The Journal of Geometric Analysis (2022) 32:250
https://doi.org/10.1007/s12220-022-00988-y

Rotational Symmetry of Solutions of Mean Curvature Flow Coming Out of a Double Cone

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Received: 3 June 2021 / Accepted: 28 June 2022 / Published online: 27 July 2022

Abstract
We show that any smooth solution to the mean curvature flow equations coming out of a rotationally symmetric double cone is also rotationally symmetric.

Keywords Mean curvature flow · Self-expanders · Moving plane method

Mathematics Subject Classification 53E10

1 Introduction

We say a family of properly embedded smooth hypersurface $\{\Sigma_t\}_{t\in I} \subset \mathbb{R}^{n+1}$ is a solution of the mean curvature flow (MCF) equations if

$$\left(\frac{\partial x}{\partial t}\right)^{\perp} = H_{\Sigma_t}(x). \tag{1}$$

Here $H_{\Sigma_t}(x)$ denotes the mean curvature vector of Σ_t at x, and $x^{\perp}$ is the normal component of x.

In this article, we are interested in solutions of MCF coming out of a rotationally symmetric double cone, by which we mean a (hyper)cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ whose link $\mathcal{L}(\mathcal{C}) = \mathcal{C} \cap \mathbb{S}^n$ is a smooth hypersurface of $\mathbb{S}^n$ and has two connected components lying in two separate hemispheres. More explicitly, we consider a cone of the form (up to an ambient rotation so that the axis of symmetry is x_1-axis)

$$x_1^2 = \begin{cases} m_1(x_2^2 + x_3^2 + \cdots + x_{n+1}^2) & x_1 \geq 0 \\ m_2(x_2^2 + x_3^2 + \cdots + x_{n+1}^2) & x_1 < 0 \end{cases} \tag{2}$$

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where $m_1, m_2 > 0$ are constants related to the aperture of the cone. Solutions coming out of cones arise naturally in the singularity analysis of MCF. In particular, the self-expanders, which are special solutions of the MCF satisfying $\Sigma_t = \sqrt{t}\Sigma$ for some hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, are often thought of as models of MCF flowing out of a conical singularity (see for example [1]). Self-expanders satisfy the elliptic equation:

$$H_\Sigma(x) = \frac{x^\perp}{2}, \tag{3}$$

which is the Euler-Lagrange equation of the functional $\int_\Sigma e^{|x|^2/4} d\mathcal{H}^n$. We can, therefore, talk about the Morse index of a given self-expander, and the Morse flow lines between two self-expanders (asymptotic to the same cone $\mathcal{C}$) are examples of non-self-similar solutions coming out of the cone.

We show that, given a smooth double cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ and a smooth solution to the MCF, $\{\Sigma_t\}_{t\in[0,T]}$, asymptotic to $\mathcal{C}$, then the flow inherits the rotational symmetry of $\mathcal{C}$ at all times. More precisely we prove

Theorem 1 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth, rotationally symmetric double cone. Suppose $\{\Sigma_t\}_{t\in[0,T)}$ is a smooth solution to the mean curvature flow asymptotic to $\mathcal{C}$, in the sense that*

$$\lim_{t\to 0^+} \mathcal{H}^n \llcorner \Sigma_t = \mathcal{H}^n \llcorner \mathcal{C} \tag{4}$$

as Radon measures, then Σ_t is also rotationally symmetric (with the same axis of symmetry) for any $t \in [0, T)$.

Remark 1 It is likely that only a finite number of such solutions exist. These include self-expanders and Morse flow lines between two self-expanders asymptotic to the same cone, some of which can be constructed using methods from [5]. In particular, the latter solutions might develop singularities. Indeed, when the parameters m_1 and m_2 in (2) are sufficiently small, by [15], we can find an unstable (connected) catenoidal self-expander and a disconnected self-expander whose two components are given by the unique self-expanders asymptotic to the top part and bottom part of the cone. One expects that there exists a Morse flow line connecting these two self-expanders. Such a flow line will necessarily develop a neck pinch in order to become disconnected.

