.

If $g(t)$ is a solution to the Ricci flow, we have

Theorem 1.2 *Let $(M^m, g(t))$, $t \in [0, T]$ with $T > 0$, be a complete solution of the Ricci flow on a noncompact manifold. Suppose (N^n, h) is another complete manifold with $Rm(h) \le \kappa$ for some $\kappa \ge 0$. Let $f : M \to N$ be a smooth map with bounded energy density, namely, $\sup_M e(f; g(0)) \le e_0$. Assume that $|Rm(g(t))| \le a/t$ for some $a > 0$ on $M \times [0, T]$ and assume that $g(t) \ge b^{-1} g(0)$ for some $b > 0$ on $M \times [0, T]$, then there exists a smooth solution F to the heat flow for harmonic map along $g(t)$ with initial map f defined on $M \times [0, T_0]$ such that*

$$\sup_{M\times[0,T_0]} e(F) \le C_1; \ |\tau(F)|_{g(t)}(\cdot, t) \le C_1 t^{-\frac{1}{2}}$$

for some $C_1 > 0$ depending only on m, n, κ, a, b, e_0 and

$$T_0 = \min\{T, \frac{1}{2}(2\kappa b e_0)^{-1}\}.$$

In particular, if $\kappa = 0$, then the harmonic map heat flow exists on $M \times [0, T]$.

The theorem is a corollary of Theorem 1.1 by the fact that $H = -2\mathrm{Ric}$ in this case and by the covariant derivatives estimates of the curvature tensor along Ricci flow by Shi [20].

To prove our results, instead of solving Dirichlet problem as in [2], we will use the method of iteration which was introduced by Eells and Sampson in their seminal work [4] and was also used in [16]. One key point is to obtain good estimates for the fundamental solution of the heat equation. For the case of fixed metric, the estimates are contained in [15]. For the case of time-dependent metrics, we apply the estimates in [1] instead. We also obtain a new estimate, see Theorem 2.1.

Finally, we would like to point out that we are still unable to give another proof of the uniqueness result on Ricci flow as in [18]. The main difficulty is that the second fundamental form of the identity map from $(M, g(0))$ to $(M, g(T))$ may not be bounded. If the curvature of the Ricci flow $|\mathrm{Rm}(g(t))| \le at^{-1+\alpha}$ with $\alpha > \frac{1}{2}$, then one can prove that the second fundamental form mentioned above is bounded and one can obtain uniqueness. On the other hand, recently, the results in Theorem 1.1 have also been used in [14] to study of Lipschitz rigidity problem in geometry of scalar curvature.

This paper is organized as follows. In Sect. 2, we give estimates for the fundamental solution of heat operator and give a proof of a generalized maximum principle. In Sect. 3, we study linear heat equations for homogeneous and in-homogeneous cases and a semi-linear heat equation closely related to the harmonic map heat flow. In Sect. 4, we study the harmonic map heat flow and give a proof of Theorem 1.1.

2 Preliminary

In this section, we will describe some estimates of the fundamental solution (Green's function) of heat operator with time-dependent complete metrics on a noncompact manifold, which will be used later. We will also extend a maximum principle.

2.1 Estimates for Fundamental Solution

Let $g(t)$, $t \in [0, T]$ be a family of complete Riemannian metrics on a manifold M^m. We always assume M is noncompact and $g(t)$ is smooth in space and time. Recall that G is the fundamental solution of heat operator $\frac{\partial}{\partial t} - \Delta_{g(t)}$ if it satisfies

$$\begin{cases} (\partial_t - \Delta_{x,t})\, G(x, t; y, s) = 0, & \text{in } M \times M \times (s, T]; \\ \lim_{t \to s^+} G(x, t; y, s) = \delta_y(x), & \text{for } y \in M. \end{cases} \tag{2.1}$$

Let

$$H := \frac{\partial}{\partial t} g.$$

Suppose $|H(x, t)|$, $|\nabla H|(x, t)$, and $|\mathrm{Rm}(g)|(x, t)$ are uniformly bounded in space and time, where the norms and covariant derivatives are taken with respect to $g(t)$. It is known that the fundamental solution exists and is positive, see [6] for example. We have the following estimates for G, see [1].

Theorem 2.1 *Let $g(t)$, $t \in [0, T]$ be a family of smooth complete metrics on M as above with $|H|_g \leq H_0$, $|\nabla H|_g \leq H_1$, and $|Rm(g(t))| \leq k_0$. Then, we have the following:*

(a) [1, Theorem 5.5] There are constants $C, D > 0$ depending only on H_0, k_0, m, T such that

$$G(x, t; y, s) \leq \frac{C}{V_x^{\frac{1}{2}}(\sqrt{t-s})V_y^{\frac{1}{2}}(\sqrt{t-s})} \exp\left(-\frac{r^2(x, y)}{D(t-s)}\right)$$

for any $0 \leq s < t \leq T$. Here, $r(x, y)$ is the distance and $V_x(\rho)$ is the volume of the geodesic ball of radius ρ with center at x with respect to $g(0)$.

(b) [1, Corollary 4.4] Fix $\alpha > 1$. For any $\delta > 0$, we have

$$G(p, t; y, s) \leq (1+\delta)^{m\alpha/2} \times \exp\left(A\delta t + \frac{B\alpha}{\delta t} r_t^2(p, q)\right) \times G(q, (1+\delta)t; y, s),$$

where $A > 0$ depends only on $m, T, H_0, H_1, k_0, \alpha$, and B depends only on H_0, T.

(c) For any $x \in M$, $0 \leq s < t \leq T$, we have

$$\int_M G(x, t; y, s) dV_s(y) = 1.$$

Proof (a) and (b) are from [1]. It remains to prove (c). We use similar idea as in the proof of Lemma 5.1 in [1]. Because the curvature of $g(0)$ is bounded, we can find a smooth function ρ so that

$$C^{-1}(r(x)+1) \le \rho(x) \le C(r(x)+1), |\nabla_{g(0)}\rho| + |\nabla^2_{g(0)}\rho| \le C$$

for some $C > 0$ depending only on k_0 and m [21]. Here, $r(x)$ is the distance from a fixed point p with respect to $g(0)$. Let η be a smooth cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, +\infty)$, $\eta > 0$ on $[0, 2)$, $0 \ge \eta'/\eta^{\frac{1}{2}} \ge -C_0$ and $\eta'' \ge -C_0$ on $[0, +\infty)$ with C_0 being a positive absolutely constant. Let $\phi = \eta(\rho/R)$.

For $0 \le s_1 < s_2 < t$, we have

$$\begin{aligned}
&\left|\int_M \phi G(x,t;y,s_2)\mathrm{d}V_{s_2}(y) - \int_M \phi G(x,t;y,s_1)\mathrm{d}V_{s_1}(y)\right| \\
=&\left|\int_{s_1}^{s_2} (\frac{\partial}{\partial s}\int_M \phi G(x,t;y,s)\mathrm{d}V_s(y))\mathrm{d}s\right| \\
=&\left|\int_{s_1}^{s_2}\int_M \phi\left(\frac{\partial}{\partial s}G(x,t;y,s) + h(y,s)G(x,t;y,s)\right)\mathrm{d}V_s(y)\mathrm{d}s\right| \\
=&\left|\int_{s_1}^{s_2}\int_M \phi\Delta_{s,y}G(x,t;y,s)\mathrm{d}V_s(y)\mathrm{d}s\right| \\
=&\left|\int_{s_1}^{s_2}\int_M G(x,t;y,s)\Delta_{s,y}\phi\mathrm{d}V_s(y)\mathrm{d}s\right|,
\end{aligned}$$

where $h = \frac{1}{2}\operatorname{tr}_g H$, and we have used the fact that G is also the fundamental solution of the conjugate heat equation i.e., $(-\frac{\partial}{\partial s} - \Delta_{s,y} - h(y,s))G = 0$. Now

$$\Delta_{s,y}\phi = \frac{1}{R}\phi'\Delta_{s,y}\rho + \frac{1}{R^2}\phi''|\nabla_{g(s)}\rho|^2.$$

Since $|H|, |\nabla H|$ are uniformly bounded, we conclude that

$$|\Delta_{s,y}\phi| \le C$$

for some constant independent of R and s. This implies that

$$\left|\int_M \phi G(x,t;y,s_2)\mathrm{d}V_{s_2}(y) - \int_M \phi G(x,t;y,s_1)\mathrm{d}V_{s_1}(y)\right| \le \frac{C'}{R}$$

for some constant C' independent of s_1, s_2, where we have also used [1, Corollary 5.2] so that

$$\int_M \phi G(x,t;y,s)\mathrm{d}V_s(y) \le c$$

for some constant c independent of s. Let $R \to \infty$ and note that

$$\lim_{s\to t^-}\int_M G(x,t;y,s)\mathrm{d}V_s(y)=1,$$

we obtain

$$\int_M G(x,t;y,s)\mathrm{d}V_s(y)=1.$$

The result follows. □

By the theorem, we can proceed as in the proof of [16, Lemma 2.1] to have the following:

Corollary 2.1 *Under the assumptions in Theorem 2.1, we have*

$$\int_M |G(p,t;y,s)-G(q,t;y,s)|\,dV_s(y)\le C\times\frac{r_t(p,q)}{\sqrt{t-s}} \tag{2.2}$$

for any $p,q\in M$ and $0\le s<t\le T$. Here, C is a constant depending only on m, H_0, H_1, k_0, and T. Here, r_t is the distance function with respect to $g(t)$.

