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On Boundedness of Oscillating Multipliers on Stratified Lie Groups

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Abstract
In this paper, we study the oscillating spectral multipliers associated with the sub-Laplacian L on an arbitrary stratified Lie group G. We prove the boundedness of the operators $m_{\alpha,\beta,t}(L) = \psi(L)L^{-\beta/2}e^{itL^{\alpha/2}}$ on Hardy spaces $H^p(G)$ for all $p \in (0, \infty)$ and $\beta/\alpha \geq Q|1/p - 1/2|$, where ψ is a smooth function on $[0, \infty)$ vanishing on $[0, a]$ and equal to 1 on $[b, \infty)$ for some $0 < a < b < \infty$, and Q is the homogeneous dimension of G. This extends the existing results and can be applied to obtain L^p estimates for Riesz means of the Schrödinger operators associated with the fractional powers of L.

Keywords Stratified Lie group · Sub-Laplacian · Oscillating multiplier · Hardy space

Mathematics Subject Classification 42B15 · 42B30 · 43A22

1 Introduction and Statement of Main Results

Given a nonnegative self-adjoint operator L, the oscillating spectral multiplier operators associated with L are operators of the form

$$m_{\alpha,\beta}(L) = \psi(L)L^{-\beta/2}e^{iL^{\alpha/2}}. \tag{1.1}$$

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where $\alpha, \beta > 0$ and ψ is a smooth function on $[0, \infty)$ which vanishes on $[0, a]$ and is equal to 1 on $[b, \infty)$ for some $0 < a < b < \infty$. These operators has a strong connection with the Cauchy problem for Schrödinger and wave equations associated with fractional powers of L. In many situations, they also provide examples of the so-called strongly singular integral operators.

Oscillating multipliers in the context of $\mathbb{R}^n$ have been studied extensively; see [18, 19, 32–34]. Then some of the results have been extended to more abstract settings such as Lie groups, Riemannian manifolds, symmetric spaces and metric measure spaces. We refer to [1, 3, 5, 12, 13, 15, 17, 22, 27, 31, 35, 36] and the references therein. We now give a brief discussion on the work [15] which is closely related to our paper. In [15], the endpoint estimates of the oscillating multipliers on an arbitrary stratified Lie group G were investigated. More precisely, they introduced a class of spectral multipliers on G, which covers not only the Mihlin-Hörmander-type spectral multipliers studied by Folland, Hulanicki and Stein [20], Mauceri and Meda [31] and Christ [14], but also the oscillating multipliers of the form (1.1). Their main result implies that, if $\alpha > 0$ and $\beta/\alpha \geq Q/2$, then

$$\|m_{\alpha,\beta}(L)f\|_{L^1(G)} \leq C\|f\|_{H^1(G)}, \tag{1.2}$$

where L is the sub-Laplacian on G, and Q is the homogeneous dimension of G. By using the complex interpolation theorem they also proved that for $p \in (1, \infty)$, $\alpha > 0$ and $\beta/\alpha \geq Q|1/p - 1/2|$,

$$\|m_{\alpha,\beta}(L)f\|_{L^p(G)} \leq C\|f\|_{L^p(G)}. \tag{1.3}$$

It is worth pointing out that the estimate above improves the those in (see [1, 31]), in which it was proved that (1.3) holds true if $\beta/\alpha > Q|1/p - 1/2|$. In the special case of the Heisenberg type groups, sharp estimates of oscillating multipliers with minimal smoothness (depending on the topological dimensions rather than the homogeneous ones) were obtained recently in [3, 36].

In the present paper, we consider oscillating multipliers on general stratified Lie groups. Our main aim is to extend the estimates (1.2) and (1.3) to the full range $p \in (0, \infty)$. Before stating our results, let us briefly recall some basic concepts concerning stratified Lie groups. For more details, we refer to the monographs [2, 20]. A Lie group G is said to be stratified if it is connected and simply connected, and its Lie algebra $\mathfrak{g}$ can be decomposed as a direct sum $\mathfrak{g} = V_1 \oplus \cdots \oplus V_\tau$, with $[V_1, V_k] = V_{k+1}$ for $1 \leq k \leq \tau - 1$ and $[V_1, V_\tau] = 0$. Such a group G is necessarily nilpotent, and the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism which takes the Lebesgue measure on $\mathfrak{g}$ to a bi-invariant Haar measure μ on G. The number