As an easy corollary we obtain the following rotational symmetry result:

Corollary 1 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth, rotationally symmetric double cone, then any smooth self-expander Σ asymptotic to $\mathcal{C}$ is also rotationally symmetric (with the same axis of symmetry).*

Remark 2 Singular self-expanders asymptotic to $\mathcal{C}$ do exist, but our theorem only applies in the smooth case. The smoothness assumption is in place to avoid further technicality introduced by the moving plane method, see Sect. 2.4.

The rotational symmetry is known in many other cases. Fong and McGrath [13] showed that same conclusion holds if the cone is rotationally symmetric and the expander is mean convex. Bernstein-Wang (Lemma 8.3 in [4]) later showed that same conclusion holds if the cone is rotationally symmetric and the expander is weakly stable (in particular, mean convexity implies weak stability so this generalizes the Fong-cGrath result). In contrast, our result applies to all solutions coming out of the cone and does not assume any extra condition about the flow other than smoothness. For other geometric flows, Chodosh [8] proved rotational symmetry of expanding asymptotically conical Ricci solitons with positive sectional curvature.

It is also worth mentioning that, although in general given a rotationally symmetric smooth cone $\mathcal{C}$, there could be multiple self-expanders asymptotic to $\mathcal{C}$, if there exists a unique self-expander asymptotic to $\mathcal{C}$, it must inherit the rotational symmetry. Uniqueness holds, for example, when the link of $\mathcal{C}$, $\mathcal{L}(C)$, is connected, or, in the double cone case, when the parameters m_1, m_2 in (2) are sufficiently large [4]. It is interesting to determine whether the rotational symmetry holds when the link $\mathcal{L}(C)$ has 3 or more connected components. We suspect that counterexamples exist. We refer to [3, 4, 6, 7] and [11] for more information on self-expanders.

The proof of Theorem 1 relies on the moving plane method pioneered by Alexandrov to prove that embedded compact constant mean curvature hypersurfaces are round spheres. The method was further employed to minimal surfaces by Schoen [20] to prove certain uniqueness theorems for catenoids. More recently, Martín-Savas-Halilaj-Smoczyk [19] showed uniqueness of translators (that is, solutions of the MCF equation that evolve by translating along one fixed direction) with one asymptotically paraboloidal end. Choi–Haslhofer–Hershkovits [9] and Choi–Haslhofer–Hershkovits–White [10] used a parabolic variant of the method to deduce rotational symmetry of certain ancient solutions to the MCF equation (that is, solutions of the MCF equation which exist on $(-\infty, 0)$). These methods were further generalized to non-smooth settings very recently by Haslhofer–Hershkovitz–White [14] and by Bernstein–Maggi [2].

Although a self-expander Σ satisfies an elliptic PDE, the hypersurface obtained after reflecting a self-expander with respect to a hyperplane does not satisfy the above equation anymore (it is rather a translated self-expander). For this reason, we could not directly apply the usual elliptic maximum principle and Hopf lemma, and we need to work in spacetime $\mathbb{R}^{n+1} \times [0, T]$ and use the MCF equations directly with a parabolic version of the maximum principles, which will lead to the more general Theorem 1. Consequently, our method is in spirit closer to that used by [9].

2 Preliminaries

2.1 Notations

Throughout the paper, $B_r(x)$ will denote the Euclidean ball of radius r centered at a point $x \in \mathbb{R}^{n+1}$. By a (smooth) MCF in $\mathbb{R}^{n+1}$, we mean a family of embedded hypersurfaces $\{\Sigma_t\}_{t\in I}$ for some interval I such that

$$\left(\frac{\partial x}{\partial t}\right)^{\perp} = H_{\Sigma_t}(x) \tag{5}$$

for all $x \in \Sigma_t, t \in I$. Given an open set $U \subset \mathbb{R}^{n+1}$, we say $\{\Sigma_t\}_{t\in I}$ is a MCF in U if the above equation is satisfied locally (given a local parametrization of the hypersurface) at every $x \in \Sigma_t \cap U$ and $t \in I$.

