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Global Axisymmetric Solutions to the 3D MHD Equations with Nonzero Swirl

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Abstract

This paper studies sufficient conditions under which axisymmetric solutions with nonzero swirl components to the Cauchy problem of the 3D incompressible magnetohydrodynamic (MHD) equations are globally well-posed. We first establish a Serrin-type regularity criterion via the swirl component of velocity for the MHD equations without magnetic diffusion. Some new estimates were introduced to overcome the difficulty caused by the absence of magnetic diffusion. Moreover, we prove the global existence of axisymmetric solutions in the presence of magnetic diffusion provided that the scaling-invariant smallness condition was prescribed only on the swirl component of initial velocity while the initial magnetic field can be arbitrarily large.

Keywords MHD equations · Axisymmetric solutions · Global existence

Mathematics Subject Classification 35Q30 · 76D03

1 Introduction and the Main Results

In this paper, we are concerned with the following Cauchy problem of the 3D incompressible MHD equations:

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$$
\begin{cases}
\partial_t u + (u\cdot\nabla)u - (b\cdot\nabla)b - \nu\Delta u + \nabla p = 0,\\
\partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u - \mu\Delta b = 0,\\
\nabla\cdot u = \nabla\cdot b = 0,\\
(u,b)|_{t=0} = (u_0,b_0)\,,
\end{cases}
\tag{1.1}
$$

where $u(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ denotes the velocity field, $b(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ stands for the magnetic field, and $p(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ is the scalar pressure. The non-negative parameters ν and μ are the kinematic viscosity coefficient and the magnetic resistivity coefficient, respectively. The MHD equations describe the motion of electrically conducting fluids (e.g. Plasma fluid, liquid-metal fluid, etc.), which have a wide range of applications in astrophysics and plasma physics.

When the magnetic field $b=0$, (1.1) reduces to the classical incompressible Navier–Stokes equations. The existence of global weak solutions in the energy space for the Navier–Stokes equations goes back to Leray [29]. However, the fundamental question regarding the global regularity of the 3D Navier–Stokes equations with large initial data remains open, and it is generally viewed as one of the most important open problems in mathematics [12]. Therefore, there are some studies on conditional regularity [4, 5, 17–21, 26] and references therein. Furthermore, many efforts are made to study the solutions with some special structures such as oscillations or slow variations in one direction, see [6, 7], for example. In a similar way, many mathematicians are interested in the axisymmetric flows whose definition is given in Sect. 2. For an axisymmetric Navier–Stokes flow without swirl ($u^\theta=0$), Ukhovskii et al. [40] and Ladyzhenskaya [33] independently proved the global well-posedness of 3D axisymmetric Navier–Stokes equations. In this case, the quantity ω^θ/r solves the transport-diffusion equation (see for instance [36])

$$
\partial_t\left(\frac{\omega^\theta}{r}\right) + u\cdot\nabla\left(\frac{\omega^\theta}{r}\right) - \left(\Delta + \frac{2}{r}\partial_r\right)\frac{\omega^\theta}{r} = 0,
$$

from which one has for every $p\in[1,\infty]$ that

$$
\left\|\frac{\omega^\theta(t)}{r}\right\|_{L^p(\mathbb{R}^3)} \le \left\|\frac{\omega_0^\theta}{r}\right\|_{L^p(\mathbb{R}^3)}. \tag{1.2}
$$

The proofs in [33, 40] were exactly based on the above global a priori estimate and the maximum principle for vorticity. Later, Leonardi et al. [34] provided a refined proof. Recently, Abidi [1] further improved the initial regularity assumption to $u_0\in H^{\frac{1}{2}}(\mathbb{R}^3)$. If $u^\theta\neq 0$ (with non-trivial swirl), global well-posedness for the axisymmetric Navier–Stokes equations becomes much more difficult. There are many important progresses on this problem, see [2, 10, 27, 30, 37, 44] and references therein.

While for the MHD equations, Duvaut and Lions [11] (see also Sermange and Temam [39]) established the global existence of weak solutions and the local well-posedness of strong solutions for (1.1) in the classical Sobolev space $H^s(\mathbb{R}^3)$, $s\ge 3$. However, the global regularity of strong solutions for the 3D MHD equations is still a challenging open problem. It is worthy to mention that Fefferman et al. [13, 14] and

Chemin et al. [3] proved the local existence in various classical function spaces for (1.1) without resistivity. There are also many progresses on various Serrin-type regularity criteria; see, example [8, 9, 16, 22, 23, 41, 43, 45–47] and references therein. Motivated by the results on the axisymmetric Navier–Stokes equations, if one considers the 3D axisymmetric MHD equations without swirl component of velocity ($u^\theta = 0$), then it is obvious to see that the equation of ω^θ/r becomes

$$\partial_t\left(\frac{\omega^\theta}{r}\right) + u\cdot\nabla\left(\frac{\omega^\theta}{r}\right) - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\frac{\omega^\theta}{r} = -\frac{b^r j^\theta}{r^2} - \partial_z\left(\frac{b^\theta}{r}\right)^2 + \frac{1}{r}\left(b^r\partial_r + b^z\partial_z\right) j^\theta, \tag{1.3}$$

where j^θ is the swirl component of $\nabla\times b$. Note that the right-hand side of (1.3) includes b^r and b^z except for the term $\partial_z\left(\frac{b^\theta}{r}\right)^2$, this complicated coupling structure causes much trouble in the analysis and one cannot find an efficient way to deal with (1.3) for the moment. Thus, looking for solutions with some special structure is desirable. In 2015, Lei [31] considered a family of special axisymmetric initial data with $u_0^\theta = b_0^r = b_0^z = 0$, precisely

$$u_0 = u_0^r e_r + u_0^z e_z, \quad b_0 = b_0^\theta e_\theta.$$

Then, (1.3) reduces to

$$\partial_t\left(\frac{\omega^\theta}{r}\right) + u\cdot\nabla\left(\frac{\omega^\theta}{r}\right) - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\frac{\omega^\theta}{r} = -\partial_z\left(\frac{b^\theta}{r}\right)^2.$$

Moreover, b^θ/r solves the following homogeneous equation

$$\partial_t\left(\frac{b^\theta}{r}\right) + u\cdot\nabla\left(\frac{b^\theta}{r}\right) - \mu\left(\Delta + \frac{2}{r}\partial_r\right)\frac{b^\theta}{r} = 0.$$

It is proved in [31] that there exists a unique global axisymmetric solution for (1.1) with $\nu > 0$ and $\mu = 0$ if the initial data is smooth enough. Later, Jiu et al. proved the global well-posedness of the 3D axisymmetric MHD equations with horizontal dissipation and vertical magnetic diffusion in $H^2(\mathbb{R}^3)$ in [24] while the case for vertical velocity dissipation and vertical magnetic diffusion was investigated in [42] and with only vertical dissipation in $H^s(\mathbb{R}^3)$ with $s > 5/2$ in [25]. Similar as the swirl case ($u^\theta \neq 0$) for the Navier–Stokes equations, the global well-posedness for MHD equations becomes much more difficult in the presence of swirl components. If the initial data (u_0, b_0) is assumed to be axisymmetric, $b_0^r = b_0^z = 0$ and the scaling-invariant norms $\|ru_0^\theta\|_{L^\infty}$ and $\left\|b_0^\theta/r\right\|_{L^{3/2}}$ are small enough, Liu [35] proved the global well-posedness of the 3D axisymmetric MHD equations with $\nu > 0$ and $\mu > 0$. In this paper, we investigate the global well-posedness for the axisymmetric MHD equations in the presence of non-trivial swirl components, and the axisymmetric solutions possess the following form:

$$u(t,x) = u^r\,(t,r,z)\,e_r + u^\theta\,(t,r,z)\,e_\theta + u^z\,(t,r,z)\,e_z,$$
$$b(t,x) = b^\theta\,(t,r,z)\,e_\theta. \tag{1.4}$$

As discussed above, the situation becomes much more difficult for general axisymmetric magnetic field $b(x,t)$ (in the presence of b^r and b^z). The main obstacle lies in the strong coupling effect between velocity and magnetic fields (see for example (1.3)). On the other hand, the general axisymmetric magnetic field will prevent us from obtaining the global a priori estimates for b^θ/r and ru^θ which are the key ingredients of the analysis (see the discussion in [10]). Therefore, with the particular structure of axisymmetric solutions (1.4), equations (1.1) with $\nu = 1$ and $\mu = 0$ is equivalent to

$$\begin{cases} \partial_t u^r + u\cdot\nabla u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r + \partial_r p = \frac{(u^\theta)^2}{r} - \frac{(b^\theta)^2}{r}, \\ \partial_t u^\theta + u\cdot\nabla u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta = -\frac{u^r u^\theta}{r}, \\ \partial_t u^z + u\cdot\nabla u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)u^z + \partial_z p = 0, \\ \partial_t b^\theta + u\cdot\nabla b^\theta = \frac{u^r b^\theta}{r}, \\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \\ (u^r, u^\theta, u^z, b^\theta)\big|_{t=0} = (u_0^r, u_0^\theta, u_0^z, b_0^\theta). \end{cases} \tag{1.5}$$

Then, the vorticity equations in the cylindrical coordinates can be written as

$$\begin{cases} \partial_t \omega^r + u\cdot\nabla\omega^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^r = \left(\omega^r\partial_r + \omega^z\partial_z\right)u^r, \\ \partial_t \omega^\theta + u\cdot\nabla\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^\theta = \frac{u^r}{r}\omega^\theta + \partial_z\frac{(u^\theta)^2}{r} - \partial_z\frac{(b^\theta)^2}{r}, \\ \partial_t \omega^z + u\cdot\nabla\omega^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^z = \left(\omega^r\partial_r + \omega^z\partial_z\right)u^z, \end{cases} \tag{1.6}$$

where

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}.$$

Now, we introduce the following new variables:

$$\Pi := \frac{b^\theta}{r}, \quad \Omega := \frac{\omega^\theta}{r}, \quad \Phi := \frac{\omega^r}{r}, \quad \Gamma := ru^\theta, \quad \Lambda := \frac{u^\theta}{\sqrt{r}}.$$

Thus, one can easily check that the equations of $(\Pi, \Omega, \Phi, \Gamma, \Lambda)$ satisfy that

$$\begin{cases} \partial_t \Pi + u \cdot \nabla \Pi = 0, \\ \partial_t \Omega + u \cdot \nabla \Omega - \left(\Delta + \frac{2}{r} \partial_r \right) \Omega = -\partial_z \Pi^2 - 2 \frac{u^\theta}{r} \Phi, \\ \partial_t \Phi + u \cdot \nabla \Phi - \left(\Delta + \frac{2}{r} \partial_r \right) \Phi = \left(\omega^r \partial_r + \omega^z \partial_z \right) \frac{u^r}{r}, \\ \partial_t \Gamma + u \cdot \nabla \Gamma - \left(\Delta - \frac{2}{r} \partial_r \right) \Gamma = 0, \\ \partial_t \Lambda + u \cdot \nabla \Lambda - \left(\Delta + \frac{\partial_r}{r} - \frac{3}{4r^2} \right) \Lambda = -\frac{3}{2} \frac{u^r}{r} \Lambda. \end{cases} \tag{1.7}$$

Let us explain why these unknowns are introduced. Γ and Ω can be found in [36] which were introduced to study the general inviscid vortex dynamics in the presence of the swirl component of the velocity, where Γ is only transported by the velocity field u, and the equation of Γ in [36] (note that it is the inviscid case) implies conservation of circulation on material circles centered on the axis of symmetry, the vortex-stretching term in Ω-equation is absent (similar to the 2D Navier–Stokes vorticity equation), it is found that the quantity Ω is not conserved along particle trajectories, and it changes in response to the swirl component of velocity. In our case, the situation for Ω becomes much more difficult. The presence of u^θ gives the additional term $u^\theta \Phi / r$, this forces us to consider the equation for Φ, and the estimate for Ω is related to the property of Λ (see also the discussions in [2, 30]). Moreover, the right-hand term $\partial_z \Pi^2$ can be viewed as an external force due to the coupling effect of the magnetic field, then it is necessary to study the equation for Π. Fortunately, the important feature of the axisymmetric solutions is that the equations for Π and Γ imply the uniform L^p bounds for L^p initial data. The key observation is that the axisymmetric MHD equations exhibit nice properties once they are formulated in terms of these new unknowns, if a closed a priori estimate for Ω is derived (see Step 3 in the proof of Theorem 1.1), then the global regularity follows simultaneously. Therefore, these new unknowns are of great importance in this paper, and their properties allow us to prove the global well-posedness for the axisymmetric MHD equations.