Proof The proof is exactly as in [16]. We sketch the argument here for the sake of completeness. Let $\delta>0$ and $1<\alpha<4$ to be determined later.

$$\begin{aligned}&\int_M |G(p,t;y,s)-G(q,t;y,s)|\mathrm{d}V_s(y)\\ \le&\int_M |G(q,(1+\delta)t;y,s)-G(q,t;y,s)|\mathrm{d}V_s(y)\\ &+\int_M |G(p,t;y,s)-G(q,(1+\delta)t;y,s)|\mathrm{d}V_s(y)\\ =&(\mathrm{I})+(\mathrm{II}).\end{aligned}$$

By Theorem 2.1**(b)** and **(c)**:

$$\begin{aligned}(\mathrm{I})\le&\int_M |(1+\delta)^{m\alpha/2}\times\exp(A\delta t)\times G(q,(1+\delta)t;y,s)-G(q,t;y,s)|\mathrm{d}V_s(y)\\ &+\int_M |(1+\delta)^{m\alpha/2}\times\exp(A\delta t)\times G(q,(1+\delta)t;y,s)-G(q,(1+\delta)t;y,s)|\mathrm{d}V_s(y)\\ \le&\int_M (1+\delta)^{m\alpha/2}\times\exp(A\delta t)\times G(q,(1+\delta)t;y,s)-G(q,t;y,s)\mathrm{d}V_s(y)\\ &+\int_M [(1+\delta)^{m\alpha/2}\times\exp(A\delta t)-1]\times G(q,(1+\delta)t;y,s)\mathrm{d}V_s(y)\\ =&2[(1+\delta)^{m\alpha/2}\times\exp(A\delta t)-1].\end{aligned}$$

Here, A is a constant in the theorem. By Theorem 2.1**(b)** and **(c)** again,

$$(\mathrm{II})\le 2\left[(1+\delta)^{m\alpha/2}\times\exp(A\delta t+\frac{B\alpha}{\delta t}r_t^2(p,q))-1\right].$$

Here, B is also the constant in the theorem. Here, A and B are independent of δ. Hence,

$$\int_M |G(p,t;y,s) - G(q,t;y,s)| \mathrm{d}V_s(y) \leq 4\left[(1+\delta)^{m\alpha/2} \times \exp\left(A\delta t + \frac{B\alpha}{\delta t} r_t^2(p,q)\right) - 1\right].$$

Let $r = r_t(p,q)$. If $\frac{r}{\sqrt{t}} > \frac{1}{2\sqrt{B}}$, we have

$$\int_M |G(p,t;y,s) - G(q,t;y,s)| \mathrm{d}V_s(y) \leq 2 \leq \frac{2r}{\sqrt{t}}.$$

If $\frac{r}{\sqrt{t}} \leq \frac{1}{2\sqrt{B}}$, then let $\delta = \frac{r\sqrt{\alpha B}}{\sqrt{t}}$ and $\alpha = 2$. So $\delta^2 = \frac{r^2 \alpha B}{t} \leq \frac{1}{4}\alpha < 1$, this means that $\delta < 1$. So

$$\int_M |G(p,t;y,s) - G(q,t;y,s)| \mathrm{d}V_s(y) \leq C_1 \left(\exp(C_2\delta) - 1\right) \leq C_3\delta \leq C_4 \frac{r}{\sqrt{t}}.$$

Here, $C_1 - -C_4$ are positive constants depending only on m, H_0, H_1, k_0, and T.

Therefore, we complete the proof of this corollary. □

2.2 A Generalized Maximum Principle

In this subsection, we want to show the following generalized maximum principle which will be used later frequently. This type of maximum principle was originated in the work of Karp and Li [9]. Different variants were obtained before, see [3, 17, 19]. We will use a trick in [3] to prove the following generalization of the maximum principle in [19].

Theorem 2.2 *Let $g(x,t), t \in [0, T_1]$ be a family of smooth Riemannian metrics on M^m, with $\frac{\partial}{\partial t} g = H$, so that $\sup_{M \times [0,T_1]} |H| \leq R_0$. Suppose $f(x,t)$ is a smooth function such that $\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f \leq 0$ whenever $f > 0$, and*

$$\int_0^{T_1} \int_M \exp\left(-a r_0^2(x)\right) f_+^2(x,t) dV_0 dt < \infty \tag{2.3}$$

for some constant $a > 0$, where $r_0(x)$ is the distance function to a fixed point p with respect to $g(0)$ and $f_+ = \max\{f, 0\}$ is the positive part of f. If $f(x,0) \leq 0$ for all $x \in M$, then $f(x,t) \leq 0$ for all $(x,t) \in M \times [0, T_1]$.

Proof In [19], it was assumed that $\frac{\partial}{\partial t} g \leq 0$. To prove the result in our setting, let $F(x,t)$ be such that $\mathrm{d}V_t = e^{F(x,t)} \mathrm{d}V_0$. For $0 < T \leq T_1$, which will be specified later, let

$$h(x,t) = -\frac{\theta r_t^2(x)}{4(2T - t)}$$

for $0 \leq t \leq T$. Here, $\theta > 0$ is a constant which will be chosen later and $r_t(x)$ is the distance function to a fixed point p with respect to $g(t)$. Then,

$$\begin{aligned}\frac{\partial}{\partial t}h =& -\frac{\theta r_t^2(x)}{4(2T-t)^2} - \frac{\theta r_t(x)}{2(2T-t)} \times (\frac{\partial}{\partial t}r_t)\\ =& -\theta^{-1}|\nabla h|^2 - \frac{\theta r_t(x)}{2(2T-t)} \times (\frac{\partial}{\partial t}r_t)\\ \leq& -\theta^{-1}|\nabla h|^2 + \theta^{-1}(2T-t)R_0|\nabla h|^2\end{aligned}$$

because

$$|\nabla h|^2 = \frac{\theta^2 r_t^2(x)}{4(2T-t)^2}$$

and $|\frac{\partial}{\partial t}r_t| \leq \frac{1}{2}R_0 r_t$.

Now we assume $T \leq \min\{T_1, \frac{1}{4R_0}\}$ and choose $\theta = \frac{1}{4}$, we obtain

$$\frac{\partial}{\partial t}h \leq -2|\nabla h|^2 \tag{2.4}$$

for $0 \leq t \leq T$.

Next, let $\beta > 0$ be a constant and $0 \leq \phi(x) \leq 1$ be be the smooth function such that $\phi = 1$ in $B_0(p, R)$, $\phi = 0$ outside $B_0(p, 2R)$ and $|\widetilde{\nabla}\phi| \leq \frac{2}{R}$, where $\widetilde{\nabla}$ denotes the gradient with respect to $g(0)$. We have

$$\begin{aligned}0 \geq& \int_0^T e^{-\beta t}\int_M \phi^2 e^h f_+ \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f \mathrm{d}V_t \mathrm{d}t\\ =& \frac{1}{2}\int_0^T e^{-\beta t}\int_M \phi^2 e^h \frac{\partial}{\partial t}(f_+^2)\mathrm{d}V_t\mathrm{d}t - \int_0^T e^{-\beta t}\int_M \phi^2 e^h f_+(\Delta_t f)\mathrm{d}V_t\mathrm{d}t.\end{aligned} \tag{2.5}$$

Now let us estimate the last two terms in (2.5).

By integration by parts and Cauchy–Schwartz inequality, we have

$$\begin{aligned}\int_M \phi^2 e^h f_+(\Delta_t f)\mathrm{d}V_t \leq& \int_M e^h f_+^2|\nabla\phi|^2\mathrm{d}V_t + \int_M \phi^2 e^h f_+^2|\nabla h|^2\mathrm{d}V_t\\ \leq& \int_M e^h f_+^2|\nabla\phi|^2\mathrm{d}V_t - \int_M \phi^2 e^h f_+^2\frac{\partial}{\partial t}h\mathrm{d}V_t.\end{aligned} \tag{2.6}$$

On the other hand, since Then $\frac{\partial}{\partial t}\mathrm{d}V_t = (\frac{\partial}{\partial t}F)\mathrm{d}V_t = \frac{\mathrm{tr}_g H}{2}\mathrm{d}V_t$, we have

$$\begin{aligned}&\frac{1}{2}\int_0^T e^{-\beta t}\int_M \phi^2 e^h\frac{\partial}{\partial t}(f_+^2)\mathrm{d}V_t\mathrm{d}t\\ &= \frac{1}{2}\Big[(e^{-\beta t}\int_M \phi^2 e^h f_+^2\mathrm{d}V_t)|_0^T - \int_0^T e^{-\beta t}\int_M \phi^2 e^h(\frac{\partial}{\partial t}h)f_+^2\mathrm{d}V_t\mathrm{d}t\\ &\quad - \int_0^T e^{-\beta t}\int_M \phi^2 e^h f_+^2(\frac{\partial}{\partial t}F)\mathrm{d}V_t\mathrm{d}t + \beta\int_0^T e^{-\beta t}\int_M \phi^2 e^h f_+^2\mathrm{d}V_t\mathrm{d}t\Big].\end{aligned} \tag{2.7}$$

