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Symplectic Flatness and Twisted Primitive Cohomology

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Abstract

We introduce the notion of symplectic flatness for connections and fibre bundles over symplectic manifolds. Given an A_∞-algebra, we present a flatness condition that enables the twisting of the differential complex associated with the A_∞-algebra. The symplectic flatness condition arises from twisting the A_∞-algebra of differential forms constructed by Tsai, Tseng and Yau. When the symplectic manifold is equipped with a compatible metric, the symplectic flat connections represent a special subclass of Yang–Mills connections. We further study the cohomologies of the twisted differential complex and give a simple vanishing theorem for them.

Keywords Symplectic geometry · Flat bundles · A-infinity algebra

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1 Introduction

For a vector bundle E over a smooth manifold M of dimension d, we can study differential forms $\Omega^*(M, E)$ taking values in the fibre space of E. For these forms, we can write a de Rham-type complex for $\Omega^*(M, E)$:

$$0 \longrightarrow \Omega^0(M, E) \xrightarrow{d_A} \Omega^1(M, E) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Omega^d(M, E) \xrightarrow{d_A} 0 \tag{1.1}$$

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where locally $d_A = d + A$ and $A \in \Omega^1(M, \mathrm{End}\, E)$ is the connection acting on E. The above complex is only a differential complex if the curvature of the connection, $F = (d_A)^2 = dA + A \wedge A = 0$. This is the well-known fact that the de Rham complex can be *twisted* by a flat bundle, i.e. a bundle E that allows for a connection whose curvature vanishes.

Now suppose M is a complex manifold of dimension $d = 2n$ and E is a complex vector bundle over M. On complex manifolds, the differential forms can be decomposed into (p, q) components, that is

$$\Omega^k(M, E) = \bigoplus_{k=p+q} \mathcal{A}^{p,q}(M, E),$$

and the exterior derivative decomposes into the Dolbeault operators, $d = \partial + \bar{\partial}$. The decompositions of forms and exterior derivative allow us to consider a more refined complex, the Dolbeault complex, which can be twisted as well:

$$0 \longrightarrow \mathcal{A}^{0,0}(M, E) \xrightarrow{\bar{\partial}_A} \mathcal{A}^{0,1}(M, E) \xrightarrow{\bar{\partial}_A} \cdots \xrightarrow{\bar{\partial}_A} \mathcal{A}^{0,n}(M, E) \xrightarrow{\bar{\partial}_A} 0 \tag{1.2}$$

where locally $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ and $A^{0,1}$ is the (0, 1) component of the connection form A. That the above complex is actually a differential complex imposes the condition

$$\left(\bar{\partial}_A\right)^2 = (\bar{\partial} + A^{0,1})^2 = \bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = F^{0,2} = 0. \tag{1.3}$$

Notice that this is a weakening of the smooth flatness condition $F = 0$ in that (1.3) requires only that the $F^{0,2}$ component of the curvature vanishes. But a complex vector bundle with a connection such that $F^{0,2} = 0$ is well known to be equivalent to the bundle E having the structure of a holomorphic vector bundle. And so in the complex case, *complex flat* bundles which can twist the Dolbeault complex are just holomorphic vector bundles.

Flat bundles and holomorphic vector bundles are basic and important objects on smooth and complex manifolds, respectively. In this paper, we are interested in exploring special vector bundles on symplectic manifolds. Specifically, we ask a simple question: is there a *symplectic flatness* condition for a vector bundle E over a symplectic manifold, (M^{2n}, ω)?

We can answer this question by proceeding in a similar way as in the smooth and complex cases described above. Let us first recall that on a symplectic manifold, (M^{2n}, ω), the differential forms can be expressed in a polynomial expansion in powers of ω. This decomposition of forms is commonly called the Lefschetz decomposition and given by

$$\Omega^k(M) = \bigoplus_{k=2r+s} \omega^r \wedge P^s(M),$$

where $P^s(M)$ for $s = 0, 1, \ldots, n$, denotes the space of primitive forms, i.e. forms that we can not extract an ω from them. More precisely, a form $\beta \in P^s(M)$ if there does not exist an $\xi \in \Omega^{s-2}(M)$ such that $\beta = \omega \wedge \xi$. And besides forms, the exterior derivative, like in the complex case, also has a decomposition into two linear differential operators (∂_+, ∂_-) dependent on the symplectic structure [7]

$$d = \partial_+ + \omega \wedge \partial_- \tag{1.4}$$

and with desirable properties: $(\partial_+)^2 = (\partial_-)^2 = 0$ and $\omega \wedge \partial_+\partial_- = -\omega \wedge \partial_-\partial_+$. Together, they lead to a differential complex that is elliptic (see [7] and references therein)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P^0(M) & \xrightarrow{\partial_+} & P^1(M) & \xrightarrow{\partial_+} & \ldots & \xrightarrow{\partial_+} & P^{n-1}(M) & \xrightarrow{\partial_+} & P^n(M) \\ &&&&&&&&&& \big\downarrow{\scriptstyle -\partial_+\partial_-} \\ 0 & \xleftarrow{-\partial_-} & P^0(M) & \xleftarrow{-\partial_-} & P^1(M) & \xleftarrow{-\partial_-} & \ldots & \xleftarrow{-\partial_-} & P^{n-1}(M) & \xleftarrow{-\partial_-} & P^n(M) \end{array} \tag{1.5}$$

Hence, we can try to extend this complex to one acting on $P^*(M, E)$, the space of primitive forms with values in E and also twist the operators $(\partial_+, \partial_-, \partial_+\partial_-)$ with a connection form as in (1.1) and (1.2). However, the peculiar definitions of the differentials in (1.5) raise immediate issues. In particular, note that both $\partial_- : P^s \to P^{s-1}$ and $\partial_+\partial_- : P^n \to P^n$ do not increase the degree of the form by one. Can we simply twist these operators in the primitive complex above by a connection one-form?

The twisting procedure in the symplectic case turns out to be a bit more subtle. Nevertheless, twisting the primitive complex can still be achieved because the complex has an A_∞-algebra structure [6]. We will show in this paper, that given any A_∞-module structure of differential forms taking values in E, there is a natural twisting of the differentials of the complex by a connection one-form. The new twisted differentials, however, do not represent a deformation of the A_∞-module and hence does not lead to a new A_∞-module structure.

Our result in the symplectic case is the following. We can write down a twisted primitive complex of forms taking values in E, if the connection one-form on E satisfies the following *symplectic flatness* condition:

Definition 1.1 For (M^{2n}, ω) a symplectic manifold, let $\pi : E \to M$ be a vector bundle with a connection, d_A the corresponding covariant derivative and F the curvature two-form. We call the connection **symplectically flat** if

$$\begin{aligned} F &= \Phi\, \omega \\ d_A \Phi &= d\Phi + [A, \Phi] = 0, \end{aligned} \tag{1.6}$$

where $\Phi \in \Omega^0(M, \mathrm{End}\, E)$. If such a connection exists on E, we say E is a symplectically flat bundle.

Table 1 A comparison of smooth, complex and symplectic flat bundles

	Smooth M	Complex (M^{2n}, J)	Symplectic (M^{2n}, ω)
Forms	Ω^k	$\Omega^k = \oplus\, \mathcal{A}^{p,q}$	$\Omega^k = \oplus\, \omega^r \wedge P^s$
Differential	d	$d = \partial + \bar{\partial}$	$d = \partial_+ + \omega \wedge \partial_-$
Flatness condition	$F = 0$	$F^{0,2} = 0$	$F = \Phi\, \omega,\ d_A \Phi = 0$
Special local frame	$A = 0$	$A^{0,1} = 0$	$A = \Phi\lambda\,,\ \Phi = \text{const.}, d\lambda = \omega$

Remark 1.2 For a principal bundle P over (M^{2n}, ω), we say P is symplectically flat if there exists a connection form on P whose curvature satisfies (1.6).

Though written as two equations, condition (1.6) for symplectically flat is effectively just a single equation in all $2n$ dimensions. For when $n = 1$, the first equation, $F = \Phi\, \omega$, is trivial and gives no condition. On the other hand, when $n \geq 2$, the second equation, $d_A \Phi = 0$, becomes unnecessary as it is implied by the first equation of (1.6) and the Bianchi identity.

To arrive at the symplectic flatness condition, we shall begin in Sect. 2 by reviewing the A_∞-algebra structure of the primitive cochain complex of Tsai, Tseng and Yau [6]. We will present a general procedure to twist the differential complex associated with any A_∞-algebra. This general procedure gives the flatness conditions for de Rham, Dolbeault and the primitive symplectic complex and results in the twisted primitive complex given in (2.15).

With the symplectic flatness condition established, it is helpful to have examples of symplectic flat bundles and understand some of their properties. We will do this in Sect. 3. Indeed, we will show that symplectically flat connections are a special type of Yang–Mills connections in the presence of a compatible metric on (M^{2n}, ω). Also, as shown in Table 1, one can choose special local frames such that the local connection form takes the form $A = \Phi\, \lambda$, where Φ is a constant matrix and $d\lambda = \omega$. We also point out an interesting relationship: when ω is an integral class and we can define a circle bundle X over M whose Euler class is given by ω, (i.e. the prequantum circle bundle of M), the symplectic flat bundle lifts to a flat bundle on X.

Finally, having twisted the primitive elliptic complex, we analyse the resulting cohomologies on $P^*(M, E)$ in Sect. 4. We will calculate these twisted primitive cohomologies on $\mathbb{R}^{2n}$ and also prove a simple vanishing theorem for the twisted cohomologies when Φ is invertible.

In this paper, we will mainly focus on symplectically flat vector bundles. An extensive discussion of symplectically flat principal bundles including their classification will be given in a companion work [8].

This paper is written for a special issue of the Journal of Geometric Analysis in honour of Peter Li. We are fortunate to have had the opportunity to interact closely with Peter Li through our affiliations with the UC Irvine Geometry/Topology group which Peter Li shaped and built-up over twenty plus years as a leading, senior faculty at UC Irvine. Peter Li has without fail been supportive and generous with kind advice to us. We are grateful to him for his strong encouragement in our research work.

2 Twisting the Differential Complex of an A_∞-algebra

On symplectic manifolds, Tsai–Tseng–Yau [6] showed that the primitive cochain complex in (1.5) can be extended to an A_∞-algebra. (More precisely, it is a commutative A_3 algebra.) We will first describe this algebra below. For ease, we will call this algebra of differential forms on symplectic manifolds the **TTY algebra**. We then proceed to give a heuristic description of how to twist the differential of the primitive TTY algebra. Going further, we show in generality how the differential of any A_∞-module can be twisted. The twisted differential of the TTY algebra is just a special case of this general A_∞-module twisting.