2.2 Pseudolocality for MCF

We will be frequently using the following pseudolocality result of Ilmanen–Neves–Schulze (see also [12]):

Theorem 2 (Theorem 1.5 of [16]) *Let $\{\Sigma_t\}_{t\in(0,T]}$ be a mean curvature flow in $\mathbb{R}^{n+1}$. Given any $\eta > 0$, there is $\delta, \varepsilon > 0$ such that if $x \in \Sigma_0$ and $\Sigma_0 \cap C_1(x)$ is a graph over $B_1(x) \cap \Sigma_0$ with Lipschitz constant bounded by ε, then $\Sigma_t \cap C_\delta(x)$ can be written as a graph over $B_\delta(x) \cap T_x\Sigma_0$ with Lipschitz bounded by η for any $t \in [0, \delta^2) \cap [0, T)$.*

Here, for $x = (x, x_{n+1}) \in \mathbb{R}^{n+1}$,

$$C_r(x) = \{(y, y_{n+1}) \in \mathbb{R}^{n+1} \mid |x - y| < r, |y_{n+1} - x_{n+1}| < r\} \tag{6}$$

is the closed cylinder centered at x. Roughly speaking, this theorem says that if the initial data of our MCF are graphical in some cylinder centered at x, then at least for a short time, the evolution of the hypersurface stays graphical in a possibly smaller cylinder. We will primarily use this theorem to show that our flow is graphical outside of a large ball for a short time, although strictly speaking we sometimes need to apply the above theorem in the context of integral Brakke flow.

2.3 Parabolic Maximum Principles

In this section, $Z_r(x, t)$ will denote the spacetime cylinder of radius r centered at $(x, t) \in \mathbb{R}^n \times \mathbb{R}$; that is,

$$Z_r(x, t) = \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid |y - x| < r, s < t < s + r^2\}. \tag{7}$$

To carry out the moving plane method, the most important ingredients are the maximum principle and Hopf lemma. In our case, we need a version of those theorems applicable to graphical solutions of MCF; that is, functions $u : Z_r(0, 0) \to \mathbb{R}$ satisfying the following parametrized PDE:

$$u_t = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \tag{8}$$

Observe that the difference of two graphical solutions to MCF satisfies a second-order linear parabolic PDE (provided the gradients are bounded a priori, which will be the case since our solutions are asymptotically conical), so by standard theory of linear parabolic PDEs [18], we have (cf. Sect. 6.2 in [9]):

Lemma 1 (Maximum Principle)
Suppose u, v are graphical solutions to the MCF in a parabolic cylinder $Z_r(0,0)$ with $u(0,0) = v(0,0)$. If $u \leq v$ in $Z_r(0,0)$, then $u = v$ in $Z_r(0,0)$.

Lemma 2 (Hopf Lemma) *Suppose u, v are graphical solutions to the MCF in a half parabolic cylinder $Z_r(0,0) \cap \{x_1 \geq 0\}$ with $u(0,0) = v(0,0)$ and $\frac{\partial u}{\partial x_1}(0,0) = \frac{\partial v}{\partial x_1}(0,0)$. If $u \leq v$ in $Z_r(0,0) \cap \{x_1 \geq 0\}$, then $u = v$ in $Z_r(0,0) \cap \{x_1 \geq 0\}$.*

2.4 Asymptotically Conical Mean Curvature Flow

Here, we will briefly discuss the class of MCFs we consider in Theorem 1. Given a smooth cone $\mathcal{C} \subset \mathbb{R}^n$ and a hypersurface Σ, we say Σ is $C^{k,\alpha}$-asymptotic to $\mathcal{C}$ if

$$\lim_{\rho \to 0^+} \rho\Sigma = \mathcal{C} \text{ in } C^{k,\alpha}_{loc}(\mathbb{R}^{n+1} \setminus \{0\}).$$

We need our MCF to be at least $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ in order to apply the maximum principle and Hopf Lemma stated above. This will almost not be an issue if we assume our cone is at least C^3, as the following proposition shows.