Since $|\frac{\partial}{\partial t}F| \le C_1(n, R_0)$ for some constant C_1 depending only on n, R_0, if we choose $\beta = C_1(n, R_0)$, then by by (2.4), (2.5), (2.6), (2.7), we have

$$\begin{aligned}\int_M \phi^2(x)e^{h(x,T)} f_+^2(x, T)\mathrm{d}V_T \le& 4e^{\beta T}\int_0^T e^{-\beta t}\int_M e^h f_+^2|\nabla\phi|^2\mathrm{d}V_t\mathrm{d}t\\ \le& C(n, R_0, T_1)e^{\beta T}\int_0^T\int_M e^h f_+^2|\widetilde{\nabla}\phi|^2\mathrm{d}V_0\mathrm{d}t.\end{aligned} \tag{2.8}$$

Let $R \to \infty$ in (2.8), we have

$$\begin{aligned}&\int_M e^{h(x,T)} f_+^2(x, T)\mathrm{d}V_T\\ &\le \liminf_{R\to\infty}\frac{C(n, R_0, T_1)e^{\beta T}}{R^2}\int_0^T\int_{B_0(p,2R)-B_0(p,R)} e^{-\frac{r_0^2(x)}{C(R_0,T_1)T}} f_+^2\mathrm{d}V_0\mathrm{d}t.\end{aligned}$$

Hence, if $T < \frac{1}{aC(R_0,T_1)}$, by the assumption (2.3), we have

$$\int_M e^{h(x,T)} f_+^2(x, T)\mathrm{d}V_T \le 0.$$

This implies $f(x, T) \le 0$ for all $x \in M$. We can repeat the argument above to show that $f \le 0$ in $[0, T)$ if $T < \frac{2}{aC(R_0,T_1)}$. One then can start with T and show that $f \le 0$ in $[0, 2T)$ as long as $2T < T_1$. From this, it is easy to see that the theorem is true. □

3 Results on Heat Equation

3.1 Linear Equation

To prepare the construction of harmonic map heat flow, we first study the linear heat equation. Let M^m be a noncompact smooth manifold with dimension $m \ge 3$ and let $g(t)$ be a family of smooth complete Riemannian metrics on M, $0 \le t \le T$ for some $T > 0$. This means that $g(t)$ is smooth both in space and time on $M \times [0, T]$. Denote

$$H(x, t) := \frac{\partial}{\partial t}g(x, t). \tag{3.1}$$

Let $F(x, t)$ be a bounded smooth function on $M \times [0, T]$ and $f(x)$ be a bounded smooth function on M. We want to study the following problems:

$$\begin{cases}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)u = F \text{ in } M \times [0, T];\\ u(x, 0) = 0;\end{cases} \tag{3.2}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) v = 0 \text{ in } M \times [0, T]; \\ v(x, 0) = f(x). \end{cases} \tag{3.3}$$

Here, $\Delta_{g(t)}$ is the Laplacian operator with respect to $g(t)$.

Proposition 3.1 *With the above notation and assumptions, there is a solution u of (3.2) and a solution v of (3.3) so that both u and v are smooth in $M \times [0, T]$. Moreover,*

$$\begin{cases} \sup_{M\times[0,T]} |u| \le T \sup_{M\times[0,T]} |F|; \\ \sup_{M\times[0,T]} |v| \le \sup_{M} |f|. \end{cases}$$

Proof This is standard. For any $R >> 1$, let $0 \le \phi_R \le 1$ be a smooth function on M so that $\phi_R = 1$ in $B_p(R)$ and $\phi_R = 0$ outside $B_p(2R)$ where $p \in M$ is a fixed point and $B_p(r)$ is the geodesic ball of radius r with respect to $g(0)$. By [5, Theorems 7, 12, Chapter 3], there is a smooth solution u_R of the following initial-boundary value problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) u_R = \phi_R F & \text{in } \Omega_R \times [0, T]; \\ u_R(x, 0) = 0 & \text{in } x \in \Omega_R; \\ u_R(x, t) = 0 & \text{in } (x, t) \in \partial\Omega_R \times [0, T]. \end{cases}$$

where Ω_R is a bounded domain in M with smooth boundary and with $B_p(2R) \Subset \Omega_R$. Since

$$\left|\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) u_R\right| \le \sup_{M\times[0,T]} |F| =: \mathfrak{m},$$

we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(u_R - t\mathfrak{m}) \le 0, \quad \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(u_R + t\mathfrak{m}) \ge 0.$$

By the maximum principle, one can conclude that

$$\sup_{\Omega_R\times[0,T]} |u_R| \le T \sup_{M\times[0,T]} |F|.$$

From this one may conclude that for any bounded domain D in M, and for any $k \ge 1$, the derivatives of u_R with respect to space up to order k and the derivatives with respect to t up to order $[k/2]$ are bounded in $D \times [0, T]$ by a constant independent of R, provided R is large enough. Here, $[k/2]$ is the integral part of $k/2$. See [11, Chapter 4] for example. From this, by taking a convergent subsequence, one can find a smooth solution of (3.2) so that

$$\sup_{M\times[0,T]} |u(x, t)| \le T \sup_{M\times[0,T]} |F|.$$

Similarly, one can construct solution v to (3.3) with the following estimate:

$$\sup_{M\times[0,T]}|v|\le\sup_M|f|.$$

□

To construct harmonic map heat flow, we also need some estimates of the gradients of the solutions obtained in the previous proposition. In order to obtain the estimates, we need more conditions on $g(t)$. As before, let

$$H=\frac{\partial}{\partial t}g.$$

Proposition 3.2 *With the notation and assumptions as in Proposition 3.1. Moreover, assume that*

$$|H|_{g(t)},|\nabla H|_{g(t)},|Rm(g(t))|_{g(t)}\le K$$

for some $K>0$ on $M\times[0,T]$.

(i) The solutions u,v obtained in Proposition 3.1 satisfy the following gradient estimates:

$$\sup_M|\nabla u|(\cdot,t)\le C(m,K,T)\left(\sup_{M\times[0,t]}|F|\right)t^{\frac{1}{2}}$$

and

$$\sup_M|\nabla v|(\cdot,t)\le e^{C(m,K)t}\sup_M|\nabla f|$$

for all $0\le t\le T$, for some constants $C(m,K)$ depending only on m,K and $C(m,K,T)$ depending only on m,K,T.

(ii) The solution v obtained in Proposition 3.1 satisfies the following estimate:

$$|v(x,t)-f(x)|\le C(m,K,T)t^{\frac{1}{2}}\sup_M|\nabla f|$$

for all $(x,t)\in M\times[0,T]$ for some constant $C(m,K,T)$ depending only on m, K, and T.

Proof (i) Let us prove the estimate of $|\nabla v|$ first. Obviously, we may assume that $|\nabla f|$ is uniformly bounded. Otherwise, the estimate is obvious. By the Bochner formula and the fact that $|H|$ and $|\mathrm{Rm}(g(t))|$ are uniformly bounded by K, one can conclude that

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla v|\le C(m,K)|\nabla v|,$$

whenever $|\nabla v| > 0$. So we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left(e^{-C(m,K)t}|\nabla v|\right) \le 0.$$

On the other hand, since v is bounded and

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) v = 0,$$

one can conclude by using cutoff functions and integrating by parts that

$$\int_0^T \int_M \exp(-ar_0^2(x))|\nabla v|^2 \mathrm{d}V_0 \mathrm{d}t < \infty$$

for some $a > 0$. Here, we have used the fact that $|H|$ is bounded so that $g(t)$ and g_0 are uniformly equivalent and volume comparison because $|\mathrm{Rm}(g(0))|$ is bounded. Apply the maximum principle Theorem 2.2 to the function

$$e^{-C(m,K)t}|\nabla v| - \sup_M |\nabla f|,$$

one can conclude that

$$|\nabla v|(x,t) \le e^{C(m,K)t} \sup_M |\nabla f|.$$

in $M \times [0, T]$.

Next we want to estimate $|\nabla u|$. Since $|H|$, $|\nabla H|$, $|\mathrm{Rm}(g(t))|$ are bounded by K, we can construct the fundamental solution $G(x,t;y,s)$ of the heat operator as in Sect. 2.1 with estimates as in [1]. If we let

$$w(x,t) = \int_0^t \int_M G(x,t;y,s)F(y,s)\mathrm{d}V_s(y)\mathrm{d}s,$$

then $\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) w = F$ in $M \times (0, T]$ which is continuous up to $t = 0$ so that $w(x,0) = 0$. Moreover, w is bounded by Theorem 2.1. By the maximum principle Theorem 2.2, we conclude that $u \equiv w$. Hence,

$$u(x,t) = \int_0^t \int_M G(x,t;y,s)F(y,s)\mathrm{d}V_s(y)\mathrm{d}s.$$

Then, by Corollary 2.1 we have

$$\begin{aligned}|u(x,t)-u(x',t)| &\le \int_0^t \mathrm{d}s\int_M |G(x,t;y,s)-G(x',t;y,s)|\times|F(y,s)|\mathrm{d}V_s(y)\\&\le (\sup_{M\times[0,t]}|F|)\times\int_0^t \mathrm{d}s\int_M |G(x,t;y,s)-G(x',t;y,s)|\mathrm{d}V_s(y)\\&\le C(\sup_{M\times[0,t]}|F|)\times\int_0^t \frac{r_t(x,x')}{\sqrt{t-s}}\mathrm{d}s\\&\le Cr_t(x,x')\times(\sup_{M\times[0,t]}|F|)\times t^{\frac{1}{2}}.\end{aligned}$$

From this, it is easy to see that the estimate for $|\nabla u|$ is true.