The contributions of this paper are two-fold: Global regularity follows by only controlling the swirl component of the velocity field, which implies the dominant role of velocity field in magnetohydrodynamics. Moreover, global regularity also follows provided that dimensionless smallness conditions were only restricted on the swirl component of velocity field, which again confirms the dominant role of velocity and gives some new insights in studying the motion of magnetohydrodynamics. Now, let us state our first result.

Theorem 1.1 *Let* $(u_0, b_0) \in H^2\left(\mathbb{R}^3\right)$ *be axisymmetric divergence-free vector fields such that* $u_0 = u_0^r e_r + u_0^\theta e_\theta + u_0^z e_z$, $b_0 = b_0^\theta e_\theta$, $\Pi_0 \in L^\infty(\mathbb{R}^3)$ *and* $\nabla b_0 \in L^\infty(\mathbb{R}^3)$. *Suppose*

$$u^\theta \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\beta} \leq 1, \quad 3 < \beta \leq \infty, \quad 2 \leq \alpha < \infty, \tag{1.8}$$

then the corresponding solution (u, b) *of (1.5) can be extended beyond* T.

Remark 1.1 If we further assume that $\Gamma_0 \in L_T^\infty L^\infty(\mathbb{R}^3)$, one can establish the following weighted Serrin-type regularity criterion for $d < 1$:

$$r^d u^\theta \in L^t(0, T; L^s(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{t} + \frac{3}{s} \le 1 - d, \quad \frac{3}{1-d} < s \le \infty.$$

As a matter of fact, using Hölder's inequality,

$$\begin{aligned} \|u^\theta\|_{L_T^\alpha L^\beta} &= \left\| \left(r^d u^\theta\right)^\lambda \left(r u^\theta\right)^\xi \right\|_{L_T^\alpha L^\beta} \\ &\le \left\| r^d u^\theta \right\|_{L_T^t L^s}^\lambda \|r u^\theta\|_{L_T^\infty L^\infty}^\xi \le \left\| r^d u^\theta \right\|_{L_T^t L^s}^\lambda \|\Gamma_0\|_{L_T^\infty L^\infty}^\xi, \end{aligned}$$

where

$$\lambda + \xi = 1, \quad \xi + d\lambda = 0, \quad \frac{1}{\alpha} = \frac{\lambda}{t} + \frac{\xi}{\infty}, \quad \frac{1}{\beta} = \frac{\lambda}{s} + \frac{\xi}{\infty}.$$

Then,

$$\frac{2}{\alpha} + \frac{3}{\beta} = \frac{2\lambda}{t} + \frac{3\lambda}{s} \le \lambda(1-d) = 1,$$

which implies, under the condition (1.8), the corresponding solution can be extended beyond T.

Remark 1.2 For the endpoint case $u^\theta \in L_T^\infty L^3$, the global regularity also follows if smallness condition is prescribed. Once the regularity criterion on the swirl component u^θ is established, one can also establish the Serrin-type criterion in terms of the component ω^z or u^z without much difficulty .

Let's go back to (1.1) with $\nu = 1$, $\mu = 1$ and the axisymmetric solution (u, b) of the form (1.4), then (1.1) in the cylindrical coordinates can be written as

$$\begin{cases} \partial_t u^r + u \cdot \nabla u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) u^r + \partial_r p = \frac{(u^\theta)^2}{r} - \frac{(b^\theta)^2}{r}, \\ \partial_t u^\theta + u \cdot \nabla u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) u^\theta = -\frac{u^r u^\theta}{r}, \\ \partial_t u^z + u \cdot \nabla u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right) u^z + \partial_z p = 0, \\ \partial_t b^\theta + u \cdot \nabla b^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) b^\theta = \frac{u^r b^\theta}{r}, \\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \\ (u^r, u^\theta, u^z, b^\theta)\big|_{t=0} = (u_0^r, u_0^\theta, u_0^z, b_0^\theta). \end{cases} \tag{1.9}$$

Now, we have the following global existence result for the MHD equations (1.9).

Theorem 1.2 *Let the initial data* $(u_0, b_0) \in H^2(\mathbb{R}^3)$ *be axisymmetric divergence-free vector fields such that* $u_0 = u_0^r e_r + u_0^\theta e_\theta + u_0^z e_z$, $b_0 = b_0^\theta e_\theta$. *Suppose that* $\epsilon > 0$, $\Gamma_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\Pi_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *and* $\nabla b_0 \in L^\infty(\mathbb{R}^3)$, *if there exists a small constant* $\delta > 0$ *such that*

$$\left(\|\Omega_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + \|\Pi_0\|_{L^2}^2 \|\Pi_0\|_{L^3}^2\right)^{\frac{1}{2}} \|\Gamma_0\|_{L^2} \|\Gamma_0\|_{L^\infty} \le \delta, \tag{1.10}$$

or

$$\Psi_0 \cdot \|\Gamma_0\|_{L^2} \sup_{t>0} \|\Gamma\|_{L^\infty(r\le\epsilon)} \le \delta, \tag{1.11}$$

where

$$\Psi_0 := \left(\|\Omega_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + \frac{1}{\epsilon^4}\left(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right) \|\Gamma_0\|_{L^\infty}^3 + \|\Pi_0\|_{L^2}^2 \|\Pi_0\|_{L^3}^2\right)^{\frac{1}{2}}.$$

Then, system (1.9) is globally well-posed.

Remark 1.3 Note that (1.11) verifies the significant partial regularity results in [22], which asserts that the one-dimensional Hausdorff measure of the singular set is zero. This implies that the singularity of axisymmetric solutions can only happen at the axis of z, once we have good control of swirl component of velocity at the z-axis, the singularity will vanish.

Remark 1.4 We would have expected the validity of Theorem 1.2 for (1.5) (the case without magnetic resistivity). In fact, one can't establish the crucial uniform estimate for $\|\Pi\|_{L_T^4 L^4}$ as discussed in (4.9) in the case of non-resistivity, while it is necessary for us to show the global existence.

Remark 1.5 If $b_0^\theta = 0$, then condition (1.10) reduces to

$$\left(\|\Omega_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4\right)^{\frac{1}{2}} \|\Gamma_0\|_{L^2} \|\Gamma_0\|_{L^\infty} \le \delta,$$

and condition (1.11) reduces to

$$\left(\|\Omega_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + \frac{1}{\epsilon^4} \|u_0\|_{L^2}^2 \|\Gamma_0\|_{L^\infty}^3\right)^{\frac{1}{2}} \|\Gamma_0\|_{L^2} \sup_{t>0} \|\Gamma\|_{L^\infty(r\le\epsilon)} \le \delta,$$

which are exactly the smallness conditions established in [30] for the 3D axisymmetric Navier–Stokes equations.

Remark 1.6 We note that if (u, b, p) is a solution to system (1.1) with $\nu > 0$ and $\mu \ge 0$, so does

$$u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x),\ b^\lambda(t,x) = \lambda b(\lambda^2 t, \lambda x) \text{ and } p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

for all $\lambda > 0$. Direct computation implies that conditions (1.10) and (1.11) are scaling-invariant.

We now give the outline of the proofs for Theorem 1.1 and Theorem 1.2. To prove the global regularity, the system (1.7) plays an important role. We first introduce a quantity $\mathcal{A}(T) = \|\Omega\|^2_{L^\infty_T L^2} + \|\nabla\Omega\|^2_{L^2_T L^2}$, then we prove the bounds for $\|u\|_{L^\infty_T L^\infty}$ and $\|\nabla\omega\|_{L^4_T L^{12}}$ via the estimates of $\|\omega\|_{L^\infty_T L^4}$ and $\|\nabla\omega^2\|_{L^2_T L^2}$. The second step is to give the estimates of ∇u and ∇b, which is different from the techniques used in [2]. Here, a new strategy for the L^p_T-L^q_x estimates for parabolic version of singular integrals and potentials is applied (see Lemma 2.4). Then, we show that the solution (u, b) can be extended beyond T once the boundedness of $\mathcal{A}(T)$ is obtained, while its bound can be guaranteed by the conditions of Theorem 1.1. For the proof of Theorem 1.2, it is sufficient to show that for any $T < \infty$, $\mathcal{A}(T) < \infty$ under the prescribed smallness conditions, and then global existence follows in a similar fashion of Theorem 1.1.

Throughout this paper, C stands for some real positive constant, which may be different in each case. Sometimes, we shall alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$. Finally, we introduce the space $L^\alpha_T L^\beta \equiv L^\alpha(0, T; L^\beta(\mathbb{R}^3))$ as follows:

$$\|u\|_{L^\alpha_T L^\beta} = \begin{cases} \left(\int_0^T \|u(t,\cdot)\|^\alpha_{L^\beta} dt\right)^{\frac{1}{\alpha}}, & \text{if } 1 \leq \alpha < \infty, \\ \operatorname{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^\beta}, & \text{if } \alpha = \infty, \end{cases}$$

where

$$\|u(t,\cdot)\|_{L^\beta} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(t,x)|^\beta dx\right)^{\frac{1}{\beta}}, & \text{if } 1 \leq \beta < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |u(t,\cdot)|, & \text{if } \beta = \infty. \end{cases}$$

The rest of this paper is organized as follows. In Sect. 2, we present some basic estimates and useful lemmas which are important for the analysis in the rest of the paper. Section 3 is devoted to proving Theorem 1.1 once the elementary estimates are prepared. Then we prove Theorem 1.2 in Sect. 4.