2.1 Preliminaries: TTY Algebra

We mostly follow the notations of [6].

As mentioned, differential forms on a symplectic manifold (M^{2n}, ω) has a Lefschetz decomposition. Any $\eta_k \in \Omega^k(M)$ can be expressed as a polynomial in ω:

$$\eta_k = \beta_k + \omega \wedge \beta_{k-2} + \ldots + \omega^p \wedge \beta_{k-2p} + \ldots \tag{2.1}$$

where $\{\beta_k, \beta_{k-2}, \ldots, \beta_{k-2p}, \ldots\}$ are all primitive forms in $P^*(M)$ and are determined uniquely by η_k and ω. The non-degeneracy of ω also allows us to define the following three operators on differential forms [6]:

1. L^p: When $p = k \geq 0$, $L^k = \omega^k\wedge$. When $p = -k < 0$, L^{-k} removes k powers of ω from a form. For example, acting on the Lefschetz component $\omega^r \wedge \beta_s$ for $\beta_s \in P^s$, $L^{-k}(\omega^r \wedge \beta_s) = \omega^{r-k} \wedge \beta_s$ if $k \leq r < n - s + 1$. When $k > r$, then $L^{-k}(\omega^r \wedge \beta_s) = 0$.
2. $*_r$: For $\alpha \in \Omega^k$, $*_r\, \alpha = L^{n-k}\alpha$.
3. Π^p: By Lefschetz decomposition, every $\eta \in \Omega^k$ can be uniquely written as $\beta + \omega^{p+1} \wedge \gamma$, where $\beta = \beta_k + \omega \wedge \beta_{k-2} + \ldots + \omega^p \wedge \beta_{k-2p}$ in the decomposition of (2.1). Then, $\Pi^p\eta = \beta$.

The third operator, Π^p, is a projection operator and defines the space of p-filtered forms: $F^p\Omega^k(M) = \Pi^p\left[\Omega^k(M)\right]$ for $0 \leq p \leq n$. Note that the Lefschetz decomposition of p-filtered forms has at most terms of order ω^p.

For each $p = 0, \ldots, n$, there is a TTY algebra consisting of forms in $F^p\Omega^*(M)$. In this paper, we are mainly concerned with the $p = 0$ filtered case $F^0\Omega^*(M) = P^*(M)$ which consist of just primitive forms. To simplify notation, we will write $\Pi = \Pi^0$, i.e. the projection onto the primitive component of the Lefschetz decomposition.

The TTY algebra is an A_∞-algebra . Let us recall the definition of an A_∞-algebra (for a reference, see [3]).

Definition 2.1 An A_∞-**algebra** is a $\mathbb{Z}$-graded vector space $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}^k$ endowed with graded linear maps

$$m_k : \mathcal{A}^{\otimes k} \to \mathcal{A}, \qquad k \geq 1$$

of degree $2 - k$ satisfying

$$\sum_{r+s+t=k} (-1)^{r+st} m_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0. \tag{2.2}$$

The first three relations of (2.2) are the following:

$$m_1 m_1 = 0 \tag{2.3}$$
$$m_1 m_2 = m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1) \tag{2.4}$$
$$m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1). \tag{2.5}$$

By the third relation, if $m_3 = 0$, then m_2, which acts as a product, is associative. An A_∞-algebra with only $\{m_1, m_2\}$ non-zero is simply a differential graded algebra (DGA). The TTY algebra is, however, generally non-associative with $m_3 \neq 0$, but m_k for $k \geq 4 = 0$ can be set to zero. Hence, it can be more precisely called an A_3 algebra.

We now define the TTY-algebra $(\mathcal{F}, m_1, m_2, m_3)$ on primitive forms $F^0\Omega^*(M) = P^*(M)$. First, the elements of the primitive TTY-algebra are those of the differential complex (1.5). The grading follows that of the complex and to help distinguish the two sets of primitive forms, we use the $\pm$ subscript

$$\mathcal{F} = \left\{P^0_+, P^1_+, \ldots, P^n_+, P^n_-, \ldots, P^1_-, P^0_-\right\} \tag{2.6}$$

where for $k = 0, 1, \ldots, n$, $P^k_+ = P^k$ have grading k, and $P^k_- = P^k$ have grading $2n+1-k$. As for the m_k maps, the first map, m_1, is just the differential of the complex (1.5).

The m_1 map.

$$m_1\beta = \begin{cases} \partial_+\beta, & \text{for } \beta \in P^k_+, \quad k < n, \\ -\partial_+\partial_-\beta, & \text{for } \beta \in P^n_+, \\ -\partial_-\beta, & \text{for } \beta \in P^k_-. \end{cases} \tag{2.7}$$

The m_2 map is the product operation and is dependent on the pair of primitive spaces that it acts on. At times, we will denote it by the product symbol and write $m_2(\beta, \gamma)$ as $\beta \times \gamma$. This $\times$ operation is graded commutative.

The m_2 map.

1. For $\beta \in P^j_+, \gamma \in P^k_+$, set

   $$\beta \times \gamma = \Pi(\beta \wedge \gamma) + \Pi *_r [-dL^{-1}(\beta \wedge \gamma) + (\partial_-\beta) \wedge \gamma + (-1)^j \beta \wedge (\partial_-\gamma)].$$

   Note that when $j + k \leq n$, the second term is trivial, and when $j + k > n$, the first term is trivial.
2. For $\beta \in P^j_+, \gamma \in P^k_-$, set

   $$\beta \times \gamma = (-1)^j *_r [\beta \wedge (*_r\gamma)].$$

3. For $\beta \in P_-^j$, $\gamma \in P_+^k$, set

$$\beta \times \gamma = *_r[(*_r\beta) \wedge \gamma].$$

4. For $\beta \in P_-^j$, $\gamma \in P_-^k$, set

$$\beta \times \gamma = 0.$$

The m_3 maps measures the non-associativity of the product $\times$. The non-associativity only arises when all three forms in the input come from P_+^*.

The m_3 map.

1. For $\beta \in P_+^i$, $\gamma \in P_+^j$, $\sigma \in P_+^k$ and $i + j + k \geq n + 2$, we set

$$m_3(\beta, \gamma, \sigma) = \Pi *_r [\beta \wedge L^{-1}(\gamma \wedge \sigma) - L^{-1}(\beta \wedge \gamma) \wedge \sigma].$$

2. For all other cases, we set $m_3(\beta, \gamma, \sigma) = 0$.

Finally, $m_k = 0$ for $k \geq 4$.

2.2 Twisting the Differential of the Primitive TTY Algebra

We now give a heuristic description of the twisting of the primitive elliptic complex (1.5). Let E be a vector bundle and consider $\Omega^*(M, E)$ and $P^*(M, E)$, the space of differential forms and primitive forms, respectively, taking values in E. We start with the relation of (∂_+, ∂_-) with d in (1.4). In particular, acting on primitive forms, $\partial_+ = \Pi\, d$ and $\partial_- = L^{-1}d$. Now, we can decompose the twisted exterior derivative into two components when acting on primitive forms $\beta \in P^s(M, E)$. Locally with $d_A = d + A\wedge$, we can write

$$\begin{aligned} d_A\beta &= (d + A\wedge)\,\beta = (\partial_+ + \omega \wedge \partial_-)\,\beta + (A \wedge \beta) \\ &= \Pi\,[(d + A\wedge)\,\beta] + \omega \wedge L^{-1}\,[(d + A\wedge)\,\beta] \\ &= \Pi(d_A\,\beta) + \omega \wedge L^{-1}(d_A\,\beta) \end{aligned}$$

where we have noted in the second line that a primitive form wedge a one-form has a Lefschetz decomposition into two terms and in the third line, that the decomposition is independent of the choice of local frames. This allows us to define the global twisted operators:

$$\begin{aligned} \partial_{+A} &= \Pi\, d_A : P^i(M, E) \to P^{i+1}(M, E) \\ \partial_{-A} &= L^{-1} d_A : P^i(M, E) \to P^{i-1}(M, E) \end{aligned}$$

such that

$$d_A = \partial_{+A} + \omega \wedge \partial_{-A} \qquad \text{acting on } P^k(M, E) \tag{2.8}$$

which gives a twisted version of (1.4). Locally, we have the expressions

$$\partial_{+A}\,\beta_i = \Pi\,[(d + A\,\wedge)\,\beta_i]\,, \qquad \partial_{-A}\,\beta_i = L^{-1}\,[(d + A\,\wedge)\,\beta_i]\,. \tag{2.9}$$

Now we can express the action of $(d_A)^2$ on a primitive form in two ways. First, note that the commutator

$$[d_A, \omega] = [d + A, \omega] = 0. \tag{2.10}$$

Therefore, we have

$$\begin{aligned}(d_A)^2\beta &= d_A\,(\partial_{+A}\,\beta + \omega \wedge \partial_{-A}\,\beta)\\ &= \partial_{+A}\,\partial_{+A}\,\beta + \omega \wedge \partial_{-A}\,\partial_{+A}\,\beta + \omega \wedge d_A\partial_{-A}\,\beta\\ &= \partial_{+A}\,\partial_{+A}\,\beta + \omega \wedge (\partial_{-A}\,\partial_{+A} + \partial_{+A}\,\partial_{-A})\,\beta + \omega^2 \wedge \partial_{-A}\,\partial_{-A}\,\beta \end{aligned}\tag{2.11}$$

Alternatively, we can also write

$$\begin{aligned}(d_A)^2\beta &= F \wedge \beta\\ &= (F_0 + \omega\,\Phi) \wedge \beta\\ &= F_0 \wedge \beta + \omega \wedge \Phi\beta \end{aligned}\tag{2.12}$$

where in the second line, we have Lefschetz decomposed the curvature $F = F_0 + \omega\,\Phi$ with $F_0 = \Pi\,F \in P^2(M, \mathrm{End}\,E)$ and $\Phi \in \Omega^0(M, \mathrm{End}\,E)$. If $F_0 = 0$, then comparing (2.11) with (2.12) and matching the Lefschetz components, we find

$$(\partial_{+A})^2 = (\partial_{-A})^2 = 0, \tag{2.13}$$

$$\omega\,(\partial_{+A}\,\partial_{-A} + \partial_{-A}\,\partial_{+A}) = \omega\,\Phi. \tag{2.14}$$