To prove (ii), for $x\in M$,

$$\begin{aligned}&|v(x,t)-f(x)|\\&=\left|\int_M G(x,t;y,0)f(y)\mathrm{d}V_0(y)-f(x)\right|\\&=\left|\int_M G(x,t;y,0)(f(y)-f(x))\mathrm{d}V_0(y)\right|\\&\le \sup_M|\nabla f|\int_M G(x,t;y,0)r(x,y)\mathrm{d}V_0(y)\end{aligned}$$

where $r(x,y)$ is the distance between x,y with respect to $g(0)$ and we have used Theorem 2.1. By Theorem 2.1 and volume comparison, one can proceed as in [16] to conclude that

$$\int_M G(x,t;y,0)r(x,y)\mathrm{d}V_0(y)\le C_1t^{\frac{1}{2}}$$

for some constant C_1 depending only on m,K,T. From this, (ii) follows. □

3.2 A Semi-linear Heat Equation

We want to use the results in Sect. 3.1 to study the following semi-linear equation. Let $g(t)$ be a smooth family of complete metrics defined on M with $t\in[0,T]$. We want to consider the following system of semi-linear equation which is closely related to harmonic map heat flow:

$$\begin{cases}\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)u=F_{BC}(u)\langle\nabla u^B,\nabla u^C\rangle \text{ in } M\times[0,T];\\ u(0,x)=f(x),\end{cases}\tag{3.4}$$

where $u=(u^A):M\times[0,T]\to\mathbb{R}^q$ is a vector-valued function and $f=(f^A):M\to\mathbb{R}^q$ and $F_{BC}=(F^A_{BC}):\mathbb{R}^q\to\mathbb{R}^q$ are smooth functions. The ∇u^B and the inner product $\langle \nabla u^B, \nabla u^C \rangle$ are taken with respect to $g(t)$. As before, let

$$H := \frac{\partial}{\partial t} g.$$

Lemma 3.1 *Assume*

$$|H|_{g(t)}, |\nabla H|_{g(t)}, |Rm(g(t))|_{g(t)} \leq K$$

for some $K > 0$ on $M \times [0, T]$ and $|F| \leq L$. Suppose f is a smooth function so that f and $|\nabla f|$ are bounded with

$$\mathfrak{m} := \sup_M \left(\sum_A |\nabla f^A|^2 \right)^{\frac{1}{2}} < \infty.$$

Then, there is a constant $T_1 > 0$ depending only on m, q, K, L, T, and $\mathfrak{m}$ so that (3.4) has a smooth solution in $M \times [0, T_1]$ with u and $|\nabla u|$ uniformly bounded.

Proof We use iteration as in [4, 16]. Define $u^{-1} = 0$ and define u^k inductively: u^k is the solution of the following linear equation:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) u^k = F_{BC}(u^{k-1}) \langle \nabla u^{k-1,B}, \nabla u^{k-1,C} \rangle \text{ in } M \times [0, T]; \\ u^k(0, x) = f(x), \end{cases} \tag{3.5}$$

for $k \geq 0$ where $u^k = (u^{k,A})$. The equation for each component is

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)} \right) u^{k,A} = F^A_{BC}(u^{k-1}) \langle \nabla u^{k-1,B}, \nabla u^{k-1,C} \rangle.$$

First we want to show that u^k is well defined and smooth in $M \times [0, T]$ for all $k \geq 0$. Suppose u^{k-1} is well defined and smooth so that

$$\sup_{M \times [0,T]} |\nabla u^{k-1}|^2 < \infty.$$

Note that this is true for $k = 1$, by Propositions 3.1, 3.2 and assumptions on f. Since $|F|$ is bounded and the inductive hypothesis, by Propositions 3.1 and 3.2, then for all $k \geq 1$ (3.5) has a solution u^k which is smooth in $M \times [0, T]$, is uniformly bounded and

$$\sup_{M \times [0,T]} |\nabla u^k|^2 < \infty.$$

Next we want to show that if $0 < T_1 \leq T$ is small enough, then $|\nabla u^k|$ will be uniformly bounded independent of k in $M \times [0, T_1]$. By Proposition 3.2, we have

$$|\nabla u^{k,A}|(\cdot, t) \leq C(m, K, T) t^{\frac{1}{2}} \sup_{M \times [0,t]} |F^A_{BC}(u^{k-1})| |\nabla u^{k-1,B}| |\nabla u^{k-1,C}| + e^{C(m,K)t} \sup_M |\nabla f^A|$$

Let

$$p_k(t) := \sup_{M\times[0,t]} \left(\sum_A |\nabla u^{k,A}|^2(\cdot,t)\right)^{\frac{1}{2}}.$$

We have

$$p_k(t) \le C_1(t^{\frac{1}{2}} p_{k-1}^2 + \mathfrak{m})$$

for some constant C_1 depending only on m, q, K, T, L. So

$$C_1 t^{\frac{1}{2}} p_k(t) \le \left(C_1 t^{\frac{1}{2}} p_{k-1}(t)\right)^2 + C_1^2 t^{\frac{1}{2}} \mathfrak{m}.$$

Suppose T_1 is such that $C_1^2 T_1^{\frac{1}{2}} \mathfrak{m} \le \frac{1}{4}$, then for $0 < t \le T_1$

$$C_1 t^{\frac{1}{2}} p_0(t) \le \left(C_1 t^{\frac{1}{2}} p_{-1}(t)\right)^2 + C_1^2 t^{\frac{1}{2}} \mathfrak{m} = C_1^2 t^{\frac{1}{2}} \mathfrak{m} \le \frac{1}{2}$$

because $u^{-1} = 0$. Inductively, we conclude that

$$C_1 t^{\frac{1}{2}} p_k(t) \le \left(C_1 t^{\frac{1}{2}} p_{k-1}(t)\right)^2 + C_1^2 t^{\frac{1}{2}} \mathfrak{m} \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Hence, we let $T_1 > 0$ so that $T_1^{\frac{1}{2}} = \min\{T^{\frac{1}{2}}, \frac{1}{4} C_1^{-2} (1+\mathfrak{m})^{-1}\}$, then

$$p_k(T_1) \le \frac{1}{2} C_1^{-1} T_1^{-\frac{1}{2}}$$

for all k. From this and by the proof of Proposition 3.1, we also conclude that u^k are uniformly bounded on $M \times [0, T_1]$.

We claim that in any bounded coordinate neighborhood U, for any $l \ge 1$, there is a constant C independent of k so that $|D_t^\alpha D_x^\beta u^k| \le C$ if $2\alpha + \beta \le l$. Here D_t and D_x are partial derivatives with respect to t and local coordinates x. If the claim is true, then by a diagonal process, we can find a smooth solution of (3.4) in $M \times [0, T_1]$, so that $|u|$ and $|\nabla u|$ are uniform bounded in space and time.

The idea of the claim is as follows. For each k, the right hand side of (3.5) is uniformly bounded. By standard theory, we have some Hölder norm of the ∇u^k being bounded. This will imply bounds of higher derivatives for u^{k+1} etc. We sketch the proof as follows. Let ϕ be a smooth cutoff function with support inside a bounded coordinate neighborhood U so that it is 1 in an open set $V \Subset U$. Then, one can check that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(\phi(u^k - f)) = G^k$$

where G^k is uniformly bounded by a constant independent of k and is zero outside U. Moreover, $\phi(u^k - f) = 0$ at $t = 0$. By [5, Theorem 4, p. 191], we have

$$|u^k|_\delta + |D_x u^k|_\delta \le C$$

in $V \times [0, T_1]$ for some constant C and $\delta > 0$ independent of k. Here, $|\cdot|_\delta$ is the Hölder norm in $V \times [0, T_1]$ with respect to the distance function $d(P, Q) = \left(|x - x'|^2 + |t - t'|\right)^{\frac{1}{2}}$ for $P = (x, t)$, $Q = (x', t')$. From this and the Schauder estimates, one may get $|u^{k+1}|_{2+\delta}$ being uniformly bounded in $V' \times [0, T_1]$ for any $V' \Subset V$. Then, $|u^{k+2}|_{3+\delta}$ is uniformly bounded in $V'' \Subset V'$ and so on. This proves the claim. □

4 Short-Time Existence of Harmonic Map Heat Flow

We will obtain a short-time existence result for harmonic map heat flow coupled with a smooth family of complete metrics in the domain manifold. First, let us recall the basic facts about the harmonic map heat flow.

4.1 The Harmonic Map Heat Flow

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f : (M^m, g) \to (N^n, h)$ be a smooth map. Let ∇, $\widetilde{\nabla}$ be Riemannian connections on M, N respectively. Consider the vector bundle $T^*(M) \otimes f^{-1}(T(N))$ over M. Let D be the connection on this bundle defined as (for ω a 1-form and Y a vector field along f):

$$D_X(\omega \otimes Y) = \nabla_X \omega \otimes Y + \omega \otimes \widetilde{\nabla}_{f_* X} Y.$$

In general, one can extend the connection to $\otimes^k(T^*(M)) \otimes f^{-1}(T(N))$. If in local coordinates x in M, y in N, a section of this bundle is given by

$$s = u^\alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \partial_{y^\alpha},$$

then

$$\begin{aligned} s_{|p} &:= D_{\partial_{x^p}} s \\ &= u^\alpha_{i_1 \ldots i_k ; p} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \partial_{y^\alpha} + u^\alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \widetilde{\nabla}_{f_*(\partial_{x^p})} \partial_{y^\alpha} \\ &= u^\alpha_{i_1 \ldots i_k ; p} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \partial_{y^\alpha} + f^\beta_p u^\alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \widetilde{\nabla}_{\partial_{y^\beta}} \partial_{y^\alpha}. \end{aligned}$$

Here and in the following, ‘ ; ’ will denote the covariant derivative with respect to ∇ and ‘ | ’ will denote the covariant derivative with respect to the connection D on the bundle $T^*(M) \otimes f^{-1}(T(N))$.