$$Q = \sum_{j=1}^{\tau} \dim(V_j)$$

is called the homogeneous dimension of G. One can define, in a canonical (natural) way, a family of dilations $\{D_t\}_{t>0}$ on G that adapted to the stratification. A homogeneous quasi-norm on G is a function $x \mapsto |x|$ from G to $[0, \infty)$ which vanishes only at the group identity e and satisfies that $|x^{-1}| = |x|$ and $|D_t x| = t|x|$ for all $x \in G$ and $t > 0$. Note that there exits at least one homogeneous quasi-norm on G; moreover, any two homogeneous quasi-norms on G are equivalent (see [20]). Henceforth we fix a homogeneous quasi-norm on G. It satisfies a quasi-triangle inequality: there exists a constant $\gamma \geq 1$ such that

$$|xy| \leq \gamma(|x| + |y|) \tag{1.4}$$

for all $x, y \in G$.

Let $n_1 = \dim(V_1)$, and fix a basis $X_1, \cdots, X_{n_1}$ for V_1. Identifying each X_j with a left-invariant vector field on G, we consider the sub-Laplacian

$$L = -\sum_{j=1}^{n_1} X_j^2.$$

For any bounded Borel measurable function m on $\mathbb{R}_+ := [0, \infty)$, we can define the spectral multiplier operator

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda,$$

where $\{E_\lambda\}_{\lambda \geq 0}$ is the spectral resolution of L. The operator $m(L)$ is bounded on $L^2(G)$ with operator norm bounded by $\|m\|_{L^\infty(\mathbb{R}_+)}$.

For any $\alpha, \beta > 0$ and $t \in \mathbb{R}$, we defined the functions $m_{\alpha,\beta,t}$ and $\widetilde{m}_{\alpha,\beta,t}$ on $\mathbb{R}_+$ by

$$m_{\alpha,\beta,t}(\lambda) := \psi(\lambda)\lambda^{-\beta/2} e^{it\lambda^{\alpha/2}},$$
$$\widetilde{m}_{\alpha,\beta,t}(\lambda) := (1+\lambda)^{-\beta/2} e^{it\lambda^{\alpha/2}},$$

respectively, where, as before, ψ is a smooth function on $[0, \infty)$ which vanishes on $[0, a]$ and is equal to 1 on $[b, \infty)$ for some $0 < a < b < \infty$.

The main results of the present paper are the following two theorems.

Theorem 1.1 *Let G be a stratified Lie group and let L be the sub-Laplacian on G. Then for $p \in (0, 1)$, $\alpha \in (0, \infty)$ and $\beta/\alpha \geq Q(1/p - 1/2)$,*

$$\left\|\widetilde{m}_{\alpha,\beta,t}(L) f\right\|_{H^p(G)} \leq C(1+|t|)^{Q(1/p-1/2)} \|f\|_{H^p(G)}, \quad \forall t \in \mathbb{R}, \tag{1.5}$$

where $H^p(G)$ are the Hardy spaces on G, and C is a constant independent of t.

Theorem 1.2 *Let G be a stratified Lie group and let L be the sub-Laplacian on G. Then for $p \in (0, \infty)$, $\alpha \in (0, \infty)$ and $\beta/\alpha \geq Q|1/p - 1/2|$, we have*

$$\left\|m_{\alpha,\beta,t}(L) f\right\|_{H^p(G)} \leq C(1+|t|)^{Q|1/p-1/2|} \|f\|_{H^p(G)}, \quad \forall t \in \mathbb{R}, \tag{1.6}$$

where C is a constant independent of t.

Some comments on Theorems 1.1 and 1.2 are in order.

(i) In the context of $\mathbb{R}^n$, the estimates (1.5) and (1.6) were proved by Miyachi [32]. By using the complex interpolation theorem of Calderón and Torchinsky [9], he showed that (1.5) also holds for $p \in [1, \infty)$. However, it is unclear to us whether such a complex interpolation theorem is valid for Hardy spaces on stratified Lie groups. Thus, we do not know whether (1.5) holds for $p \in [1, \infty)$.

(ii) In contrast with the work [15], we consider time-dependent oscillating multipliers, and derive the bound $(1 + |t|)^{Q|1/p-1/2|}$ for the operators $m_{\alpha,\beta,t}(L)$ in Theorem 1.2. It is worth pointing out that the approach in [15] along with a homogeneity argument can only give the bound $(1 + t)^{Q|1/p-1/2|+\varepsilon}$ (where ε is any positive number) for $m_{\alpha,\beta,t}(L)$. Thus, our results can be applied to get a better estimate for the Riesz means associated with the fractional powers of L (see Corollary 1.3 below).