This suggests the following twisted primitive complex

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P^0(M,E) & \xrightarrow{\partial_{+A}} & P^1(M,E) & \xrightarrow{\partial_{+A}} & \cdots & \xrightarrow{\partial_{+A}} & P^{n-1}(M,E) & \xrightarrow{\partial_{+A}} & P^n(M,E)\\ &&&&&&&&&& \Big\downarrow{\scriptstyle -\partial_{+A}\,\partial_{-A}+\Phi}\\ 0 & \xleftarrow{-\partial_{-A}} & P^0(M,E) & \xleftarrow{-\partial_{-A}} & P^1(M,E) & \xleftarrow{-\partial_{-A}} & \cdots & \xleftarrow{-\partial_{-A}} & P^{n-1}(M,E) & \xleftarrow{-\partial_{-A}} & P^n(M,E) \end{array} \tag{2.15}$$

This is a differential complex if F satisfies the symplectically flat condition of $F = \Phi\omega$ (i.e. $F_0 = 0$) and $d_A\Phi = 0$ given in Definition 1.6. In particular, we write out the composition of the differential operators in the middle of the complex:

$$\begin{aligned}(-\partial_{+A}\,\partial_{-A} + \Phi)\,\partial_{+A}\,\beta_{n-1} &= \left[\partial_{+A}\,(\partial_{+A}\,\partial_{-A} - \Phi) + \Phi\partial_{+A}\right]\beta_{n-1}\\ &= \left[-\Phi\partial_{+A} + \partial_{+A}\,\Phi\right]\beta_{n-1}\\ &= \Pi\left[(-\Phi(d + A) + (d + A)\Phi)\,\beta_{n-1}\right]\end{aligned}$$

$$= \Pi\left[(d\Phi + [A, \Phi])\, \beta_{n-1}\right] = \Pi\left[(d_A\Phi) \wedge \beta_{n-1}\right]$$

$$\begin{aligned}
\partial_{-A}\ (-\partial_{+A}\,\partial_{-A}\ + \Phi)\, \beta_n &= (\partial_{+A}\,\partial_{-A}\ - \Phi)\, \partial_{-A}\, \beta_n + \partial_{-A}\ (\Phi\beta_n) \\
&= (-\Phi\partial_{-A}\ + \partial_{-A}\,\Phi)\, \beta_n \\
&= -\Phi L^{-1}(d + A)\beta_n + L^{-1}(d + A)(\Phi\beta_n) \\
&= L^{-1}\,(d\Phi + [A, \Phi])\, \beta_n = L^{-1}\,((d_A\Phi) \wedge \beta_n)
\end{aligned}$$

which both vanish since $d_A\Phi = d\Phi + [A, \Phi] = 0$.

In the next subsection, we will give a more systematic description of how to obtain the twisted differentials. We will show how all A_∞-algebras can be twisted and that the symplectic flat condition needed above matches exactly the required condition for general twisting.

2.3 Twisting the Differential of an A_∞-module

We are interested to twist the primitive elliptic complex (1.5) in a similar manner to how the de Rham complex is twisted in (1.1). Prior to twisting, the untwisted de Rham complex together with the wedge product gives the de Rham DGA: $(\Omega^*(M), m_1 = d, m_2 = \wedge)$. In the presence of a vector bundle E with fibre V, twisting the complex consists of locally tensoring by V, i.e. $\Omega^*(U) \otimes V$ and modifying the differential $m_1 = d$ to $m_1' = d + A : \Omega^k(U) \otimes V \to \Omega^{k+1} \otimes V$. In order that the twisted complex remains a differential complex, we obtain the condition

$$m_1' \circ m_1' = (d + A)(d + A) = dA + A \wedge A = F = 0.$$

Let us make two observations. First, modifying $m_1 \to m_1'$ does not result in a new DGA consisting of $(\Omega^*(M) \otimes V, m_1' = d + A, m_2')$. Indeed, we have not and do not need to define a new product m_2' on $\Omega^*(M) \otimes V$. So twisting the de Rham complex does not represent a deformation preserving the DGA structure. Second, without modifying the maps (m_1, m_2), we can tensor the de Rham DGA by matrices: $(\Omega^*(U) \otimes \mathrm{End}\, V, m_1 = d, m_2 = \wedge)$. This is still a DGA with

$$m_2(\eta_1 \otimes e_1, \eta \otimes e_2) = m_2(\eta_1, \eta_2) \otimes (e_1 \cdot e_2)$$

Note that in the context of this tensored de Rham DGA, the flatness condition can be written simply in terms of the deformed m_1' map:

$$F = dA + A \wedge A = m_1(A) + m_2(A, A) = m_1'(A) = 0.$$

Now to twist the primitive elliptic complex, we first observe that any A_∞-algebra also has a natural tensor product with matrices. Let Ω be an A_∞-algebra. We twist $\mathcal{A} = \Omega \otimes \mathrm{End}\ V$ where V, a vector space, is the fibre of E. m_k acting on $\Omega^{\otimes k}$ can be extended to m_k acting on $\mathcal{A}^{\otimes k}$ where

$$m_k(a_1 \otimes e_1, \ldots, a_k \otimes e_k) = m_k(a_1, \ldots, a_k) \otimes (e_1 \circ \cdots \circ e_k).$$

Elements of the complex are vector valued. They are elements of $\mathcal{B} = \Omega \otimes V$ which to be precise is an A_∞-module over $\mathcal{A}$. (See for example [3] for the definition of an A_∞ module.) The m_k acting on $\mathcal{B}$ is given by

$$m_k(a_1 \otimes e_1, \ldots, a_{k-1} \otimes e_{k-1}, a_k \otimes v) = m_k(a_1, \ldots, a_k) \otimes (e_1 \circ \cdots \circ e_{k-1})v.$$

The above twisting for differential graded algebras can be generalized to twist any A_∞-module $\mathcal{B}$. In [2], Gibson described the case for A_3-algebra (i.e. $m_k = 0$ for all $k \geq 4$) and showed that $m_1'(-) = m_1 + m_2(A, -) - m_3(A, A, -)$ squares to zero when $m_1'(A) = 0$. Here, we give the general statement for any A_∞-module $\mathcal{B}$.

Definition 2.2 Let $\{\mathcal{A}, m_k\}$ be an A_∞-algebra, $\mathcal{B}$ an A_∞-module over $\mathcal{A}$ and A an element of $\mathcal{A}$ of grading one. We define the operator m_1' as

$$\begin{aligned} m_1'(B) &= m_1(B) + m_2(A, B) - m_3(A, A, B) - m_4(A, A, A, B) + \ldots \\ &= \sum \delta_k m_k(A^{\otimes(k-1)} \otimes B) \end{aligned}$$

where

$$\delta_k = (-1)^{\frac{(k-1)(k-2)}{2}} = \begin{cases} 1, & k = 4m+1 \text{ or } 4m+2, \\ -1, & k = 4m+3 \text{ or } 4m. \end{cases}$$

Here, we use the notation m_1' since it is obtained by twisting m_1, but m_1' is generally not the differential of an A_∞-module. As mentioned above, we do not need morphisms m_k' for $k \geq 2$ as we are not aiming to obtain a deformed A_∞-structure. The theorem below will give a sufficient condition that ensures that $m_1' \circ m_1' = 0$.

Theorem 2.3 $m_1' \circ m_1' = 0$ *if* $m_1'(A) = 0$.

We first note a simple relation for the δ_k's.

Lemma 2.4 *For any* $r \geq 0$, $s \geq 1$, *we have*

$$\delta_{r+1}\delta_s = (-1)^{r(s-1)}\delta_{r+s}.$$

Proof Observe that $\delta_{r+1}\delta_s = (-1)^{\frac{r(r-1)+(s-1)(s-2)}{2}}$ and the power

$$\frac{r(r-1) + (s-1)(s-2)}{2} = \frac{(r+s-1)(r+s-2)}{2} - r(s-1).$$

□

Proof of Theorem 2.3. For any $B \in \mathcal{B}$,

$$m_1'^2 B = \sum_{i,j\geq 1} \delta_i \delta_j m_i(A^{\otimes(i-1)} \otimes m_j(A^{\otimes(j-1)} \otimes B)).$$

Let $r = i - 1$ and $s = j$. Then $r \geq 0$, $s \geq 1$ and the sum can be written as

$$\sum_{k=1}^{\infty} \sum_{r+s=k} \delta_{(r+1)} \delta_s m_{r+1}[A^{\otimes r} \otimes m_s(A^{\otimes(s-1)} \otimes B)].$$

Rewrite every term as

$$\begin{aligned}&\delta_{(r+1)}\delta_s m_{r+1}[A^{\otimes r} \otimes m_s(A^{\otimes(s-1)} \otimes B)] = (-1)^{rs}\delta_{r+1}\delta_s m_{r+1}(1^{\otimes r} \otimes m_s)\\ &(A^{\otimes(r+s-1)} \otimes B)\\ &\quad = (-1)^r \delta_k m_{r+1}(1^{\otimes r} \otimes m_s)(A^{\otimes(k-1)} \otimes B)\end{aligned}$$

By the definition of an A_∞-algebra, we have

$$\sum_{\substack{r+s=k\\ r\geq 0, s\geq 1}} (-1)^r m_{r+1}(1^{\otimes r} \otimes m_s) + \sum_{\substack{r+s+t=k\\ r\geq 0,s,t\geq 1}} (-1)^{r+st} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

So $m_1'^2 B$ can be described as

$$\begin{aligned}m_1'^2 B &= -\sum_{k=1}^{\infty} \sum_{\substack{r+s+t=k\\ r\geq 0,s,t\geq 1}} (-1)^{r+st} \delta_k m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})(A^{\otimes(k-1)} \otimes B)\\ &= \sum_{r\geq 0,s,t\geq 1} (-1)^{r+s(r+t)+1} \delta_{r+s+t} m_{r+t+1}(A^{\otimes r} \otimes m_s(A^{\otimes s}) \otimes A^{\otimes(t-1)} \otimes B)\end{aligned}$$

Since $\delta_{r+t+1}\delta_s = (-1)^{(r+t)(s+1)}\delta_{r+s+t}$, we have $(-1)^{r+s(r+t)+1}\delta_{r+s+t} = (-1)^{t+1}\delta_{r+t+1}\delta_s$. When we take the sum over s first, we get

$$m_1'^2 B = \sum_{r\geq 0,t\geq 1} (-1)^{t+1}\delta_{r+t+1} m_{r+t+1}\left(A^{\otimes r} \otimes \sum_{s\geq 1} \delta_s m_s(A^{\otimes s}) \otimes A^{\otimes(t-1)} \otimes B\right).$$

By assumption $\sum_{s\geq 1} \delta_s m_s(A^{\otimes s}) = m_1' A = 0$. Thus, $m_1'^2 B = 0$.