2 Preliminaries

For $(x_1, x_2, x_3) \in \mathbb{R}^3$, let us introduce the cylindrical coordinates

$$r = \sqrt{(x_1)^2 + (x_2)^2}, \quad \theta = \arctan\frac{x_2}{x_1}, \quad z = x_3,$$

and denote e_r, e_θ, e_z the standard basis vectors in the cylindrical coordinate system

$$e_r(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad e_\theta(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A function f or a vector field $u = (u^r, u^\theta, u^z)$ is said to be axisymmetric if f, u^r, u^θ and u^z do not depend on θ:

$$u(x_1, x_2, x_3) = u^r(r, z)e_r + u^\theta(r, z)e_\theta + u^z(r, z)e_z.$$

The following lemma shows that if the initial data is axisymmetric, then the local strong solution of (1.1) is also axisymmetric. As a matter of fact, this argument has been shown in [34] (with zero swirl) and then in [38] (for generally non-zero swirl) by using the method of Banach fixed point theorem. One can see also an alternative proof in [15].

Lemma 2.1 *Assume that the initial data (u_0, b_0) is axisymmetric. Then, the local strong solution (u, b) to (1.1) is also axisymmetric.*

The next lemma gives the boundary information as r goes to zero which was used to deal with the boundary terms after performing integration by parts.

Lemma 2.2 *[32, Corollary 1] Let $k, l, m \in \mathbb{N}$, $u \in C^k\left(\mathbb{R}^3, \mathbb{R}^3\right)$ be an axisymmetric vector field, $u = u^z(z, r)e_z + u^r(z, r)e_r + u^\theta(z, r)e_\theta$. Then $u^z, u^r, u^\theta \in C^k\left(\mathbb{R} \times \overline{\mathbb{R}^+}\right)$ and*

$$\begin{aligned} &\partial_r^{2\ell+1} u^z\left(z, 0^+\right) = 0, \quad 1 \le 2\ell + 1 \le k, \\ &\partial_r^{2m} u^r\left(z, 0^+\right) = \partial_r^{2m} u^\theta\left(z, 0^+\right) = 0, \quad 0 \le 2m \le k. \end{aligned}$$

The following lemma plays an important role in obtaining the estimates for axisymmetric functions.

Lemma 2.3 *[2] For smooth axisymmetric vector filed u, its vorticity $\omega = \nabla \times u$, for any $1 < p < \infty$ and $\tilde\nabla = (\partial_r, \partial_z)$, there holds*

$$\begin{aligned} &\left\|\tilde\nabla u^r\right\|_{L^p} + \left\|\tilde\nabla u^z\right\|_{L^p} + \left\|\frac{u^r}{r}\right\|_{L^p} \le C\left\|\omega^\theta\right\|_{L^p}, \\ &\left\|\tilde\nabla\left(\frac{u^r}{r}\right)\right\|_{L^p} \le C\left\|\frac{\omega^\theta}{r}\right\|_{L^p}, \quad \left\|\tilde\nabla\tilde\nabla\left(\frac{u^r}{r}\right)\right\|_{L^p} \le C\left\|\partial_z\left(\frac{\omega^\theta}{r}\right)\right\|_{L^p}. \end{aligned}$$

In order to obtain the higher order estimates of vorticity, we need to introduce the following lemma which states the maximal L^p_T-L^q_x regularity for the heat kernel.

Lemma 2.4 *[28, Theorem 7.3] The operator A defined by $f(x,t) \longmapsto Af(x,t) = \int_0^t e^{(t-s)\Delta}\Delta f ds$ is bounded from $L^p\left((0,T), L^q\left(\mathbb{R}^3\right)\right)$ to $L^p\left((0,T), L^q\left(\mathbb{R}^3\right)\right)$ for every $T \in (0, \infty]$, $1 < p < \infty$, $1 < q < \infty$.*

3 Proof of Theorem 1.1

Before showing the proof, we present some facts which will be used frequently in the sequel without mention. Let (u, b) be the smooth solution to (1.1) with $\nu = 1$ and $\mu = 0$ corresponding to the initial data $(u_0, b_0) \in L^2$. Then, for any $t \geq 0$, it is easy to get

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\|\nabla u\|_{L_t^2 L^2}^2 \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{3.1}$$

The equation of Π in (1.7) satisfies homogeneous transport equation, then one can easily derive for $p \in [2, \infty]$ that

$$\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}.$$

On the other hand, it should be noted that Chae and Lee in [10] proved that for each $p \in [2, \infty]$,

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma_0\|_{L^p}.$$

The proof is divided into 4 steps.

Step 1: Bound for $\|\omega\|_{L_T^\infty L^4} + \|\nabla \omega^2\|_{L_T^2 L^2}$

We present some elementary estimates, which depend on $\mathcal{A}(T)$, once the bound for $\mathcal{A}(T)$ is obtained, then some uniform bounds for vorticity immediately follow.

Lemma 3.1 *Let (u, b) be the smooth axisymmetric solution of (1.5) on $[0, T)$, for some $T < \infty$, then*

$$\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty}^4 dt \leq C\mathcal{A}^2(T).$$

Proof By Gagliardo–Nirenberg inequality, one has

$$\left\|\frac{u^r}{r}\right\|_{L^\infty} \leq C\left\|\nabla\left(\frac{u^r}{r}\right)\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2\left(\frac{u^r}{r}\right)\right\|_{L^2}^{\frac{1}{2}}.$$

It follows from Lemma 2.3 that

$$\left\|\nabla\left(\frac{u^r}{r}\right)\right\|_{L^2} \leq C\|\Omega\|_{L^2}, \quad \left\|\nabla^2\left(\frac{u^r}{r}\right)\right\|_{L^2} \leq C\|\nabla\Omega\|_{L^2}.$$

This implies that

$$\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty}^4 dt \leq C\|\Omega\|_{L_T^\infty L^2}^2\|\nabla\Omega\|_{L_T^2 L^2}^2 \leq C\left(\|\Omega\|_{L_T^\infty L^2}^2 + \|\nabla\Omega\|_{L_T^2 L^2}^2\right)^2 = C\mathcal{A}^2(T).$$

□

The next lemma concerns some bounds of b^θ and Λ.

Lemma 3.2 *Assume* $(u_0, b_0) \in H^2(\mathbb{R}^3)$. *Let* (u, b) *be the corresponding axisymmetric weak solution of system (1.5) with the form (1.4) on* $[0, T)$, *for some* $T < \infty$, *then we have*

$$\|b^\theta\|_{L_T^\infty L^\infty} \leq C_1, \tag{3.2}$$

$$\|\Lambda\|_{L_T^\infty L^4}^4 + 3\|\nabla \Lambda^2\|_{L_T^2 L^2}^2 + 3\left\|\frac{u^\theta}{r}\right\|_{L_T^4 L^4}^4 \leq C_2, \tag{3.3}$$

$$\|\Lambda\|_{L_T^\infty L^8}^8 + \|\nabla \Lambda^4\|_{L_T^2 L^2}^2 + \int_0^T \int_{\mathbb{R}^3} \frac{\Lambda^8}{r^2} dxdt \leq C_3, \tag{3.4}$$

where the constants C_1, C_2, C_3 *depend on the initial data,* T *and* $\mathcal{A}(T)$.

Proof Multiplying the b^θ equation of (1.5) by $|b^\theta|^{p-2} b^\theta$, $2 \leq p < \infty$ and performing integration in space, one can get

$$\frac{1}{p}\frac{d}{dt}\|b^\theta\|_{L^p}^p = \int_{\mathbb{R}^3} \frac{u^r}{r}|b^\theta|^p dx \leq \left\|\frac{u^r}{r}\right\|_{L^\infty} \|b^\theta\|_{L^p}^p.$$

Therefore

$$\frac{d}{dt}\|b^\theta\|_{L^p} \leq \left\|\frac{u^r}{r}\right\|_{L^\infty} \|b^\theta\|_{L^p}.$$

The Gronwall's inequality implies

$$\|b^\theta\|_{L_T^\infty L^p} \leq \|b_0^\theta\|_{L^p} \exp\left\{\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty} dt\right\}.$$

Taking $p \to +\infty$, by Lemma 3.1, one has

$$\begin{aligned}
\|b^\theta\|_{L_T^\infty L^\infty} &\leq \|b_0^\theta\|_{L^\infty} \exp\left\{\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty} dt\right\} \\
&\leq \|b_0^\theta\|_{L^\infty} \exp\left\{\left(\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty}^4 dt\right)^{\frac{1}{4}} T^{\frac{3}{4}}\right\} \\
&\leq \|b_0^\theta\|_{L^\infty} \exp\left\{C\mathcal{A}^{\frac{1}{2}}(T) T^{\frac{3}{4}}\right\}.
\end{aligned}$$

This is (3.2).

Multiplying the Λ equation of (1.5) by Λ^3 and integrating the resulting equation over $\mathbb{R}^3$, one has

$$\frac{1}{4}\frac{d}{dt}\|\Lambda\|_{L^4}^4 + \frac{3}{4}\|\nabla \Lambda^2\|_{L^2}^2 + \frac{3}{4}\left\|\frac{u^\theta}{r}\right\|_{L^4}^4 = \frac{3}{2}\int_{\mathbb{R}^3} \frac{u^r}{r}\Lambda^4 dx \leq \frac{3}{2}\left\|\frac{u^r}{r}\right\|_{L^\infty} \|\Lambda\|_{L^4}^4.$$

Then, it follows by Gronwall's inequality and Lemma 3.1 that

$$\begin{aligned}\|\Lambda\|_{L_T^\infty L^4}^4+3\|\nabla\Lambda^2\|_{L_T^2L^2}^2+3\left\|\frac{u^\theta}{r}\right\|_{L_T^4L^4}^4 &\le \|\Lambda_0\|_{L^4}^4\exp\left\{C\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\}\\ &\le C\|u_0\|_{H^2}^2\exp\left\{C\mathcal{A}^{\frac12}(T)T^{\frac34}\right\},\end{aligned}$$

where

$$\|\Lambda_0\|_{L^4}^4\le\|u_0^\theta\|_{L^\infty}^2\left\|\frac{u_0^\theta}{r}\right\|_{L^2}^2\le C\left(\|\nabla u_0^\theta\|_{L^2}^{\frac12}\left\|\nabla^2u_0^\theta\right\|_{L^2}^{\frac12}\right)^2\|\nabla u_0\|_{L^2}^2\le C\|u_0\|_{H^2}^2,$$

which implies (3.3).