In case we have a smooth map $f : M \times [0, T] \to N$, we may also consider $D_t = D_{\partial_t}$. If

$$s = u^\alpha_{i_1 \dots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \partial_{y^\alpha}$$

then

$$s_{|t} = D_t(s) = \partial_t u^\alpha_{i_1 \dots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \partial_{y^\alpha} + f^\beta_t u^\alpha_{i_1 \dots i_k} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \widetilde{\nabla}_{\partial_{y^\beta}} \partial_{y^\alpha}.$$

Now consider a smooth map $f : M \times [0, T] \to N$ and its derivative

$$s := df = f^\alpha_i \mathrm{d}x^i \otimes \partial_{y^\alpha}.$$

The energy density of f is defined by

$$e(f) := |s|^2_{g,h} := g^{ij} f^\alpha_i f^\beta_j h_{\alpha\beta}$$

in local coordinates. The second fundamental form of f is defined by

$$Ds := Ddf.$$

In local coordinates,

$$\begin{aligned} Ds = Ddf =& s^\alpha_{i|j} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \partial_{y^\alpha} \\ =& f^\alpha_{;ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \partial_{y^\alpha} + f^\alpha_i f^\beta_j \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \widetilde{\nabla}_{\partial_{y^\beta}} \partial_{y^\alpha}. \end{aligned}$$

Note that $s^\alpha_{i|j} = s^\alpha_{j|i}$. The tension field of f is defined by

$$\tau(f) := \mathrm{tr}_g(Ds),$$

which is the trace of the second fundamental form and is a vector field along f. In local coordinates

$$\tau(f)^\alpha := g^{ij} s^\alpha_{i|j}.$$

Suppose $g(t)$ is a smooth family of metrics on M, $t \in [0, T]$. Then, the harmonic map heat flow $f(x, t)$ coupled with varying metrics $g(t)$ is defined by

$$\frac{\partial}{\partial t} f = \tau(f). \tag{4.1}$$

Here, $f : M \times [0, T] \to N$ is a smooth map and the tension field on the right is computed with respect to $g(t)$. See the seminal paper by Eells and Sampson [4]. Note that in [4], the metric g is fixed.

In local coordinates, (4.1) can be expressed as

$$\frac{\partial}{\partial t} f^\alpha(x,t) = g^{ij}(x,t)(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \widetilde{\Gamma}^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j). \tag{4.2}$$

Here, f^α_i, f^α_{ij} denote the partial derivatives of f^α and Γ, $\widetilde{\Gamma}$ are the connections of $g(t)$ and h, respectively.

4.2 A Priori Estimates

We want to obtain some a priori estimates for the energy density and the norm of the tension field for solutions of harmonic map heat flow. Let us first estimate the energy density. Let $g(t)$ be a smooth family of complete metrics on M^m which is noncompact, $t \in [0, T]$ and let (N^n, h) be another complete Riemannian manifold. Suppose

$$F : M \times [0, T] \to N$$

is a solution to the harmonic map heat flow. As before, let

$$H = \frac{\partial}{\partial t} g.$$

Direct computations give:

Lemma 4.1 *In local coordinates of x^i in M and y^α in N,*

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) e(F) = -\, g^{il} g^{kj} \left(H_{kl} + 2R_{kl}\right) F^\alpha_i F^\beta_j h_{\alpha\beta} - 2|DdF|^2 + 2g^{pq} g^{ij} F^\sigma_p F^\gamma_q F^\tau_i F^\beta_j S_{\gamma\tau\beta\sigma}.$$

where R_{ij} is Ricci tensor of $g(t)$ and S is the curvature tensor of N.

Proof This is well known [4]. The only difference is that g also depends on t, and we have a term involving $\partial_t g = H$. □

Lemma 4.2 *Let $(M^m, g(t))$, (N, h) and F be as in the previous lemma so that $e(F)$ is uniformly bounded in space and time. Suppose that $|H|_{g(t)}$ is uniformly bounded by L. Suppose*

$$2Ric(g(t))(x,t) + H(x,t) \geq -K(t) g(x,t)$$

for some $K(t) \geq 0$ so that

$$K_0 =: \int_0^T K(t)dt < \infty,$$

and suppose $Rm(h) \leq \kappa$ *for some* $\kappa \geq 0$. *Let*

$$e_0 = \sup_M e(F)(\cdot, 0).$$

Then,

$$e(F)(\cdot, t) \leq \exp(\lambda(t))v(t)$$

on $[0, T_1]$ *where*

$$v(t) = \left(e_0^{-1} - 2\kappa \exp(K_0)t\right)^{-1}, \quad \lambda(t) := \int_0^t K(\tau)d\tau$$

and $T_1 = \min\{T, \frac{1}{2}(2\kappa e_0 \exp(K_0))^{-1}\}$. *Hence,* $e(F)(\cdot, t) \leq 2e_0 \exp(K_0)$ *for* $t \in [0, T_1]$. *In particular if* $\kappa = 0$, *then* $T_1 = T$.

Proof Let $s = dF = F_i^\alpha \mathrm{d}x^i \otimes \partial_{y^\alpha}$ in local coordinates of $x \in M$ and $F(x, \cdot)$ in N. Let S be the curvature tensor of (N, h), then by Lemma 4.1, we have

$$(\frac{\partial}{\partial t} - \Delta_{g(t)})e(F) \leq K(t)e(F) + 2\kappa e^2(F).$$

Let $\lambda(t) := \int_0^t K(\tau)\mathrm{d}\tau \leq K_0 < +\infty$. Then,

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta_{g(t)})(\exp(-\lambda)e(F)) &\leq 2\kappa \exp(\lambda)(\exp(-\lambda)e(F))^2 \\ &\leq 2\kappa \exp(K_0)(\exp(-\lambda)e(F))^2\end{aligned}$$

Let $v(t)$ be the solution of the ODE

$$v' = 2\kappa \exp(K_0)v^2$$

with $v(0) = e_0$. Then,

$$v(t) = \left(e_0^{-1} - 2\kappa \exp(K_0)t\right)^{-1}.$$

$v(t)$ is well defined if $t < (2\kappa e_0 \exp(K_0))^{-1}$. Let $T_1 = \min\{T, \frac{1}{2}(2\kappa e_0 \exp(K_0))^{-1}\}$.

Let $\Theta(x, t) := \exp(-\lambda(t))e(F)$. Then in $M \times [0, T_1]$, we have

$$(\frac{\partial}{\partial t} - \Delta_{g(t)})(\Theta - v) \leq 2\kappa \exp(K_0)(\Theta + v) \times (\Theta - v) \leq C_1(\Theta - v)$$

for some constant $C_1 > 0$ whenever $\Theta - v > 0$. This implies that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(\exp(-C_1 t)(\Theta - v)) \leq 0,$$

whenever $\exp(-C_1 t)(\Theta - v) > 0$. Since $g(t)$ is uniformly equivalent to $g(0)$ and the Ricci curvature of $g(t)$ is bounded from below, one may apply the maximum principle Theorem 2.2 to conclude that $\exp(-C_1 t)(\Theta - v) \le 0$ in $M \times [0, T_1]$. From this it is easy to see that the result follows. □

We should remark that the bounds of $e(F)$ and T_1 do not depend on L.

In order to study the distance between $F(x, t)$ and the initial map $F(x, 0)$, we need to estimate the norm of the tension field. Again by direct computations we have, see [8]:

Lemma 4.3

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\tau(F)|^2 = 2S_{\delta\alpha\gamma\beta} F_k^\delta F_k^\gamma F_t^\alpha F_t^\beta - 2F_{tk}^\alpha F_{tk}^\alpha - 2F_t^\alpha H_{kl} F_{kl}^\alpha - F_t^\alpha F_k^\alpha (2\nabla_l H_{lk} + \nabla_k H_{ll}).$$

Here, the computation is at x and $F(x, t)$ under normal coordinates with respect to $g(t)$ and h.

Using this we obtain the following:

Lemma 4.4 *With same notation and assumptions as in Lemma 4.2, assume $|H| \le at^{-1}$, $|\nabla H| \le at^{-\frac{3}{2}}$ for some $a > 0$. Suppose*

$$e(F) \le \mathfrak{m}.$$

in $M \times [0, T]$. Then, there is a constant $C > 0$ depending only on $m, n, T, a, K_0, \kappa, \mathfrak{m}$ such that

$$|\tau(F)|(x, t) \le Ct^{-\frac{1}{2}}$$

on $M \times [0, T]$.

