The Journal of Geometric Analysis (2022) 32:238
https://doi.org/10.1007/s12220-022-00976-2

Existence and Asymptotical Behavior of Multiple Solutions for the Critical Choquard Equation

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Received: 4 April 2022 / Accepted: 14 June 2022 / Published online: 15 July 2022

Abstract
We deal with the following critical Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = P(x)|u|^{p-2}u + \varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^3}\frac{Q(y)|u(y)|^{6-\mu}}{|x-y|^{\mu}}\mathrm{d}y\Big)Q(x)|u|^{4-\mu}u, \quad x\in\mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter, $0 < \mu < 3$, $p \in (4, 6)$. Under some conditions on the potential functions $V(x)$, $P(x)$, and $Q(x)$, we obtain the existence of multiple solutions and their asymptotical behavior as $\varepsilon \to 0$.

Keywords Choquard equation · Critical exponent · Variational methods

Mathematics Subject Classification 35A15 · 35B40 · 35J20

1 Introduction and Main Results

In this paper, we consider the following critical Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = P(x)|u|^{p-2}u + \varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^3}\frac{Q(y)|u(y)|^{6-\mu}}{|x-y|^{\mu}}\mathrm{d}y\Big)Q(x)|u|^{4-\mu}u, \quad x\in\mathbb{R}^3, \quad (1.1)$$

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where $\varepsilon > 0$ is a parameter, $0 < \mu < 3$, $p \in (4, 6)$. The potential functions $V(x)$, $P(x)$, and $Q(x)$ are three bounded and continuous functions in $\mathbb{R}^3$ satisfying $\inf\limits_{x\in\mathbb{R}^3} V(x) > 0$, $\inf\limits_{x\in\mathbb{R}^3} P(x) > 0$ and $\inf\limits_{x\in\mathbb{R}^3} Q(x) > 0$.

The Choquard equation

$$-\Delta u + u = \Big(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \mathrm{d}y\Big)u, \quad x \in \mathbb{R}^3,$$

was used by Pekar [17] to describe the quantum theory of polaron at rest. Then it was introduced by Choquard [10] as an approximation to Hartree–Fock theory of one-component plasma. Penrose [18] also derived it as a model of self-gravitating matter, in which quantum state reduction is understood as a gravitational phenomenon. Lieb [10] proved the existence and uniqueness (up to translations) of solutions by using symmetric decreasing rearrangement inequalities. Lions [11] obtained the existence of infinitely many spherically symmetric solutions. Ma and Zhao [14] showed the positive solutions of this equation must be radially symmetric and monotone decreasing about some fixed point by the method of moving planes. Moroz and Van Schaftingen [15] studied the generalized Choquard equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^3,$$

where I_α is a Riesz potential and $p > 1$. For an optimal range of parameters, they showed the regularity, positivity, and radial symmetry of the ground states and derived decay property at infinity as well.

Gao and Yang [6] studied the Brezis–Nirenberg type problem of the nonlinear Choquard equation

$$-\Delta u - \lambda u = \Big(\int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}y\Big)|u|^{2^*_\mu - 2}u, \quad x \in \mathbb{R}^3,$$

where Ω is a bounded domain and λ is a parameter, $N \geq 3$ and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent under the sense of Hardy–Littlewood–Sobolev inequality. They established some existence results for this equation. Shen, Gao, and Yang [19] investigated the critical Choquard equation with potential well

$$-\Delta u + (\lambda V(x) - \beta)u = (|x|^{-\mu} * |u|^{2^*_\mu})|u|^{2^*_\mu - 2}u, \quad x \in \mathbb{R}^N,$$

where $\lambda, \beta > 0$, $0 < \mu < N$, $N \geq 4$, 2^*_μ is the critical exponent. They proved the existence of ground state solutions which localize near the potential well $\inf V^{-1}(0)$ and also characterize the asymptotic behavior as $\lambda \to \infty$. Furthermore, the multiple solutions were also established by Lusternik–Schnirelmann category theory.

For the semiclassical problem, Liu and Tang [12] studied the following subcritical equation

$$-\varepsilon^2 \Delta w + V(x)w = \varepsilon^{-\theta} W(x)(I_\theta * (W|w|^p))|w|^{p-2}w, \quad w \in H^1(\mathbb{R}^N),$$

where $\varepsilon > 0$, $N > 2$, $\theta \in [2.\frac{N+\theta}{N-2})$. The potential functions $V(x)$, $W(x)$ are bounded positive functions. By using pseudo-index theory, they established the multiplicity of solutions. Alves et al. [1] studied the following critical equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^3}\frac{Q(y)(|u(y)|^{6-\mu}+F(u(y)))}{|x-y|^{\mu}}\mathrm{d}y\Big)$$
$$\Big(Q(x)(|u|^{4-\mu}u+\frac{1}{6-\mu}f(u))\Big),\quad x\in\mathbb{R}^3,$$

where $\varepsilon > 0$ is a parameter, $0 < \mu < 3$. The potential functions $V(x)$ and $Q(x)$ are two bounded and continuous functions in $\mathbb{R}^3$ satisfying $\inf_{x\in\mathbb{R}^3} V(x) > 0$ and $\inf_{x\in\mathbb{R}^3} Q(x) > 0$. When $Q(x) \equiv 1$ and $V(x)$ satisfies

$$\min_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x),$$

they proved the existence of ground state solution and multiple solutions. Moreover, the concentration phenomenon was also considered. Zhang and Zhang [29] considered the following critical Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^3}\frac{|u(y)|^{6-\mu}+Q(y)F(u(y))}{|x-y|^{\mu}}\mathrm{d}y\Big)$$
$$\Big(|u|^{4-\mu}u+\frac{1}{6-\mu}Q(x)f(u)\Big),\quad x\in\mathbb{R}^3,$$

where $\varepsilon > 0$ is a parameter, $0 < \mu < 3$. The potential functions $V(x)$ and $Q(x)$ are two bounded and continuous functions. Under the condition,

$$Q(x) \geq \lim_{|x|\to\infty} Q(x),\ x\in\mathbb{R}^3,$$

and

$$\mathcal{V}\cap\mathcal{Q} = \{x\in\mathbb{R}^3 : V(x) = V_{\min}, Q(x) = Q_{\max}\} \neq \varnothing,$$

they established a relationship between the category of the set $\mathcal{V}\cap\mathcal{Q}$ and the number of solutions by employing the Lusternik–Schnirelmann category theory.

On the other hand, the reduction methods are also used to study the Choquard equation. Wei and Winter [22] considered

$$-\varepsilon^2\Delta u + V(x)u = \frac{1}{8\pi\varepsilon^2}(\frac{1}{|x|}*u^2)u,\quad x\in\mathbb{R}^3,$$

where $\varepsilon > 0$, $V \in C^2(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$. They proved that for any given positive integer K, if $P_1, P_2, ..., P_K \in \mathbb{R}^3$ were given nondegenerate critical points of $V(x)$, then for ε sufficiently small, there existed a positive solution for the equation and this solution had exactly K local maximum points $Q_i^{\varepsilon}(i = 1, 2, ..., K)$ with $Q_i^{\varepsilon} \to P_i$

as $\varepsilon \to 0$. Luo, Peng and Wang [13] also investigated the above problem. For ε small enough, by using a local Pohozaev type of identity, blow-up analysis, and the maximum principle, they showed the uniqueness of positive solutions concentrating at the nondegenerate critical points of $V(x)$. For more results about Choquard equations, we refer to [5, 7–9, 16, 23, 26, 28, 31] and the references therein.

Motivated by the above works, we are concerned with the existence and concentration behavior of positive solutions for (1.1). We note that (1.1) involves three different potentials. This brings a competition between the potentials V, P, and Q: each one would like to attract ground states to their minimum or maximum points, respectively. It makes difficulties in determining the concentration position of solutions. This kind of problem can be traced back to [20, 21] for the semilinear Schrödinger equation. See also [24, 25, 27, 30] for other related results. We first recall the following famous Hardy–Littlewood–Sobolev inequality.

Proposition 1.1 (Hardy–Littlewood–Sobolev inequality). *Let $t, r > 1$ and $0 < \mu < 3$ with $\frac{1}{t} + \frac{\mu}{3} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^3)$ and $h \in L^r(\mathbb{R}^3)$. There exists a sharp constant $C(t, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le C(t,\mu,r)|f|_t|h|_r.$$

Remark 1.2 By Proposition 1.1, the term

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^r|u(y)|^r}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$

is well defined if $|u|^r \in L^s(\mathbb{R}^3)$ satisfies $\frac{2}{s} + \frac{\mu}{3} = 2$. Therefore, for $u \in H^1(\mathbb{R}^3)$, we will require $sr \in [2, 6]$. Then $\frac{6-\mu}{3} \le r \le 6-\mu$. Here, $\frac{6-\mu}{3}$ is called the lower critical exponent and $6-\mu$ is called the upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality.

Proposition 1.3 (Optimizers for $S_{H,L}$). *[6] Define*

$$S_{H,L} = \inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{6-\mu}|u(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{6-\mu}}}.$$

Then $S_{H,L}$ is achieved if and only if

$$u = C\left(\frac{b}{b^2+|x-a|^2}\right)^{\frac{1}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^3$ and $b > 0$ are parameters.

Remark 1.4 [6] In fact,

$$U(x) = \frac{3^{\frac{1}{4}}}{(1+|x|^2)^{\frac{1}{2}}}$$

is a minimizer for S, the best Sobolev constant, and is also the minimizer for $S_{H,L}$. Moreover,

$$S_{H,L} = \frac{S}{C(3,\mu)^{\frac{1}{6-\mu}}},$$

where $C(3,\mu)$ is the sharp constant in Proposition 1.1.

To state our main results, some hypotheses about the potential functions are needed as follows:

(H_1) $V_\infty > V_{\min}$ or $P_{\max} > P^\infty$,
(H_2) $Q(x) \le Q^\infty$ for $x \in \mathbb{R}^3$,
(H_3) $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} = \{x \in \mathbb{R}^3 : V(x) = V_{\min}, P(x) = P_{\max}, Q(x) = Q_{\max}\} \ne \emptyset$,

where

$$V_{\min} := \min_{x\in\mathbb{R}^3} V(x), \quad \mathcal{V} := \{x \in \mathbb{R}^3 : V(x) = V_{\min}\}, \quad V_\infty := \liminf_{|x|\to\infty} V(x),$$
$$P_{\max} := \max_{x\in\mathbb{R}^3} P(x), \quad \mathcal{P} := \{x \in \mathbb{R}^3 : P(x) = P_{\max}\}, \quad P^\infty := \limsup_{|x|\to\infty} P(x),$$
$$Q_{\max} := \max_{x\in\mathbb{R}^3} Q(x), \quad \mathcal{Q} := \{x \in \mathbb{R}^3 : Q(x) = Q_{\max}\}, \quad Q^\infty := \limsup_{|x|\to\infty} Q(x).$$

Obviously, under the assumptions (H_1), the set $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$ is bounded.

Our main results are as follows:

Theorem 1.5 *Suppose that the potentials $V(x)$, $P(x)$, $Q(x)$ satisfy conditions (H_1), (H_2) and (H_3). Then*

(i) For any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (1.1) has at least $cat_{(\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q})_\delta}(\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})$ solutions for $\varepsilon \in (0, \varepsilon_\delta)$, where $(\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta = \{x \in \mathbb{R}^3 : dist(x, \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}) \le \delta\}$.