The above prescription for A_∞ twisting motivated by twisting the de Rham DGA gives the standard twisting for the Dolbeault complex and the primitive TTY-algebra.

Example 2.5 (Twisted Dolbeault complex) Let E be a vector bundle over a complex manifold M. Given a local trivialization over $U \subset M$, the $(0, 1)$ part of a connection acting on $\Omega^{0,*}(U, E)$ can be represented as $\bar{\partial} + A^{0,1}$ with $A \in \Omega^1(U, \mathrm{End}\, E)$.

$m_1'(A^{0,1}) = \bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0$ over any U if and only if $F^{0,2} = 0$, i.e. this connection is holomorphically flat.

Example 2.6 (Twisted primitive TTY-algebra) Let E be a vector bundle over a symplectic manifold (M^{2n}, ω). Given a local trivialization over $U \subset M$, a connection can be represented locally as $d + A$ where $A \in \Omega^1(U, \mathrm{End}\, E) = P^1(U, \mathrm{End}\, E)$. The primitive complex $P^*(U, E)$ is an A_∞-module over the TTY-algebra $P^*(U, \mathrm{End}\, E)$. Hence, we can define the operator m_1' as in Definition 2.2 and is explicitly given by

$$m_1'\beta = \begin{cases} \partial_{+A}\, \beta & \text{for } \beta \in P_+^k(U, E), \quad k < n, \\ (-\partial_{+A}\, \partial_{-A} + \Phi)\beta & \text{for } \beta \in P_+^n(U, E), \\ -\partial_{-A}\, \beta & \text{for } \beta \in P_-^k(U, E). \end{cases} \tag{2.16}$$

In details for the $\beta \in P_+^n(U, E)$ case, we have

$$\begin{aligned} m_1'\beta &= m_1(\beta) + m_2(A, \beta) - m_3(A, A, \beta) \\ &= -\partial_+\partial_-\beta_n + \Pi\left[-dL^{-1}(A \wedge \beta_n) + \partial_- A \wedge \beta_n - A \wedge \partial_-\beta_n\right] \\ &\quad - \Pi\left[A \wedge L^{-1}(A \wedge \beta_n) - L^{-1}(A \wedge A)\beta_n\right] \\ &= -\partial_{+A}\, \partial_{-A}\, \beta_n + \Pi\, L^{-1}(dA + A \wedge A)\beta_n = -\partial_{+A}\, \partial_{-A}\, \beta_n + \Phi\, \beta_n. \end{aligned}$$

Furthermore,

$$m_1'(A) = \begin{cases} \Pi(dA + A \wedge A) & \text{for } n \geq 2 \\ -dL^{-1}(dA + A \wedge A) + A \wedge L^{-1}(dA + A \wedge A) - L^{-1}(dA + A \wedge A) \wedge A & \text{for } n = 1 \end{cases}$$

Therefore, $m_1'(A) = 0$ if and only if the curvature has no primitive component, i.e. $F = \Phi\, \omega$ and also $d\Phi + [A, \Phi] = 0$, having noted that $L^{-1}F = \Phi$. The second equation means that Φ is covariantly constant which implies the global condition $d_A \Phi = 0$.

Remark 2.7 For the higher p-filtered TTY algebra (i.e. $p > 0$), if we define m_1' as in Definition 2.2, then $m_1'(A) = 0$ is just the the usual flat connection condition.

3 Examples and Properties of Symplectically Flat Bundles

We first give some simple examples of symplectically flat bundles.

Example 3.1 When the principal bundle P is rank 1, the condition of symplectically flat becomes $F = dA = c\, \omega$ for some constant c. Specifically, a circle bundle whose Euler class is $c\, \omega$ would be symplectically flat.

Example 3.2 A projectively flat bundle has curvature

$$F = c\,\omega\,\mathbf{I}$$

where c is a constant and $\mathbf{I}$ is the identity map over the fibre. Hence, projectively flat bundles are symplectically flat.

Example 3.3 When $\dim M = 2$, the symplectically flat condition is exactly identical to satisfying the Yang–Mills equations in the presence of a compatible metric. More generally, for $\dim M \geq 2$, a symplectically flat connection is always a critical point of the Yang–Mills functional with respect to a compatible metric

$$\int_M \mathrm{tr}\,(F \wedge *F).$$

Using the relation $*\,\omega = \frac{1}{(n-1)!}\,\omega^{n-1}$, it is straightforward to see that a symplectically flat curvature satisfies the Yang–Mills equation

$$d_A^* F = -*\,d_A * (\Phi\,\omega) = -\frac{1}{(n-1)!}\,*\left((d_A\Phi)\,\omega^{n-1}\right) = 0.$$

Hence, symplectically flat connections are a special subset of Yang–Mills solutions.

Example 3.4 When the symplectic manifold has dimension $\dim M = 4$, a symplectically flat connection satisfies the self-dual condition with respect to a compatible metric

$$F = *F.$$

Clearly here, the symplectically flat condition $F = \Phi\,\omega$ is a stronger condition than the self-dual condition.

Below we give some properties of symplectically flat bundles.

Proposition 3.5 *Suppose there is a symplectically flat connection on a manifold M with curvature $F = \Phi\,\omega$. Locally, there exists a trivialization such that Φ is represented as a constant matrix, and the covariant derivative can be written as $d + \Phi\lambda$ for some local 1-form λ satisfying $d\lambda = \omega$.*

The proof of the theorem is based on the following lemma (see for example, [4] Proposition 5.8):

Lemma 3.6 *Locally, if $d + \tilde{A}$ is a flat covariant derivative, then $\tilde{A} = g^{-1}dg$ for some matrix valued function g.*

Proof of Proposition 3.5. First choose local sections $\{s_1, \ldots, s_r\}$ forming a frame of $\Gamma(U, E)$, where r is the rank of vector bundle E. Then d_A can be written as $d + A$, and Φ can be represented by a matrix Φ_s with respect to this frame. Take $\lambda \in \Omega^1(U)$ such that $d\lambda = \omega$. Since $d + A$ is a symplectically flat covariant derivative, a straightforward calculation shows that

$$d(A - \Phi_s\lambda) + (A - \Phi_s\lambda) \wedge (A - \Phi_s\lambda) = 0.$$

i.e. $d + A - \Phi_s\lambda$ is a local flat covariant derivative.

By Lemma 3.6 there exists some invertible g such that

$$A - \Phi_s\lambda = g^{-1}dg.$$

Thus,

$$gAg^{-1} + gdg^{-1} = g\Phi_s g^{-1}\lambda.$$

Then we have

$$[gAg^{-1} + gdg^{-1}, g\Phi_s g^{-1}] = g\Phi_s^2 g^{-1}\lambda - g\Phi_s^2 g^{-1}\lambda = 0.$$

On the other hand,

$$d(g\Phi_s g^{-1}) + [gAg^{-1} + gdg^{-1}, g\Phi_s g^{-1}] = g(d\Phi_s + [A, \Phi_s])g^{-1} = 0.$$

Therefore, $d(g\Phi_s g^{-1}) = 0$, i.e. $g\Phi_s g^{-1}$ is a constant. Let

$$[s_1' \; \ldots \; s_r'] = [s_1 \; \ldots \; s_r]g^{-1}$$

be another local frame. Then Φ will be represented by $g\Phi_s g^{-1}$ with respect to this new frame. And the local covariant derivative becomes

$$d + gAg^{-1} + gdg^{-1} = d + g\Phi_s g^{-1}\lambda.$$

□

As an application of Proposition 3.5, suppose (M, ω) is a Kähler manifold and its Levi-Civita connection is symplectically flat. Then the Ricci curvature $\mathrm{Ric}(u, v)$ satisfies

$$\mathrm{Ric}(u, v) = i\,\mathrm{tr}_{\mathbb{C}} F(u, v) = \frac{1}{2}\omega(u, v)(\mathrm{tr}\, J\Phi + i\,\mathrm{tr}\,\Phi).$$

By Theorem 3.5 around every point in M, we can find some g such that $g\Phi g^{-1}$ is a constant. That implies that $\mathrm{tr}\,\Phi = \mathrm{tr}\, g\Phi g^{-1}$ is a constant. For the same reason, $\mathrm{tr}\, J\Phi$ is also a constant. Therefore, the Ricci tensor $r(u, v)$ has the following property:

$$r(u, v) = \mathrm{Ric}(-Ju, v) = c\,\omega(-Ju, v) = c\, g(u, v),$$

where $c = \frac{1}{2}(\mathrm{tr}\, J\Phi + i\,\mathrm{tr}\,\Phi)$. In other words, we have

Proposition 3.7 *If the Levi-Civita connection of a Kähler manifold is symplectically flat, then the manifold is Kähler–Einstein.*

Symplectically flat bundles also have a simple alternative description. When ω is integral, it induces a flat connection on the prequantum circle bundle, i.e. the circle bundle over M with Euler class given by ω.

To show this, we recall a result of Tanaka-Tseng [5] that the primitive TTY algebra is A_∞-quasi-isomorphic to the cone algebra $\mathcal{C}^*(M) = \Omega^*(M) \oplus \theta\, \Omega^*(M)$ where $d\theta = \omega$. Furthermore, when ω is integral and we can consider a circle bundle X over M whose Euler class is ω, then the de Rham DGA of the circle bundle $\Omega^*(X)$ is quasi-isomorphic to both the $\mathcal{C}^*(M)$ algebra and the primitive TTY algebra.

We can extend these quasi-isomorphism relations between algebras to include symplectically flat connections. Let E be a vector bundle over M with a connection, and d_A the corresponding covariant derivative. On the twisted cone algebra $\mathcal{C}^*(M, E) = \Omega^*(M, E) \oplus \theta\, \Omega^*(M, E)$ with $d\theta = \omega$, we define the operator

$$D_{\mathcal{C}} = d_A - \theta\, \Phi.$$

Proposition 3.8 *The above connection is symplectically flat if and only if* $D_{\mathcal{C}}^2 = 0$.