Proposition 1 (cf. Proposition 3.3 in [6]) *Let $\mathcal{C}$ be a C^3 cone and suppose $\{\Sigma_t\}_{t\in(0,T]}$ is a MCF such that*

$$\lim_{t \to 0^+} \mathcal{H}^n \llcorner \Sigma_t = \mathcal{H}^n \llcorner \mathcal{C}, \tag{9}$$

then we have for $\alpha \in [0,1)$ and $t \in (0,T)$,

$$\lim_{\rho \to 0^+} \rho\Sigma_t = \mathcal{C} \text{ in } C^{2,\alpha}_{loc}(\mathbb{R}^{n+1} \setminus \{0\}). \tag{10}$$

Proof In this proof, $[f]_{\alpha;A}$ will denote the α-Hölder seminorm of f on the set A, where A can be a subset the space $\mathbb{R}^{n+1}$ or time $(0,T)$.

It is enough to prove that locally Σ_t is a $C^{2,\alpha}$ normal graph over $\mathcal{C}$ outside of a large ball. By Theorem 2 (strictly speaking we need to consider the flow together with the initial condition $\Sigma_0 = \mathcal{C}$ as an integral Brakke flow and apply the theorem for Brakke flows), there is $\delta > 0$ such that $\Sigma_t \cap C_\delta(x_0)$ can be written as a normal graph over $C^n_\delta(x)$. This induces a map $u_{x_0} : [0,\delta^2) \times C^n_\delta(x_0) \to \mathbb{R}$ whose graph describes part of the flow in the spacetime of the flow:

$$\mathcal{M} = \mathcal{C} \times \{0\} \cup \bigcup_{t\in(0,T)} \Sigma_t \times \{t\}. \tag{11}$$

It follows from interior estimates of [12] that for sufficiently small δ, u_{x_0} satisfies the estimate

$$\delta^{-1} \sup_{C^n_\delta(x_0)} \left|u_{x_0}(0,\cdot)\right| + \sup_{C^n_\delta(x_0)} \left|\nabla u_{x_0}(0,\cdot)\right| \leq 1 \tag{12}$$

Since $\{\Sigma_t\}$ is a MCF, u_{x_0} satisfies the parametrized equation

$$\frac{\partial u_{x_0}}{\partial t} = \sqrt{1+\left|\nabla_x u_{x_0}\right|^2}\operatorname{div}\left(\frac{\nabla_x u_{x_0}}{\sqrt{1+\left|\nabla_x u_{x_0}\right|^2}}\right) \tag{13}$$

with initial conditions $u_{x_0}(0, x_0) = \left|\nabla_x u_{x_0}(0, x_0)\right| = 0$. This is a quasilinear parabolic PDE in divergence form, so Hölder estimates (see e.g., Chapter 6 of [17]) imply that, given $\alpha \in (0, 1)$, there exists some constant C such that

$$\sup_{[0,\delta^2]} [\nabla_x u_{x_0}(s, \cdot)]_{\alpha; C^n_{\delta/2}(x_0)} + \sup_{C^n_{\delta/2}(x_0)} [\nabla_x u_{x_0}(\cdot, x)]_{\alpha/2;[0,\delta^2)} \leq C\delta^{-\alpha} \tag{14}$$

Schauder estimates (see e.g., Chapter 4 of [17] or [18]) then give higher-order estimates of the form:

$$\sum_{i=0}^{2}(\delta/4)^{i-1} \sup_{C^n_{\delta/4}(x_0))} \left|\nabla_x^i u_{x_0}(s, \cdot)\right| + (\delta/4)^{1+\alpha}[\nabla_x^2 u_{x_0}(s, \cdot)]_{\alpha; C^n_{\delta/4}(x_0)} \leq C \tag{15}$$

for $s \in [0, \delta^2)$, and

$$\sup_{C^n_{\delta/4}(x_0)} [\nabla_x u_{x_0}(\cdot, x)]_{\frac{1}{2};[0,\delta^2)} \leq C(\delta/4)^{-1}. \tag{16}$$