Multiplying the Λ equation of (1.5) by Λ^7 and integrating the resulting equation over $\mathbb{R}^3$, one obtains

$$\frac18\frac{d}{dt}\|\Lambda\|_{L^8}^8+\frac{7}{16}\|\nabla\Lambda^4\|_{L^2}^2+\frac34\int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}dx=-\frac32\int_{\mathbb{R}^3}\frac{u^r}{r}\Lambda^8dx.$$

Obviously

$$\frac{d}{dt}\|\Lambda\|_{L^8}^8+\|\nabla\Lambda^4\|_{L^2}^2+\int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}dx\le C\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^8}^8.$$

By Gronwall's inequality and Lemma 3.1, one has

$$\begin{aligned}\|\Lambda\|_{L_T^\infty L^8}^8+\|\nabla\Lambda^4\|_{L_T^2L^2}^2+\int_0^T\int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}dxdt &\le C\|\Lambda_0\|_{L^8}^8\exp\left\{C\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\}\\ &\le C\|u_0\|_{H^2}^8\exp\left\{C\mathcal{A}^{\frac12}(T)T^{\frac34}\right\},\end{aligned}$$

where

$$\begin{aligned}\|\Lambda_0\|_{L^8}^8 &\le \left\|u_0^\theta\right\|_{L^\infty}^4\left\|\frac{u_0^\theta}{r}\right\|_{L^4}^4\\ &\le\left(\left\|\nabla u_0^\theta\right\|_{L^2}^{\frac14}\left\|\nabla^2u_0^\theta\right\|_{L^2}^{\frac34}\right)^4\left(\left\|\frac{u_0^\theta}{r}\right\|_{L^2}^{\frac14}\left\|\nabla\frac{u_0^\theta}{r}\right\|_{L^2}^{\frac34}\right)^4\\ &\le C\|u_0\|_{H^2}^8.\end{aligned}$$

□

The following lemma gives the estimates for vorticity.

Lemma 3.3 *Assume $(u_0,b_0)\in H^2(\mathbb{R}^3)$ and $\Pi_0\in L^\infty(\mathbb{R}^3)$. Let (u,b) be the corresponding axisymmetric weak solution of system (1.5) satisfying (1.4) on $[0,T)$, for some $T<\infty$, then we have*

$$\left\|\omega^{\theta}\right\|_{L_T^{\infty}L^4}^4+\left\|\nabla(\omega^{\theta})^2\right\|_{L_T^2L^2}^2+\left\|\frac{\omega^{\theta}}{\sqrt{r}}\right\|_{L_T^4L^4}^4\le C_4, \tag{3.5}$$

$$\left\|\omega^{\theta}\right\|_{L_T^{\infty}L^2}^2+\left\|\nabla\omega^{\theta}\right\|_{L_T^2L^2}^2+\left\|\frac{\omega^{\theta}}{r}\right\|_{L_T^2L^2}^2\le C_5, \tag{3.6}$$

$$\left\|\omega^{r}\right\|_{L_T^{\infty}L^4}^4+\left\|\omega^{z}\right\|_{L_T^{\infty}L^4}^4+\left\|\nabla(\omega^{r})^2\right\|_{L_T^2L^2}^2+\left\|\nabla(\omega^{z})^2\right\|_{L_T^2L^2}^2+\left\|\frac{\omega^{r}}{\sqrt{r}}\right\|_{L_T^4L^4}^4\le C_7, \tag{3.7}$$

where the constants C_4, C_5 and C_7 depend on the initial data, T and $\mathcal{A}(T)$.

Proof Multiplying the ω^{θ} equation of (1.6) by $|\omega^{\theta}|^2\omega^{\theta}$ and then integrating the resulting equation over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left\|\omega^{\theta}\right\|_{L^4}^4+\frac{3}{4}\left\|\nabla(\omega^{\theta})^2\right\|_{L^2}^2+\left\|\frac{\omega^{\theta}}{\sqrt{r}}\right\|_{L^4}^4\\
&=\int_{\mathbb{R}^3}\frac{u^r}{r}(\omega^{\theta})^4dx+\int_{\mathbb{R}^3}\partial_z\left(\frac{(u^{\theta})^2}{r}\right)\cdot|\omega^{\theta}|^2\omega^{\theta}dx-\int_{\mathbb{R}^3}\partial_z\left(\frac{(b^{\theta})^2}{r}\right)\cdot|\omega^{\theta}|^2\omega^{\theta}dx\\
&:=A_1+A_2+A_3.
\end{aligned} \tag{3.8}$$

For the first term A_1, it follows that

$$A_1\le\left\|\frac{u^r}{r}\right\|_{L^{\infty}}\left\|\omega^{\theta}\right\|_{L^4}^4. \tag{3.9}$$

As for the second term A_2, by integrating by parts, we have

$$\begin{aligned}
A_2&=-3\int_{\mathbb{R}^3}\frac{(u^{\theta})^2}{r}\cdot(\omega^{\theta})^2\cdot\partial_z\omega^{\theta}dx=-\frac{3}{2}\int_{\mathbb{R}^3}\frac{(u^{\theta})^2}{r}\cdot\omega^{\theta}\cdot\partial_z(\omega^{\theta})^2dx\\
&=-\frac{3}{2}\int_{\mathbb{R}^3}\left(\frac{u^{\theta}}{\sqrt{r}}\right)^2\cdot\omega^{\theta}\cdot\partial_z(\omega^{\theta})^2dx.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
|A_2|&\le C\left\|\frac{u^{\theta}}{\sqrt{r}}\right\|_{L^8}^2\left\|\omega^{\theta}\right\|_{L^4}\left\|\partial_z(\omega^{\theta})^2\right\|_{L^2}\\
&\le C\left\|\frac{u^{\theta}}{\sqrt{r}}\right\|_{L^8}^8+\left\|\omega^{\theta}\right\|_{L^4}^4+\frac{1}{4}\left\|\partial_z(\omega^{\theta})^2\right\|_{L^2}^2.
\end{aligned} \tag{3.10}$$

For the last term A_3, by integration by parts, Hölder's inequality and Young's inequality, one has

$$
\begin{aligned}
A_3 &= 3\int_{\mathbb{R}^3} \frac{(b^\theta)^2}{r}\cdot(\omega^\theta)^2\cdot\partial_z\omega^\theta dx = \frac{3}{2}\int_{\mathbb{R}^3}\frac{(b^\theta)^2}{r}\cdot\omega^\theta\cdot\partial_z(\omega^\theta)^2 dx\\
&\le \frac{3}{2}\|\Pi\|_{L^4}\left\|b^\theta\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^4}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}\\
&\le C\|\Pi_0\|_{L^4}^4\left\|b^\theta\right\|_{L^\infty}^4+\left\|\omega^\theta\right\|_{L^4}^4+\frac{1}{4}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}^2.
\end{aligned}
\tag{3.11}
$$

Plugging (3.9), (3.10) and (3.11) into (3.8), one may conclude that

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left\|\omega^\theta\right\|_{L^4}^4+\frac{1}{4}\left\|\nabla(\omega^\theta)^2\right\|_{L^2}^2+\left\|\frac{\omega^\theta}{\sqrt{r}}\right\|_{L^4}^4\\
&\le \left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^4}^4+C\left\|\Lambda\right\|_{L^8}^8+2\left\|\omega^\theta\right\|_{L^4}^4+C\|\Pi_0\|_{L^4}^4\left\|b^\theta\right\|_{L^\infty}^4.
\end{aligned}
$$

Then by Gronwall's inequality, Lemma 3.1, (3.2) and (3.4), it follows

$$
\begin{aligned}
&\left\|\omega^\theta\right\|_{L_T^\infty L^4}^4+\left\|\nabla(\omega^\theta)^2\right\|_{L_T^2L^2}^2+4\left\|\frac{\omega^\theta}{\sqrt{r}}\right\|_{L_T^4L^4}^4\\
&\le C\left(\left\|\omega_0^\theta\right\|_{L^4}^4+\int_0^T\|\Lambda\|_{L^8}^8dt+\|\Pi_0\|_{L^4}\int_0^T\left\|b^\theta\right\|_{L^\infty}^4dt\right)\exp\left(C\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty}dt+CT\right)\\
&\le C\left(\left\|\omega_0^\theta\right\|_{L^4}^4+\|\Lambda\|_{L_T^\infty L^8}^8T+\|\Pi_0\|_{L^4}\left\|b^\theta\right\|_{L_T^\infty L^\infty}^4T\right)\exp\left(C\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty}dt+CT\right)\\
&\le C_4,
\end{aligned}
$$

where the constant C_4 depends on the initial data, C_1, C_3, T and $\mathcal{A}(T)$. Then this gives (3.5).

Multiplying the ω^θ equation of (1.6) by ω^θ and then integrating the resulting equation over $\mathbb{R}^3$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\omega^\theta\right\|_{L^2}^2+\left\|\nabla\omega^\theta\right\|_{L^2}^2+\left\|\frac{\omega^\theta}{r}\right\|_{L^2}^2=\int_{\mathbb{R}^3}\left(\frac{\omega^\theta}{r}u^r\omega^\theta-\partial_z\omega^\theta\frac{(u^\theta)^2}{r}+\partial_z\omega^\theta\frac{(b^\theta)^2}{r}\right)dx\\
&\le\left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^2}^2+C\left\|\frac{u^\theta}{\sqrt{r}}\right\|_{L^4}^4+\left\|b^\theta\right\|_{L^\infty}^2\|\Pi\|_{L^2}^2+\frac{1}{2}\left\|\partial_z\omega^\theta\right\|_{L^2}^2.
\end{aligned}
$$

Thus,

$$
\frac{d}{dt}\left\|\omega^\theta\right\|_{L^2}^2+\left\|\nabla\omega^\theta\right\|_{L^2}^2+2\left\|\frac{\omega^\theta}{r}\right\|_{L^2}^2\le C\left\|\frac{u^\theta}{\sqrt{r}}\right\|_{L^4}^4+2\left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^2}^2+2\|\Pi_0\|_{L^2}^2\left\|b^\theta\right\|_{L^\infty}^2.
$$

By Gronwall's inequality and (3.2), it follows that

$$\begin{aligned}
&\left\|\omega^{\theta}\right\|_{L_T^{\infty}L^2}^2+\left\|\nabla\omega^{\theta}\right\|_{L_T^2L^2}^2+2\left\|\frac{\omega^{\theta}}{r}\right\|_{L_T^2L^2}^2\\
&\le\left(\left\|\omega_0^{\theta}\right\|_{L^2}^2+2\left\|\Pi_0\right\|_{L^2}^2\int_0^T\left\|b^{\theta}\right\|_{L^{\infty}}^2dt\right)\exp\left\{\int_0^T\left\|\frac{u^r}{r}\right\|_{L^{\infty}}dt\right\}\\
&\le\left(\left\|\omega_0^{\theta}\right\|_{L^2}^2+2\left\|\Pi_0\right\|_{L^2}^2\left\|b^{\theta}\right\|_{L_T^{\infty}L^{\infty}}^2T\right)\exp\left\{\int_0^T\left\|\frac{u^r}{r}\right\|_{L^{\infty}}dt\right\}\\
&\le C_5,
\end{aligned}$$

where the constant C_5 depends on the initial data, C_1, T and $\mathcal{A}(T)$.