Proof By Lemma 4.3, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\tau(F)|^2 \le& 2S_{\delta\alpha\gamma\beta} F_k^\delta F_k^\gamma F_t^\alpha F_t^\beta - 2F_{tk}^\alpha F_{tk}^\alpha - 2F_t^\alpha H_{kl} F_{kl}^\alpha \\ &- F_t^\alpha F_k^\alpha (2\nabla_l H_{lk} + \nabla_k H_{ll}) \\ \le& C(m, n)\left(\kappa e(F)|\tau(F)|^2 + at^{-1}|\tau(F)||DdF| \right. \\ &\left. + at^{-\frac{3}{2}} e(F)^{\frac{1}{2}}|\tau(F)|\right) \\ &- 2F_{tk}^\alpha F_{tk}^\alpha.\end{aligned}$$

So at the point where $|\tau(F)| > 0$,

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\tau(F)| \le C(m, n)\left(\kappa e(F)|\tau(F)| + at^{-1}|DdF| + at^{-\frac{3}{2}} e(F)^{\frac{1}{2}}\right)$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)(t|\tau(F)|)\\&\le C(m,n)\left(\kappa e(F)t|\tau(F)|+a|DdF|+at^{-\frac{1}{2}}e(F)^{\frac{1}{2}}\right)+|\tau(F)|\\&\le C_1(|DdF|+t^{-\frac{1}{2}}).\end{aligned}$$

because $|\tau(F)|\le|DdF|$ and $t\le T$. Here and below C_i will denote a positive constant depending only on $m,n,T,K_0,\kappa,a,\mathfrak{m}$.

On the other hand, by Lemma 4.1, we have

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)e(F)\le K(t)e(F)+2\kappa e^2(F)-2|DdF|^2.$$

Let $\lambda(t)=\int_0^t K(s)\mathrm{d}s$ and $\Theta=\exp(-\lambda(t))e(F)$, then

$$\left(\frac{\partial}{\partial t}-\Delta\right)\Theta\le -{C_2}^{-1}|DdF|^2+C_2.$$

So

$$\left(\frac{\partial}{\partial t}-\Delta\right)t^{\frac{1}{2}}\Theta\le -C_2^{-1}t^{\frac{1}{2}}|DdF|^2+C_3t^{-\frac{1}{2}}.$$

This implies

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(t|\tau(F)|+t^{\frac{1}{2}}\Theta\right)\le C_4t^{-\frac{1}{2}}$$

Hence,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(t|\tau(F)|+t^{\frac{1}{2}}\Theta-C_5t^{\frac{1}{2}}\right)\le 0.$$

Since we do not assume $|\tau(F)|$ is bounded, we need to estimate the integral of $|\tau(F)|^2$ in order to apply the maximum principle. Recall

$$\left(\frac{\partial}{\partial t}-\Delta\right)\Theta\le -{C_2}^{-1}|DdF|^2+C_2$$

Multiplying a cutoff function to the above inequality and then integrating by part, one can prove that

$$\int_0^{T_1}\int_{B_0(R)}|DdF|^2\mathrm{d}V_0\mathrm{d}t\le CV_{g(0)}(2R)$$

for some constant C independent of R. Here, we have used the fact that Θ is uniformly bounded and $g(t)$ is uniformly equivalent to $g(0)$. Using the fact that $V_{g(0)}(2R)\le\exp(C'(R+1))$, for some $C'>0$ independent of R. The lemma follows from the maximum principle Theorem 2.2. □

Remark 4.1 In the above lemma, we do not assume that $|\tau(F)|$ is bounded. In particular, we do not assume the tension field of the initial data is bounded. □

4.3 Short-Time Existence

Let M^m be a noncompact manifold and let $g(t)$ be a smooth family of complete metrics defined on $M \times [0, T]$ so that

$$\frac{\partial}{\partial t} g(x,t) = H(x,t). \tag{4.3}$$

Let (N^n, h) be another complete Riemannian manifold. Recall the following assumptions.

(a1) $2\mathrm{Ric}(g(t)) + H(t) \geq -K(t)g(t)$ in $M \times [0, T]$ where $K(t) \geq 0$ and

$$K_0 =: \int_0^T K(t)\mathrm{d}t < \infty.$$

(a2) $|H| \leq at^{-1}$ and $|\nabla H| \leq at^{-\frac{3}{2}}$ for some $a > 0$. Here, the norm and the covariant derivative are with respect to $g(t)$.

(a3) The curvature of h is bounded from above: $\mathrm{Rm}(h) \leq \kappa$ for some $\kappa \geq 0$..

We will prove the main short-time existence result of harmonic map heat flow Theorem 1.1 in this subsection.

As a corollary, we remove a condition that the image of the initial map is bounded in the short-time existence result [16, Theorem 3.4], provided there is a suitable exhaustion function and curvature of the target manifold is bounded from above.

Corollary 4.1 *Let (M^m, g) be a complete noncompact Riemannian manifold with Ric$(g) \geq -Kg$ for some $K \geq 0$ and let (N^n, h) be another complete noncompact manifold with Rm$(h) \leq \kappa$ for some $\kappa \geq 0$. Suppose there is a smooth function γ on M satisfying:*

$$C_0^{-1}(d(p,x)+1) \leq \gamma(x) \leq C_0(d(p,x)+1)$$

and

$$|\nabla^k \gamma| \leq C_0, \quad k = 1, 2$$

for some $C_0 > 0$ where $d(p,x)$ is the distance function on M and $p \in M$ is a fixed point. Then for any smooth map $f : M \to N$ with energy density uniformly bounded by e_0, there exists a solution to the harmonic map heat flow F from $M \times [0, T_0] \to N$ with initial value $F(x,0) = f(x)$, where $T_0 = C_1\kappa^{-1}$ for some C_1 depending only on e_0, K. Moreover,

$$\sup_M e(F(\cdot,t)) \leq 2e_0 \exp(Kt).$$

In particular, if $\kappa = 0$, then the heat flow has long time solution.

As mentioned in the introduction, Theorem 1.2 which is a result on short-time existence of the harmonic map heat flow coupled with the Ricci flow also follows immediately from Theorem 1.1 as a corollary.

Before we prove Theorem 1.1, we need the following extension lemma which will be used later.

Lemma 4.5 *Let $g(t)$ be a smooth family of complete metrics on M, $t \in [0, T]$, and (N, h) is another smooth complete manifold. Let $0 < T_1 < T$. Suppose F_1 is a smooth solution to the harmonic map heat flow from $M \times [0, T_1]$ to N and F_2 is a smooth solution to the harmonic map heat flow from $M \times [T_1, T]$ to N. Suppose $F_1 = F_2$ at $t = T_1$. Let*

$$F(x,t) = \begin{cases} F_1(x,t), \text{ if } (x,t) \in M \times [0, T_1]; \\ F_2(x,t), \text{ if } (x,t) \in M \times [T_1, T]. \end{cases}$$

Then, F is a smooth solution to the harmonic map heat flow from $M \times [0, T]$ to N.

Proof It is sufficient to show that F is smooth near (p, T_1) for all p. Consider local coordinates x^i near a point p in M and y^α near the point $F(p, T_1)$ in N. Near (p, T_1),

$$\begin{aligned}\frac{\partial}{\partial t} F_1^\alpha(x,t) =& g^{ij}(x,t)\big((F_1^\alpha)_{ij}(x,t) \\ &- \Gamma_{ij}^k(x,t)(F_1^\alpha)_k(x,t) + \widetilde{\Gamma}_{\beta\gamma}^\alpha(F_1(x,t))(F_1^\beta)_i(x,t)(F_1^\gamma)_j(x,t)\big).\end{aligned}$$

Here, $(F_1^\alpha)_i$, $(F_1^\alpha)_{ij}$ denote the partial derivatives of F_1^α and $\Gamma, \widetilde{\Gamma}$ are the Levi-Civita connections of Riemannian manifolds $(M, g(t))$ and (N, h), respectively. Similarly, we have the corresponding equations for F_2. Since F_1, F_2 are smooth up to T_1, we conclude that as $(x,t), (x', t') \to (p, T_1)$ with $t > T_1 > t'$, all the corresponding space derivatives of $F_2(x,t)$ and $F_1(x', t')$ will converge to the same limit. From the equations, we conclude that $\partial_t F_2(x,t)$ and $\partial_t F_1(x', t')$ will converge to the same limit. Differentiate the equation with respect to x, we can conclude that $\partial_t \partial_x^l F_1(x', t')$ and $\partial_t \partial_x^l F_2(x,t)$ will converge to the same limit. Differential the equation with respect to t, we conclude that $\partial_t \partial_t F_2(x,t)$ and $\partial_t \partial_t F_1(x', t')$ will converge to the same limit. Continue in this way, one can see that the lemma is true. □

The proof of Theorem 1.1 follows from the following special case so that condition **(a1)** is replaced by a condition on $\text{Ric}(g(t))$ and **(a2)** is replaced by the conditions that $|H|, |\nabla H|$ are uniformly bounded.

Proposition 4.1 *Let M^m be a noncompact manifold and let $g(t)$ be a smooth family of complete metrics defined on $M \times [0, T]$ so that*

$$\frac{\partial}{\partial t} g(x,t) = H(x,t).$$

Let (N^n, h) be another complete manifold. Suppose **(a3)** *is satisfied. Assume $|H|_{g(t)} \le L$, $|\nabla H|_{g(t)} \le L$ in $M \times [0, T]$ for some constant $L > 0$ and $Ric(g(t)) \ge -K(t)g(t)$*

for some $K(t) \geq 0$ *so that*

$$K_0 := \int_0^T K(t)dt < \infty.$$

Moreover, assume there exists a smooth exhaustion function γ *on* M *and* $C_0 > 0$ *such that*

$$C_0^{-1}(d_T(p,x)+1) \leq \gamma(x) \leq C_0(d_T(p,x)+1)$$

and

$$|\nabla_T^k \gamma| \leq C_0$$

for $1 \leq k \leq 2$*, where* d_T *is the distance function and* ∇_T *is the covariant derivative with respect to* $g(T)$*. Let* $f : (M, g_0) \to (N, h)$ *be a smooth map such that*

$$\sup_M e(f) \leq e_0.$$

Then, there exists a smooth solution F *to the heat flow for harmonic map defined on* $M \times [0, T_0]$ *with initial map* f *such that*

$$\sup_{M\times[0,T_0]} e(F) < \infty$$

for some $0 < T_0 \leq T$ *depending only on* $m, n, T, K_0, \kappa, L, e_0, C_0$*.*

Let us prove Theorem 1.1 assuming the proposition is true.