We may now estimate for $s \in [0, \delta^2)$ that

$$\begin{aligned} \left|u_{x_0}(s, x) - u_{x_0}(0, x_0)\right| &\leq s \sup_{[0,\delta^2)} \left|\partial_\tau u_{x_0}(\tau, \cdot)\right| + |x - x_0|^2 \sup_{C^n_{\delta/4}(x_0)} \left|\nabla^2 u_{x_0}(0, \cdot)\right| \\ &\leq C(\delta/4)^{-1}(|x - x_0|^2 + s) \end{aligned} \tag{17}$$

where we used triangle inequality and the fact that $\left|\nabla_x u_{x_0}(0, x_0)\right| = 0$ in the first inequality, and the MCF equation (13), and the Schauder estimate (15) in the second inequality. Consequently by triangle inequality again, given $0 < \rho < 1/4$,

$$(\rho\delta)^{-1} \sup_{C^n_{\rho\delta}(x_0)} \left|u_{x_0}(s, \cdot)\right| \leq C\rho. \tag{18}$$

Similarly we can estimate the first-order term as follows:

$$\begin{aligned} &\left|\nabla_x u_{x_0}(s, x) - \nabla_x u_{x_0}(0, x_0)\right| \\ &\leq \sqrt{s}[\nabla_x u_{x_0}(\cdot, x_0)]_{\frac{1}{2};[0,\delta^2)} + |x - x_0| \sup_{C^n_{\delta/4}(x_0)} \left|\nabla^2 u_{x_0}(0, \cdot)\right| \\ &\leq C(\delta/4)^{-1}(|x - x_0| + \sqrt{s}) \end{aligned} \tag{19}$$

where we used Schauder estimates (15) and (16) in the second inequality. Consequently, we also have

$$\sup_{C^n_{\rho\delta}(x_0)} \left|\nabla_x u_{x_0}(s,\cdot)\right| \le C\rho. \tag{20}$$

Combining the above two estimates and (15) with $\delta/4$ replaced by $\rho\delta$ implies that

$$\sum_{i=0}^{2}(\rho\delta)^{i-1} \sup_{C^n_{\rho\delta}(x_0))} \left|\nabla^i_x u_{x_0}(s,\cdot)\right| + (\rho\delta)^{1+\alpha}[\nabla^2_x u_{x_0}(s,\cdot)]_{\alpha;C^n_{\rho\delta}(x_0)} \le C(\rho+\rho^{1+\alpha}) \tag{21}$$

which can be made less than one by picking ρ sufficiently small depending on C and α. This proves that $C_{\rho\delta}(x_0)$ is a normal graph over a neighborhood of $C^n_{\rho\delta}(x_0)$ if x_0 is sufficiently far away.

Unfortunately pseudolocality only gives normal graphicality outside of a large compact set, and so we cannot conclude that the entire flow will be of class $C^{2,\alpha}$. For this reason, it is assumed that the MCF is smooth to begin with in Theorem 1. We note that it might be possible to remove this assumption using a moving plane method in non-smooth settings such as those presented in [2, 10] or [14]. Since we are assuming that the flow and the cone are both smooth, in the proof below, we will say Σ is asymptotically conical if Σ is C^∞-asymptotic to $\mathcal{C}$.

3 Rotational Symmetry

In this section, we prove Theorem 1. A typical picture of the moving plane method is illustrated in Fig. 1. As claimed before, a direct consequence of the pseudolocality Theorem 2 is the graphicality of the immortal solution outside of a large ball. For the next lemma, we denote $\Sigma^+ = \Sigma \cap \{x_{n+1} > 0\}$ and $\Sigma^- = \Sigma \cap \{x_{n+1} < 0\}$ for $\Sigma \subset \mathbb{R}^{n+1}$.

Lemma 3 *Let $\mathcal{C}$, $\{\Sigma_t\}_{t\in(0,T)}$ be as in Theorem 1. For each $t \in [0,T)$, there is $R = R(\mathcal{C},\Sigma,t)$ such that $(\Sigma_t)^+ \setminus B_R(0)$ is graphical over $\Pi_0 \setminus B_R(0)$, where $\Pi_0 = \{x_{n+1} = 0\}$; that is, the projection $\pi : (\Sigma_t)^+ \setminus B_R(0) \to \Pi_0$ is injective. The same holds for $(\Sigma_t)^-$.*