(ii) For $\varepsilon_n \to 0$ as $n \to \infty$, up to a subsequence, there exists y_n such that $u_{\varepsilon_n}(x + y_n)$, where u_{ε_n} is a solution in (i), converges in $H^1(\mathbb{R}^3)$ to a ground state solution u of

$$-\Delta u + V_{\min} u = P_{\max}|u|^{p-2}u + \left(\int_{\mathbb{R}^3} \frac{Q_{\max}^2 |u(y)|^{6-\mu}}{|x-y|^\mu} \mathrm{d}y\right)|u|^{4-\mu}u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

The proof of our main results is based on the variational method. The main difficulties lie in two aspects: (i) The unboundedness of the domain $\mathbb{R}^3$ and the critical exponent under the sense of Hardy–Littlewood–Sobolev inequality lead to the lack of compactness. Some arguments developed by Brezis and Nirenberg [3] can be applied to prove that the functional associated with (1.1) satisfies the Palais-Smale (PS) condition under some energy level. (ii) When the critical term has a potential $Q(x)$, the proof of the existence of multiple solutions become more complicated. As far as we know, there are no results about this problem. By using Lusternik–Schnirelmann theory, we establish the relationship between the category of the set $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$ and the number of solutions.

This paper is organized as follows. In the forthcoming section, we collect some necessary preliminary lemmas which will be used later. In Sect. 3, we are devoted to the energy functional with constant coefficients. In Sect. 4, the PS condition is given. In Sect. 5, the Lusternik–Schnirelmann theory is applied to prove the existence of multiple solutions.

Notation. In this paper, we make use of the following notations.

- For any $R > 0$ and $x \in \mathbb{R}^3$, $B_R(x)$ denotes the open ball of radius R centered at x.
- The letter C stands for positive constants (possibly different from line to line).
- ”$\to$” denotes the strong convergence and "$\rightharpoonup$" denotes the weak convergence.
- $|u|_q = (\int_{\mathbb{R}^3} |u|^q \mathrm{d}x)^{\frac{1}{q}}$ denotes the norm of u in $L^q(\mathbb{R}^3)$ for $2 \le q \le 6$.

2 Preliminaries

The standard norm of $E := H^1(\mathbb{R}^3)$ is given by

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x\Big)^{1/2}.$$

Since $V(x)$ is bounded and $\inf_{x\in\mathbb{R}^3} V(x) > 0$, we have the following equivalent norm

$$\|u\|_\varepsilon = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\mathrm{d}x\Big)^{1/2}.$$

For $f \in L^1_{loc}(\mathbb{R}^3)$, define

$$I_\mu * f(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^\mu}\mathrm{d}y,$$

and this integral converges in the classical Lebesgue sense for a.e. $x \in \mathbb{R}^3$ if and only if $f \in L^1(\mathbb{R}^3, (1+|x|)^{-\mu}\mathrm{d}x)$.

Remark 2.1 By Hardy–Littlewood–Sobolev inequality, I_μ defines a linear continuous map from $L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$ to $L^{\frac{6}{\mu}}(\mathbb{R}^3)$.

Define $F : E \to \mathbb{R}$ by

$$F(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q(\varepsilon x)Q(\varepsilon y)|u(x)|^{6-\mu}|u(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.$$

To prove the properties about $F(\cdot)$, for simplicity, we assume that $Q(x) \equiv 1$ in the following three Lemmas.

Lemma 2.2 *Let $u_n \rightharpoonup u$ in E and $u_n \to u$, a.e. in $\mathbb{R}^3$. Then*

$$I_\mu * |u_n|^{6-\mu} \to I_\mu * |u|^{6-\mu}, \text{ a.e. in } \mathbb{R}^3, \quad \text{as } n \to \infty.$$

Proof By Hardy–Littlewood–Sobolev inequality, $I_\mu * |u_n|^{6-\mu} \in L^{\frac{6}{\mu}}(\mathbb{R}^3)$. Choose a function $v \in L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$ satisfying $v > 0$ in $\mathbb{R}^3$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left| I_\mu * |u_n|^{6-\mu} - I_\mu * |u|^{6-\mu} \right| v \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| |u_n(y)|^{6-\mu} - |u(y)|^{6-\mu} \right| v(x)}{|x-y|^\mu} \mathrm{d}x \mathrm{d}y \\
&= \int_{\mathbb{R}^3} I_\mu * v \left| |u_n(y)|^{6-\mu} - |u(y)|^{6-\mu} \right| \mathrm{d}y.
\end{aligned}$$

Since $I_\mu * v \in L^{\frac{6}{\mu}}(\mathbb{R}^3)$, and $\left| |u_n(y)|^{6-\mu} - |u(y)|^{6-\mu} \right| \rightharpoonup 0$ in $L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$, we can obtain

$$\int_{\mathbb{R}^3} \left| I_\mu * |u_n|^{6-\mu} - I_\mu * |u|^{6-\mu} \right| v \mathrm{d}x \to 0, \text{ as } n \to \infty.$$

It follows from $v > 0$ that the result holds. □

Lemma 2.3 *Let $u_n \rightharpoonup u$ in E and $u_n \to u$, a.e. in $\mathbb{R}^3$. Then*

$$\int_{\mathbb{R}^3} (I_\mu * |u_n - u|^{6-\mu} |u|^{5-\mu})^{\frac{6}{5}} \mathrm{d}x \to 0, \text{ a.e. in } \mathbb{R}^3, \quad \text{as } n \to \infty,$$

and

$$\int_{\mathbb{R}^3} (I_\mu * |u|^{6-\mu} |u_n - u|^{5-\mu})^{\frac{6}{5}} \mathrm{d}x \to 0, \text{ a.e. in } \mathbb{R}^3, \quad \text{as } n \to \infty.$$

Proof Let $v_n = u_n - u$, then

$$\int_{\mathbb{R}^3} (I_\mu * |u_n - u|^{6-\mu} |u|^{5-\mu})^{\frac{6}{5}} \mathrm{d}x = \int_{\mathbb{R}^3} (I_\mu * |v_n|^{6-\mu})^{\frac{6}{5}} |u|^{\frac{6(5-\mu)}{5}} \mathrm{d}x.$$

By Hardy–Littlewood–Sobolev inequality, $(I_\mu * |v_n|^{6-\mu})^{\frac{6}{5}} \in L^{\frac{5}{\mu}}(\mathbb{R}^3)$, and is bounded in $L^{\frac{5}{\mu}}(\mathbb{R}^3)$. From Lemma 2.2, $I_\mu * |v_n|^{6-\mu} \to 0$, a.e. in $\mathbb{R}^3$, as $n \to \infty$. Then, we have $(I_\mu * |v_n|^{6-\mu})^{\frac{6}{5}} \rightharpoonup 0$ in $L^{\frac{5}{\mu}}(\mathbb{R}^3)$. It follows from $|u|^{\frac{6(5-\mu)}{5}} \in L^{\frac{5}{5-\mu}}(\mathbb{R}^3)$ that the first result holds. Similarly, the second limit can be obtained. □

Lemma 2.4 *Let $u_n \rightharpoonup u$ in E and $u_n \to u$, a.e. in $\mathbb{R}^3$. Then*

(i) $F(u_n - u) = F(u_n) - F(u) + o_n(1)$;
(ii) $F'(u_n - u) = F'(u_n) - F'(u) + o_n(1)$, *in* $(H^1(\mathbb{R}^3))^{-1}$.

Proof The first part (i) has been proved in [6]. We just prove the second part (ii). In fact, for any $\phi \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
&\Big|\langle F'(u_n - u), \phi\rangle - \langle F'(u_n), \phi\rangle + \langle F'(u), \phi\rangle\Big| \\
&\le C\Big|\int_{\mathbb{R}^3} \big(I_\mu * |u_n - u|^{6-\mu}|u_n - u|^{4-\mu}(u_n - u) - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n \\
&\quad + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\big)\phi\Big| \\
&\le C\Big|I_\mu * |u_n - u|^{6-\mu}|u_n - u|^{4-\mu}(u_n - u) - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n \\
&\quad + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\Big|_{\frac{6}{5}}|\phi|_6 \\
&\le C\Big|I_\mu * |u_n - u|^{6-\mu}|u_n - u|^{4-\mu}(u_n - u) - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n \\
&\quad + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\Big|_{\frac{6}{5}}\|\phi\|.
\end{aligned}$$

Next, we prove that

$$\begin{aligned}
&\Big|I_\mu * |u_n - u|^{6-\mu}|u_n - u|^{4-\mu}(u_n - u) - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n \\
&\quad + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\Big|_{\frac{6}{5}} = o_n(1).
\end{aligned}$$

Let $v_n = u_n - u$. Then, for any small $\delta > 0$,

$$\begin{aligned}
&\Big|I_\mu * |u_n - u|^{6-\mu}|u_n - u|^{4-\mu}(u_n - u) - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n\Big| \\
&= \Big|I_\mu * |v_n|^{6-\mu}|v_n|^{4-\mu}v_n - I_\mu * |v_n + u|^{6-\mu}|v_n + u|^{4-\mu}(v_n + u)\Big| \\
&= \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(I_\mu * |v_n + tu|^{6-\mu}|v_n + tu|^{4-\mu}(v_n + tu)\Big)\mathrm{d}t\Big| \\
&\le C\Big[I_\mu * (|v_n|^{5-\mu}|u| + |u|^{6-\mu})(|v_n|^{5-\mu} + |u|^{5-\mu}) + I_\mu * (|v_n|^{6-\mu} \\
&\quad + |u|^{6-\mu})(|v_n|^{4-\mu}|u| + |u|^{5-\mu})\Big] \\
&\le C\Big[(\delta I_\mu * |v_n|^{6-\mu} + C(\delta)I_\mu * |u|^{6-\mu})(|v_n|^{5-\mu} + |u|^{5-\mu}) \\
&\quad + I_\mu * (|v_n|^{6-\mu} + |u|^{6-\mu})(\delta|v_n|^{5-\mu} + C(\delta)|u|^{5-\mu})\Big] \\
&\le C\Big[\delta(I_\mu * |v_n|^{6-\mu}|v_n|^{5-\mu} + I_\mu * |v_n|^{6-\mu}|u|^{5-\mu} + I_\mu * |u|^{6-\mu}|v_n|^{5-\mu}) \\
&\quad + C(\delta)(I_\mu * |u|^{6-\mu}|v_n|^{5-\mu} + I_\mu * |v_n|^{6-\mu}|u|^{5-\mu}) + C(\delta)I_\mu * |u|^{6-\mu}|u|^{5-\mu}\Big] \\
&= C\Big[\delta f_n + C(\delta)g_n + C(\delta)I_\mu * |u|^{6-\mu}|u|^{5-\mu}\Big],
\end{aligned}$$

where

$$f_n = I_\mu * |v_n|^{6-\mu}|v_n|^{5-\mu} + I_\mu * |v_n|^{6-\mu}|u|^{5-\mu} + I_\mu * |u|^{6-\mu}|v_n|^{5-\mu},$$

and

$$g_n = I_\mu * |u|^{6-\mu}|v_n|^{5-\mu} + I_\mu * |v_n|^{6-\mu}|u|^{5-\mu}.$$

Therefore,

$$\begin{aligned}&\left|I_\mu * |v_n|^{6-\mu}|v_n|^{4-\mu}v_n - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\right|^{\frac{6}{5}}\\ &\leq C\Big[\delta(f_n)^{\frac{6}{5}} + C(\delta)(g_n)^{\frac{6}{5}} + C(\delta)(I_\mu * |u|^{6-\mu}|u|^{5-\mu})^{\frac{6}{5}}\Big].\end{aligned}$$

Define

$$\begin{aligned}G_{\delta,n}(x) = \max\Big\{&\left|I_\mu * |v_n|^{6-\mu}|v_n|^{4-\mu}v_n - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\right|^{\frac{6}{5}}\\ &-C\delta(f_n)^{\frac{6}{5}} - CC(\delta)(g_n)^{\frac{6}{5}}, 0\Big\}.\end{aligned}$$

It is easy to see that

$$0 \leq G_{\delta,n}(x) \leq C(\delta)(I_\mu * |u|^{6-\mu}|u|^{5-\mu})^{\frac{6}{5}} \in L^1(\mathbb{R}^3).$$