Proof of Theorem 1.1. Fix any small $T > s > 0$. Let $g^{(s)}(t) := g(t+s)$, $0 \leq t \leq T-s$. Then,

$$\frac{\partial}{\partial t} g^{(s)}(t) = H(s+t) =: H^{(s)}(t).$$

By condition **(a2)**, there is a constant $C_1 = C_1(a, s)$ such that $|H^{(s)}(t)|$, $|\nabla H^{(s)}(t)|$ are uniformly bounded by C_1 on $M \times [0, T-s]$. Here, the norm and covariant derivative are with respect to $g^s(t)$. By conditions **(a1)**, we have

$$2\text{Ric}(g(t)) \geq -(K(t) + C_1)g(t).$$

Together with **(a3)** and the condition on γ, by Proposition 4.1, for any smooth map $\widetilde{f} : M \to N$ with energy density bounded by $\widetilde{e}_0$, and for any $T - s > t_0 > 0$, the harmonic map heat flow has a solution $\widetilde{F}$ in $M \times [t_0, t_0 + \widetilde{T}_0]$ with initial map $\widetilde{f}$ so that

$$\sup_{M\times[t_0,t_0+\widetilde{T}_0]} e(\widetilde{F}) < \infty.$$

Here, $\widetilde{T}_0$ depends only on $m, n, T-s, K_0, \kappa, \widetilde{e}_0, C_1, C_0$ as long as $t_0+\widetilde{T}_0 \le T-s$.

In particular, the harmonic map heat flow has a solution $F^{(s)}$ in $M\times[0,\widetilde{T}_1]$ with initial map f so that

$$\sup_{M\times[0,\widetilde{T}_1]} e(F^{(s)}) < \infty.$$

Here, $\widetilde{T}_1$ depends only on $m, n, T-s, K_0, C_1, \kappa, e_0, C_0$ as long as $\widetilde{T}_1 \le T-s$. By Lemma 4.2, we conclude that

$$\sup_{M\times[0,\widetilde{T}_1]} e(F^{(s)}) \le 2e_0\exp(K_0)$$

provided $\widetilde{T}_1 \le \frac{1}{2}(2\kappa e_0\exp(K_0))^{-1}$. If this is the case, then one can extend the solution to $[0,\widetilde{T}_1+\widetilde{T}_0]$ by Lemma 4.5, provided $\widetilde{T}_1+\widetilde{T}_0 \le T-s$, where $\widetilde{T}_0$ depends only on $m, n, T-s, K_0, C_1, \kappa, C_0$, and

$$\widetilde{e}_0 := 2e_0\exp(K_0).$$

Continue in this way, we conclude that the harmonic map heat flow has a solution $F^{(s)}$ in $M\times[0,T_s]$ with initial map f so that

$$\sup_{M\times[0,T_s]} e(F^{(s)}) \le 2e_0\exp(K_0)$$

where

$$T_s = \min\{T-s, \frac{1}{2}(2\kappa e_0\exp(K_0))^{-1}\}.$$

By Lemma 4.4, we have

$$|\tau(F^{(s)})|_{g^{(s)}(t)} \le C_2 t^{-\frac{1}{2}}$$

where C_2 depends only on $m, n, e_0, a, K_0, \kappa$. Hence, $d_N(F^{(s)}(x,t), f(x)) \le C_3$ in $M\times[0,T_s]$ for some C_3 independent of s. In local coordinates, $F^{(s)}$ satisfies a system of semi-linear equations

$$\frac{\partial}{\partial t}(F^{(s)})^\alpha(x,t) = g_s^{ij}(x,t)\left((F^{(s)})^\alpha_{ij} - \Gamma^k_{ij}(g_s(x,t))(F^{(s)})^\alpha_k + \widetilde{\Gamma}^\alpha_{\beta\gamma}(F^{(s)})^\beta_i(F^{(s)})^\gamma_j\right).$$

Moreover, $|\nabla(F^{(s)})^\alpha|$ are uniformly bounded. Then, we can argue as in the proof of Lemma 3.1 to conclude that for any precompact domain $\Omega\subset M$, all orders of derivatives of $F^{(s)}$ are uniformly bounded in $\Omega\times[0,T_s]$. Passing to a subsequence, we conclude that $F^{(s)}$ will converge on $M\times[0,T_0]$ to a solution of the harmonic map heat flow coupled with $g(t)$ on $M\times[0,T_0]$ with initial map being f such that $e(F)$ and $|\tau(F)|$ have bounds as stated in the theorem. □

4.4 Proof of Proposition 4.1

Our method is to use conformal change to find solutions on compact domains. We then obtain estimates for the energy density and the norm of the tension field in order to take limit as in the proof of Theorem 1.1 under the assumption of Proposition 4.1. Let $\chi \in (0, \frac{1}{8})$, $\Phi : [0, 1) \to [0, \infty)$ be the function:

$$\Phi(s) = \begin{cases} 0, & s \in [0, 1-\chi]; \\ -\log\left[1 - \left(\dfrac{s-1+\chi}{\chi}\right)^2\right], & s \in (1-\chi, 1). \end{cases} \tag{4.4}$$

Let $\varphi \geq 0$ be a smooth function on $\mathbb{R}$ such that $\varphi(s) = 0$ if $s \leq 1 - \chi + \chi^2$, $\varphi(s) = 1$ for $s \geq 1 - \chi + 2\chi^2$

$$\varphi(s) = \begin{cases} 0, & s \in [0, 1-\chi+\chi^2]; \\ 1, & s \in (1-\chi+2\chi^2, 1). \end{cases} \tag{4.5}$$

such that $\dfrac{2}{\chi^2} \geq \varphi' \geq 0$. Define

$$\mathfrak{F}(s) := \int_0^s \varphi(\tau)\Phi'(\tau)d\tau.$$

From [13], we have:

Lemma 4.6 *Suppose* $0 < \chi < \frac{1}{8}$. *Then, the function* $\mathfrak{F} \geq 0$ *defined above is smooth and satisfies the following:*

(i) $\mathfrak{F}(s) = 0$ *for* $0 \leq s \leq 1 - \chi + \chi^2$.
(ii) $\mathfrak{F}' \geq 0$ *and for any* $k \geq 1$, $\exp(-k\mathfrak{F})\mathfrak{F}^{(k)}$ *is uniformly bounded.*

Let γ be the exhaustion function as in the assumption of the proposition. Let $\chi = \frac{1}{16}$. For $\rho > 1$, let U_ρ be the component of $\gamma^{-1}([0, \rho))$ containing a fixed point p. Note that U_ρ exhausts M as $\rho \to \infty$. Now we consider a function on U_ρ defined by

$$\phi(x) := \mathfrak{F}(\frac{\gamma(x)}{\rho}).$$

and let

$$\widetilde{g}(t) := \exp(2\phi)g(t).$$

Then, $\widetilde{g}$ is a smooth family of complete metrics on U_ρ so that

$$\frac{\partial}{\partial t}\widetilde{g} = \widetilde{H}$$

where $\widetilde{H} = \exp(2\phi)H$.

Lemma 4.7 *With the above notations, under the assumptions as in Proposition 4.1, we have in $U_\rho \times [0, T]$:*

(i) $|\widetilde{H}(t)|_{\widetilde{g}(t)}, |\widetilde{\nabla}\widetilde{H}(t)|_{\widetilde{g}(t)} \le C$ *for some constant C depending only on L, T, C_0, where $\widetilde{\nabla}$ is the derivative with respect to $\widetilde{g}(t)$.*

(ii) $2Ric(\tilde{g}(t)) + \tilde{H}(t) \ge -\tilde{K}(t)\tilde{g}(t)$ *for some $\tilde{K} \ge 0$ so that $\int_0^T \tilde{K}(t)dt \le \tilde{K}_0$ for some constant $\tilde{K}_0$ depending only on L, T, K_0, C_0.*

(iii) $|Rm(\tilde{g}(t))|$ *is uniformly bounded.*

Proof of Theorem 1.1. Since $|H| \le L$, we have

$$C_1^{-1} g(t) \le g(T) \le C_1 g(t) \tag{4.6}$$

for some $C_1 = C_1(L, T)$. On the other hand, since $|\nabla H| \le L$ and $\frac{\partial}{\partial t} g = H$, if we let Γ and $\bar{\Gamma}$ be the Christoffel symbols of $g(t)$ and $\bar{g} = g(T)$, respectively, and let $A = \Gamma - \bar{\Gamma}$, we have $|A|_{g(t)}$ is bounded by a constant depending only on L, T. Since

$${}^t\nabla^2\gamma = \nabla_T^2\gamma + A * \nabla_T\gamma,$$

we have

$$|\nabla\gamma|_{g(t)} \le C_2, \quad |\nabla^2\gamma|_{g(t)} \le C_2, \tag{4.7}$$

for some constant $C_2 = C_2(L, T, C_0)$. Next we want to compute the gradient and Hessian of ϕ. Let the covariant derivative with respect to $g(t)$ be denoted by ;, then

$$\begin{aligned} \phi_i &= \rho^{-1}\mathfrak{F}'\gamma_i, \\ \phi_{;ij} &= \rho^{-1}\mathfrak{F}'\gamma_{;ij} + \rho^{-2}\mathfrak{F}''\gamma_i\gamma_j. \end{aligned}$$