Proof Fix a time $t_0 \in [0,T)$. By Proposition 1, there exists $R = R(\Sigma,t_0) > 0$ such that $\Sigma^+_{t_0} \setminus B_R(0)$ is asymptotically conical to $\mathcal{C}$. Since $\Sigma_0 = \mathcal{C}$, we can treat the flow as an integral Brakke flow starting from $\mathcal{C}$. By the pseudolocality theorem for Brakke flows, i.e., Theorem 2, given $\eta > 0$, there exists t_1 such that for $0 < t < t_1$ and $x \in \mathcal{C}\setminus B_1(0)$, $\Sigma^+_t \cap C_{\sqrt{t_1}}(x)$ can be written as a normal graph over $B^n_{\sqrt{t_1}}(x)\cap T_x\mathcal{C}$ with Lipschitz constant bounded by η. By parabolic rescaling, we see that, for $0 < t < 2t_0$ and $x \in \mathcal{C} \setminus B_{\sqrt{2t_0t_1^{-1}}}(0)$, $\Sigma^+_t \cap C_{\sqrt{2t_0}}(x)$ can be written as a normal graph over $B^n_{\sqrt{2t_0}}(x) \cap T_x\mathcal{C}$ with Lipschitz constant bounded by η. In particular, putting $t = t_0$ gives the desired graphicality.

Thus, we have proven that for every given $t \in (0,T)$ and $\eta > 0$, there is $R = R(\Sigma,t,\eta)$ such that $\Sigma^+_t \setminus B_R(0)$ is a normal graph over $\mathcal{C}$ with Lipschitz constant

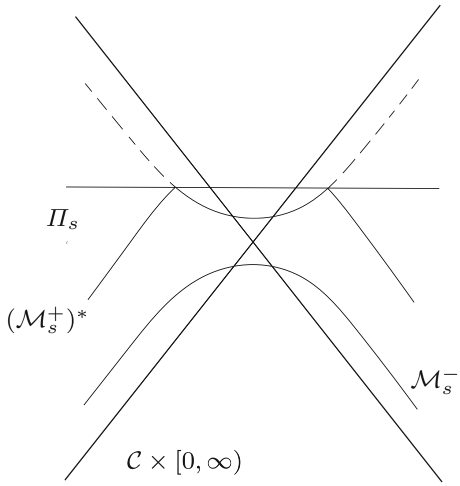


Fig. 1 A typical picture of the moving plane

bounded by η. Since $\mathcal{C} \cap \{x_{n+1} > 0\}$ is a Lipschitz graph over Π_0, the unit normal vector ν_{Π_0} is not contained in any tangent space to $x' \in (\mathcal{C} \cap \{x_{n+1} > 0\}) \setminus \{0\}$. Therefore, by taking η sufficiently small, we may make sure that ν_{Π_0} is also not contained in any tangent space to $x \in \Sigma_t^+ \setminus B_R(0)$ (here $R = R(t, \eta)$), but of course, η in turn depends on t). This proves that $\Sigma_t^+ \setminus B_R(0)$ is graphical over Π_0 as well.

For the rest of the section, let

$$\Pi_s = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+1} \times [0, \infty) \tag{22}$$

be the hyperplane at level s in spacetime. Given a set $A \subset \mathbb{R}^{n+1} \times [0, \infty)$ and $t, s \in [0, \infty)$, we let

$$A^t = \{(x, x_{n+1}, t') \in A \mid t' = t\} \tag{23}$$

be the time t slice of A,

$$A_s^+ = \{(x, x_{n+1}, t) \in A \mid x_{n+1} > s\} \tag{24}$$

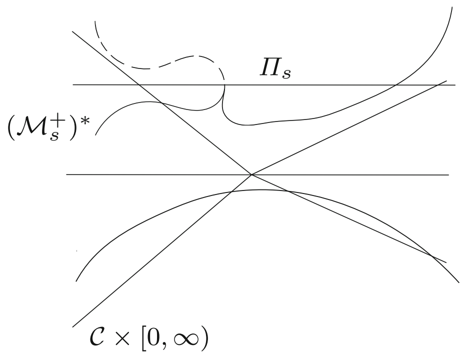
be the part of A lying above Π_s, A_s^- be the part of A lying below Π_s and finally

$$A_s^* = \{(x, x_{n+1}, t) \mid (x, 2s - x_{n+1}, t) \in A\} \tag{25}$$

be the reflection of A across Π_s, but we will often drop the subscript s when it is understood to avoid excessive subscripts. Given two sets $A, B \subset \mathbb{R}^{n+1} \times [0, \infty)$ we say $A > B$ if for any $(x, x_{n+1}, t) \in A$ we have $x_{n+1} > y_{n+1}$ for any $(x, y_{n+1}, t) \in B$ (if there is any such point).