Proof Write $d_A = d + A$ locally, then $D_{\mathcal{C}} = d + A - \theta\Phi$.

$$\begin{aligned} & D_{\mathcal{C}}^2 = d(A - \theta\Phi) + (A - \theta\Phi) \wedge (A - \theta\Phi) = 0 \\ \iff\quad & (dA + A \wedge A - \Phi\omega) + \theta(d\Phi + [A, \Phi]) = 0 \\ \iff\quad & \begin{cases} F = dA + A \wedge A = \Phi\omega \\ d_A\Phi = d\Phi + [A, \Phi] = 0 \end{cases} \end{aligned}$$

$\square$

Corollary 3.9 *Suppose* $\omega \in \Omega^2(M)$ *is an integral closed 2-form on a manifold* M*, and* $\pi : X \to M$ *be the circle bundle whose Euler class is* ω*. If* E *is a symplectically flat bundle over* M*, then* $\pi^* E$ *is a flat bundle over* X*.*

4 Calculation of Twisted Cohomology

In this section, we consider a vector bundle E over a symplectic manifold (M, ω) with a symplectically flat connection whose curvature $F = \Phi\, \omega$. The twisted primitive elliptic complex (2.15) gives the following cohomologies:

$$PH_+^k(M, E) = \frac{\ker \partial_{+A}}{\operatorname{im} \partial_{+A}} \qquad PH_+^n(M, E) = \frac{\ker(\partial_{+A}\, \partial_{-A} - \Phi)}{\operatorname{im} \partial_{+A}}$$
$$PH_-^k(M, E) = \frac{\ker \partial_{-A}}{\operatorname{im} \partial_{-A}} \qquad PH_-^n(M, E) = \frac{\ker \partial_{-A}}{\operatorname{im}(\partial_{+A}\, \partial_{-A} - \Phi)}$$

where $k = 0, \ldots, n-1$. To simplify notation below, we use m_1' as defined in (2.16) to denote the differentials $\{\partial_{+A}, -(\partial_{+A}\, \partial_{-A} - \Phi), -\partial_{-A}\}$ in the complex (2.15).

By (2.14), we obtain a condition on the elements of the $PH^*(M, E)$ cohomology.

Lemma 4.1 *Let $\beta \in PH^*(M, E)$. Then $\Phi\beta$ is trivial in $PH^*(M, E)$ cohomology class. Specifically, for $\beta_k \in P^k_+(M, E)$ and $0 \le k \le n$,*

$$\Phi\beta_k = \partial_{+A}\,\partial_{-A}\,\beta_k, \tag{4.1}$$

and for $\bar{\beta}_k \in P^k_-(M, E)$,

$$\Phi\bar{\beta}_k = \begin{cases} \partial_{-A}\,\partial_{+A}\,\bar{\beta}_k, & 0 \le k < n \\ (-\partial_{+A}\,\partial_{-A}\, + \Phi)\beta_k, & k = n \end{cases} \tag{4.2}$$

Proof For $k < n$, note that (2.14) simplifies to just $\partial_{+A}\,\partial_{-A}\, + \partial_{-A}\,\partial_{+A}\, = \Phi$ as ω is an injective map when acting on forms of degree less than n. Using this, (4.1) and (4.2) then follow immediately after imposing $m_1'(\beta_k) = \partial_{+A}\,\beta_k = 0$ or $m_1'(\bar{\beta}_k) = -\partial_{-A}\,\bar{\beta}_k = 0$. When $k = n$, (4.1) is just $m_1'(\beta_n) = 0$ and (4.2) is also trivial since $m_1'(\bar{\beta}_n) = -\partial_{-A}\,\bar{\beta}_n = 0$. $\square$

Theorem 4.2 *If Φ is invertible, then $PH^*(M, E) = 0$.*

Proof Let Φ^{-1} be the inverse of Φ. We note that $d_A\Phi = 0$ implies $d_A\Phi^{-1} = 0$ since

$$0 = d_A\left(\Phi\Phi^{-1}\right) = d\Phi\,\Phi^{-1} + \Phi\,d\Phi^{-1} + [A, \Phi]\,\Phi^{-1} + \Phi\,[A, \Phi^{-1}] = \Phi\,d_A\Phi^{-1}.$$

Therefore, for arbitrary $\alpha \in \Omega^*(M, E)$, we have that

$$(d + A)(\Phi^{-1}\alpha) = \Phi^{-1}(d + A)\alpha$$

It follows that both ∂_{+A} and ∂_{-A} also commute with Φ^{-1}. So for any $\beta \in PH^*_\pm(M, E)$, if $m_1'\beta = 0$, then $m_1'(\Phi^{-1}\beta) = \Phi^{-1}m_1'\beta = 0$. By Lemma 4.1, $\beta = \Phi(\Phi^{-1}\beta)$ must be m_1'-exact. $\square$

Corollary 4.3 *When* rank $E = 1$ *and E is non-flat, then $PH^*(M, E) = 0$.*

The above Theorem 4.2 is a vanishing statement for $PH^*(M, E)$ of which Φ plays a central role.

4.1 Local Cohomologies

For arbitrary $x \in M$, we have a neighbourhood U of x isomorphic to $\mathbb{R}^{2n}$ such that $E|_U \simeq U \times V$. There exist $\lambda \in \Omega^1(U)$ such that $d\lambda = \omega$. According the proof of Proposition 3.5, we can find a frame $\{e_i\}$ on $E|_U$ such that $Ae_i = \Phi\lambda e_i$ and $\Phi \in \operatorname{End} V$ is a constant. Locally, we have $A = \Phi\lambda$ and we obtain the following result regarding the twisted primitive cohomologies:

Theorem 4.4

$$PH_+^k(U,E) = \begin{cases} \ker\Phi, & k=0 \\ \lambda \operatorname{coker}\Phi, & k=1 \\ 0, & k\geq 2 \end{cases}$$
$$PH_-^k(U,E) = 0.$$

Here, $\ker\Phi$ *and* $\operatorname{coker}\Phi$ *are subspaces of* V *and can be represented by constant sections.*

We will make use for the proof of the theorem the local Poincare' lemmas for the $\{\partial_+, \partial_+\partial_-, \partial_-\}$ operators in [7]. Before giving the proof, let us first write down some expressions that are m_1-closed, where m_1 refers to the the untwisted differentials of (2.7), instead of m_1'-closed. Below, we will often use the simpler product notation $\times$ to denote the m_2 map as described in Sec. 2.1.

Lemma 4.5 *Let* $\beta_k \in P_+^k$ *such that* $m_1'(\beta_k) = 0$ *and* $\bar\beta_k \in P_-^k$ *such that* $m_1'(\bar\beta_k) = 0$. *Then for a symplectically flat connection of the form* $A = \Phi\lambda$ *where* Φ *is a constant,*

$$\begin{aligned} \partial_+\left(\beta_k - \lambda\times\partial_{-A}\,\beta_k\right) &= 0, \qquad k = 0,1,\ldots,n-1, \\ -\partial_+\partial_-\left(\beta_n - \lambda\times\partial_{-A}\,\beta_n\right) &= 0, \\ -\partial_-\left(\bar\beta_k + \lambda\times\partial_{+A}\,\bar\beta_k\right) &= 0, \qquad k = 0,1,\ldots,n-1. \end{aligned}$$

Proof For $\beta_k \in P_+^k$ and $m_1'(\beta_k) = 0$, (4.1) implies $\Phi\beta_k = (\partial_+ + \Phi\lambda\times)\partial_{-A}\,\beta_k$, or equivalently,

$$\Phi\left(\beta_k - \lambda\times\partial_{-A}\,\beta_k\right) = \partial_+\partial_{-A}\,\beta_k. \tag{4.3}$$

By direct computation, we will show $\beta_k - \lambda\times\partial_{-A}\,\beta_k$ is m_1-closed. When $k<n$, we have

$$\begin{aligned} \partial_+\left(\beta_k - \lambda\times\partial_{-A}\,\beta_k\right) &= \partial_+\beta_k + \lambda\times\partial_+\partial_{-A}\,\beta_k \\ &= \partial_+\beta_k + \Phi\lambda\times\left(\beta_k - \lambda\times\partial_{-A}\,\beta_k\right) \\ &= \partial_{+A}\,\beta_k = 0 \end{aligned}$$

having used Leibniz rule in the first line and noting that $\lambda\times(\lambda\times\partial_{-A}\,\beta_k) = (\lambda\times\lambda)\times\partial_{-A}\,\beta_k = 0$. When $k=n$, we have

$$\begin{aligned} -\partial_+\partial_-\left(\beta_n - \lambda\times\partial_{-A}\,\beta_n\right) &= -\partial_+\partial_-\beta_n + \lambda\times\partial_+\partial_{-A}\,\beta_n \\ &= -\partial_+\partial_-\beta_n + \Phi\lambda\times\beta_n - \Phi\lambda\times\left(\lambda\times\partial_{-A}\,\beta_n\right) \\ &= m_1(\beta_n) + m_2(\Phi\lambda,\beta_n) - m_3(\Phi\lambda,\Phi\lambda,\beta_n) \\ &= -(\partial_+\partial_-)_A\beta_n = 0 \end{aligned}$$

which follows from the relation

$$\begin{aligned}\Phi\lambda \times (\lambda \times \partial_{-A}\,\beta_n) &= (\Phi\lambda \times \lambda) \times \partial_{-A}\,\beta_n + m_3(\Phi\lambda, \lambda, \partial_+\partial_{-A}\,\beta_n)\\ &= m_3(\Phi\lambda, \Phi\lambda, \beta_n - \lambda \times \partial_{-A}\,\beta_n) = m_3(\Phi\lambda, \Phi\lambda, \beta_n)\end{aligned}$$

since

$$m_3(\Phi\lambda, \Phi\lambda, m_2(\lambda, \partial_{-A}\,\beta_n)) = m_2(\Phi\lambda, m_3(\Phi\lambda, \lambda, \partial_{-A}\,\beta_n)) = 0,$$

having used the following A_∞ relation involving m_4 (with $m_4 = 0$ for the TTY algebra):

$$\begin{aligned}&m_2(m_3 \otimes \mathbf{1}) + m_2(\mathbf{1} \otimes m_3) - m_3(m_2 \otimes \mathbf{1}^{\otimes 2}) + m_3(\mathbf{1} \otimes m_2 \otimes \mathbf{1}) - m_3(\mathbf{1}^{\otimes 2} \otimes m_2)\\ &= m_1 m_4 - m_4\left(m_1 \otimes \mathbf{1}^{\otimes 3} + \mathbf{1} \otimes m_1 \otimes \mathbf{1}^{\otimes 2} + \mathbf{1}^{\otimes 2} \otimes m_1 \otimes \mathbf{1} + \mathbf{1}^{\otimes 3} \otimes m_1\right) = 0.\end{aligned}$$

For $\bar{\beta}_k \in P^k_-$ with $k \neq n$ and $m_1(\bar{\beta}_k) = 0$, (4.2) implies $\Phi\bar{\beta}_k = (\partial_- - \Phi\lambda\times)\partial_{+A}\,\bar{\beta}_k$, or equivalently,

$$\Phi\left(\bar{\beta}_k + \lambda \times \partial_{+A}\,\bar{\beta}_k\right) = \partial_-\partial_{+A}\,\bar{\beta}_k. \tag{4.4}$$

Similar to above, it then follows

$$\begin{aligned}\partial_-\left(\bar{\beta}_k + \lambda \times \partial_{+A}\,\bar{\beta}_k\right) &= \partial_-\bar{\beta}_k - \lambda \times \partial_-\partial_{+A}\,\bar{\beta}_k\\ &= \partial_-\bar{\beta}_k - \Phi\lambda \times \bar{\beta}_k - \Phi\lambda \times (\lambda \times \partial_{+A}\,\bar{\beta}_k) = \partial_{-A}\,\bar{\beta}_k = 0\end{aligned}$$

having noted that $\lambda \times (\lambda \times \partial_{+A}\,\bar{\beta}_k) = (\lambda \times \lambda) \times \partial_{+A}\,\bar{\beta}_k = 0$. $\square$

We now give a proof of Theorem 4.4.