(iii) Letting $t = 1$ and $p = 1$ in (1.6), we have $H^1(G) \to H^1(G)$ estimate for $m_{\alpha,\beta}(L)$, which is stronger than the $H^1(G) \to L^1(G)$ estimate in (1.2) due to the fact that $H^1(G) \hookrightarrow L^1(G)$. Moreover, Theorem 1.2 completes the scale of the estimates of $m_{\alpha,\beta,t}(L)$ for all $p \in (0, \infty)$ and $t \in \mathbb{R}$, while (1.2) and (1.3) only provide estimates of $m_{\alpha,\beta}(L)$ for $p \in [1, \infty)$.

We now discuss an application of Theorem 1.2 to the study of Riesz means associated with the fractional powers of L. For $k, \alpha, t > 0$, defined the operators

$$I_{k,\alpha,t}(L) = kt^{-k} \int_0^t (t-s)^{-k-1} e^{-isL^{\alpha/2}} ds.$$

We extend the definition of $I_{k,\alpha,t}(L)$ to $t < 0$ by setting

$$I_{k,\alpha,t}(L) = \bar{I}_{k,\alpha,-t}(L), \quad t < 0.$$

See [33, 38] for the study of these operators on $\mathbb{R}^n$ and [1, 8, 30] for their generalizations to more general contexts.

By using Theorem 1.2, the spectral theorem for $H^p(G)$ (see, e.g., [31]), and a standard argument from [38], we can derive the following result.

Corollary 1.3 *For $p \in (0, \infty)$ and $k \geq Q|1/p - 1/2|$, there exists a constant C such that*

$$\left\| I_{k,\alpha,t}(L) f \right\|_{H^p(G)} \leq C \|f\|_{H^p(G)}$$

for all $t \neq 0$.

Remark 1.4 In the context of general Lie groups of polynomial growth, the best known result concerning boundedness of $I_{k,\alpha,t}(L)$ so far is that for $p \in [1, \infty)$

$$\left\| I_{k,\alpha,t}(L) f \right\|_{L^p(G)} \leq C \|f\|_{L^p(G)} \tag{1.7}$$

provided (i) $0 < \alpha \leq 1$ and $k > d|1/p - 1/2|$, or (ii) $\alpha > 1$ and $k > \max(d, D)|1/p - 1/2|$, where d and D are the local dimension and dimension at infinity of G, respectively (see [1,Theorem 3]). In the particular case of stratified Lie groups, Corollary 1.3 not only sharpens (1.7) by allowing $k = Q|1/p - 1/2|$, but also extends it to all $p \in (0, \infty)$.

A few words about our proofs are in order. The proofs of our main results are inspired by the ideas and techniques developed in [8, 10, 11]. Note that in [8, 10, 11], the boundedness of the Schrödinger group corresponding to the multipliers $\widetilde{m}_{\alpha,\beta,t}(L)$ with $\alpha = 2$. However, it is not clear if the approaches in [8, 10, 11] can be applicable to the general case $\alpha > 0$. To deal with the general case of α, we shall employ an interesting weighted L^2 estimate on stratified groups due to Sikora [37]. In order to derive the $H^p(G) \to H^p(G)$ boundedness of $\widetilde{m}_{\alpha,\beta,t}(L)$ for $0 < p < 1$, we will utilize several equivalent characterizations of Hardy spaces on stratified Lie groups, including the characterizations via the radial maximal function, the Littlewood–Paley square function, the Lusin function and the atomic decomposition. The $H^p(G) \to H^p(G)$ boundedness of $m_{\alpha,\beta,t}(L)$ for $0 < p < 1$ then follows from that of $\widetilde{m}_{\alpha,\beta,t}(L)$ and a spectral multiplier theorem for $H^p(G)$. Finally, to prove the $H^p(G) \to H^p(G)$ boundedness of $m_{\alpha,\beta,t}(L)$ for $1 \leq p < \infty$, we will identify $H^p(G)$ with the homogeneous Triebel-Lizorkin spaces $\dot{F}^0_{p,2}(G)$, and use complex interpolation of the spaces $\dot{F}^s_{p,q}(G)$. Similar ideas were also used in [7] to prove the boundedness of Schrödinger groups associated with fractional powers of the Hermite operators on $\mathbb{R}^n$.