By Lemma 2.2, we can obtain

$$\begin{aligned}&f_n \to 0, \text{ a.e. in } \mathbb{R}^3, \text{ as } n \to \infty,\\ &g_n \to 0, \text{ a.e. in } \mathbb{R}^3, \text{ as } n \to \infty,\end{aligned}$$

and

$$\begin{aligned}&\left|I_\mu * |v_n|^{6-\mu}|v_n|^{4-\mu}v_n - I_\mu * |u_n|^{6-\mu}|u_n|^{4-\mu}u_n + I_\mu * |u|^{6-\mu}|u|^{4-\mu}u\right| \to 0,\\ &\text{a.e. in } \mathbb{R}^3, \text{ as } n \to \infty.\end{aligned}$$

Thus,

$$G_{\delta,n}(x) \to 0, \text{ a.e. in } \mathbb{R}^3, \text{ as } n \to \infty.$$

Then, we have

$$\int_{\mathbb{R}^3} G_{\delta,n}(x) \to 0, \text{ as } n \to \infty. \tag{2.1}$$

By the definition of $G_{\delta,n(x)}$ and the boundedness of f_n in $L^{\frac{6}{5}}(\mathbb{R}^3)$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big|I_\mu*|v_n|^{6-\mu}|v_n|^{4-\mu}v_n-I_\mu*|u_n|^{6-\mu}|u_n|^{4-\mu}u_n+I_\mu*|u|^{6-\mu}|u|^{4-\mu}u\Big|^{\frac{6}{5}}\mathrm{d}x\\ &\le C\Big[\delta\int_{\mathbb{R}^3}(f_n)^{\frac{6}{5}}\mathrm{d}x+C(\delta)\int_{\mathbb{R}^3}(g_n)^{\frac{6}{5}}\mathrm{d}x\Big]+\int_{\mathbb{R}^3}G_{\delta,n}(x)\mathrm{d}x\\ &\le C\Big[C\delta+C(\delta)\int_{\mathbb{R}^3}(g_n)^{\frac{6}{5}}\mathrm{d}x\Big]+\int_{\mathbb{R}^3}G_{\delta,n}(x)\mathrm{d}x\end{aligned}$$

Thus, by Lemma 2.2 and (2.1),

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\Big|I_\mu*|v_n|^{6-\mu}|v_n|^{4-\mu}v_n-I_\mu*|u_n|^{6-\mu}|u_n|^{4-\mu}u_n+I_\mu*|u|^{6-\mu}|u|^{4-\mu}u\Big|^{\frac{6}{5}}\mathrm{d}x\le C\delta.$$

It follows from the arbitrariness of δ that

$$\Big|I_\mu*|v_n|^{6-\mu}|v_n|^{4-\mu}v_n-I_\mu*|u_n|^{6-\mu}|u_n|^{4-\mu}u_n+I_\mu*|u|^{6-\mu}|u|^{4-\mu}u\Big|_{\frac{6}{5}}=o_n(1).$$

□

Making the change of variable $x\to\varepsilon x$, we can rewrite problem (1.1) as

$$-\Delta u+V(\varepsilon x)u=P(\varepsilon x)|u|^{p-2}u+\Big(\int_{\mathbb{R}^3}\frac{Q(\varepsilon y)|u(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}y\Big)Q(\varepsilon x)|u|^{4-\mu}u,\quad x\in\mathbb{R}^3. \tag{2.2}$$

Thus, the corresponding energy functional is

$$I_\varepsilon(u)=\frac{1}{2}\|u\|_\varepsilon^2-\frac{1}{p}\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x-\frac{1}{2(6-\mu)}F(u).$$

It is easy to check that I_ε is well defined on E and $I_\varepsilon\in C^1(E,\mathbb{R})$. Then we can define the Nehari manifold

$$\mathcal{N}_\varepsilon=\{u\in E\setminus\{0\}\mid\langle I_\varepsilon'(u),u\rangle=0\}.$$

Lemma 2.5 *There exists* $C_0>0$ *which is independent of* ε *such that*

$$\|u\|_\varepsilon>C_0\quad\text{and}\quad I_\varepsilon(u)\ge\frac{p-2}{2p}C_0^2,\quad\text{for all }u\in\mathcal{N}_\varepsilon.$$

Proof For any $u\in\mathcal{N}_\varepsilon$, we have

$$\|u\|_\varepsilon^2=\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x+F(u).$$

It follows from the Hardy–Littlewood–Sobolev inequality and Sobolev embedding theorem that

$$F(u)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{Q(\varepsilon x)Q(\varepsilon y)|u(x)|^{6-\mu}|u(y)|^{6-\mu}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\le C|u|_6^{2(6-\mu)}\le C\|u\|_\varepsilon^{2(6-\mu)}.$$

Without loss of generality, we assume that $\|u\|_\varepsilon\le 1$. Then

$$\|u\|_\varepsilon^2\le C(\|u\|_\varepsilon^p+\|u\|_\varepsilon^{2(6-\mu)})\le C\|u\|_\varepsilon^p.$$

Thus, the first desired result follows. On the other hand, we have

$$\begin{aligned}
I_\varepsilon(u)&=\frac{1}{2}\|u\|_\varepsilon^2-\frac{1}{p}\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x-\frac{1}{2(6-\mu)}F(u)\\
&\ge\frac{1}{2}\|u\|_\varepsilon^2-\frac{1}{p}\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x-\frac{1}{p}F(u)\\
&=\frac{1}{2}\|u\|_\varepsilon^2-\frac{1}{p}(\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x+F(u))\\
&\ge(\frac{1}{2}-\frac{1}{p})\|u\|_\varepsilon^2\\
&\ge\frac{p-2}{2p}C_0^2.
\end{aligned}$$

□

Lemma 2.6 *For any* $u\in E\backslash\{0\}$, *there exists a unique* $t(u)>0$ *such that* $t(u)u\in(\mathcal{N})_\varepsilon$ *and*

$$I_\varepsilon(t(u)u)=\max_{t\ge 0}I_\varepsilon(tu).$$

Proof For any $u\in E\setminus\{0\}$, define $g(t)=I_\varepsilon(tu),\ t\in[0,+\infty)$. Then

$$g(t)=\frac{t^2}{2}\|u\|_\varepsilon^2-\frac{t^p}{p}\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x-\frac{t^{2(6-\mu)}}{2(6-\mu)}F(u).$$

It is easy to see that $g(t)>0$ for $t>0$ small and $g(t)<0$ for $t>0$ large enough, so there exists $t_0>0$ such that

$$g'(t_0)=0\quad\text{and}\quad g(t_0)=\max_{t\ge 0}g(t)=\max_{t\ge 0}I_\varepsilon(tu).$$

It follows from $g'(t_0)=0$ that $t_0u\in\mathcal{N}_\varepsilon$.

If there exist $0<t_1<t_2$ such that $t_1u\in\mathcal{N}_\varepsilon$ and $t_2u\in\mathcal{N}_\varepsilon$. Then

$$\frac{1}{t_1^{p-2}}\|u\|_\varepsilon^2=\int_{\mathbb{R}^3}P(\varepsilon x)|u|^p\mathrm{d}x+t_1^{2(6-\mu)-p}F(u),$$

and

$$\frac{1}{t_2^{p-2}}\|u\|_\varepsilon^2 = \int_{\mathbb{R}^3} P(\varepsilon x)|u|^p \mathrm{d}x + t_2^{2(6-\mu)-p} F(u).$$

It follows that

$$\left(\frac{1}{t_1^{p-2}} - \frac{1}{t_2^{p-2}}\right)\|u\|_\varepsilon^2 = \left(t_1^{2(6-\mu)-p} - t_2^{2(6-\mu)-p}\right) F(u),$$

which is a contradiction. □

Lemma 2.7 *For any* $\varepsilon > 0$, *let*

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u), \quad c_\varepsilon^* = \inf_{u\in E\setminus\{0\}} \max_{t\geq 0} I_\varepsilon(tu), \quad c_\varepsilon^{**} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_\varepsilon(\gamma(t)),$$

where

$$\Gamma_\varepsilon = \{\gamma(t) \in C([0,1], E) \mid \gamma(0) = 0,\ I_\varepsilon(\gamma(1)) < 0\}.$$

Then, $c_\varepsilon = c_\varepsilon^* = c_\varepsilon^{**}$.

Proof We divide the proof into three steps.

Step1. $c_\varepsilon^* = c_\varepsilon$. By Lemma 2.6, we have

$$c_\varepsilon^* = \inf_{u\in E\setminus\{0\}} \max_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in E\setminus\{0\}} I_\varepsilon(t(u)u) = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) = c_\varepsilon.$$

Step2. $c_\varepsilon^* \geq c_\varepsilon^{**}$. For any $u \in E \setminus \{0\}$, there exists T large enough, such that $I_\varepsilon(Tu) < 0$. Define $\gamma(t) = tTu$, $t \in [0,1]$. Then we have $\gamma(t) \in \Gamma_\varepsilon$ and, therefore,

$$c_\varepsilon^{**} = \inf_{\gamma\in\Gamma_\varepsilon} \sup_{t\in[0,1]} I_\varepsilon(\gamma(t)) \leq \sup_{t\in[0,1]} I_\varepsilon(\gamma(t)) \leq \max_{t\geq 0} I_\varepsilon(tu).$$

It follows that $c_\varepsilon^* \geq c_\varepsilon^{**}$.

Step3. $c_\varepsilon^{**} \geq c_\varepsilon$. For any $u \in E \setminus \{0\}$ with $\|u\|_\varepsilon$ small, we know

$$\|u\|_\varepsilon^2 > \int_{\mathbb{R}^3} P(\varepsilon x)|u|^p \mathrm{d}x + F(u). \tag{2.3}$$

We claim that every $\gamma(t) \in \Gamma_\varepsilon$ has to cross $\mathcal{N}_\varepsilon$. Otherwise, by the continuity of $\gamma(t)$, (2.3) still holds when u is replaced by $\gamma(1)$. Then, we can obtain

$$\begin{aligned} I_\varepsilon(\gamma(1)) &= \frac{1}{2}\|\gamma(1)\|_\varepsilon^2 - \frac{1}{p}\int_{\mathbb{R}^3} P(\varepsilon x)|\gamma(1)|^p \mathrm{d}x - \frac{1}{2(6-\mu)}F(\gamma(1)) \\ &\geq \frac{1}{2}\|\gamma(1)\|_\varepsilon^2 - \frac{1}{p}\int_{\mathbb{R}^3} P(\varepsilon x)|\gamma(1)|^p \mathrm{d}x - \frac{1}{p}F(\gamma(1)) \\ &\geq \frac{1}{2}\|\gamma(1)\|_\varepsilon^2 - \frac{1}{p}\|\gamma(1)\|_\varepsilon^2 \\ &> 0, \end{aligned}$$

which contradicts the definition of $\gamma(1)$. It follows from the claim that $c_\varepsilon^{**} \geq c_\varepsilon$. □

One can easily check that the functional I_ε satisfies the mountain-pass geometry that is the following lemma holds ([26]).

Lemma 2.8 *I_ε has the mountain geometry structure.*

(i) There exist $a_0, r_0 > 0$ independent of ε, such that $I_\varepsilon(u) \geq a_0$, for all $u \in E$ with $\|u\|_\varepsilon = r_0$.
(ii) For any $u \in E \setminus \{0\}$, $\lim_{t\to\infty} I_\varepsilon(tu) = -\infty$.