In the following, we estimate $\int_0^T\|(u^r,u^z)\|_{L^{\infty}}^2dt$. By Gagliardo-Nirenberg inequality, Lemma 2.3 and (3.6), it follows that

$$\begin{aligned}
\int_0^T\|(u^r,u^z)\|_{L^{\infty}}^2dt&\le C\int_0^T\left(\|\nabla(u^r,u^z)\|_{L^2}^{\frac{1}{2}}\|\nabla^2(u^r,u^z)\|_{L^2}^{\frac{1}{2}}\right)^2dt\\
&\le C\int_0^T\|\nabla u\|_{L^2}\left(\left\|\nabla\omega^{\theta}\right\|_{L^2}+\left\|\frac{\omega^{\theta}}{r}\right\|_{L^2}\right)dt\\
&\le C\|\nabla u\|_{L_T^2L^2}\left(\left\|\nabla\omega^{\theta}\right\|_{L_T^2L^2}+\left\|\frac{\omega^{\theta}}{r}\right\|_{L_T^2L^2}\right)\\
&\le C\|u_0\|_{L^2}\left(\left\|\nabla\omega^{\theta}\right\|_{L_T^2L^2}+\left\|\frac{\omega^{\theta}}{r}\right\|_{L_T^2L^2}\right)\\
&\le C_6.
\end{aligned}\tag{3.12}$$

where the constant C_6 depends on C_5.

Multiplying the ω^r equation in (1.6) by $|\omega^r|^2\omega^r$ and integrating the resulting equation over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|\omega^r\|_{L^4}^4+\|\omega^z\|_{L^4}^4\right)+\frac{3}{4}\left\|\nabla(\omega^r)^2\right\|_{L^2}^2+\frac{3}{4}\left\|\nabla(\omega^z)^2\right\|_{L^2}^2+\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4\\
&=\int_{\mathbb{R}^3}\omega^r\partial_r u^r|\omega^r|^2\omega^r dx+\int_{\mathbb{R}^3}\omega^z\partial_z u^r|\omega^r|^2\omega^r dx\\
&\quad+\int_{\mathbb{R}^3}\omega^r\partial_r u^z|\omega^z|^2\omega^z dx+\int_{\mathbb{R}^3}\omega^z\partial_z u^z|\omega^z|^2\omega^z dx\\
&:=H_1+H_2+H_3+H_4.
\end{aligned}\tag{3.13}$$

For the first term H_1, by integration by parts and Lemma 2.2, it follows that $u^r\big|_{r=0} = u^\theta\big|_{r=0} = 0$, which implies that $\partial_z u^\theta(t,0,z) = 0$, therefore $\omega^r(t,0,z) = 0$. Thus, one gets that

$$\begin{aligned}
H_1 &= \int_{\mathbb{R}^3} \partial_r u^r (\omega^r)^4 dx \\
&= 2\pi \int_{-\infty}^{\infty} u^r(\omega^r)^4 \Big|_{r=0}^{r=\infty} dz - 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} \left(4u^r(\omega^r)^3 \cdot \partial_r\omega^r \cdot r + u^r \cdot (\omega^r)^4\right) drdz \\
&\le 2\|u^r\|_{L^\infty} \left\|\omega^r\right\|_{L^4}^2 \left\|\nabla(\omega^r)^2\right\|_{L^2} + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\omega^r\|_{L^4}^4 \\
&\le C\|u^r\|_{L^\infty}^2 \|\omega^r\|_{L^4}^4 + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\omega^r\|_{L^4}^4 + \frac{1}{8}\|\nabla(\omega^r)^2\|_{L^2}^2.
\end{aligned} \tag{3.14}$$

Note the fact that $\partial_r(r\omega^r) + \partial_z(r\omega^z) = 0$, we can show that

$$\begin{aligned}
H_2 &= \int_{\mathbb{R}^3} \omega^z \partial_z u^r |\omega^r|^2 \omega^r dx = -2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^r \cdot \partial_z \left(|\omega^r|^2 \cdot \omega^r \cdot \omega^z \cdot r\right) drdz \\
&= -3\int_{\mathbb{R}^3} u^r \cdot (\omega^r)^2 \partial_r \omega^r \cdot \omega^z dx + 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^r \cdot (\omega^r)^3 \cdot \partial_r(r\omega^r) drdz \\
&= -\frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \partial_r(\omega^r)^2 \cdot \omega^r \cdot \omega^z dx + \int_{\mathbb{R}^3} u^r \cdot (\omega^r)^2 \cdot \left(\frac{\omega^r}{\sqrt{r}}\right)^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} u^r \cdot \nabla(\omega^r)^2 \cdot (\omega^r)^2 dx \\
&\le \frac{3}{2}\|u^r\|_{L^\infty} \|\omega^r\|_{L^4} \|\omega^z\|_{L^4} \left\|\nabla(\omega^r)^2\right\|_{L^2} \\
&\quad + \|u^r\|_{L^\infty} \|\omega^r\|_{L^4}^2 \left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^2 + \|u^r\|_{L^\infty} \|\omega^r\|_{L^4}^2 \|\nabla(\omega^r)^2\|_{L^2} \\
&\le C\|u^z\|_{L^\infty}^2 \left(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4\right) + \frac{1}{2}\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4 + \frac{1}{8}\|\nabla(\omega^r)^2\|_{L^2}^2.
\end{aligned} \tag{3.15}$$

We deal with the third term H_3 which is similar to H_1. By Lemma 2.2 and the regularity of local strong solutions, one obtains

$$\begin{aligned}
H_3 &= \int_{\mathbb{R}^3} \omega^r \partial_r u^z |\omega^z|^2 \omega^z dx = -2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^z \cdot \partial_r(\omega^r \cdot |\omega^z|^2 \cdot \omega^z r) drdz \\
&= 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^z \cdot \partial_z(r\omega^z) \cdot |\omega^z|^2 \cdot \omega^z drdz - \frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \omega^r \cdot \omega^z \cdot \partial_r(\omega^z)^2 dx \\
&= \int_{\mathbb{R}^3} u^z \partial_z \omega^z |\omega^z|^3 dx - \frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \omega^r \cdot \omega^z \cdot \partial_r(\omega^z)^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} u^z (\omega^z)^2 \partial_z(\omega^z)^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \omega^r \cdot \omega^z \cdot \partial_r(\omega^z)^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u^z\|_{L^\infty}\|\omega^z\|_{L^4}^2\|\nabla(\omega^z)^2\|_{L^2} + C\|u^z\|_{L^\infty}\|\omega^r\|_{L^4}\|\omega^z\|_{L^4}\|\nabla(\omega^z)^2\|_{L^2} \\
&\le C\|u^z\|_{L^\infty}^2\|\omega^z\|_{L^4}^4 + \frac{1}{16}\|\nabla(\omega^z)^2\|_{L^2}^2 \\
&\quad + C\|u^z\|_{L^\infty}^2\left(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4\right) + \frac{1}{16}\|\nabla(\omega^z)^2\|_{L^2}^2 \\
&\le C\|u^r\|_{L^\infty}^2\left(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4\right) + \frac{1}{8}\|\nabla(\omega^z)^2\|_{L^2}^2. \qquad (3.16)
\end{aligned}
$$

Finally, for the last term H_4, one can get

$$
\begin{aligned}
H_4 = &-\int_{\mathbb{R}^3} u^z\partial_z\left(|\omega^z|^4\right)dx = -2\int_{\mathbb{R}^3} u^z(\omega^z)^2\cdot\partial_z(\omega^z)^2 dx \\
&\le C\|u^z\|_{L^\infty}^2\|\omega^z\|_{L^4}^4 + \frac{1}{8}\|\nabla(\omega^z)^2\|_{L^2}^2.
\end{aligned} \qquad (3.17)
$$

Plugging (3.14), (3.15), (3.16) and (3.17) into (3.13), yields

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4\right) + \frac{1}{2}\left\|\nabla(\omega^r)^2\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla(\omega^z)^2\right\|_{L^2}^2 + \frac{1}{2}\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4 \\
&\le C\left(\|u^r\|_{L^\infty}^2 + \|u^z\|_{L^\infty}^2 + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)\left(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4\right).
\end{aligned}
$$

Thanks to Lemma 3.1 and (3.12), one has by Gronwall's inequality that

$$
\begin{aligned}
&\|\omega^r\|_{L_T^\infty L^4}^4 + \|\omega^z\|_{L_T^\infty L^4}^4 + 2\left\|\nabla(\omega^r)^2\right\|_{L_T^2L^2}^2 + 2\left\|\nabla(\omega^z)^2\right\|_{L_T^2L^2}^2 + 2\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L_T^4L^4}^4 \\
&\le \left(\|\omega_0^r\|_{L^4}^4 + \|\omega_0^z\|_{L^4}^4\right)\exp\left\{C\int_0^T \|(u^r,u^z)\|_{L^\infty}^2 dt + C\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\} \\
&\le C_7, \qquad (3.18)
\end{aligned}
$$

where the constant C_7 depends on the initial data, C_6 and $\mathcal{A}(T)$. The estimate (3.7) immediately follows from (3.18). □

From (3.5) and (3.7), one can easily obtain that

$$
\|\omega\|_{L_T^\infty L^4}^4 + \left\|\nabla\omega^2\right\|_{L_T^\infty L^2}^2 < \infty. \qquad (3.19)
$$

Using Gagliardo–Nirenberg inequality, we have

$$
\|\omega\|_{L_T^4 L^{12}} \le C\|\nabla\omega^2\|_{L_T^2L^2}^{\frac{1}{2}} < \infty. \qquad (3.20)
$$

On the other hand, by the Gagliardo–Nirenberg inequality, one has

$$
\|u\|_{L^\infty} \le C\|u\|_{L^2}^{\frac{1}{7}}\|\nabla u\|_{L^4}^{\frac{6}{7}}, \qquad (3.21)
$$

then by Young's inequality, (3.1) and (3.19), one obtains that

$$\|u\|_{L_T^\infty L^\infty} \le C\left(\|u\|_{L_T^\infty L^2} + \|\omega\|_{L_T^\infty L^4}\right) \le C_8, \tag{3.22}$$

where the constant C_8 depends on C_4 and C_7.

Step 2: Estimates for ∇u and ∇b

In the following, we focus on the estimates of ∇u and ∇b.

Lemma 3.4 *Assume $(u_0, b_0) \in H^2(\mathbb{R}^3)$, $\Pi_0 \in L^\infty(\mathbb{R}^3)$ and $\nabla b_0 \in L^\infty(\mathbb{R}^3)$. Let (u, b) be the axisymmetric solution of system (1.5) satisfying (1.4) on $[0, T)$, for some $T < \infty$, then we have*

$$\|\nabla u\|_{L_T^4 L^\infty} \le C_9, \tag{3.23}$$

$$\|\nabla b\|_{L_T^\infty L^\infty} \le C_{10}, \tag{3.24}$$

where the constants C_9 and C_{10} depend on the initial data, T and $\mathcal{A}(T)$.