By Lemma 4.6,

$$\begin{cases} |\nabla\phi|_{g(t)} \le C_3 \exp(\phi); \\ |\nabla^2\phi|_{g(t)} \le C_3 \exp(2\phi). \end{cases} \tag{4.8}$$

for some $C_3 = C_3(L, T, C_0)$ because $\phi \ge 0$.

$$\begin{aligned} |\widetilde{H}|^2_{\widetilde{g}(t)} =& \widetilde{g}^{ij}\widetilde{g}^{kl}\widetilde{H}_{ik}\widetilde{H}_{jl} \\ =& e^{-4\phi}g^{ij}g^{kl}e^{4\phi}H_{ik}H_{ij} \\ =& |H|^2_{g(t)}. \end{aligned}$$

Since

$$\begin{aligned} \tilde{\nabla}\tilde{H} =& (\tilde{\nabla} - \nabla)\tilde{H} + \nabla\tilde{H} \\ =& (\tilde{\Gamma} - \Gamma) * e^{2\phi} * H + 2e^{2\phi}\phi'\rho^{-1}\nabla\gamma * H \\ &+ e^{2\phi} * H \qquad\qquad , \\ =& (2\phi'\rho^{-1}\nabla\gamma * g * g^{-1}) * e^{2\phi} * H \\ &+ 2e^{2\phi}\phi'\rho^{-1}\nabla\gamma * H + e^{2\phi} * H \end{aligned}$$

we have

$$|\tilde{\nabla}\tilde{H}|_{\tilde{g}} \leq C_4 \tag{4.9}$$

for some $C_4 = C_4(L, T, C_0)$. These prove (i).

To prove (ii), by (4.8), denote the Ricci tensor of $\tilde{g}$ by $\tilde{R}_{ij}$ and the Ricci tensor of $g(t)$ by R_{ij}, then

$$\begin{aligned}\tilde{R}_{ij} =& R_{ij} + (m-2)\phi_{;ij} + (m-2)\phi_i\phi_j - [\Delta_{g(t)}\phi + (m-2)|\nabla\phi|^2]g_{ij} \\ \geq& -K(t)g_{ij} - C_5\exp(2\phi)g_{ij} \\ \geq& -\tilde{K}(t)\tilde{g}_{ij}\end{aligned}$$

where $\tilde{K}(t) = K(t) + C_5(L, T, C_0)$, because $\phi \geq 0$.

To prove (iii),

$$|\widetilde{\mathrm{Rm}}|_{\tilde{g}} \leq C_6 \exp(-2\phi)\left(|\mathrm{Rm}|_g + |\nabla^2\phi|_g + |\nabla\phi|_g^2\right)$$

for some $C_6 = C_6(m)$. By using (4.8) and the fact that the curvature of g is uniformly bounded in $U_\rho \times [0, T]$ the result follows. □

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1 Let $\phi, \rho, U_\rho, \tilde{g}$ be as above. We claim that there is $T_0 > 0$ depending only on $m, n, K_0, \kappa, e_0, L, C_0, T$ such that the heat flow for harmonic map from $(U_\rho, \tilde{g}(t))$ to N with initial map $f|_{U_\rho}$ has a solution $F^{(\rho)}$ on $U_\rho \times [0, T_0]$ so that

$$\sup_{U_\rho\times[0,T_0]} e(F^{(\rho)}) < \infty.$$

Suppose the claim is true, using the fact that $2\mathrm{Ric}(\tilde{g}(t)) + \widetilde{H}(t) \geq -\tilde{K}(t)\tilde{g}(t)$ for some $\tilde{K}$ with

$$\tilde{K}_0 := \int_0^T \tilde{K} < \infty$$

where $\tilde{K}_0$ depends only on K_0, L, C_0, T, one can proceed as in the proof of Theorem 1.1 to conclude the proposition is true by taking limit of a subsequence of $F^{(\rho)}$ with $\rho \to \infty$.

In order to prove the claim, we use the method as in [4] and [16]. Since the image of $f|_{U_\rho}$ is bounded in N. Let $f|_{U_\rho} \Subset \Omega \Subset N$. Here, Ω is a bounded domain in N and let Ω_1 be another bounded domain with $\Omega \Subset \Omega_1$. Isometrically embed a neighborhood O of Ω_1 into $\mathbb{R}^q$ for some $q \in \mathbb{N}$. Let W be a bounded tubular neighborhood of O in $\mathbb{R}^q$. Let $\pi : W \to O$ be the nearest point projection. Write

$$\pi = (\pi^1, \pi^2, \cdots, \pi^q) = (\pi^A)_{1\leq A\leq q}.$$

We can extend π smoothly from a possible smaller tubular neighborhood V of Ω_1 to the whole $\mathbb{R}^q$ such that each π^A is compactly supported and π is not changed in V. Hence, π^A, $\pi^A_B := \frac{\partial \pi^A}{\partial z^B}$, $\pi^A_{BC} := \frac{\partial \pi^A}{\partial z^B \partial z^C}$ etc are bounded, where $z = (z^A)$ are the standard coordinates of $\mathbb{R}^q$.

Let $f : U_\rho \to N$ so that $f(U_\rho) \Subset \Omega$. Then we can write

$$f(x) = (f^A(x)) \in \mathbb{R}^q.$$

Note that $e(f) = \sum_A |\nabla f^A(x)|^2$. By Lemma 4.7,

$$|\tilde{H}|_{\tilde{g}(t)}, |\widetilde{\nabla} \tilde{H}|_{\tilde{g}(t)} \leq C_1$$

for some $C_1 = C_1(L, T, C_0)$. $|\mathrm{Rm}(\tilde{g}(t))| \leq Q$ which may also depend on ρ. We may assume

$$|\pi^A_{BC}| \leq D.$$

Consider the following system of equations:

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)} \right) F^A = -\pi^A_{BC}(F) \langle \nabla F^B, \nabla F^C \rangle \tag{4.10}$$

in $U_\rho \times (0, T]$ and $F^A(x, 0) = f^A(x)$ in U_ρ for $A = 1, 2, \cdots, q$. By Lemma 3.1, the system has a smooth solution on $U_\rho \times [0, T_1]$, where T_1 depends only on $m, q, D, L, T, C_0, Q, \sup_{U_\rho} e(f)$. Moreover, F and $|\nabla F|$ are uniformly bounded by a constant C depending only on $m, q, D, L, T, C_0, Q, \sup_{U_\rho} e(f)$ and Ω. On the other hand,

$$F^A(x, t) = \int_M G(x, t; y, 0) f^A(y) \mathrm{d}y + \int_0^t \int_M G(x, t; y, s) Q^A(y, s) \mathrm{d}V_s(y) \mathrm{d}s$$

where G is the fundamental solution to the heat equation coupled with $g(t)$ and Q^A is the right hand side of (4.10). By Proposition 3.2, we conclude that

$$|F^A(x, t) - f^A(x)| \leq C_2 t^{\frac{1}{2}}$$

for some constant $C_2 = C_2(m, q, D, Q, T, \sup_{U_\rho} e(f))$. In particular, there exists $0 < T_2 \leq T_1$ depending only on $m, q, D, Q, T, \sup_{U_\rho} e(f)$ such that $F(x, t)$ will be inside the tubular neighborhood W of O for $0 \leq t \leq T_2$. By the proof of [16, Lemma 3.2] and the maximum principle Theorem 2.2, we conclude $F(x, t) \in N$ for $(x, t) \in U_\rho \times [0, T_2]$. This implies $F(x, t)$ satisfies the harmonic map heat flow on $U_\rho \times [0, T_2]$ to N.

Up to now, we have proved the following: If $f : U_\rho \to N$ is a smooth bounded map with energy density bounded, then there is a smooth solution F to the heat flow for harmonic map with initial data f on $U_\rho \times [0, T_2]$ so that the image and the energy density of F are uniformly bounded. By Lemma 4.5, we conclude that as long as the energy of F is uniformly bounded on $U_\rho \times [0, T']$ and F has bounded image, then F can be extended beyond T' as a solution to the harmonic map heat flow so that the energy is uniformly bounded.

Recall that $2\mathrm{Ric}(\tilde{g}) + \tilde{H} \geq -\tilde{K}(t)\tilde{g}(t)$ with $\tilde{K}_0 = \int_0^T \tilde{K} < \infty$. Using this condition, by Lemma 4.2, we conclude that

$$e(F)(\cdot, t) \leq 2e_0 \exp(\tilde{K}_0)$$

as long as $0 \leq t \leq T_0 := \min\{T, \frac{1}{2}\left(2\kappa e_0 \exp(\tilde{K}_0)\right)^{-1}\}$. By Lemma 4.4, there is constant C_3 depending only on $m, n, e_0, L, \tilde{K}_0, T, \kappa$ such that

$$|\tau(F)|(x, t) \leq C_3 t^{-\frac{1}{2}}$$

as long as $0 \leq t \leq T_0$. Since $\tau(F) = F_t$, we have $d_N(f(x), F(x,t)) \leq C_5 t^{\frac{1}{2}}$. In particular, the image of $U_\rho \times [0, T_0]$ is bounded because $f(U_\rho)$ is bounded. This completes the proof of the claim and hence the proposition. □

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