The organization of this paper is as follows. In Sect. 2, we recall the definition of the Hardy spaces $H^p(G)$, and collect some of their equivalent characterizations, which will be needed in establishing the $H^p(G) \to H^p(G)$ boundedness of $\widetilde{m}_{\alpha,\beta,t}(L)$. In particular, the Littlewood–Paley characterization of $H^p(G)$ implies that $H^p(G)$ can be identified with the homogeneous Triebel-Lizorkin space $\dot{F}^0_{p,2}(G)$. The proofs of our main results Theorems 1.1 and 1.2 will be given in Sects. 3 and 4, respectively.

Notation Throughout this paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers, while $\mathbb{N}$ denotes the set of all positive integers. We always use C to denote positive constants, which are independent of the main parameters involved and whose values may vary at every occurrence. By writing $f \lesssim g$, we mean that $f \leq Cg$. We also use $f \sim g$ to denote that $C^{-1}g \leq f \leq Cg$.

2 Hardy Spaces on Stratified Lie Groups and Their Characterizations

Throughout this section, G is a stratified Lie group and L is the sub-Laplacian on G. Our purpose in this section is to recall the definition of Hardy spaces $H^p(G)$ and give several equivalent characterizations of these spaces.

2.1 Definition of $H^p(G)$ Via Radial Maximal Function

Hardy spaces on general homogeneous groups were introduced and studied by Folland and Stein in [20]. They proved several equivalent characterizations of these spaces, including the radial maximal function characterization, the nontangential maximal function characterization, the grand maximal function characterization and atomic decomposition. For the sake of simplicity, we take the radial maximal function characterization as the definition of $H^p(G)$. Following [20, p.140], a Schwartz function Φ on G is said to be a *commutative approximate identity*, if $\int_G \Phi(x)d\mu(x) = 1$ and $\Phi_t * \Phi_s = \Phi_s * \Phi_t$ holds for all $t, s > 0$, where $\Phi_t(x) := t^{-Q}\Phi(t^{-1}x)$.

Definition 2.1 ([20]) Let $\Phi \in \mathcal{S}(G)$ be a commutative approximate identity on G. For $0 < p < \infty$, the Hardy space $H^p(G)$ is defined as the set of all $f \in \mathcal{S}'(G)$ such that

$$\|f\|_{H^p(G)} := \left\|M_\Phi^0 f\right\|_{L^p(X)} < \infty,$$

where $M_\Phi^0 f$ is the radial maximal function defined by

$$M_\Phi^0 f(x) = \sup_{t>0} |f * \Phi_t f(x)|.$$

Remark 2.2 The definition of $H^p(G)$ is independent of the choice of the commutative approximate identity Φ. Moreover, the spaces $H^p(G)$ are also characterized in terms of the nontangential and grand maximal functions. For these results, see [20,Corollary 4.17].

In the context of stratified Lie groups, it is convenient to construct commutative approximate identity via sub-Laplacians. Indeed, Hulanicki's theorem says that if $\phi \in \mathcal{S}(\mathbb{R}_+)$, then the convolution kernel of $\phi(L)$, denoted by Φ, is in $\mathcal{S}(G)$ (see [26]). Furthermore, since the convolution kernel of $\phi(t^2L)$ is Φ_t and since $\phi(s^2L)\phi(t^2L) = \phi(t^2L)\phi(s^2L)$, we have $\Phi_t * \Phi_s = \Phi_s * \Phi_t$. Hence, if $\phi \in \mathcal{S}(\mathbb{R}_+)$ such that $\phi(0) = 1$, then for $p \in (0, \infty)$,

$$f \in H^p(G) \Longleftrightarrow M_{\phi,L}^0 f \in L^p(G), \tag{2.1}$$

where

$$M_{\phi,L}^0 f(x) := \sup_{t>0} |\phi(t^2L) f(x)|.$$

Remark 2.3 For $1 < p < \infty$, we have $H^p(G) = L^p(G)$; see [20, p. 75].

2.2 Characterization of $H^p(G)$ Via the Lusin Functions

The theory of Hardy spaces associated with nonnegative self-adjoint operators satisfying Davies-Gaffney estimates were studied in [23] (for $p \geq 1$) and [16, 28] (for $0 < p < 1$). The Hardy space theory developed in these works can be applied to our setting. Given a function $f \in L^2(G)$, consider the following Lusin function associated with the sub-Laplacian L

$$\mathcal{S}_L f(x) := \left(\int_0^\infty \int_{|y^{-1}x|<t} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \frac{d\mu(y)dt}{t^{Q+1}}\right)^{1/2}. \tag{2.2}$$