Lemma 2.9 *For any $\varepsilon > 0$ and $Q(x) \equiv q$, we have $c_\varepsilon < \frac{5-\mu}{2(6-\mu)} S_{H,L}^{\frac{6-\mu}{5-\mu}} q^{\frac{-2}{5-\mu}}$, where q is a positive constant.*

Proof For any $\epsilon > 0$, define

$$U_\epsilon(x) = \frac{1}{\sqrt{\epsilon}} U\left(\frac{x}{\epsilon}\right),\ u_\epsilon(x) = \phi(x)U_\epsilon(x),\ x \in \mathbb{R}^3,$$

where $\phi(x) \in C_0^\infty(\mathbb{R}^3)$ is such that $\phi = 1$ on $B_1(0)$ and $\phi = 0$ on $B_2^c(0)$. From Lemma 2.6 in [1], we know that

$$\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 \mathrm{d}x = C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon), \tag{2.4}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\epsilon(x)|^{6-\mu}|u_\epsilon(y)|^{6-\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \geq C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}}), \tag{2.5}$$

and

$$\int_{\mathbb{R}^3} |u_\epsilon|^t \mathrm{d}x = \begin{cases} O(\epsilon^{\frac{6-t}{2}}), & t \in (3,6), \\ O(\epsilon^{\frac{3}{2}}|ln\epsilon|), & t = 3, \\ O(\epsilon^{\frac{t}{2}}), & t \in [2,3). \end{cases} \tag{2.6}$$

Then, for $t > 0$,

$$\begin{aligned}
I_\varepsilon(tu_\epsilon) =& \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla u_\epsilon|^2 + V(\varepsilon x)u_\epsilon^2)\mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{R}^3} P(\varepsilon x)|u_\epsilon|^p \mathrm{d}x \\
& - \frac{q^2 t^{2(6-\mu)}}{2(6-\mu)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\epsilon(x)|^{6-\mu}|u_\epsilon(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
\leq& \frac{t^2}{2}(C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)) - \frac{q^2 t^{2(6-\mu)}}{2(6-\mu)}(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}})) \\
& + C(t^2 O(\epsilon) - t^p O(\epsilon^{\frac{6-p}{2}})) := h(t).
\end{aligned}$$

It is easy to see that $h(t) \to -\infty$ as $t \to +\infty$, $h(0) = 0$ and $h(t) > 0$ as t is small. Therefore, there exists $t_\varepsilon > 0$ such that $h(t)$ attains its maximum. Then, differentiating h at t_ϵ, we can obtain

$$\begin{aligned}
&(C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)) - t_\epsilon^{2(6-\mu)-2} q^2 (C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}})) \\
&= -C(O(\epsilon) - t_\epsilon^{p-2} O(\epsilon^{\frac{6-p}{2}})).
\end{aligned}$$

When ϵ is small enough, it follows from the above expression that there exist $t_1, t_2 > 0$ independent of ϵ such that $t_1 < t_\epsilon < t_2$. Noting

$$\frac{t^2}{2}(C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)) - \frac{q^2 t^{2(6-\mu)}}{2(6-\mu)}(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}}))$$

attains its maximum at

$$\left(\frac{C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)}{q^2(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}}))}\right)^{\frac{1}{2(6-\mu)-2}}.$$

Then, we have

$$\begin{aligned}
h(t_\epsilon) \leq& \frac{5-\mu}{2(6-\mu)} q^{\frac{-2}{5-\mu}} \left(\frac{C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)}{(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}}))^{\frac{1}{6-\mu}}}\right)^{\frac{6-\mu}{5-\mu}} + C\left(t_\epsilon^2 O(\epsilon) - t_\epsilon^p O(\epsilon^{\frac{6-p}{2}})\right) \\
\leq& \frac{5-\mu}{2(6-\mu)} q^{\frac{-2}{5-\mu}} \left(\frac{C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}} + O(\epsilon)}{(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}} - O(\epsilon^{\frac{6-\mu}{2}}))^{\frac{1}{6-\mu}}}\right)^{\frac{6-\mu}{5-\mu}} + C\left(t_2^2 O(\epsilon) - t_1^p O(\epsilon^{\frac{6-p}{2}})\right) \\
\leq& \frac{5-\mu}{2(6-\mu)} q^{\frac{-2}{5-\mu}} \left(\frac{C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}})}{(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}})^{\frac{1}{6-\mu}}}\right)^{\frac{6-\mu}{5-\mu}} + O(\epsilon) + C\left(t_2^2 O(\epsilon) - t_1^p O(\epsilon^{\frac{6-p}{2}})\right).
\end{aligned}$$

Since $p \in (4, 6)$, then $0 < \frac{6-p}{2} < 1$. Thus, as ϵ is small enough, we have

$$O(\epsilon) + C(t_2^2 O(\epsilon) - t_1^p O(\epsilon^{\frac{6-p}{2}})) < 0.$$

Then, we can get

$$h(t_\epsilon) < \frac{5-\mu}{2(6-\mu)} q^{\frac{-2}{5-\mu}} \left(\frac{C(3,\mu)^{\frac{3}{2(6-\mu)}} S_{H,L}^{\frac{3}{2}})}{(C(3,\mu)^{\frac{3}{2}} S_{H,L}^{\frac{6-\mu}{2}})^{\frac{1}{6-\mu}}} \right)^{\frac{6-\mu}{5-\mu}} = \frac{5-\mu}{2(6-\mu)} q^{\frac{-2}{5-\mu}} S_{H,L}^{\frac{6-\mu}{5-\mu}}.$$

By Lemma 2.7, the proof is completed. □

Lemma 2.10 *Any $(PS)_c$ sequence $\{u_n\}$ for I_ε is bounded, and*

$$\limsup_{n\to\infty} \|u_n\|_\varepsilon \le \sqrt{\frac{2pc}{p-2}}.$$

Proof Suppose that $\{u_n\}$ is a $(PS)_c$ sequence of I_ε, we have

$$I_\varepsilon(u_n) \to c, \quad I'_\varepsilon(u_n) \to 0.$$

Thus

$$\begin{aligned} c + o_n(1) + o_n(1)\|u_n\|_\varepsilon &= I_\varepsilon(u_n) - \frac{1}{p}\langle I'_\varepsilon(u_n), v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\varepsilon^2 + \left(\frac{1}{p} - \frac{1}{2(6-\mu)}\right) F(u). \end{aligned}$$

It follows that

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\varepsilon^2 \le c + o_n(1) + o_n(1)\|u_n\|_\varepsilon.$$

Then $\{u_n\}$ is bounded in E, and the second result holds. □

Lemma 2.11 *If u is a critical point of I_ε on $\mathcal{N}_\varepsilon$, then u is a critical point of I_ε in E.*

Proof Since u is a critical point of I_ε on $\mathcal{N}_\varepsilon$, there exists $\theta \in \mathbb{R}$ such that

$$I'_\varepsilon(u) = \theta J'_\varepsilon(u),$$

where $J_\varepsilon(u) = \langle I'_\varepsilon(u), u\rangle$.

It follows from $u \in \mathcal{N}_\varepsilon$ that

$$\begin{aligned} \langle J'_\varepsilon(u), u\rangle &= 2\|u\|_\varepsilon^2 - p\int_{\mathbb{R}^3} P(\varepsilon x)|u|^p \mathrm{d}x - 2(6-\mu)F(u) \\ &= (2-p)\|u\|_\varepsilon^2 + (p - 2(6-\mu))F(u) < 0. \end{aligned}$$

Then, by $0 = \langle I'_\varepsilon(u), u\rangle = \theta\langle J'_\varepsilon(u), u\rangle$, we have $I'_\varepsilon(u) = 0$. □

3 The Energy Functional with Constant Coefficients

We need some results about Eq. (2.2) with constant coefficients. Consider the following problem

$$-\Delta u + ku = \tau|u|^{p-2}u + \nu^2\Big(\int_{\mathbb{R}^3}\frac{|u(y)|^{6-\mu}}{|x-y|^{\mu}}\mathrm{d}y\Big)|u|^{4-\mu}u, \quad x\in\mathbb{R}^3, \tag{3.1}$$

where k, τ, and ν are positive constants. The associated energy functional is

$$I_{k\tau\nu}(u) = \frac{1}{2}\|u\|_k^2 - \frac{\tau}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x - \frac{\nu^2}{2(6-\mu)}\widetilde{F}(u),$$

where

$$\|u\|_k = \Big(\int_{\mathbb{R}^3}(|\nabla u|^2 + ku^2)\mathrm{d}x\Big)^{1/2},$$

$$\widetilde{F}(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{6-\mu}|u(y)|^{6-\mu}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y.$$

By Lemma 2.7, we have

$$m_{k\tau\nu} := \inf_{u\in\mathcal{N}_{k\tau\nu}} I_{k\tau\nu}(u) = \inf_{u\in E\setminus\{0\}}\max_{t\geq 0} I_{k\tau\nu}(tu),$$

where $\mathcal{N}_{k\tau\nu} = \big\{u\in E\setminus\{0\}\,|\,\langle I'_{k\tau\nu}(u),u\rangle = 0\big\}$. Especially, $I_\infty(u)$, m_∞, and $\mathcal{N}_\infty$ mean $I_{V_\infty P^\infty Q^\infty}(u)$, $m_{V_\infty P^\infty Q^\infty}$, and $\mathcal{N}_{V_\infty P^\infty Q^\infty}$, respectively.

Lemma 3.1 *Problem* (3.1) *has at least one ground state solution.*

Proof By Lemma 2.7 and Lemma 2.8, there exits a sequence $\{u_n\}$ which is a $(PS)_{m_{k\tau\nu}}$ sequence of $I_{k\tau\nu}$. By Lemma 2.10, we know that $\{u_n\}$ is bounded in E. Hence, up to a subsequence, we have

$$u_n \rightharpoonup u \quad \text{in } E,$$
$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3,$$
$$u_n \rightharpoonup u \quad \text{in } L^q(\mathbb{R}^3),\ \text{for}\ 2\leq q\leq 6.$$

It is easy to verify that $I'_{k\tau\nu}(u) = 0$.

Case1. $u\neq 0$.

For this case, we have $u \in \mathcal{N}_{k\tau\nu}$. Therefore, $I_{k\tau\nu}(u) \geq m_{k\tau\nu}$. Then we get

$$\begin{aligned} m_{k\tau\nu} = \lim_{n\to\infty} I_{k\tau\nu}(u_n) &= \lim_{n\to\infty}\left[I_{k\tau\nu}(u_n) - \frac{1}{p}\langle I'_{k\tau\nu}(u_n), u_n\rangle\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_k^2 + \left(\frac{1}{p}-\frac{1}{2(6-\mu)}\right)\nu^2\widetilde{F}(u_n)\right] \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_k^2 + \left(\frac{1}{p}-\frac{1}{2(6-\mu)}\right)\nu^2\widetilde{F}(u) \\ &= I_{k\tau\nu}(u) - \frac{1}{p}\langle I'_{k\tau\nu}(u), u\rangle \\ &= I_{k\tau\nu}(u) \geq m_{k\tau\nu}. \end{aligned}$$

Thus, $I_{k\tau\nu}(u) = m_{k\tau\nu}$. Moreover, we have $u_n \to u$ in E.

Case2. $u = 0$.