Proof Note that the equation of u

$$u_t + u\cdot\nabla u + \nabla p = \Delta u + b\cdot\nabla b, \tag{3.25}$$

where $u\cdot\nabla u$ can be rewritten as

$$u\cdot\nabla u = (\nabla\times u)\times u + \nabla\left(\frac{|u|^2}{2}\right).$$

Therefore, by taking curl operator to (3.25), one can get

$$\omega_t - \Delta\omega = -\nabla\times(\omega\times u) + \nabla\times(b\cdot\nabla b),$$

it follows that

$$\omega = e^{t\Delta}\omega_0 - \int_0^t e^{(t-s)\Delta}\left(\nabla\times(\omega\times u) - \partial_z\left(\Pi b^\theta e_\theta\right)\right)ds. \tag{3.26}$$

Therefore, by combining Lemma 2.4, interpolation inequality and (3.26), we deduce that

$$\begin{aligned}\|\nabla\omega\|_{L_T^4 L^{12}} &\lesssim \|\omega\times u\|_{L_T^4 L^{12}} + \|\Pi\cdot b^\theta\|_{L_T^4 L^{12}}\\ &\lesssim \|u\|_{L_T^\infty L^\infty}\|\omega\|_{L_T^4 L^{12}} + \|\Pi_0\|_{L^{12}}\left\|b^\theta\right\|_{L_T^\infty L^\infty} T^{\frac{1}{4}}\\ &\lesssim \|\omega\|_{L_T^4 L^{12}}\|u\|_{L_T^\infty L^\infty} + \|\Pi_0\|_{L^\infty}^{\frac{5}{6}}\|\Pi_0\|_{L^2}^{\frac{1}{6}}\left\|b^\theta\right\|_{L_T^\infty L^\infty} T^{\frac{1}{4}}\end{aligned}$$

$$\lesssim \|\omega\|_{L^4_T L^{12}} \|u\|_{L^\infty_T L^\infty} + \|\Pi_0\|^{\frac{5}{6}}_{L^\infty} \|b_0\|^{\frac{1}{6}}_{H^2} \left\|b^\theta\right\|_{L^\infty_T L^\infty} T^{\frac{1}{4}}. \tag{3.27}$$

On the other hand, by the Gagliardo–Nirenberg inequality, one has

$$\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|^{\theta}_{L^4} \|\nabla^2 u\|^{1-\theta}_{L^{12}},$$

where

$$\frac{1}{\infty} - \frac{1}{3} = \left(\frac{1}{4} - \frac{1}{3}\right)\theta + \left(\frac{1}{12} - \frac{2}{3}\right)(1-\theta),$$

it's easy to deduce that $\theta = \dfrac{1}{2}$, then it is straightforward to verify that

$$\begin{aligned} \|\nabla u\|^4_{L^4_T L^\infty} &\leq C \|\nabla u\|^2_{L^\infty_T L^4} \|\nabla^2 u\|^2_{L^2_T L^{12}} \\ &\leq C \|\omega\|^2_{L^\infty_T L^4} \|\nabla \omega\|^2_{L^2_T L^{12}} \\ &\leq C \|\omega\|^2_{L^\infty_T L^4} \|\nabla \omega\|^2_{L^4_T L^{12}} T^{\frac{1}{2}}. \end{aligned} \tag{3.28}$$

Inserting (3.27) into (3.28), thanks to Lemmas 3.2, (3.20) and 3.3, (3.7), we have

$$\|\nabla u\|^4_{L^4_T L^\infty} \lesssim \|\omega\|^2_{L^\infty_T L^4} \left(\|u\|^2_{L^\infty_T L^\infty} \|\nabla(\omega)^2\|_{L^2_T L^2} + \|\Pi_0\|^{\frac{5}{3}}_{L^\infty} \|b_0\|^{\frac{1}{3}}_{H^2} \left\|b^\theta\right\|^2_{L^\infty_T L^\infty} \right) T^{\frac{1}{2}}. \tag{3.29}$$

Therefore, (3.23) follows immediately from (3.29).

Now taking ∇ operator to the equations $b^\theta e_\theta$. Thence,

$$\frac{d}{dt}\nabla b + u \cdot \nabla\nabla b = -\nabla u \cdot \nabla b + \frac{u^r}{r}\nabla b + \nabla u^r \otimes \Pi e_\theta - \frac{u^r}{r}\Pi e_\theta \otimes e_r,$$

Multiplying the above equation by $|\nabla b|^{p-2}\nabla b$ and integrating the resulting equation over $\mathbb{R}^3$ yields

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\nabla b\|^p_{L^p} &\leq \|\nabla u\|_{L^\infty} \|\nabla b\|^p_{L^p} + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\nabla b\|^p_{L^p} \\ &\quad + \|\nabla u^r\|_{L^\infty} \|\Pi\|_{L^p} \|\nabla b\|^{p-1}_{L^p} + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\Pi\|_{L^p} \|\nabla b\|^{p-1}_{L^p}. \end{aligned}$$

It immediately implies that

$$\begin{aligned} \frac{d}{dt}\|\nabla b\|_{L^p} &\leq \|\nabla u\|_{L^\infty} \|\nabla b\|_{L^p} + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\nabla b\|_{L^p} + \|\nabla u^r\|_{L^\infty} \|\Pi\|_{L^p} \\ &\quad + \left\|\frac{u^r}{r}\right\|_{L^\infty} \|\Pi\|_{L^p}. \end{aligned}$$

Applying Gronwall's inequality and taking $p \to \infty$, we obtain

$$\begin{aligned}
&\|\nabla b\|_{L^\infty_T L^\infty} \\
&\quad \le \left(\|\nabla b_0\|_{L^\infty} + \int_0^T \left(\|\nabla u\|_{L^\infty} + \left\| \frac{u^r}{r} \right\|_{L^\infty} \right) \|\Pi\|_{L^\infty} dt \right) \\
&\quad \exp\left(\int_0^T \left(\|\nabla u\|_{L^\infty} + \left\| \frac{u^r}{r} \right\|_{L^\infty} \right) dt \right) \\
&\le \left(\|\nabla b_0\|_{L^\infty} + \|\Pi_0\|_{L^\infty} \|\nabla u\|_{L^\infty_T L^\infty} T + \|\Pi_0\|_{L^\infty} \int_0^T \left\| \frac{u^r}{r} \right\|_{L^\infty} dt \right) \\
&\cdot \exp\left(\|\nabla u\|_{L^\infty_T L^\infty} T + \int_0^T \left\| \frac{u^r}{r} \right\|_{L^\infty} dt \right).
\end{aligned}$$

At last, by Lemma 3.1, (3.23) and Gronwall's inequality, we have

$$\|\nabla b\|_{L^\infty_T L^\infty} \le C_{10},$$

where the constant C_{10} depends on the initial data, C_9, T and $\mathcal{A}(T)$.

Step 3: $H^2(\mathbb{R}^3)$ Estimates of (u, b)

The following lemma shows that the boundedness of $\mathcal{A}(T)$ guarantees the smoothness of axisymmetric solutions to (1.5).

Lemma 3.5 *Assume* $(u_0, b_0) \in H^2(\mathbb{R}^3)$, $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, $\Pi_0 \in L^\infty(\mathbb{R}^3)$ *and* $\nabla b_0 \in L^\infty(\mathbb{R}^3)$. *If*

$$\mathcal{A}(T) < \infty, \tag{3.30}$$

for some $0 < T < \infty$, *then the corresponding solution of system (1.5) remains smooth on* $[0, T]$.

Proof Applying "Δ" operator to the equation $(1.1)_{1,2}$, and then taking the inner product, we have for any $t \in [0, T)$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u(t,\cdot)\|_{L^2}^2 + \|\Delta b(t,\cdot)\|_{L^2}^2 \right) + \left\| \nabla^3 u(t,\cdot) \right\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \Delta u \cdot \Delta(u\cdot\nabla u) dx + \int_{\mathbb{R}^3} \Delta u \cdot \Delta(b\cdot\nabla b) dx \\
&\quad - \int_{\mathbb{R}^3} \Delta b \cdot \Delta(u\cdot\nabla b) dx + \int_{\mathbb{R}^3} \Delta b \cdot \Delta(b\cdot\nabla u) dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the first term I_1, one has

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} \Delta u \cdot (\Delta u \cdot \nabla u) dx - \int_{\mathbb{R}^3} \Delta u \cdot (u \cdot \nabla \Delta u) dx - 2\int_{\mathbb{R}^3} \Delta u \cdot (\nabla u : \nabla^2 u) dx \\
&\le 3\|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2}^2 + \|u\|_{L^\infty} \|\Delta u\|_{L^2} \|\Delta \nabla u\|_{L^2} \\
&\le 3\|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2}^2 + C\|u\|_{L^\infty}^2 \|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\Delta \nabla u\|_{L^2}^2.
\end{aligned}
$$

For the second term I_2, by utilizing the integration by parts and the fact $\operatorname{div} b = 0$, we have

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} \Delta u \cdot (\Delta b \cdot \nabla b) dx + \int_{\mathbb{R}^3} \Delta u \cdot (b \cdot \nabla \Delta b) dx + 2\int_{\mathbb{R}^3} \Delta u \cdot (\nabla b : \nabla^2 b) dx \\
&= \int_{\mathbb{R}^3} \Delta u \cdot (\Delta b \cdot \nabla b) dx - \int_{\mathbb{R}^3} \Delta b \cdot (b \cdot \nabla \Delta u) dx + 2\int_{\mathbb{R}^3} \Delta u \cdot (\nabla b : \nabla^2 b) dx \\
&\le 3\|\nabla b\|_{L^\infty} \|\Delta b\|_{L^2} \|\Delta u\|_{L^2} + \|b\|_{L^\infty} \|\Delta b\|_{L^2} \|\Delta \nabla u\|_{L^2} \\
&\le C\|\nabla b\|_{L^\infty} \left(\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \right) + C\|b\|_{L^\infty}^2 \|\Delta b\|_{L^2}^2 + \frac{1}{8}\|\Delta \nabla u\|_{L^2}^2.
\end{aligned}
$$

The third term I_3 can be estimated as following

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}^3} \Delta b \cdot \Delta (u \cdot \nabla b) dx \\
&= -\int_{\mathbb{R}^3} \Delta b \cdot (\Delta u \cdot \nabla b) dx - \int_{\mathbb{R}^3} \Delta b \cdot (u \cdot \nabla \Delta b) dx - 2\int_{\mathbb{R}^3} \Delta b \cdot (\nabla u : \nabla^2 b) dx.
\end{aligned}
$$

By using integration by parts and thanks to divergence-free of u, we see that

$$
\int_{\mathbb{R}^3} \Delta b \cdot (u \cdot \nabla \Delta b) dx = 0,
$$

it immediately implies

$$
I_3 \le \|\nabla b\|_{L^\infty} \|\Delta u\|_{L^2} \|\Delta b\|_{L^2} \le \|\nabla b\|_{L^\infty} \left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \right).
$$

The last term I_4, similarly to I_1, one obtains that

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} \Delta b \cdot \Delta (b \cdot \nabla u) dx \\
&= \int_{\mathbb{R}^3} \Delta b \cdot (\Delta b \cdot \nabla u) dx + \int_{\mathbb{R}^3} \Delta b \cdot (b \cdot \nabla \Delta u) dx + 2\int_{\mathbb{R}^3} \Delta b \cdot (\nabla b \cdot \Delta u) dx \\
&\le \|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2}^2 + \|b\|_{L^\infty} \|\Delta b\|_{L^2} \|\nabla \Delta u\|_{L^2} + \|\nabla b\|_{L^\infty} \|\Delta b\|_{L^2} \|\Delta u\|_{L^2}
\end{aligned}
$$

$$\leq \|\nabla u\|_{L^\infty}\|\Delta u\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\Delta b\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta u\|_{L^2} + \|\nabla b\|_{L^\infty} \left(\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right).$$