Proof of Theorem 4.4 The proof of the theorem is divided into five cases.

Case 1 $PH^0_+(U, E)$. Let $\beta_0 \in PH^0_+(U, E)$. By (4.1), we have $\Phi\beta_0 = 0$ and hence, $\beta_0 \in \ker\Phi$. Imposing $\partial_{+A}\,\beta_0 = 0$, we find

$$0 = \partial_{+A}\,\beta_0 = (d + \Phi\lambda)\beta_0 = d\beta_0. \tag{4.5}$$

Therefore, β_0 must be constant and an element of $\ker\Phi$.

Case 2 $PH^1_+(U, E)$. Let $\beta_1 \in PH^1_+(U, E)$. By (4.1), $\Phi\beta_1 = (d + \Phi\lambda)\partial_{-A}\,\beta_1$ which we can write as

$$\Phi\,(\beta_1 - \lambda\,\partial_{-A}\,\beta_1) = d\,\partial_{-A}\,\beta_1. \tag{4.6}$$

We can show that $(\beta_1 - \lambda\,\partial_{-A}\,\beta_1)$ is d-closed:

$$\begin{aligned}d(\beta_1 - \lambda\,\partial_{-A}\,\beta_1) &= d\beta_1 - \omega\,\partial_{-A}\,\beta_1 + \lambda \wedge d\,\partial_{-A}\,\beta_1\\ &= d\beta_1 - \omega\,\partial_{-A}\,\beta_1 + \Phi\lambda \wedge (\beta_1 - \lambda\,\partial_{-A}\,\beta_1)\end{aligned}$$

$$= \partial_{+A}\,\beta_1 = 0$$

By Poincaré Lemma, there exists $\xi_0 \in P^0_+$ such that

$$\beta_1 - \lambda\,\partial_{-A}\,\beta_1 = d\xi_0. \tag{4.7}$$

Together with (4.6), this implies

$$\partial_{-A}\,\beta_1 = \Phi\xi_0 + \sigma_0 \tag{4.8}$$

where σ_0 is a constant matrix. Substituting (4.8) into (4.7) gives

$$\beta_1 = d\xi_0 + \lambda(\Phi\xi_0 + \sigma_0) = \partial_{+A}\,\xi_0 + \lambda\,\sigma_0.$$

But if $\sigma_0 \in \operatorname{im}\Phi$, then there exists some constant $\tilde{\sigma_0}$ such that $\lambda\,\sigma_0 = (d + \Phi\lambda)\tilde{\sigma_0}$. Therefore, $\sigma_0 \in \operatorname{coker}\Phi$ if $\beta_1 = \lambda\,\sigma_0$ represents a non-trivial class.

Case 3 $PH^k_+(U, E)$ for $k = 2, \ldots, n$. Let $\beta_k \in PH^k_+(U, E)$ for $k = 2, \ldots, n$. By Lemma 4.5 and local Poincaré lemmas for ∂_+ and $\partial_+\partial_-$ operators [7], there exists a $\xi_{k-1} \in P^{k-1}_+$ such that

$$\beta_k - \lambda \times \partial_{-A}\,\beta_k = \partial_+\xi_{k-1}. \tag{4.9}$$

Inserting this into (4.3) then implies

$$\partial_{-A}\,\beta_k = \Phi\xi_{k-1} + \partial_+\sigma_{k-2} \tag{4.10}$$

for some $\sigma_{k-2} \in P^{k-2}_+$. Together, (4.9)–(4.10) give us

$$\begin{aligned} \beta_k &= \partial_+\xi_{k-1} + \Phi\lambda \times \xi_{k-1} + \lambda \times \partial_+\sigma_{k-2} \\ &= \partial_{+A}\,\xi_{k-1} - \partial_{+A}\,(\lambda \times \sigma_{k-2}) + \Phi\lambda \times (\lambda \times \sigma_{k-2}) \\ &= \partial_{+A}\,(\xi_{k-1} - \lambda \times \sigma_{k-2}) \end{aligned}$$

after noting that $\lambda \times (\lambda \times \sigma_{k-2}) = (\lambda \times \lambda) \times \sigma_{k-2} = 0$.

Case 4 $PH^k_-(U, E)$ for $k = 0, 1, \ldots, n-1$. Let $\bar{\beta}_k \in PH^k_+(U, E)$ for $k = 0, 1, \ldots, n-1$. By Lemma 4.5 and the local Poincaré lemmas for ∂_- operator [7], there exists a $\bar{\xi}_{k+1} \in P^{k+1}_-$ such that

$$\bar{\beta}_k + \lambda \times \partial_{+A}\,\bar{\beta}_k = \partial_-\bar{\xi}_{k+1}. \tag{4.11}$$

Inserting this into (4.4) then implies

$$\partial_{+A}\,\bar{\beta}_k = \begin{cases} \Phi\bar{\xi}_{k+1} + \partial_-\bar{\sigma}_{k+2} & k = 0, 1, \ldots, n-2 \\ \Phi\bar{\xi}_{k+1} + \partial_+\partial_-\sigma_n & k = n-1 \end{cases} \tag{4.12}$$

for some $\bar{\sigma}_{k+2} \in P_-^{k+2}$ and $\sigma_n \in P_+^n$. For $k < n-1$, (4.11)–(4.12) imply

$$\begin{aligned}\bar{\beta}_k &= \partial_- \bar{\xi}_{k+1} - \Phi\lambda \times \bar{\xi}_{k+1} - \lambda \times \partial_- \bar{\sigma}_{k+2}\\ &= \partial_{-A}\, \bar{\xi}_{k+1} + \partial_{-A}\, (\lambda \times \bar{\sigma}_{k+2}) + \Phi\lambda \times (\lambda \times \bar{\sigma}_{k+2})\\ &= \partial_{-A}\, (\bar{\xi}_{k+1} + \lambda \times \bar{\sigma}_{k+2})\end{aligned}$$

since $\lambda \times (\lambda \times \bar{\sigma}_{k+2}) = (\lambda \times \lambda) \times \bar{\sigma}_{k+2} = 0$. Similarly, when $k = n-1$, we obtain

$$\begin{aligned}\bar{\beta}_k &= \partial_- \bar{\xi}_n - \Phi\lambda \times \bar{\xi}_n - \lambda \times \partial_+\partial_- \sigma_n\\ &= \partial_{-A}\, \bar{\xi}_n + \partial_{-A}\, (\lambda \times \sigma_n) + \Phi\lambda \times (\lambda \times \sigma_n)\\ &= \partial_{-A}\, (\bar{\xi}_n + \lambda \times \sigma_n)\end{aligned}$$

having noted that $\lambda \times (\lambda \times \sigma_n) = (\lambda \times \lambda) \times \sigma_n + m_3(\lambda, \lambda, -\partial_+\partial_-\sigma_n)$=0.

Case 5 $PH_-^n(U, E)$. Let $\bar{\beta}_n \in PH_-^n(U, E)$. Hence, $0 = \partial_{-A}\, \bar{\beta}_n = \partial_- \bar{\beta}_n - \Phi\lambda \times \bar{\beta}_n$, or equivalently,

$$\partial_- \bar{\beta}_n = \Phi\lambda \times \bar{\beta}_n \tag{4.13}$$

Further, by Leibniz rule, we find that $\lambda \times \bar{\beta}_n$ is ∂_--closed.

$$\partial_-(\lambda \times \bar{\beta}_n) = -\lambda \times \partial_- \bar{\beta}_n = -\lambda \times (\Phi\lambda \times \bar{\beta}_n) = 0.$$

Since $\lambda \times \bar{\beta}_n \in P_-^{n-1}$ is ∂_--closed, local Poincaré lemma implies there exists an $\bar{\xi}_n \in P_-^n$ such that

$$\lambda \times \bar{\beta}_n = \partial_- \bar{\xi}_n. \tag{4.14}$$

Together with (4.13), this implies

$$\bar{\beta}_n = \Phi\bar{\xi}_n + \partial_+\partial_-\sigma_n \tag{4.15}$$

for some $\sigma_n \in P_+^n$. Now let $\xi_n \in P_+^n$ be the same n-form as $\bar{\xi}_n$, i.e. $\xi_n = \bar{\xi}_n$ as primitive form. We will show that in fact $\bar{\beta}_n = m_1'(\xi_n - \lambda \times \partial_-\sigma_n) = (-\partial_{+A}\, \partial_{-A} + \Phi)(\xi_n - \lambda \times \partial_-\sigma_n)$ with $(\xi_n - \lambda \times \partial_-\sigma_n) \in P_+^n$. To do so, we write $\lambda \times \partial_-\sigma_n = \Pi(\lambda \wedge \partial_-\sigma_n) = \lambda \wedge \partial_-\sigma_n - \omega L^{-1}(\lambda \wedge \partial_-\sigma_n)$.

$$\begin{aligned}&(-\partial_{+A}\, \partial_{-A} + \Phi)(\xi_n - \lambda \times \partial_-\sigma_n)\\ &= \Big[-(d + \Phi\lambda\wedge)L^{-1}(d + \Phi\lambda\wedge) + \Phi\Big]\Big[\xi_n - (\lambda \wedge \partial_-\sigma_n - \omega L^{-1}(\lambda \wedge \partial_-\sigma_n))\Big]\\ &= \Big[-(d + \Phi\lambda\wedge)L^{-1}(d + \Phi\lambda\wedge) + \Phi\Big](\xi_n - \lambda \wedge \partial_-\sigma_n)\\ &\quad + \Big[-(d + \Phi\lambda\wedge)L^{-1}(d + \Phi\lambda\wedge) + \Phi\Big]\omega L^{-1}(\lambda \wedge \partial_-\sigma_n)\end{aligned}$$

Note first that the second term vanishes.