Since $\{u_n\}$ is a $(PS)_{m_{k\tau\nu}}$ sequence of $I_{k\tau\nu}$, we have

$$o_n(1) = \langle I'_{k\tau\nu}(u_n), u_n\rangle = \|u_n\|_k^2 - \tau\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x - \nu^2\widetilde{F}(u_n).$$

Assume that

$$\|u_n\|_k^2 \to l \text{ and } \tau\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x + \nu^2\widetilde{F}(u_n) \to l.$$

It is easy to see that $l \neq 0$. If $\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x \to 0$, then $\nu^2\widetilde{F}(u_n) \to l$. By the definition of $S_{H,L}$, we can get

$$\nu^2\widetilde{F}(u_n) \leq \nu^2 S_{H,L}^{-(6-\mu)}\|u_n\|_k^{2(6-\mu)}.$$

Letting $n \to \infty$, we have

$$l \leq \nu^2 S_{H,L}^{-(6-\mu)} l^{6-\mu}. \tag{3.2}$$

Then,

$$l \geq \nu^{\frac{-2}{5-\mu}} S_{H,L}^{\frac{6-\mu}{5-\mu}}.$$

Thus,

$$\begin{aligned}\frac{5-\mu}{2(6-\mu)}S_{H,L}^{\frac{6-\mu}{5-\mu}}\nu^{\frac{-2}{5-\mu}} > m_{k\tau\nu} &= \lim_{n\to\infty} I_{k\tau\nu}(u_n)\\ &= \lim_{n\to\infty}\Big[\frac{1}{2}\|u_n\|_k^2 - \frac{\tau}{p}\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x - \frac{\nu^2}{2(6-\mu)}\widetilde{F}(u_n)\Big]\\ &= \frac{1}{2}l - \frac{1}{2(6-\mu)}l\\ &\geq \frac{5-\mu}{2(6-\mu)}S_{H,L}^{\frac{6-\mu}{5-\mu}}\nu^{\frac{-2}{5-\mu}},\end{aligned}$$

which is a contradiction. Therefore, $\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x \to b > 0$ as $n \to \infty$. Thus, by Lions's Lemma, there exists $\{y_n\} \subset \mathbb{R}^3$, $\rho, \eta > 0$ such that

$$\int_{B_\rho(y_n)} |u_n|^2\mathrm{d}x \geq \eta. \tag{3.3}$$

Let $\widetilde{u}_n(x) = u_n(x+y_n)$. Then $||\widetilde{u}_n|| \leq C$ in E. This implies that there exists $\widetilde{u} \in E$ such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in E and $\widetilde{u}_n \to \widetilde{u}$ a.e. in $\mathbb{R}^3$. By (3.3), we get $\widetilde{u} \neq 0$. It is easy to prove that

$$I_{k\tau\nu}(\widetilde{u}_n) \to m_{k\tau\nu} \text{ and } I'_{k\tau\nu}(\widetilde{u}_n) \to 0.$$

Thus, we have $I'_{k\tau\nu}(\widetilde{u}) = 0$ and $\widetilde{u} \in \mathcal{N}_{k\tau\nu}$. Then the proof follows from the argument used in the case of $u \neq 0$. □

Lemma 3.2 *For $k_i > 0$, $\tau_i > 0$ and $\nu_i > 0$, $i = 1, 2$. If*

$$\min\{k_2 - k_1, \tau_1 - \tau_2, \nu_1 - \nu_2\} \geq 0,$$

then $m_{k_1\tau_1\nu_1} \leq m_{k_2\tau_2\nu_2}$. Additionally, if $\max\{k_2 - k_1, \tau_1 - \tau_2, \nu_1 - \nu_2\} > 0$, then $m_{k_1\tau_1\nu_1} < m_{k_2\tau_2\nu_2}$.

Proof By Lemma 3.1, there exists $u \in E$ satisfying $I_{k_2\tau_2\nu_2}(v) = m_{k_2\tau_2\nu_2} = \max_{t\geq 0} I_{k_2\tau_2\nu_2}(tu)$. By Lemma 2.6, there exists $t_0 > 0$ such that $I_{k_1\tau_1\nu_1}(t_0u) = \max_{t\geq 0} I_{k_1\tau_1\nu_1}(tu)$. Then

$$m_{k_1\tau_1\nu_1} \leq \max_{t\geq 0} I_{k_1\tau_1\nu_1}(tv) = I_{k_1\tau_1\nu_1}(t_0v) \leq I_{k_2\tau_2\nu_2}(t_0v) \leq I_{k_2\tau_2\nu_2}(v) = m_{k_2\tau_2\nu_2}.$$

□

Lemma 3.3 *For any $\xi \in \mathbb{R}^3$, $\limsup_{\varepsilon\to 0} c_\varepsilon \leq m_{V(\xi)P(\xi)Q(\xi)}$.*

Proof For any $\xi \in \mathbb{R}^3$, by Lemma 3.1, we assume that u is a ground state solution to the equation corresponding to the functional $I_{V(\xi)P(\xi)Q(\xi)}$. Set $u_\varepsilon(x) = \varphi(\varepsilon x - \xi)u(x - \frac{\xi}{\varepsilon})$, where $\varphi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ is a cut-off function satisfying $\varphi = 1$, $|x| < 1$ and $\varphi = 0$, $|x| \geq 2$. Then, there exists T large enough, such that $I_\varepsilon(Tu_\varepsilon) < 0$. Define $\gamma_\varepsilon(t) = tTu_\varepsilon$, $t \in [0, 1]$. It is easy to see that $\gamma_\varepsilon(t) \in \Gamma_\varepsilon$ in Lemma 2.7. By direct computation, we have

$$\int_{\mathbb{R}^3}(|\nabla u_\varepsilon|^2 + V(\varepsilon x)|u_\varepsilon|^2)\mathrm{d}x = \int_{\mathbb{R}^3}(|\nabla u|^2 + V(\xi)|u|^2)\mathrm{d}x + o_\varepsilon(1),$$
$$\int_{\mathbb{R}^3} P(\varepsilon x)|u_\varepsilon|^p\mathrm{d}x = \int_{\mathbb{R}^3} P(\xi)|u|^p\mathrm{d}x + o_\varepsilon(1)$$
$$F(u_\varepsilon) = Q^2(\xi)\widetilde{F}(u) + o_\varepsilon(1),$$

Therefore,

$$\begin{aligned} I_\varepsilon(\gamma_\varepsilon(t)) &= \frac{(tT)^2}{2}\|u_\varepsilon\|_\varepsilon^2 - \frac{(tT)^p}{p}\int_{\mathbb{R}^3} P(\varepsilon x)|u_\varepsilon|^p\mathrm{d}x - \frac{(tT)^{2(6-\mu)}}{2(6-\mu)}F(u_\varepsilon) \\ &= \frac{(tT)^2}{2}\|u\|_{V(\xi)}^2 - \frac{(tT)^p}{p}\int_{\mathbb{R}^3} P(\xi)|u|^p\mathrm{d}x - \frac{(tT)^{2(6-\mu)}}{2(6-\mu)}Q^2(\xi)\widetilde{F}(u) + o_\varepsilon(1) \\ &= I_{V(\xi)P(\xi)Q(\xi)}(tTu) + o_\varepsilon(1) \\ &\leq I_{V(\xi)P(\xi)Q(\xi)}(u) + o_\varepsilon(1) = m_{V(\xi)P(\xi)Q(\xi)} + o_\varepsilon(1) \end{aligned}$$

Thus,

$$c_\varepsilon \leq \max_{0 \leq t \leq 1} I_\varepsilon(\gamma_\varepsilon(t)) \leq m_{V(\xi)P(\xi)Q(\xi)} + o_\varepsilon(1).$$

It follows that $\limsup_{\varepsilon \to 0} c_\varepsilon \leq m_{V(\xi)P(\xi)Q(\xi)}$. □

4 The Palais-Smale Condition

Lemma 4.1 *Suppose that the condition* (H_2) *holds. Let* $\{u_n\} \subset E$ *be a* $(PS)_c$ *sequence for* I_ε *with* $c < \frac{5-\mu}{2(6-\mu)} S_{H,L}^{\frac{6-\mu}{5-\mu}} (Q^\infty)^{\frac{-2}{5-\mu}}$ *and such that* $u_n \rightharpoonup 0$ *in* E*. Then, one of the following conclusions holds.*

(i) $u_n \to 0$ *in* E*;*
(ii) There exists a sequence $\{y_n\} \subset \mathbb{R}^3$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2\mathrm{d}x \geq \beta.$$

Proof Suppose that (ii) does not occur. Then, for any $R > 0$, one has

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \mathrm{d}x = 0.$$

Then, we have

$$u_n \to u \quad \text{in } L^q(\mathbb{R}^3), \text{ for } 2 < q < 6.$$

Noting $o_n(1) = \langle I'_\varepsilon(u_n), u_n\rangle$, we can obtain

$$\|u_n\|_\varepsilon^2 = F(u_n) + o_n(1).$$

By Lemma 2.10, $\{u_n\}$ is bounded in E. Up to a subsequence, we can assume that

$$\|u_n\|_\varepsilon^2 \to l \text{ and } F(u_n) \to l.$$

Assume by contradiction that $l > 0$. From condition (H_2),

$$F(u_n) \le (Q^\infty)^2 \widetilde{F}(u_n).$$

By the definition of $S_{H,L}$, we can get

$$F(u_n) \le (Q^\infty)^2 S_{H,L}^{-(6-\mu)} \|u_n\|_\varepsilon^{2(6-\mu)}.$$

It follows that

$$l \ge S_{H,L}^{\frac{6-\mu}{5-\mu}} (Q^\infty)^{\frac{-2}{5-\mu}}.$$

Since $I_\varepsilon(u_n) = c + o_n(1)$, we can deduce that

$$c \ge \frac{5-\mu}{2(6-\mu)} S_{H,L}^{\frac{6-\mu}{5-\mu}} (Q^\infty)^{\frac{-2}{5-\mu}},$$

which is a contradiction with our assumption. Therefore, $l = 0$ and the conclusion follows. □

Lemma 4.2 *Suppose that the condition (H_2) holds. Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence for I_ε with $c < m_\infty$ and $u_n \rightharpoonup 0$ in E. Then $u_n \to 0$ in E.*

Proof Assume that $u_n \nrightarrow 0$ in E. Let $\{t_n\} \subset (0, +\infty)$ be a sequence such that $\{t_n u_n\} \subset \mathcal{N}_\infty$. Then, we claim that the sequence $\{t_n\}$ satisfies that $\limsup_{n\to\infty} t_n \le 1$.

Assume by contradiction that there exists $\delta > 0$ and a subsequence still denoted by $\{t_n\}$, such that, for all $n \in \mathbb{N}$,

$$t_n \ge 1 + \delta.$$

Since $\langle I'_\varepsilon(u_n), u_n\rangle = o_n(1)$, we get

$$\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^3} P(\varepsilon x)|u_n|^p \mathrm{d}x + F(u_n) + o_n(1). \tag{4.1}$$

Using $t_n u_n \in \mathcal{N}_\infty$, we have

$$t_n^2\|u_n\|_\infty^2 = t_n^p P^\infty \int_{\mathbb{R}^3} |u_n|^p \mathrm{d}x + t_n^{2(6-\mu)}(Q^\infty)^2 \widetilde{F}(u_n).$$

Then, we can obtain

$$\begin{aligned}
&\left(\frac{1}{t_n^{p-2}} - 1\right)\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{V_\infty}{t_n^{p-2}} - V(\varepsilon x)\right)|u_n|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^3} [P^\infty - P(\varepsilon x)]|u_n|^p \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [t_n^{(12-2\mu-p)}(Q^\infty)^2 - Q(\varepsilon x)Q(\varepsilon y)]\frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1).
\end{aligned}$$

By the definition of V_∞ and P^∞, for any $\sigma > 0$, there exists $R = R(\sigma) > 0$, such that, for $|\varepsilon x| \geq R$,

$$V(\varepsilon x) > V_\infty - \sigma > \frac{V_\infty}{t_n^{p-2}} - \sigma \tag{4.2}$$

and

$$P(\varepsilon x) < P^\infty + \sigma. \tag{4.3}$$

Moreover, $\|u_n\|_\varepsilon$ is bounded and $u_n \to 0$ in $L^q_{loc}(\mathbb{R}^3)$, for $2 \leq q < 6$. Then, we can obtain

$$\int_{\mathbb{R}^3}\left(\frac{V_\infty}{t_n^{p-2}} - V(\varepsilon x)\right)|u_n|^2 \mathrm{d}x \leq C\sigma + o_n(1)$$

and

$$\int_{\mathbb{R}^3}[P^\infty - P(\varepsilon x)]|u_n|^p \mathrm{d}x \geq -C\sigma + o_n(1).$$

Therefore,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[t_n^{2(6-\mu)-p}(Q^\infty)^2 - Q(\varepsilon x)Q(\varepsilon y)]\frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \leq C\sigma + o_n(1).$$

Since $t_n > 1+\delta$ and $Q(\varepsilon x) \leq Q^\infty$, it follows from the above inequality that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \leq C\sigma + o_n(1).$$

By the arbitrariness of σ, we can obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 0. \tag{4.4}$$

Since $u_n \nrightarrow 0$ in E, by Lemma 4.1, we know that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \beta. \tag{4.5}$$

Set $v_n(x) = u_n(x+y_n)$. Then $\{v_n(x)\}$ is a bounded sequence in E. Therefore, there exists $v \in E$ such that

$$v_n \to v, \text{ a.e. in } \mathbb{R}^3.$$

By (4.5), $v \neq 0$ in E. Then, it follows from Fatou Lemma and (4.4) that

$$\begin{aligned}0 < \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^{6-\mu}|v(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y &\leq \lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_n(x)|^{6-\mu}|v_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 0,\end{aligned}$$

which is a contradiction.