Proof of Theorem 1 Without loss of generality assume that $\mathcal{C}$'s axis of symmetry is the x_1-axis. Evidently it suffices to show that the flow preserves the reflection symmetry

Fig. 2 Boundary touching



across any hyperplane containing the x_1-axis, which without loss of generality, we will take to be $\{x_{n+1} = 0\}$.

We will use the moving plane method on the spacetime $\mathbb{R}^{n+1} \times [0, \infty)$. The spacetime track

$$\mathcal{M} = \bigcup_{t \in [0,T)} \Sigma_t \times \{t\} \tag{26}$$

is a properly embedded hypersurface in $\mathbb{R}^{n+1} \times [0, T)$ asymptotic to $\mathcal{C} \times [0, T)$, in the sense that at each time slice t, Σ_t is $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ as we have demonstrated in Proposition 1. Let

$$S = \{s \in [0, \infty) \mid (\mathcal{M}_s^+)^* > \mathcal{M}_s^-, (\mathcal{M}_s^+)^t \text{ is graphical over } (\Pi_s)^t \text{ for } t \in [0, T]\}. \tag{27}$$

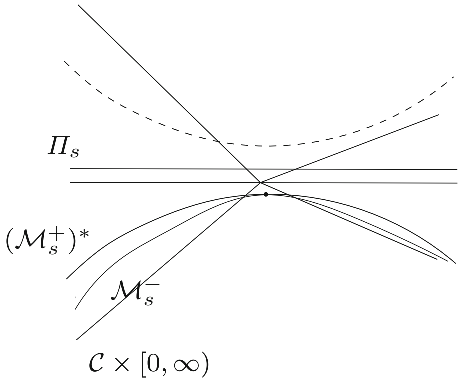
Here by graphical, we meant that $(\mathcal{M}_s^+)^t$ can be written as a normal graph over $(\Pi_s)^t$. Alternatively, since our solution is smooth, we can require that the vertical vector $e_{n+1} = (0, \ldots, 0, 1)$ is not contained in the tangent space of any point $p \in (\mathcal{M}_s^+)^t$. We first note that since the cone is symmetric across Π_0, we have $(\mathcal{C}_s^+)^* > \mathcal{C}_s^-$ for every $s > 0$. It is not hard to see that S is an open set. In fact, we just need to show that e_{n+1} is not in the tangent space at infinity for $(\mathcal{M}_s^+)^t$. By Proposition 1,

$$\lim_{\rho \to 0^+} \rho (\mathcal{M}_s^+)^t = \mathcal{C} \tag{28}$$

in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\})$, so eventually the tangent space at a point $p \in (\mathcal{M}_s^+)^t$ will lie close to the tangent space of $\mathcal{C}$. Since the cone is not vertical, e_{n+1} is clearly not contained in the tangent space of any point on $\mathcal{C}$, so by the convergence, there is $\varepsilon > 0$ such that e_{n+1} is not in the tangent space of any point $p \in (\mathcal{M}_{s-\varepsilon}^+)^t$ (Figs. 2, 3).

By Lemma 3, there is $R > 0$ such that $\Sigma_1 \setminus B_R(0) = \mathcal{M}^1 \setminus B_R(0)$ is graphical over $(\Pi_0)^1$. Moreover, this graphical scale scales parabolically, so for $s > T^2 R$ we have $(\mathcal{M}_s^+)^t$ is graphical over $(\Pi_s)^t$ for each $t \in [0, T]$. It is also evident that $(\mathcal{M}_s^+)^t$ is asymptotic to the translated cone $\mathcal{C} + 2s e_{n+1}$. This fact together with the decay estimate along asymptotically conical MCF (Lemma 5.3(1) of [5]) shows that when s is large enough the reflected part is disjoint from $(\mathcal{M}_s^-)_0^+$ (that is, the part of $\mathcal{M}$ that lies

Fig. 3 Interior touching



below level s and above 0). Together with graphicality, this implies $(\mathcal{M}_s^+)^* > \mathcal{M}_s^-$ for sufficiently large s, so S is not empty.