Combining the above estimates, it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Delta u(t,\cdot)\|_{L^2}^2 + \|\Delta b(t,\cdot)\|_{L^2}^2\right) + \|\nabla\Delta u(t,\cdot)\|_{L^2}^2 \\
&\quad \leq C\left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\right) + C\|\nabla u\|_{L^\infty}^2\|\Delta u\|_{L^2}^2 \\
&\qquad + C\|\nabla b\|_{L^\infty}^2\|\Delta b\|_{L^2}^2.
\end{aligned}$$

It then follows from Lemmas 3.2 and 3.4, (3.22), Gronwall's inequality, and thanks to $\mathcal{A}(T) < \infty$, one has that

$$\begin{aligned}
&\|\Delta u\|_{L_T^\infty L^2}^2 + \|\Delta b(t,\cdot)\|_{L_T^\infty L^2}^2 + \left\|\nabla^3 u(t,\cdot)\right\|_{L_T^2 L^2}^2 \\
&\lesssim \exp\left\{\int_0^T \left(\|u(t,\cdot)\|_{L^\infty}^2 + \|b(t,\cdot)\|_{L^\infty}^2 + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla b(t,\cdot)\|_{L^\infty}\right)dt\right\} \\
&\lesssim \exp\left\{\|u\|_{L_T^\infty L^\infty}^2 T + \|b\|_{L_T^\infty L^\infty}^2 T + \|\nabla u\|_{L_T^4 L^\infty} T^{\frac{3}{4}} + \|\nabla b\|_{L_T^\infty L^\infty} T\right\} \\
&\leq C_{11} < \infty,
\end{aligned}$$

where the constant C_{11} depends on the initial data, C_1, C_8, C_9, C_{10}, which together with (3.1) ensures that

$$\|u\|_{L_T^\infty H^2} + \|u\|_{L_T^2 H^3} < \infty, \quad \|b\|_{L_T^\infty H^2} < \infty.$$

Therefore, the smoothness follows from the classical local well-posedness theory of the 3D MHD system (1.1) with $\nu > 0$, $\mu = 0$ (see [13] for instance). The proof of Lemma 3.5 is complete.

Step 4: Bound of $\mathcal{A}(T)$.

This step gives closed estimates for the above steps by showing the bound of $\mathcal{A}(T)$. As mentioned by [34], we can not directly take the $L^2(\mathbb{R}^3)$ energy estimates for Ω and Φ due to the fact that the singularity coming from the change of variables on the z-axis is quite high. This difficulty can be overcame with the techniques introduced in [34]. More precisely, for any $\epsilon > 0$, by multiplying the equations of $(1.7)_2$ and $(1.7)_3$ by $\frac{\omega^\theta}{r^{1-\epsilon}}$ and $\frac{\omega^r}{r^{1-\epsilon}}$ respectively, integrating over $\mathbb{R}^3$ and adding them together, then by integration by parts there will appear the following boundary terms

$$\int_{-\infty}^{\infty} \partial_r\Omega\frac{\omega^\theta}{r^{1-\epsilon}}\Big|_{r=0}^{r=\infty} r\,dz \quad \text{and} \quad \int_{-\infty}^{\infty} \partial_r\Phi\frac{\omega^r}{r^{1-\epsilon}}\Big|_{r=0}^{r=\infty} r\,dz.$$

The first one can be rewritten as follows:

$$\int_{-\infty}^{\infty} \partial_r \Omega \frac{\omega^\theta}{r^{1-\epsilon}}\Big|_{r=0}^{r=\infty} r dz = -\lim_{r\to 0}\int_{-\infty}^{\infty} \partial_r \omega^\theta \frac{\omega^\theta}{r^{1-\epsilon}} dz + \lim_{r\to 0}\int_{-\infty}^{\infty} \left(\frac{\omega^\theta}{r^{1-\frac{\epsilon}{2}}}\right)^2 dz,$$

the term $\int_{-\infty}^{\infty} \partial_r \omega^\theta \frac{\omega^\theta}{r^{1-\epsilon}} dz$ tends to zero as $r \to 0$ (see, for instance, Corollary 1 in [34]). Similarly, one has

$$\int_{-\infty}^{\infty} \partial_r \Phi \frac{\omega^r}{r^{1-\epsilon}}\Big|_{r=0}^{r=\infty} r dz = \lim_{r\to 0}\int_{-\infty}^{\infty} \left(\frac{\omega^r}{r^{1-\frac{\epsilon}{2}}}\right)^2 dz.$$

Finally, by taking $\epsilon \to 0$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Phi\|_{L^2}^2 + \|\Omega\|_{L^2}^2\right) + \|\nabla\Phi\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 \\
&\quad + \int_{-\infty}^{\infty} |\Omega(t,0,z)|^2 dz + \int_{-\infty}^{\infty} |\Phi(t,0,z)|^2 dz \\
&\le \left|\int_{\mathbb{R}^3} u^\theta \partial_r \left(\frac{u^r}{r}\right) \partial_z \Phi dx\right| + \left|\int_{\mathbb{R}^3} u^\theta \partial_z \left(\frac{u^r}{r}\right) \partial_r \Phi dx\right| \\
&\quad + 2\left|\int_{\mathbb{R}^3} \frac{u^\theta}{r} \Phi\Omega dx\right| + \left|\int_{\mathbb{R}^3} \partial_z \Pi^2 \Omega dx\right| \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.31}$$

The first term J_1 can be estimated as follows

$$\begin{aligned}
J_1 &\le C\left\|u^\theta\right\|_{L^\beta}\left\|\partial_r\left(\frac{u^r}{r}\right)\right\|_{L^{\beta_1}}\|\partial_z\Phi\|_{L^2} \quad \text{where } \left(\frac{1}{\beta}+\frac{1}{\beta_1}+\frac{1}{2}=1\right) \\
&\le C\left\|u^\theta\right\|_{L^\beta}\left\|\partial_r\left(\frac{u^r}{r}\right)\right\|_{L^2}^{\sigma}\left\|\nabla\partial_r\left(\frac{u^r}{r}\right)\right\|_{L^2}^{1-\sigma}\|\partial_z\Phi\|_{L^2} \\
&\quad \text{(By Gagliardo-Nirenberg inequality)} \\
&\le C\left\|u^\theta\right\|_{L^\beta}\|\Omega\|_{L^2}^{\sigma}\|\nabla\Omega\|_{L^2}^{1-\sigma}\|\partial_z\Phi\|_{L^2} \quad \text{(By Lemma 2.3)} \\
&\le C\left\|u^\theta\right\|_{L^\beta}^{\frac{2}{\sigma}}\|\Omega\|_{L^2}^2+\frac{1}{8}\|\nabla\Omega\|_{L^2}^2+\frac{1}{8}\|\partial_z\Phi\|_{L^2}^2. \quad \text{(By Young's inequality)}
\end{aligned}$$

We note that J_2 can be estimated in a similar way as J_1 that

$$J_2 \le C\left\|u^\theta\right\|_{L^\beta}^{\frac{2}{\sigma}}\|\Omega\|_{L^2}^2+\frac{1}{8}\|\nabla\Omega\|_{L^2}^2+\frac{1}{8}\|\partial_r\Phi\|_{L^2}^2.$$

One can easily deduce that

$$\sigma = 1 - \frac{3}{\beta} \text{ and } \beta > 3.$$

The following Sobolev-Hardy inequality comes from Lemma 2.4 of [2], for any $2 \leq p \leq 3$, there holds that

$$\left\| \frac{f}{r^{\frac{1}{2}}} \right\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{3}{p}-1} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{2-\frac{3}{p}}, \quad \text{for any } f \in C_0^\infty(\mathbb{R}^3). \tag{3.32}$$

Therefore, it follows from (3.32) and Hölder's inequality that

$$\begin{aligned} J_3 &\leq C \left\|u^\theta\right\|_{L^\beta} \left\| \frac{\Omega}{r^{\frac{1}{2}}} \right\|_{L^{\frac{2\beta}{\beta-1}}} \left\| \frac{\Phi}{r^{\frac{1}{2}}} \right\|_{L^{\frac{2\beta}{\beta-1}}} \\ &\leq C \left\|u^\theta\right\|_{L^\beta} \|\Omega\|_{L^2}^{\eta} \|\nabla\Omega\|_{L^2}^{1-\eta} \|\Phi\|_{L^2}^{\eta} \|\nabla\Phi\|_{L^2}^{1-\eta} \\ &\leq C \left\|u^\theta\right\|_{L^\beta}^{\frac{1}{\eta}} \left(\|\Omega\|_{L^2}^2 + \|\Phi\|_{L^2}^2\right) \\ &\quad + \frac{1}{8}\|\nabla\Omega\|_{L^2}^2 + \frac{1}{8}\|\nabla\Phi\|_{L^2}^2, \end{aligned}$$

where

$$\eta = \frac{\beta-3}{2\beta}, \quad 3 < \beta \leq \infty.$$

For the estimate of J_4, one has

$$\begin{aligned} J_4 = \left| \int_{\mathbb{R}^3} \partial_z \Pi^2 \Omega dx \right| &\leq \left| \int_{\mathbb{R}^3} \Pi^2 \partial_z \Omega dx \right| \\ &\leq C \|\Pi_0\|_{L^4}^4 + \frac{1}{8}\|\nabla\Omega\|_{L^2}^2 . \end{aligned}$$

Plugging the above estimates into (3.31), by taking $\alpha := \frac{2}{\sigma} = \frac{1}{\eta} = \frac{2\beta}{\beta-3}$, we conclude for any $t \in [0, T)$ that

$$\mathcal{A}(T) \lesssim \left(\|\Phi_0\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2 + CT\|\Pi_0\|_{L^4}^4\right) \exp\left(CT + C\left\|u^\theta\right\|_{L_T^\alpha L^\beta}^\alpha\right). \tag{3.33}$$

Under the condition of Theorem 1.1, it follows that

$$\mathcal{A}(T) = \|\Omega\|_{L_T^\infty L^2}^2 + \|\nabla\Omega\|_{L_T^2 L^2}^2 < \infty.$$

Thanks to Lemma 3.5, then the solution (u, b) of system (1.5) can be continued beyond T, which finishes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we show that $\mathcal{A}(T)$ is uniformly bounded under the smallness conditions of Theorem 1.2, then the global existence of system (1.9) follows in the same fashion as Theorem 1.1. Similar as in (3.31), one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2+\int_{-\infty}^{\infty}|\Omega(t,r=0,z)|^2dz\\
&=-\int_{\mathbb{R}^3}\partial_z\Pi^2\Omega dx+\int_{\mathbb{R}^3}\partial_z\left(\frac{u^\theta}{r}\right)^2\Omega dx\le\|\Pi\|_{L^4}^2\|\partial_z\Omega\|_{L^2}+\left\|\frac{u^\theta}{r}\right\|_{L^4}^2\|\partial_z\Omega\|_{L^2}\\
&\le 4\|\Pi\|_{L^4}^4+4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4+\frac{1}{2}\|\nabla\Omega\|_{L^2}^2,
\end{aligned}$$

it immediately implies

$$\frac{d}{dt}\|\Omega\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\le 8\|\Pi\|_{L^4}^4+8\left\|\frac{u^\theta}{r}\right\|_{L^4}^4. \tag{4.1}$$