We next distinguish the following two cases.

Case 1: $\limsup\limits_{n\to\infty} t_n = 1$.

In this case, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$. Then,

$$\begin{aligned}&I_\infty(t_n u_n) \\ &= \frac{t_n^2}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V_\infty|u_n|^2)\mathrm{d}x - \frac{t_n^p}{p}\int_{\mathbb{R}^3} P^\infty|u_n|^p\mathrm{d}x - \frac{t_n^{2(6-\mu)}}{2(6-\mu)}(Q^\infty)^2\widetilde{F}(u_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V_\infty|u_n|^2)\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} P^\infty|u_n|^p\mathrm{d}x - \frac{1}{2(6-\mu)}(Q^\infty)^2\widetilde{F}(u_n) + o_n(1).\end{aligned}$$

Therefore,

$$
\begin{aligned}
I_\varepsilon(u_n) - I_\infty(t_n u_n) = &\frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x) - V_\infty)|u_n|^2 \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^3}(P^\infty - P(\varepsilon x))|u_n|^p \mathrm{d}x \\
&+ \frac{1}{2(6-\mu)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[(Q^\infty)^2 - Q(\varepsilon x)Q(\varepsilon y)]\frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&+ o_n(1).
\end{aligned}
$$

By (4.2) and $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$,

$$
\int_{\mathbb{R}^3}(V(\varepsilon x) - V_\infty)|u_n|^2 \mathrm{d}x \geq -C\sigma + o_n(1).
$$

Similarly,

$$
\int_{\mathbb{R}^3}(P^\infty - P(\varepsilon x))|u_n|^p \mathrm{d}x \geq -C\sigma + o_n(1).
$$

Then, noting $Q(\varepsilon x) \leq Q^\infty$, we can obtain

$$
c \geq m_\infty - C\sigma + o_n(1).
$$

By the arbitrariness of σ, we have $c \geq m_\infty$, which is a contradiction.

Case 2: $\limsup_{n\to\infty} t_n < 1$.

In this case, we may suppose that $t_n < 1$ for all $n \in \mathbb{N}$. From (4.1), we can deduce that

$$
\begin{aligned}
I_\varepsilon(t_n u_n) = &\frac{t_n^2}{2}\|u_n\|_\varepsilon^2 - \frac{t_n^p}{p}\int_{\mathbb{R}^3} P(\varepsilon x)|u_n|^p \mathrm{d}x - \frac{t_n^{2(6-\mu)}}{2(6-\mu)}F(u_n) \\
= &(\frac{t_n^2}{2} - \frac{t_n^{2(6-\mu)}}{2(6-\mu)})\|u_n\|_\varepsilon^2 + (\frac{t_n^{2(6-\mu)}}{2(6-\mu)} - \frac{t_n^p}{p})\int_{\mathbb{R}^3} P(\varepsilon x)|u_n|^p \mathrm{d}x + o_n(1) \\
\leq &(\frac{1}{2} - \frac{1}{2(6-\mu)})\|u_n\|_\varepsilon^2 + (\frac{1}{2(6-\mu)} - \frac{1}{p})\int_{\mathbb{R}^3} P(\varepsilon x)|u_n|^p \mathrm{d}x + o_n(1) \\
= &I_\varepsilon(u_n) - \frac{1}{2(6-\mu)}\langle I_\varepsilon'(u_n), u_n\rangle + o_n(1) \\
= &I_\varepsilon(u_n) + o_n(1).
\end{aligned}
$$

Using this result, we have

$$\begin{aligned}
m_\infty &\le I_\infty(t_n u_n) \\
&= I_\varepsilon(t_n u_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^3}(V_\infty - V(\varepsilon x))|u_n|^2 \mathrm{d}x \\
&\quad - \frac{t_n^p}{p}\int_{\mathbb{R}^3}(P^\infty - P(\varepsilon x))|u_n|^p \mathrm{d}x - \frac{t_n^{2(6-\mu)}}{2(6-\mu)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[(Q^\infty)^2 - Q(\varepsilon x)Q(\varepsilon y)] \\
&\quad \frac{|u_n(x)|^{6-\mu}|u_n(y)|^{6-\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\le I_\varepsilon(t_n u_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^3}(V_\infty - V(\varepsilon x))|u_n|^2 \mathrm{d}x - \frac{t_n^p}{p}\int_{\mathbb{R}^3}(P^\infty - P(\varepsilon x))|u_n|^p \mathrm{d}x \\
&\le I_\varepsilon(t_n u_n) + C\sigma + o_n(1) \\
&\le I_\varepsilon(u_n) + C\sigma + o_n(1),
\end{aligned}$$

which means that $c \ge m_\infty$, a contradiction. □

Lemma 4.3 *Suppose that the condition (H_2) holds. Then I_ε satisfies the $(PS)_c$ condition at any level $c < m_\infty$.*

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence. By Lemma 2.10, $\{u_n\}$ is bounded in E. Then there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. By standard argument, $I_\varepsilon'(u) = 0$ and $I_\varepsilon(u) \ge 0$. Set $w_n = u_n - u$. It follows from Lemma 2.4 and Brezis–Lieb' Lemma that $\{w_n\}$ is a $(PS)_{c-I_\varepsilon(u)}$ sequence. Since $c - I_\varepsilon(u) < m_\infty$, by Lemma 4.2, $w_n \to 0$ in E. Therefore, $u_n \to u$ in E. □

Lemma 4.4 *Suppose that the condition (H_2) holds. Let $\{u_n\}$ be a $(PS)_c$ sequence restricted on $\mathcal{N}_\varepsilon$ and assume $c < m_\infty$. Then $\{u_n\}$ has a convergent subsequence in E.*

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence restricted on $\mathcal{N}_\varepsilon$. Then, there exist $\{\theta_n\} \subset \mathbb{R}$ such that

$$I_\varepsilon'(u_n) = \theta_n J_\varepsilon'(u_n) + o_n(1)$$

where $J_\varepsilon(u) = \langle I_\varepsilon'(u), u\rangle$.

It follows from $u_n \in \mathcal{N}_\varepsilon$ and Lemma 2.5 that

$$\begin{aligned}
\langle J_\varepsilon'(u_n), u_n\rangle &= 2\|u_n\|_\varepsilon^2 - p\int_{\mathbb{R}^3} P(\varepsilon x)|u_n|^p \mathrm{d}x - 2(6-\mu)F(u_n) \\
&= (2-p)\|u_n\|_\varepsilon^2 + (p - 2(6-\mu))F(u_n) \\
&< (2-p)\|u_n\|_\varepsilon^2 \le (2-p)C_0^2.
\end{aligned}$$

From $0 = \langle I_\varepsilon'(u_n), u_n\rangle$ and the above inequality, we have $\theta_n = o_n(1)$. Therefore, $I_\varepsilon'(u_n) = o_n(1)$. Thus, by Lemma 4.3, the proof is completed. □

5 The Existence of Multiple Solutions

We assume that the conditions (H_1), (H_2), and (H_3) hold in this section. Let us consider a cut-off function $\eta \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that $\eta(x) = 1$ if $|x| < 1$, $\eta(x) = 0$ if $|x| > 2$ and $|\nabla \eta| \leq C$. Choose $w \in E$ with $I'_{V_{\min} P_{\max} Q_{\max}}(w) = 0$ and $I_{V_{\min} P_{\max} Q_{\max}}(w) = m_{V_{\min} P_{\max} Q_{\max}}$. For each $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$, let

$$\Psi_{\varepsilon,\xi}(x) = \eta(|\varepsilon x - \xi|) w\Big(\frac{\varepsilon x - \xi}{\varepsilon}\Big).$$

Then, there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$. Define $\Phi_\varepsilon : \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \to \mathcal{N}_\varepsilon$ by setting $\Phi_\varepsilon(\xi) = t_\varepsilon \Psi_{\varepsilon,\xi}$.

Lemma 5.1 $\lim\limits_{\varepsilon \to 0} I_\varepsilon(\Phi_\varepsilon(\xi)) = m_{V_{\min} P_{\max} Q_{\max}}$ *uniformly in* $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$.

Proof Suppose that the result is false. Then, there exists some $\alpha > 0$, $\{\xi_n\} \subset \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(\xi_n)) - m_{V_{\min} P_{\max} Q_{\max}}| \geq \alpha.$$

The compactness of $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$ implies that there exists $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$ such that $\xi_n \to \xi$, up to a subsequence if necessary. Now we claim that $\lim\limits_{n \to \infty} t_{\varepsilon_n} = 1$. Indeed, from $t_{\varepsilon_n} \Psi_{\varepsilon_n,\xi_n} \in \mathcal{N}_{\varepsilon_n}$, we have

$$t_{\varepsilon_n}^2 \|\Psi_{\varepsilon_n,\xi_n}\|_{\varepsilon_n}^2 = t_{\varepsilon_n}^p \int_{\mathbb{R}^3} P(\varepsilon_n x) |\Psi_{\varepsilon_n,\xi_n}|^p \mathrm{d}x + t_{\varepsilon_n}^{2(6-\mu)} F(\Psi_{\varepsilon_n,\xi_n}).$$

By using a change of variables and Lebesgue Dominated Convergence Theorem, we can obtain

$$\|\Psi_{\varepsilon_n,\xi_n}\|_{\varepsilon_n}^2 = \int_{\mathbb{R}^3} (|\nabla w|^2 + V(\xi) w^2) \mathrm{d}x + o_n(1),$$
$$\int_{\mathbb{R}^3} P(\varepsilon_n x) |\Psi_{\varepsilon_n,\xi_n}|^p \mathrm{d}x = \int_{\mathbb{R}^3} P(\xi) |w|^p \mathrm{d}x + o_n(1)$$

and

$$F(\Psi_{\varepsilon_n,\xi_n}) = Q^2(\xi) \widetilde{F}(w) + o_n(1).$$

Then t_n is bounded from above. Thus we can obtain

$$t_{\varepsilon_n}^2 \int_{\mathbb{R}^3} (|\nabla w|^2 + V(\xi) w^2) \mathrm{d}x = t_{\varepsilon_n}^p \int_{\mathbb{R}^3} P(\xi) |w|^p \mathrm{d}x + t_{\varepsilon_n}^{2(6-\mu)} Q^2(\xi) \widetilde{F}(w) + o_n(1).$$

It follows from Lemma 2.5 that t_n is has a positive lower bound. Without loss of generality, we assume that $t_{\varepsilon_n} \to T > 0$. Letting $n \to \infty$ in the above expression, we can get

$$T^2\int_{\mathbb{R}^3}(|\nabla w|^2+V(\xi)w^2)\mathrm{d}x=T^p\int_{\mathbb{R}^3}P(\xi)|w|^p\mathrm{d}x+T^{2(6-\mu)}Q^2(\xi)\widetilde{F}(w).$$

It follows from $w\in\mathcal{N}_{V_{\min}P_{\max}Q_{\max}}$ that $T=1$. Then, we have

$$\lim_{n\to\infty}I_{\varepsilon_n}(\Phi_{\varepsilon_n}(\xi_n))=I_{V_{\min}P_{\max}Q_{\max}}(w)=m_{V_{\min}P_{\max}Q_{\max}},$$

which is a contradiction. □

For any $\delta>0$, let $\rho=\rho(\delta)>0$ such that $(\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q})_\delta\subset B_\rho(0)$. Consider $\chi:\mathbb{R}^3\to\mathbb{R}^3$ defined as $\chi(x)=x$ for $|x|\le\rho$ and $\chi(x)=\frac{\rho x}{|x|}$ for $|x|\ge\rho$. Define $\beta_\varepsilon:\mathcal{N}_\varepsilon\to\mathbb{R}^3$ given by

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)u^2(x)\mathrm{d}x}{\int_{\mathbb{R}^3}u^2(x)\mathrm{d}x}.$$

Lemma 5.2 $\lim_{\varepsilon\to0}\beta_\varepsilon(\Phi_\varepsilon(\xi))=\xi$ *uniformly in* $\xi\in\mathbb{V}\cap\mathbb{P}\cap\mathbb{Q}$.