$$\begin{aligned}
&[-(d+\Phi\lambda\wedge)L^{-1}(d+\Phi\lambda\wedge)+\Phi]\omega L^{-1}(\lambda\wedge\partial_-\sigma_n)\\
&=-(d+\Phi\lambda\wedge)L^{-1}\omega(d+\Phi\lambda\wedge)L^{-1}(\lambda\wedge\partial_-\sigma_n)+\Phi\omega L^{-1}(\lambda\wedge\partial_-\sigma_n)\\
&=-(d+\Phi\lambda\wedge)^2L^{-1}(\lambda\wedge\partial_-\sigma_n)+\Phi\omega L^{-1}(\lambda\wedge\partial_-\sigma_n)=0
\end{aligned}$$

And the first term gives the desired result.

$$\begin{aligned}
&\Big[-(d+\Phi\lambda\wedge)L^{-1}(d+\Phi\lambda\wedge)+\Phi\Big](\xi_n-\lambda\wedge\partial_-\sigma_n)\\
&\quad=-(d+\Phi\lambda\wedge)\Big[\partial_-\xi_n+L^{-1}(\Phi\lambda\wedge\xi_n)+L^{-1}(-\omega\wedge\partial_-\sigma_n+\lambda\wedge\partial_+\partial_-\sigma_n)\Big]\\
&\qquad+\Phi\xi_n-\Phi\lambda\wedge\partial_-\sigma_n\\
&\quad=-(d+\Phi\lambda\wedge)(-\partial_-\sigma_n)+\Phi\bar{\xi}_n-\Phi\lambda\wedge\partial_-\sigma_n\\
&\quad=\Phi\bar{\xi}_n+\partial_+\partial_-\sigma_n=\bar{\beta}_n
\end{aligned}$$

where to obtain the third line, we noted $\xi_n=\bar{\xi}_n$ as forms and applied the relations in (4.14) and (4.15).

4.2 Global Cohomologies and Relation to the Twisted Cone Complex

Tanaka-Tseng in [6] gave a homotopy equivalence between the cochain complex of $P^*(M)$ and the cone complex of $\mathcal{C}^*(M)=\Omega^*(M)\oplus\theta\,\Omega^*(M)$. For differential forms taking values in E, a symplectically flat vector bundle over a symplectic manifold (M,ω), we will construct here a similar relation.

Primitive forms with values in E, denoted by $P^*(M,E)$, form a twisted cochain complex with differential m_1' as in (2.15). Similar to (2.6), we use $\mathcal{F}^*(M,E)$ to denote this complex, i.e.

$$\mathcal{F}^j(M,E)=\begin{cases}P_+^j(M,E), & 0\le j\le n,\\ P_-^{2n+1-j}(M,E), & n+1\le j\le 2n+1.\end{cases}$$

For the cone, the differential forms with values in E

$$\mathcal{C}^j(M,E)=\Omega^j(M,E)\oplus\theta\,\Omega^{j-1}(M,E) \tag{4.16}$$

also form a twisted cochain complex, with differential $D_{\mathcal{C}}=d_A-\theta\Phi$. Recall from Proposition 3.8 that $D_{\mathcal{C}}^2=0$ as long as d_A is symplectically flat.

It is useful to decompose each $\mathcal{C}^j(M,E)$ into primitive components. Specifically, let $k\le n$. For $\alpha_k\in\mathcal{C}^k(M,E)$, (4.16) implies the following decomposition

$$\begin{aligned}
\alpha_k&=\eta_k+\theta\,\eta_{k-1}\\
&=\beta_k+\omega\wedge\beta_{k-2}+\cdots+\theta\,(\beta_{k-1}+\omega\wedge\beta_{k-3}+\ldots)\,.
\end{aligned}\tag{4.17}$$

For $\alpha_{2n+1-k} \in \mathcal{C}^{2n+1-k}(M,E)$, noting that $\mathcal{C}^{2n+1-k}(M,E) = \Omega^{2n+1-k}(M,E) + \theta\big(*_r\,\Omega^k(M,E)\big)$, we have the following

$$\begin{aligned}\alpha_{2n+1-k} &= \eta_{2n+1-k} + \theta\,\eta_{2n-k}\\ &= \omega^{n-k+1}\wedge(\beta_{k-1}+\omega\wedge\beta_{k-3}+\ldots)+\theta\,\omega^{n-k}\wedge(\beta_k+\omega\wedge\beta_{k-2}+\ldots)\,.\end{aligned} \tag{4.18}$$

Throughout this subsection, all $\beta_i \in P^i(M,E)$ are primitive forms with values in E. These primitive forms can be acted upon by d_A.

We now define two maps.

Definition 4.6 In terms of the decompositions (4.17)–(4.18), we define the map $f : \mathcal{C}^j(M,E) \to \mathcal{F}^j(M,E)$

$$f(\alpha_j) = \begin{cases} \beta_j & 0 \le j \le n \\ -(\beta_k + \partial_{+A}\,\beta_{k-1}) & n+1 \le j \le 2n+1,\ k = 2n+1-j \end{cases}$$

and $g : \mathcal{F}^j(M,E) \to \mathcal{C}^j(M,E)$

$$g(\beta_j) = \begin{cases} \beta_j - \theta\,\partial_{-A}\,\beta_j & 0 \le j \le n \\ -\theta\,\omega^{n-k}\wedge\beta_k & n+1 \le j \le 2n+1,\ k = 2n+1-j \end{cases}$$

In the next two lemmas, we will show that both f and g are chain maps.

Lemma 4.7 *The map f is a chain map, i.e the following graph commutes for all $0 \le j \le 2n$.*

$$\begin{tikzcd} \mathcal{C}^j \arrow[r, "D_{\mathcal{C}}"] \arrow[d, "f"'] & \mathcal{C}^{j+1} \arrow[d, "f"] \\ \mathcal{F}^j \arrow[r, "m_1'"] & \mathcal{F}^{j+1} \end{tikzcd}$$

Proof **Case 1** $j < n$. For $\alpha_j \in \mathcal{C}^j$, we have $f(\alpha_j) = \beta_j$ and $m_1' \circ f(\alpha_j) = \partial_{+A}\,\beta_j$. On the other hand,

$$D_{\mathcal{C}}(\alpha_j) = (d_A)\eta_j + \omega\wedge\eta_{j-1} - \theta\left[(d_A)\eta_{j-1} + \Phi\,\eta_j\right]$$

Therefore, $f \circ D_{\mathcal{C}}(\alpha_j) = \Pi(d_A)\beta_j = m_1' \circ f(\alpha_j)$.
Case 2 $j = n$. For $\alpha_n \in \mathcal{C}^n$, we have $f(\alpha_n) = \beta_n$ and

$$m_1' \circ f(\alpha_j) = (-\partial_{+A}\,\partial_{-A} + \Phi)\,\beta_n.$$

On the other hand,

$$D_{\mathcal{C}}(\alpha_n) = (d_A)\eta_n + \omega\wedge\eta_{n-1} - \theta\left[(d_A)\eta_{n-1} + \Phi\,\eta_n\right]$$

$$
\begin{aligned}
&= \left[\omega \wedge (\partial_{-A}\, \beta_n + \partial_{+A}\, \beta_{n-2} + \beta_{n-1}) + \mathcal{O}(\omega^2)\right] \\
&\quad - \theta \left[(\partial_{+A}\, \beta_{n-1} + \Phi\beta_n) + \mathcal{O}(\omega)\right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f \circ D_{\mathcal{C}}(\alpha_n) &= \partial_{+A}\, \beta_{n-1} + \Phi\beta_n - \partial_{+A}\, (\partial_{-A}\, \beta_n + \partial_{+A}\, \beta_{n-2} + \beta_{n-1}) \\
&= (-\partial_{+A}\, \partial_{-A}\, + \Phi)\, \beta_n
\end{aligned}
$$

Therefore, $m_1' \circ f(\alpha_n) = f \circ D_{\mathcal{C}}(\alpha_n)$.
Case 3 $j > n$. For $\alpha_{2n+1-k} \in \mathcal{C}^j$, with $k = 2n+1-j$, we have $f(\alpha_{2n+1-k}) = -\beta_k - \partial_{+A}\, \beta_{k-1}$, and

$$
m_1' \circ f(\alpha_{2n+1-k}) = \partial_{-A}\, \beta_k + \partial_{-A}\, \partial_{+A}\, \beta_{k-1} = \partial_{-A}\, \beta_k - \partial_{+A}\, \partial_{-A}\, \beta_{k-1} + \Phi\beta_{k-1}.
$$

having used (2.14). On the other hand,

$$
\begin{aligned}
D_{\mathcal{C}}(\alpha_{2n+1-k}) &= \omega^{n-k+2}\left[(\partial_{-A}\, \beta_{k-1} + \partial_{+A}\, \beta_{k-2} + \beta_{k-2}) + \mathcal{O}(\omega)\right] \\
&\quad - \theta\, \omega^{n-k+1}\left[(\partial_{-A}\, \beta_k + \partial_{+A}\, \beta_{k-2} + \Phi\beta_{k-1}) + \mathcal{O}(\omega)\right] \\
f \circ D_{\mathcal{C}}(\alpha_{2n+1-k}) &= (\partial_{-A}\, \beta_k + \partial_{+A}\, \beta_{k-2} + \Phi\beta_{k-1}) \\
&\quad - \partial_{+A}\, (\partial_{-A}\, \beta_{k-1} + \partial_{+A}\, \beta_{k-2} + \beta_{k-2}) \\
&= \partial_{-A}\, \beta_k + \Phi\beta_{k-1} - \partial_{+A}\, \partial_{-A}\, \beta_{k-1}
\end{aligned}
$$

proving $m_1' \circ f(\alpha_{2n+1-k}) = f \circ D_{\mathcal{C}}(\alpha_{2n+1-k})$. □

Lemma 4.8 *The map g is a chain map, i.e the following graph commutes for all $0 \le j \le 2n$.*

$$
\begin{tikzcd}
\mathcal{C}^j \arrow[r, "D_{\mathcal{C}}"] & \mathcal{C}^{j+1} \\
\mathcal{F}^j \arrow[u, "g"] \arrow[r, "m_1'"] & \mathcal{F}^{j+1} \arrow[u, "g"']
\end{tikzcd}
$$

Proof **Case 1** $j < n$. For $\beta_j \in \mathcal{F}^j$, $g(\beta_j) = \beta_j - \theta\partial_{-A}\, \beta_j$, and

$$
\begin{aligned}
D_{\mathcal{C}} \circ g(\beta_j) &= (d_A)\beta_j - \omega \wedge \partial_{-A}\, \beta_j + \theta\left[(d_A)\partial_{-A}\, \beta_j - \Phi\beta_j\right] \\
&= \partial_{+A}\, \beta_j + \theta\left[\partial_{+A}\, \partial_{-A}\, \beta_j - \Phi\beta_j\right].
\end{aligned} \tag{4.19}
$$

On the other hand, $m_1'(\beta_j) = \partial_{+A}\, \beta_j$ and

$$
g \circ m_1'(\beta_j) = \partial_{+A}\, \beta_j - \theta\partial_{-A}\, \partial_{+A}\, \beta_j = \partial_{+A}\, \beta_j + \theta\left[(d_A)\partial_{-A}\, \beta_j - \Phi\beta_j\right]
$$

using (2.14). Therefore, $D_{\mathcal{C}} \circ g(\beta_j) = g \circ m_1'(\beta_j)$.