Finally we show S is closed. Suppose for a contradiction that $(s, \infty) \subset S$ (clearly $s \in S$ implies $[s, \infty) \subset S$) but $s \notin S$. At level s, either the graphicality condition or the set comparison condition $(\mathcal{M}_s)^* > \mathcal{M}_s^-$ is violated. In the first case, by parabolic rescaling, we may assume for simplicity that the nongraphicality happens first at time $t = 1$. This means that there is $p \in (\mathcal{M}_s^+)^1$ such that $e_{n+1} \in T_p(M_s^+)^1$. Thus, tangent planes of $(\mathcal{M}_s^+)^*$ and $\mathcal{M}_s^-$ at the point $(p, 1)$ must coincide. If we choose r small enough, we can ensure that $(\mathcal{M}_s^+)^*$ and $\mathcal{M}_s^-$ are graphical over $Z_r(p, 1) \cap \{x_{n+1} \leq s\}$. Since the tangent planes coincide, we can apply Hopf Lemma Lemma 2 to $(\mathcal{M}_s^+)^*$ and $\mathcal{M}_s^-$ to conclude that these hypersurfaces agree on an open neighborhood of $(p, 1)$. Moreover, the set $(\mathcal{M}_s^+)^* \cap \mathcal{M}_s^-$ is closed by definition and open by the maximum principle, so at least a connected component of $(\mathcal{M}_s^+)^*$ must coincide with a component of $\mathcal{M}_s^-$. This implies that $(\mathcal{M}_s^+)^*$ is asymptotic to both the cones $\mathcal{C} \times [0, \infty)$ and $(\mathcal{C} + 2se_{n+1}) \times [0, \infty)$, a contradiction. In the second case, s is necessarily the first level such that $(\mathcal{M}_s)^+ \cap \mathcal{M}_s^- \neq \emptyset$, and the graphicality condition implies that $(\mathcal{M}_s^+)^*$ and $\mathcal{M}_s^-$ must touch at an interior point (p, t) of the flow. Again for r small enough, they are both graphical solutions of the MCF, so the maximum principle Lemma 1 implies that $(\mathcal{M}_s^+)^*$ and $\mathcal{M}_s^-$ agree on an open neighborhood of (p, t). Since we can do this for any point $(p, t) \in (\mathcal{M}_s^+)^* \cap \mathcal{M}_s^-$, at least a connected component of $(\mathcal{M}_s^+)^*$ coincides with a component of $\mathcal{M}_s^-$, a contradiction.

We have, thus, proved that S is open, non-empty, and closed, and thus, $S = (0, \infty)$ (note that since we have a strict inequality in our setup, so we cannot conclude directly that $0 \in S$). Note that we can run a similar argument starting from the bottom half, yielding $(\mathcal{M}_s^-)^* > \mathcal{M}_s^+$ for any $s < 0$. Hence, there must be a point of touching at $s = 0$. If the intersection is in the interior we can apply the maximum principle Lemma 1 to conclude that $(\mathcal{M}_0^+)^* = \mathcal{M}_0^-$, i.e., $\mathcal{M}$ is symmetric across the reflection with respect to Π_0. The same conclusion holds if the intersection is along the boundary by using the Hopf lemma Lemma 2 instead.

Remark 3 In the proof of Theorem 1, we actually proved the stronger result that reflection symmetry is preserved for self-expanders asymptotic to a double cone. This yields, for example, that when $m_1 = m_2$ in (2), any self-expander asymptotic to $\mathcal{C}$ is symmetric across the reflection with respect to the plane $\{x_1 = 0\}$. We do not expect this to hold for cones whose links have more than 2 components.

Acknowledgements The author would like to thank his advisor, Jacob Bernstein, for numerous helpful advice and constant encouragement, especially during a period of extreme difficulty around the globe. The author would also like to thank Rory Martin-Hagemeyer for useful discussions.

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