Proof Suppose by contradiction that there exist $\delta_0>0$, $\{\xi_n\}\subset\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}$ and $\varepsilon_n\to0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(\xi_n))-\xi_n|\ge\delta_0. \tag{5.1}$$

By the definition of β_ε, we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(\xi_n))=\xi_n+\frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_nx+\xi_n)-\xi_n)|\eta(\varepsilon_nx)w(x)|^2\mathrm{d}x}{\int_{\mathbb{R}^3}|\eta(\varepsilon_nx)w(x)|^2\mathrm{d}x}.$$

Since $\{\xi_n\}\subset\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}\subset B_\rho(0)$ and $\chi\big|_{B_\rho}\equiv id$, we conclude that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(\xi_n))-\xi_n|=o_n(1),$$

which contradicts (5.1) and the desired conclusion holds. □

Define the set

$$\tilde{\mathcal{N}_\varepsilon}=\{u\in\mathcal{N}_\varepsilon:I_\varepsilon(u)\le m_{V_{\min}P_{\max}Q_{\max}}+h(\varepsilon)\}.$$

where $h(\varepsilon)=\sup_{\xi\in\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}}|I_\varepsilon(\Phi_\varepsilon(\xi))-m_{V_{\min}P_{\max}Q_{\max}}|$. We conclude from Lemma 5.1 that $h(\varepsilon)\to0$ as $\varepsilon\to0$. By the definition of $h(\varepsilon)$, for any $\xi\in\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}$ and $\varepsilon>0$, $\Phi_\varepsilon(\xi)\in\tilde{\mathcal{N}_\varepsilon}$ and $\tilde{\mathcal{N}_\varepsilon}\neq\emptyset$.

Lemma 5.3 *Let $\varepsilon_n \to 0$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ such that $I_{\varepsilon_n}(u_n) \to m_{V_{\min} P_{\max} Q_{\max}}$. Then there exists $\{y_n\} \subset \mathbb{R}^N$ such that the sequence $u_n(x + y_n)$ has a convergent subsequence in E. Moreover, up to a subsequence, $\varepsilon_n y_n \to \xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$.*

Proof Since

$$\begin{aligned} m_{V_{\min} P_{\max} Q_{\max}} &= I_{\varepsilon_n}(u_n) - \frac{1}{p}\left\langle I'_{\varepsilon_n}(u_n), u_n\right\rangle + o_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{2(6-\mu)}\right)\|u_n\|^2_{\varepsilon_n} + \left(\frac{1}{p} - \frac{1}{2(6-\mu)}\right)F(u_n) + o_n(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2(6-\mu)}\right)\|u_n\|^2_{\varepsilon_n} + o_n(1), \end{aligned}$$

then $\{u_n\}$ is bounded in E. We can have a sequence $\{y_n\} \subset \mathbb{R}^3$ and positive constants R, β such that

$$\int_{B_R(y_n)} |u_n|^2 \geq \beta > 0.$$

If not, for any $R > 0$, one has

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \mathrm{d}x = 0.$$

Then, we have

$$u_n \to u \quad \text{in } L^q(\mathbb{R}^3), \text{ for } 2 < q < 6.$$

Noting $0 = \langle I'_\varepsilon(u_n), u_n\rangle$, we can obtain

$$\|u_n\|^2_{\varepsilon_n} = F(u_n) + o_n(1).$$

Up to a subsequence, assume that

$$\|u_n\|^2_{\varepsilon_n} \to l \text{ and } F(u_n) \to l.$$

It follows from Lemma 2.5 that $l > 0$. From condition (H_2),

$$F(u_n) \leq (Q^\infty)^2 \widetilde{F}(u_n).$$

By the definition of $S_{H,L}$, we can get

$$F(u_n) \leq (Q^\infty)^2 S_{H,L}^{-(6-\mu)} \|u_n\|_\varepsilon^{2(6-\mu)}.$$

It follows that

$$l \geq S_{H,L}^{\frac{6-\mu}{5-\mu}} (Q^{\infty})^{\frac{-2}{5-\mu}}.$$

Since $I_{\varepsilon_n}(u_n) = m_{V_{\min} P_{\max} Q_{\max}} + o_n(1)$, we can deduce that

$$m_{V_{\min} P_{\max} Q_{\max}} \geq \frac{5-\mu}{2(6-\mu)} S_{H,L}^{\frac{6-\mu}{5-\mu}} (Q^{\infty})^{\frac{-2}{5-\mu}},$$

which is a contradiction with Lemma 2.9. Therefore, the conclusion follows. Denote $\tilde{u}_n(x) = u_n(x + y_n)$, going if necessary to a subsequence, we can assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \neq 0 \text{ in } E.$$

Let $t_n > 0$ be such that $t_n \tilde{u}_n \in \mathcal{N}_{V_{\min} P_{\max} Q_{\max}}$. By the definition of $I_{V_{\min} P_{\max} Q_{\max}}$ and $m_{V_{\min} P_{\max} Q_{\max}}$, we obtain

$$\begin{aligned} m_{V_{\min} P_{\max} Q_{\max}} &\leq I_{V_{\min} P_{\max} Q_{\max}}(t_n \tilde{u}_n) \\ &= I_{V_{\min} P_{\max} Q_{\max}}(t_n u_n) \\ &\leq I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) = m_{V_{\min} P_{\max} Q_{\max}} + o_n(1), \end{aligned}$$

so $I_{V_{\min} P_{\max} Q_{\max}}(t_n \tilde{u}_n) \to m_{V_{\min} P_{\max} Q_{\max}}$. Then $\{t_n \tilde{u}_n\}$ is bounded in E. Since $t_n \tilde{u}_n \in \mathcal{N}_{V_{\min} P_{\max} Q_{\max}}$, it follows from Lemma 2.5 that $\|t_n \tilde{u}_n\| \geq C_0$. Noting u_n is bounded in E, then there exists $C > 0$ such that $t_n C \geq \|t_n \tilde{u}_n\| \geq C_0$. Thus t_n has a positive lower bound. On the other hand, $\tilde{u}_n$ does not converge to 0 in E, so there exists a $\delta' > 0$ such that $\|\tilde{u}_n\| \geq \delta'$. Therefore, $t_n \delta' \leq \|t_n \tilde{u}_n\| \leq C$. Thus $\{t_n\}$ is bounded from above. Then, up to a subsequence, $t_n \to t_0 > 0$.

Denote $\hat{u}_n := t_n \tilde{u}_n$, $\hat{u} := t_0 \tilde{u}$, we have

$$I_{V_{\min} P_{\max} Q_{\max}}\left(\hat{u}_n\right) \to m_{V_{\min} P_{\max} Q_{\max}}, \ \hat{u}_n \rightharpoonup \hat{u} \text{ in } E.$$

By the Ekeland's Variational Principle, there exists a sequence $\left\{\hat{w}_n\right\} \subset \mathcal{N}_{V_{\min} P_{\max} Q_{\max}}$ satisfying

$$\hat{w}_n - \hat{u}_n \to 0 \text{ in } E, \ I_{V_{\min} P_{\max} Q_{\max}}\left(\hat{w}_n\right) \to m_{V_{\min} P_{\max} Q_{\max}}, \ I'_{V_{\min} P_{\max} Q_{\max}}\left(\hat{w}_n\right) \to 0.$$

Therefore,

$$\hat{w}_n \rightharpoonup \hat{u} \text{ in } E$$

and $\hat{u} = t_0 \tilde{u}$ is a nontrivial critical point of $I_{V_{\min} P_{\max} Q_{\max}}$. Then

$$\begin{aligned} & m_{V_{\min} P_{\max} Q_{\max}} \\ & \quad \leq I_{V_{\min} P_{\max} Q_{\max}}(\hat{u}) - \frac{1}{p} \left\langle I'_{V_{\min} P_{\max} Q_{\max}}(\hat{u}), \hat{u} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}(|\nabla \hat{u}|^2+V_{\min}|\hat{u}|^2)\mathrm{d}x+\left(\frac{1}{p}-\frac{1}{2(6-\mu)}\right)Q_{\max}^2\widetilde{F}(\hat{u})\\
&\le \liminf_{n\to\infty}\left(\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}\left(|\nabla \hat{w}_n|^2+V_{\min}|\hat{w}_n|^2\right)\mathrm{d}x+\left(\frac{1}{p}-\frac{1}{2(6-\mu)}\right)Q_{\max}^2\widetilde{F}(\hat{w}_n)\right)\\
&= \liminf_{n\to\infty}\left(I_{V_{\min}P_{\max}Q_{\max}}(\hat{w}_n)-\frac{1}{p}\left\langle I'_{V_{\min}P_{\max}Q_{\max}}\left(\hat{w}_n\right),\hat{w}_n\right\rangle\right)\\
&= m_{V_{\min}P_{\max}Q_{\max}}.
\end{aligned}
$$

Thus

$$\tilde{u}_n \to \tilde{u} \text{ in } E. \tag{5.2}$$

Now, we are going to prove that $\varepsilon_n y_n \to \xi \in \mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}$. We first claim that $\{\varepsilon_n y_n\}$ must be bounded. Otherwise, $|\varepsilon_n y_n| \to \infty$ as $n \to \infty$. For any small $\delta > 0$, there exists $\rho = \rho(\delta) > 0$, such that, for $|x| \ge \rho$,

$$V(x) > V_\infty - \delta,\ P(x) < P^\infty + \delta \text{ and } Q(x) < Q^\infty + \delta. \tag{5.3}$$

For $u \in E$, define

$$
\begin{aligned}
I_\delta(u) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+(V_\infty-\delta)|u|^2\right)\mathrm{d}x\\
&-\frac{1}{p}\int_{\mathbb{R}^3}(P^\infty+\delta)|u|^p\mathrm{d}x-\frac{1}{2(6-\mu)}(Q^\infty+\delta)^2\widetilde{F}(u).
\end{aligned}
$$

Then, we can introduce

$$m_\delta = \inf_{u\in\mathcal{N}_\delta} I_\delta(u),$$

where $\mathcal{N}_\delta = \{u \in E : \langle I'_\delta(u), u\rangle = 0\}$.