Case 2 $j = n$. For $\beta_n \in \mathcal{F}^n$, we have from (4.19) that

$$D_{\mathcal{C}} \circ g(\beta_n) = \theta \left[\partial_{+A}\, \partial_{-}\beta_n - \Phi\beta_n\right]$$

having noted that $\partial_{+A}\, \beta_n = 0$. On the other hand, $m_1'(\beta_n) = -\partial_{+A}\, \partial_{-A}\, \beta_n + \Phi\beta_n$ and we find $g \circ m_1'(\beta_n) = \theta \left[\partial_{+A}\, \partial_{-}\beta_n - \Phi\beta_n\right] = D_{\mathcal{C}} \circ g(\beta_n)$.

Case 3 $j > n$. For $\beta_k \in P_{-}^k(M, E) = \mathcal{F}^j$ where $k = 2n + 1 - j$, we have $g(\beta_k) = -\theta\, \omega^{n-k} \wedge \beta_k$ and

$$D_{\mathcal{C}} \circ g(\beta_k) = -\omega^{n-k+1} \wedge \beta_k + \theta\, \omega^{n-k} \wedge (d_A)\beta_k = \theta\, \omega^{n-k+1} \wedge \partial_{-A}\, \beta_k.$$

where we have used the primitive property that $\omega^{n-s+1} \wedge \beta_s = 0$ for $\beta_s \in P^s$. On the other hand, $m_1'(\beta_k) = -\partial_{-A}\, \beta_k$ and we find $g \circ m_1'(\beta_k) = \theta\, \omega^{n-k+1} \wedge \partial_{-A}\, \beta_k = D_{\mathcal{C}} \circ g(\beta_k)$. □

With f and g being chain maps, we now proceed to relate the cohomologies associated with $\mathcal{F}^*$ and $\mathcal{C}^*$. Define the operator $G : \mathcal{C}^j \to \mathcal{C}^{j-1}$

$$G(\eta_j + \theta\, \eta_{j-1}) = \eta_{j-1} + \theta L^{-1}\eta_j,$$

for any $\eta_j \in \Omega^j(M, E)$ and $\eta_{j-1} \in \Omega^{j-1}(M, E)$. We have the following the result.

Lemma 4.9 *The maps f, g and G are related as follows:*

$$fg = id_{\mathcal{F}}, \qquad id_{\mathcal{C}} - gf - \Phi = D_{\mathcal{C}}G + GD_{\mathcal{C}}.$$

Proof That $fg = \mathrm{id}_{\mathcal{F}}$ follows immediately from the definitions of the maps in Definition 4.6. For the second relation, we consider first the left-hand side. We have from the definitions

$$gf(\alpha_j) = \begin{cases} \beta_j - \theta\, \partial_{-A}\, \beta_j & 0 \le j \le n \\ \theta\, \omega^{n-k} \wedge \left(\beta_k + \partial_{+A}\, \beta_{k_1}\right) & n+1 \le j \le 2n+1,\ k = 2n+1-j \end{cases}$$

and therefore,

$$(\mathrm{id}_{\mathcal{C}} - gf - \Phi)(\alpha_j) = \begin{cases} \alpha_j - \Phi\alpha_j - \beta_j + \theta\, \partial_{-A}\, \beta_j & 0 \le j \le n \\ \alpha_j - \Phi\alpha_j - \theta\, \omega^{n-k} \wedge (\beta_k + \partial_{+A}\, \beta_{k-1}) & n+1 \le j \le 2n+1,\ k = 2n+1-j \end{cases}$$

As for the right-hand side, we have for all j

$$\begin{aligned} G\alpha_j &= \eta_{j-1} + \theta\, L^{-1}\eta_j \\ D_{\mathcal{C}}G\alpha_j &= (d_A)\eta_{j-1} + \omega \wedge L^{-1}\eta_j - \theta \left[(d_A)(L^{-1}\eta_j) + \Phi\eta_{j-1}\right] \\ D_{\mathcal{C}}\alpha_j &= (d_A)\eta_j + \omega \wedge \eta_{j-1} - \theta\left[(d_A)\eta_{j-1} + \Phi\eta_j\right] \\ GD_{\mathcal{C}}\alpha_j &= -(d_A)\eta_{j-1} - \Phi\eta_j + \theta\left[L^{-1}(d_A)\eta_j + L^{-1}(\omega \wedge \eta_{j-1})\right] \end{aligned}$$

which implies

$$
\begin{aligned}
(D_{\mathcal{C}}G + GD_{\mathcal{C}})\alpha_j &= \omega \wedge L^{-1}\eta_j - \Phi\alpha_j \\
&\quad + \theta\left[L^{-1}(d_A)\eta_j - (d_A)L^{-1}\eta_j + L^{-1}(\omega \wedge \eta_{j-1})\right]
\end{aligned} \tag{4.20}
$$

When $j \leq n$, this implies using the form decomposition in (4.17) and the properties of (d_A) in (2.8) and (2.10) that

$$
\begin{aligned}
(D_{\mathcal{C}}G + GD_{\mathcal{C}})\alpha_j &= \left(\eta_j - \beta_j\right) - \Phi\alpha_j + \theta\left[\partial_{-A}\,\beta_j + \eta_{j-1}\right] \\
&= (\alpha_j - \beta_j) - \Phi\alpha_j + \theta\,\partial_{-A}\,\beta_j = (\mathrm{id}_{\mathcal{C}} - gf - \Phi)(\alpha_j).
\end{aligned}
$$

And when $j > n$, with $k = 2n + 1 - j$ so that $\alpha_j = \eta_{2n+1-k} + \theta\,\eta_{2n-k}$, (4.20) together with the form decomposition in (4.18) implies

$$
\begin{aligned}
(D_{\mathcal{C}}G + GD_{\mathcal{C}})\alpha_j &= \eta_j - \Phi\alpha_j + \theta\left[-\omega^{n-k} \wedge \partial_{+A}\,\beta_{k-1} + (\eta_{j-1} - \omega^{n-k} \wedge \beta_k)\right] \\
&= \alpha_j - \Phi\alpha_j - \theta\left[\omega^{n-k} \wedge (\partial_{+A}\,\beta_{k-1} + \beta_k)\right] = (\mathrm{id}_{\mathcal{C}} - gf - \Phi)(\alpha_j).
\end{aligned}
$$

□

Lemma 4.10 *For arbitrary $\alpha \in \mathcal{C}$, if α is $D_{\mathcal{C}}$-closed, then $\Phi\alpha$ is $D_{\mathcal{C}}$-exact.*

Proof Suppose $\alpha \in \mathcal{C}^j$, we can set $\alpha = \eta_j + \theta\eta_{j-1}$ where $\eta_j, \eta_{j-1} \in \Omega^*(M, E)$. Since α is $D_{\mathcal{C}}$-closed,

$$0 = D_{\mathcal{C}}\alpha = (d_A\eta_j + \omega\eta_{j-1}) - \theta(\Phi\eta_j + d_A\eta_{j-1}).$$

So $d_A\eta_{j-1} = -\Phi\eta_j$. Then we have

$$D_{\mathcal{C}}(-\eta_{j-1}) = -d_A\eta_{j-1} + \theta\Phi\eta_{j-1} = \Phi(\eta_j + \theta\eta_{j-1}).$$

□

Now let f^* and g^* be the induced maps between $H^*_{\mathcal{C}}(M, E)$ and $PH^*(M, E)$. Then, from Lemmas 4.9 and 4.10, we have

$$
\begin{aligned}
f^*g^* &= \mathrm{id} \qquad \text{on } PH^*(M, E), \\
g^*f^* &= \mathrm{id} \qquad \text{on } \;H^*_{\mathcal{C}}(M, E),
\end{aligned}
$$

which imply the following theorem.

Theorem 4.11 *There is an isomorphism between the twisted cohomologies:*

$$PH^*(M, E) \cong H^*_{\mathcal{C}}(M, E).$$

From the local cohomologies of $PH^*(U, E)$ in Theorem 4.4, we have the corollary:

Corollary 4.12 *In a local coordinate chart U, there exists some $\lambda \in \Omega^1(U)$ such that $d\lambda = \omega$. The local cohomology*

$$H_C^j(U, E) = \begin{cases} \ker \Phi, & j = 0 \\ (\lambda - \theta) \operatorname{coker} \Phi, & j = 1 \\ 0, & j \geq 2 \end{cases}$$

And finally, with Theorem 4.2 which states the triviality of $PH^*(M, E)$ when Φ is invertible, we also have the following:

Corollary 4.13 *When Φ is invertible, $H_C^*(M, E) = 0$.*

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References

1. Bott, R., Tu, L.W.: Differential Forms in Algebraic Topology, Graduate Texts in Mathematics, 82. Springer, New York (1982)
2. Gibson, M.: Properties of the A_∞ Structure on Primitive Forms and Its Cohomology, note=Thesis (Ph.D.). University of California, Irvine, ProQuest LLC, Ann Arbor (2019)
3. Keller, B.: Introduction to A-infinity algebras and modules. Homol. Homotopy Appl. **3**(1), 1–35 (2001)
4. Ramanan, S.: Global Calculus, Graduate Studies in Mathematics, vol. 65. American Mathematical Society, Providence (2005)
5. Tanaka, H.L., Tseng, L.-S.: Odd sphere bundles, symplectic manifolds, and their intersection theory. Camb. J. Math. **6**(3), 213–266 (2018)
6. Tsai, C.-J., Tseng, L.-S., Yau, S.-T.: Cohomology and Hodge theory on symplectic manifolds: III. J. Differ. Geom. **103**(1), 83–143 (2016)
7. Tseng, L.-S., Yau, S.-T.: Cohomology and Hodge theory on symplectic manifolds: II. J. Differ. Geom. **91**(3), 417–443 (2012)
8. Tseng, L.-S., Zhou, J.: to appear

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