In the following, we need to estimate $\left\|\frac{u^\theta}{r}\right\|_{L_T^4L^4}^4$. First, we note the equation of Λ,

$$\partial_t\Lambda+u\cdot\nabla\Lambda=\left(\Delta+\frac{\partial_r}{r}-\frac{3}{4r^2}\right)\Lambda-\frac{3u^r}{2r}\Lambda, \tag{4.2}$$

multiplying the Λ equation of (4.2) by Λ^3, and integrating the resulting equation over $\mathbb{R}^3$, yields

$$\frac{1}{4}\frac{d}{dt}\|\Lambda\|_{L^4}^4+\frac{3}{4}\|\nabla\Lambda^2\|_{L^2}^2+\frac{3}{4}\left\|\frac{u^\theta}{r}\right\|_{L^4}^4=\frac{3}{2}\int_{\mathbb{R}^3}\frac{u^r}{r}|\Lambda|^4dx\le\frac{3}{2}\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4.$$

It follows that

$$4\frac{d}{dt}\|\Lambda\|_{L^4}^4+12\|\nabla\Lambda^2\|_{L^2}^2+12\left\|\frac{u^\theta}{r}\right\|_{L^4}^4\le 24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4. \tag{4.3}$$

Combining (4.1) and (4.3) leads to

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+4\|\Lambda\|_{L^4}^4\right)+2\|\nabla\Omega\|_{L^2}^2+12\|\nabla\Lambda^2\|_{L^2}^2\\
&+4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4\le 24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4+8\|\Pi\|_{L^4}^4.
\end{aligned} \tag{4.4}$$

In the following, we estimate the right-hand side term $\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4$, then we can see that with the smallness condition (1.10) in hand, $\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4$ can be absorbed by the left-hand side of (4.4). By virtue of Lemma 3.1

$$\left\| \frac{u^r}{r} \right\|_{L^\infty} \leq C \|\Omega\|_{L^2}^{\frac{1}{2}} \|\nabla\Omega\|_{L^2}^{\frac{1}{2}}, \tag{4.5}$$

and by using the Hölder's inequality, it is obvious to see

$$\begin{aligned} \|\Lambda\|_{L^4}^4 &= \int_{\mathbb{R}^3} \frac{(u^\theta)^4}{r^2} dx = \int_{\mathbb{R}^3} \left(\frac{u^\theta}{r}\right)^3 (ru^\theta) dx \\ &\leq \left\| \frac{u^\theta}{r} \right\|_{L^4}^3 \|\Gamma\|_{L^4} \leq \left\| \frac{u^\theta}{r} \right\|_{L^4}^3 \|\Gamma\|_{L^2}^{\frac{1}{2}} \|\Gamma\|_{L^\infty}^{\frac{1}{2}} \\ &\leq \left\| \frac{u^\theta}{r} \right\|_{L^4}^3 \|\Gamma_0\|_{L^2}^{\frac{1}{2}} \|\Gamma_0\|_{L^\infty}^{\frac{1}{2}}. \end{aligned} \tag{4.6}$$

Inserting (4.5) and (4.6) into (4.4), one obtains that

$$\begin{aligned} &\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + 4\|\Lambda\|_{L^4}^4\right) + \|\nabla\Omega\|_{L^2}^2 + 12\|\nabla\Lambda^2\|_{L^2}^2 + 4\left\| \frac{u^\theta}{r} \right\|_{L^4}^4 \\ &\leq C\|\Omega\|_{L^2}^{\frac{1}{2}} \|\partial_z\Omega\|_{L^2}^{\frac{1}{2}} \left\| \frac{u^\theta}{r} \right\|_{L^4}^3 \|\Gamma_0\|_{L^2}^{\frac{1}{2}} \|\Gamma_0\|_{L^\infty}^{\frac{1}{2}} + 8\|\Pi\|_{L^4}^4 \\ &\leq C\|\Omega\|_{L^2}^{\frac{1}{2}} \|\Gamma_0\|_{L^2}^{\frac{1}{2}} \|\Gamma_0\|_{L^\infty}^{\frac{1}{2}} \left(\|\nabla\Omega\|_{L^2}^2 + \left\| \frac{u^\theta}{r} \right\|_{L^4}^4 \right) + 8\|\Pi\|_{L^4}^4. \end{aligned} \tag{4.7}$$

In the following, we define a finite time T_0 as

$$\sup\left\{ t > 0 \middle| \|\Omega(t,\cdot)\|_{L^2}^2 + \|\nabla\Omega\|_{L_t^2 L^2}^2 + 4\,\|\Lambda(t,\cdot)\|_{L^4}^4 \leq 2\varsigma_0 \right\} := T_0 < \infty, \tag{4.8}$$

where ς_0 given by

$$\varsigma_0 := \|\Omega_0\|_{L^2}^2 + 4\,\|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2 \|\Pi_0\|_{L^2}^2.$$

Note that Π satisfies an advection-diffusion equation:

$$\frac{d}{dt}\Pi + u\cdot\nabla\Pi - \left(\Delta + 2\frac{\partial_r}{r}\right)\Pi = 0,$$

it is easy to get for any $t > 0$

$$\|\Pi(t,\cdot)\|_{L^2}^2 + \|\nabla\Pi\|_{L_t^2 L^2} \leq \|\Pi_0\|_{L^2}^2,$$

and for $2 \leq p \leq \infty$,

$$\|\Pi(t,\cdot)\|_{L^p} \leq \|\Pi_0\|_{L^p}.$$

On the other hand, one has the following uniform estimate

$$\int_0^T \|\Pi\|_{L^4}^4 dt \le \int_0^T \left(\|\Pi\|_{L^3}^{\frac{1}{2}}\|\Pi\|_{L^6}^{\frac{1}{2}}\right)^4 dt \le \|\Pi_0\|_{L^3}^2 \int_0^T \|\Pi\|_{L^6}^2 dt$$
$$\le \|\Pi_0\|_{L^3}^2 \int_0^T \|\nabla\Pi\|_{L^2}^2 dt \le C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2. \tag{4.9}$$

Hence, integrating (4.7) in time variable over $[0, T_0]$ yields

$$\|\Omega\|_{L^\infty_{T_0}L^2}^2 + 4\|\Lambda\|_{L^\infty_{T_0}L^4}^4 + \|\nabla\Omega\|_{L^2_{T_0}L^2}^2 + 12\|\nabla\Lambda^2\|_{L^2_{T_0}L^2}^2 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4_{T_0}L^4}^4$$
$$\le \|\Omega_0\|_{L^2}^2 + 4\|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2$$
$$+ C\left(\|\Lambda_0\|_{L^4}^4 + \|\Omega_0\|_{L^2}^2 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2\right)^{\frac{1}{4}} \|\Gamma_0\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^\infty}^{\frac{1}{2}}$$
$$\left(\|\nabla\Omega\|_{L^2_{T_0}L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^4_{T_0}L^4}^4\right).$$

By smallness condition (1.10) and taking a small δ such that

$$C\left(\|\Lambda_0\|_{L^4}^4 + \|\Omega_0\|_{L^2}^2 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2\right)^{\frac{1}{4}} \|\Gamma_0\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^\infty}^{\frac{1}{2}} \le C\delta^{\frac{1}{2}} \le \frac{1}{2},$$

which yields

$$\|\Omega\|_{L^\infty_{T_0}L^2}^2 + 4\|\Lambda\|_{L^\infty_{T_0}L^4}^4 + \|\nabla\Omega\|_{L^2_{T_0}L^2}^2 \le \|\Omega_0\|_{L^2}^2 + 4\|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2.$$

This contradicts with the definition of T_0, we in fact have completed the first part of the proof of Theorem 1.2.

In the following, we deal with (4.4) as follows:

$$\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + 4\|\Lambda\|_{L^4}^4\right) + 2\|\nabla\Omega\|_{L^2}^2 + 12\|\nabla\Lambda^2\|_{L^2}^2 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4$$
$$\le 24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4(r\le\epsilon)}^4 + 24\int_{r\ge\epsilon}\left|\frac{u^r}{r}\Lambda^4\right| dx + 8\|\Pi\|_{L^4}^4$$
$$\le C\|\Omega\|_{L^2}^{\frac{1}{2}}\|\partial_z\Omega\|_{L^2}^{\frac{1}{2}}\left\|\frac{u^\theta}{r}\right\|_{L^4}^3\|\Gamma\|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac{1}{2}} + \frac{1}{\epsilon^4}\left\|\frac{u^r}{r}\right\|_{L^2}\left\|\frac{u^\theta}{r}\right\|_{L^2}$$
$$\times \|\Gamma\|_{L^\infty(r\ge\epsilon)}^3 + 8\|\Pi\|_{L^4}^4$$

$$
\begin{aligned}
&\le C\|\Omega\|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac{1}{2}}\left(\|\partial_z\Omega\|_{L^2}^2+\left\|\frac{u^\theta}{r}\right\|_{L^4}^4\right)\\
&\quad+\frac{1}{\epsilon^4}\left\|\frac{u^r}{r}\right\|_{L^2}\left\|\frac{u^\theta}{r}\right\|_{L^2}\|\Gamma\|_{L^\infty(r\ge\epsilon)}^3+8\|\Pi\|_{L^4}^4.
\end{aligned}
\tag{4.10}
$$

We define

$$
\sup\left\{t>0\Big|\|\Omega(t,\cdot)\|_{L^2}^2+\|\nabla\Omega\|_{L_t^2L^2}^2+4\,\|\Lambda(t,\cdot)\|_{L^4}^4\le 2\Psi_0^2\right\}:=T_1<\infty. \tag{4.11}
$$

Integrating (4.10) in time variable over $[0, T_1)$ yields

$$
\begin{aligned}
&\|\Omega\|_{L_{T_1}^\infty L^2}^2+4\|\Lambda\|_{L_{T_1}^\infty L^4}^4+\|\nabla\Omega\|_{L_{T_1}^2L^2}^2+12\|\nabla\Lambda^2\|_{L_{T_1}^2L^2}^2+4\left\|\frac{u^\theta}{r}\right\|_{L_{T_1}^4L^4}^4\\
&\le C\|\Omega\|_{L_{T_1}^\infty L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^2}^{\frac{1}{2}}\sup_{t\in(0,T_1)}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac{1}{2}}\left(\|\partial_z\Omega\|_{L_{T_1}^2L^2}^2+\left\|\frac{u^\theta}{r}\right\|_{L_{T_1}^4L^4}^4\right)\\
&+\frac{1}{\epsilon^4}\left(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2\right)\|\Gamma_0\|_{L^\infty}^3+C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2+\|\Omega_0\|_{L^2}^2+4\,\|\Lambda_0\|_{L^4}^4\,.
\end{aligned}
$$

By condition (1.11) and (4.11), we obtain

$$
\|\Omega\|_{L_{T_1}^\infty L^2}^2+4\|\Lambda\|_{L_{T_1}^\infty L^4}^4\le\Psi_0^2.
$$

This implies the second part of Theorem 1.2.

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