By Lemma 3.2 and condition (H_1), we have $m_{V_{\min}P_{\max}Q_{\max}} < m_\infty$. Noting the continuity of m_δ about δ, we can obtain $m_{V_{\min}P_{\max}Q_{\max}} < m_\delta$ for δ small. For u_n, there exists $\tilde{t}_n > 0$ such that $\tilde{t}_n u_n \in \mathcal{N}_\delta$. It is easy to see that $\{\tilde{t}_n\}$ is bounded. For any small $\sigma > 0$, from (5.2), there exists $R > 0$ and N big enough, such that

$$\int_{B_R^c(0)}(|\nabla u_n(x+y_n)|^2+u_n^2(x+y_n))\mathrm{d}x < \sigma,\ \text{for any } n \ge N.$$

Thus,

$$\int_{B_R^c(y_n)}(|\nabla u_n(x)|^2+u_n^2(x))\mathrm{d}x < \sigma,\ \text{for any } n \ge N. \tag{5.4}$$

From $|\varepsilon_n y_n| \to \infty$ as $n \to \infty$, we can get $B_R(y_n) \cap B_{\frac{\rho}{\varepsilon_n}}(0) = \emptyset$. Then, by using (5.4), we have

$$\Big| \int_{B_{\frac{\rho}{\varepsilon_n}}(0)} (V(\varepsilon_n x) - V^{\infty})|\tilde{t}_n u_n|^2 \mathrm{d}x \Big| < C\sigma$$

and

$$\Big| \int_{B_{\frac{\rho}{\varepsilon_n}}(0)} (P(\varepsilon_n x) - P^{\infty})|\tilde{t}_n u_n|^p \mathrm{d}x \Big| < C\sigma.$$

Thus, noting (5.3), we can get

$$\begin{aligned}
I_{\varepsilon_n}(u_n) &\geq I_{\varepsilon_n}(\tilde{t}_n u_n) \\
&= I_\delta(\tilde{t}_n u_n) + \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon_n x) - (V^{\infty} - \delta))|\tilde{t}_n u_n|^2 \mathrm{d}x \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} (P(\varepsilon_n x) - (P^{\infty} + \delta))|\tilde{t}_n u_n|^p \mathrm{d}x \\
&\quad - \frac{1}{2(6-\mu)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(Q(\varepsilon_n x)Q(\varepsilon_n y) - (Q^{\infty}+\delta)^2)|\tilde{t}_n u_n(x)|^{6-\mu}|\tilde{t}_n u_n(y)|^{6-\mu}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y \\
&\geq I_\delta(\tilde{t}_n u_n) + \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon_n x) - (V^{\infty} - \delta))|\tilde{t}_n u_n|^2 \mathrm{d}x \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} (P(\varepsilon_n x) - (P^{\infty} + \delta))|\tilde{t}_n u_n|^p \mathrm{d}x \\
&\geq I_\delta(\tilde{t}_n u_n) + \frac{1}{2}\int_{B_{\frac{\rho}{\varepsilon_n}}(0)} (V(\varepsilon_n x) - (V^{\infty} - \delta))|\tilde{t}_n u_n|^2 \mathrm{d}x \\
&\quad - \frac{1}{p}\int_{B_{\frac{\rho}{\varepsilon_n}}(0)} (P(\varepsilon_n x) - (P^{\infty} + \delta))|\tilde{t}_n u_n|^p \mathrm{d}x \\
&\geq m_\delta - C\sigma.
\end{aligned}$$

Therefore, $m_{V_{\min}P_{\max}Q_{\max}} \geq m_\delta$, which is a contradiction.

Up to a subsequence, assume that $\varepsilon_n y_n \to \xi$. Hence, it suffices to show that $V(\xi) = V_{\min}$, $P(\xi) = P_{\max}$ and $Q(\xi) = Q_{\max}$. Arguing by contradiction again, we assume that $V(\xi) > V_{\min}$, $P(\xi) < P_{\max}$ or $Q(\xi) < Q_{\max}$. Since

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} V(\varepsilon_n y_n + \varepsilon_n x)\,\hat{u}_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} V(\xi)\hat{u}^2 \mathrm{d}x,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^3} P(\varepsilon_n y_n + \varepsilon_n x)\,|\hat{u}_n|^p \mathrm{d}x = \int_{\mathbb{R}^3} P(\xi)|\hat{u}|^p \mathrm{d}x$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q(\varepsilon_n x_n + \varepsilon_n x)Q(\varepsilon_n y_n + \varepsilon_n y)|\hat{u}_n(x)|^{6-\mu}|\hat{u}_n(y)|^{6-\mu}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y$$

$$= Q^2(\xi) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\hat{u}(x)|^{6-\mu} |\hat{u}(y)|^{6-\mu}}{|x-y|^{\mu}} \mathrm{d}x \mathrm{d}y,$$

we can obtain

$$\begin{aligned} m_{V_{\min} P_{\max} Q_{\max}} &\leq I_{V_{\min} P_{\max} Q_{\max}}(\hat{u}) \\ &< \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \hat{u}|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\xi) \hat{u}^2 - \frac{1}{p} \int_{\mathbb{R}^N} P(\xi) |\hat{u}|^p - \frac{1}{2(6-\mu)} Q^2(\xi) \tilde{F}(\hat{u}) \\ &= \lim_{n \to \infty} I_{\varepsilon_n}(t_n u_n) \leq \lim_{n \to \infty} I_{\varepsilon_n}(u_n) = m_{V_{\min} P_{\max} Q_{\max}}, \end{aligned}$$

which is a contradiction. Therefore, $V(\xi) = V_{\min}$, $P(\xi) = P_{\max}$, and $Q(\xi) = Q_{\max}$, and the proof is completed. □

Lemma 5.4 *For any* $\delta > 0$, *there holds that*

$$\lim_{\varepsilon \to 0} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta) = 0.$$

Proof Let $\{\varepsilon_n\} \subset (0, +\infty)$ be such that $\varepsilon_n \to 0$. By definition, there exists $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$dist(\beta_{\varepsilon_n}(u_n), (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta) = \sup_{u \in \tilde{\mathcal{N}}_{\varepsilon_n}} dist(\beta_{\varepsilon_n}(u), (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta) + o_n(1).$$

So, it suffices to find a sequence $\{\xi_n\} \subset (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta$ satisfying

$$\lim_{n \to +\infty} |\beta_{\varepsilon_n}(u_n) - \xi_n| = 0. \tag{5.5}$$

By Lemma 5.3, we can obtain $\tilde{u} \in E$ such that $u_n(x + y_n) \to \tilde{u}$ in E, and, up to a subsequence, $\varepsilon_n y_n \to \xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$. Thus, $\varepsilon_n y_n \in (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta$ for n large enough. It is easy to see that

$$\beta_{\varepsilon_n}(u_n) = \varepsilon_n y_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n x + \varepsilon_n y_n) - \varepsilon_n y_n) |u_n(x + y_n)|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |u_n(x + y_n)|^2 \mathrm{d}x}.$$

Set $\xi_n = \varepsilon_n y_n$. We have that the sequence $\{\xi_n\}$ satisfies (5.5). This completes the proof. □

Lemma 5.5 *[4] Let I be a C^1 functional defined on a C^1 Finsler manifold M. If I is bounded from below and satisfies the (PS) condition, then I has at least $cat_M M$ distinct critical points.*

Lemma 5.6 *[2] Let $\Gamma, \Omega^+, \Omega^-$ be closed sets with $\Omega^- \subset \Omega^+$. Let $\Phi : \Omega^- \to \Gamma$, $\beta : \Gamma \to \Omega^+$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $Id : \Omega^- \to \Omega^+$. Then $cat_\Gamma \Gamma \geq cat_{\Omega^+} \Omega^-$.*

The proof of Theorem 1.1: (i) For a fixed $\delta > 0$, by Lemmas 5.1 and 5.4, we know that there exists a $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}_\varepsilon} \xrightarrow{\beta_\varepsilon} (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta$$

is well defined. From Lemma 5.2, for ε small enough, there is a function $\lambda(\xi)$ with $|\lambda(\xi)| < \frac{\delta}{2}$ uniformly in $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$, such that $\beta_\varepsilon(\Phi_\varepsilon(\xi)) = \xi + \lambda(\xi)$ for all $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$. Define $H(t, \xi) = \xi + (1-t)\lambda(\xi)$. Then, $H : [0, 1] \times \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \to (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta$ is continuous. Obviously, $H(0, \xi) = \beta_\varepsilon(\Phi_\varepsilon(\xi))$, $H(1, \xi) = \xi$ for all $\xi \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$. That is, $H(t, \xi)$ is homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon$ and the inclusion map $id : \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \to (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta$. By Lemma 5.6, we obtain

$$cat_{\tilde{\mathcal{N}_\varepsilon}} \tilde{\mathcal{N}_\varepsilon} \geq cat_{(\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta} (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}).$$

On the other hand, using the definition of $\tilde{\mathcal{N}_\varepsilon}$ and choosing ε_δ small if necessary, we see that I_ε satisfies the (PS) condition in $\tilde{N}_\varepsilon$ recalling (H_1) and Lemma 4.4. By Lemma 5.5, we know that I_ε has at least $cat_{\tilde{\mathcal{N}_\varepsilon}}(\tilde{\mathcal{N}_\varepsilon})$ critical points on $\mathcal{N}_\varepsilon$. By Lemma 2.11, these points are critical points of I_ε in E. Consequently, we see that the problem (1.1) has at least $cat_{(\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})_\delta} (\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q})$ solutions.

(ii) By the definition of c_ε and $m_{V_{\min} P_{\max} Q_{\max}}$, we have $m_{V_{\min} P_{\max} Q_{\max}} \leq c_\varepsilon$. Then,

$$\liminf_{\varepsilon \to 0} c_\varepsilon \geq m_{V_{\min} P_{\max} Q_{\max}}.$$

It follows from Lemma 3.3 that

$$\lim_{\varepsilon \to 0} c_\varepsilon = m_{V_{\min} P_{\max} Q_{\max}}.$$

By Lemma 5.3, there exist $\{y_n\} \subset \mathbb{R}^3$ and $v \in E$ such that

$$u_n(x + y_n) \to v, \text{ in } E.$$

Now we prove that v is a ground state solution of equation 1.2. For any $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, since $I'_{\varepsilon_n}(u_n) = 0$, we have $\langle I'_{\varepsilon_n}(u_n(x)), \psi(x - y_n) \rangle = 0$. By direct computation, it is easy to get

$$\langle I'_{V_{\min} P_{\max} Q_{\max}}(v), \psi \rangle = 0.$$

On the other hand,

$$\begin{aligned} & m_{V_{\min} P_{\max} Q_{\max}} \\ & \quad = \lim_{n \to \infty} I_{\varepsilon_n}(u_n) \\ & \quad = \lim_{n \to \infty} \left[I_{\varepsilon_n}(u_n) - \frac{1}{2(6-\mu)} \langle I'_{\varepsilon_n}(u_n), u_n \rangle \right] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big[\Big(\frac{1}{2}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(\varepsilon_n x)u_n^2)\mathrm{d}x-\Big(\frac{1}{p}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}P(\varepsilon_n x)|u_n|^p\mathrm{d}x\Big]\\
&= \Big(\frac{1}{2}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}(|\nabla v|^2+V(\xi)v^2)\mathrm{d}x-\Big(\frac{1}{p}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}P(\xi)|v|^p\mathrm{d}x\\
&= \Big(\frac{1}{2}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}(|\nabla v|^2+V_{\min}v^2)\mathrm{d}x-\Big(\frac{1}{p}-\frac{1}{2(6-\mu)}\Big)\int_{\mathbb{R}^3}P_{\max}|v|^p\mathrm{d}x\\
&= I_{V_{\min}P_{\max}Q_{\max}}(v)-\frac{1}{2(6-\mu)}\langle I'_{V_{\min}P_{\max}Q_{\max}}(v),v\rangle\\
&= I_{V_{\min}P_{\max}Q_{\max}}(v).
\end{aligned}
$$

Thus, v is a ground solution of Eq. (1.2). □

Acknowledgements Y. Chen is supported by National Natural Science Foundation of China (No. 12161007), Guangxi science and technology base and talent project (AD21238019), and Doctoral Foundation of Guangxi University of Science and Technology (20Z30). Z. Yang is supported by National Natural Science Foundation of China (Nos. 11771385, 11961081).

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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