Definition 2.4 ([23, 28]) For $p \in (0, \infty)$, the Hardy space $H_L^p(G)$ associated with L is defined as the completion of $\{f \in L^2(G) : \mathcal{S}_L f \in L^p(G)\}$ with (quasi-)norm

$$\|f\|_{H_L^p(G)} := \|\mathcal{S}_L f\|_{L^p(G)}.$$

In [39], Song and Yan proved that, if (X, d, μ) is a metric measure space satisfying volume doubling condition, and $\mathcal{L}$ is a nonnegative self-adjoint operator whose heat kernel satisfies the Gaussian upper bound, then the Hardy spaces defined via the radial maximal function coincide with those defined via the Lusin function. It is well known that the heat kernel of a sub-Laplacian L on a stratified Lie group G satisfies the Gaussian upper bound (see, e.g., [40,Theorem in VIII.2.7]). Hence, combining (2.1) and [39,Theorem 1.3] (see also the final remark in [39]), we deduce the following result.

Proposition 2.5 *For $p \in (0, \infty)$, $H^p(G) = H_L^p(G)$ with equivalent (quasi-)norms.*

2.3 Characterization of $H^p(G)$ Via Littlewood–Paley Functions

In this subsection we give the Littlewood–Paley characterization of $H^p(G)$. More precisely we will identify $H^p(G)$ with the homogeneous Triebel-Lizorkin spaces $\dot{F}^0_{p,2}(G)$. Homogeneous Triebel-Lizorkin spaces on stratified Lie groups were studied in [25]. Recently, there have been also some important developments on Triebel-Lizorkin spaces associated with abstract nonnegative self-adjoint operators which cover the case of sub-Laplacians on stratified Lie groups; see, e.g., [4, 21, 29]. Based on these works, the definition of Triebel-Lizorkin spaces associated with sub-Laplacians on stratified Lie groups becomes natural.

In what follows, by a "partition of unity" we will mean a function a function $\varphi \in \mathcal{S}(\mathbb{R}_+)$ such that $\operatorname{supp} \varphi \subset [1/4, 4]$, $\int \varphi(\lambda) \frac{d\lambda}{\lambda} \neq 0$ and

$$\sum_{\ell \in \mathbb{Z}} \varphi(2^{-2\ell}\lambda) = 1 \quad \forall \lambda \in (0, \infty).$$

Definition 2.6 Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty]$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(G)$ is defined as the collection of all $f \in \mathcal{S}'(G)/\mathcal{P}$ such that

$$\|f\|_{\dot{F}^s_{p,q}(G)} := \left\| \left(\sum_{\ell \in \mathbb{Z}} \left(2^{\ell s} |\varphi(2^{-2\ell}L) f|\right)^q \right)^{1/q} \right\|_{L^p(G)} < \infty,$$

with the usual modification when $q = \infty$. Here $\mathcal{P}$ denotes the space of polynomials on G.

Remark 2.7 In Definition 2.6, $\varphi(2^{-2\ell}L)f$ is well defined for $f \in \mathcal{S}'(G)/\mathcal{P}$, since the convolution kernel of $\varphi(2^{-2\ell}L)$ is a Schwartz function on G having all vanishing moments (see [25]).

Remark 2.8 It is proved in [25] that the scale of $\dot{F}^s_{p,q}(G)$ is independent of choice of the sub-Laplacian L and the "partition of unity" function φ. This indicates that these spaces reflect properties of the group G, not of the sub-Laplacian used for the construction of the Littlewood–Paley decomposition.

Note that the spaces $\dot{F}^s_{p,q}(G)$ fall into the scope of general theory of Triebel-Lizorkin spaces associated with nonnegative self-adjoint operators. In particular, from [4,Corollary 3.15] we see that, for $p \in (0,\infty)$,

$$f \in \dot{F}^0_{p,2}(G) \Longleftrightarrow \mathcal{S}_L f \in L^p(G).$$

This, in combination with Proposition 2.5, yields the following result.

Proposition 2.9 *For $p \in (0,\infty)$, $H^p(G) = \dot{F}^0_{p,2}(G)$ with equivalent (quasi-)norms.*

2.4 Atomic Decomposition of $H^p(G)$

Folland and Stein [20] established an atomic decomposition of $H^p(G)$. Their atoms are defined in a similar manner as the classical H^p atoms in $\mathbb{R}^n$. However, in the present paper, since we are concerned with oscillating spectral multipliers, it is more convenient to use a version of atoms associated with the sub-Laplacian L.

Definition 2.10 ([16, 23, 28]) Let $p \in (0,1]$ and $M \in \mathbb{N}$. A measurable function a on G is called a (p, M, L)-atom, if there exist a function $b \in D(L^M)$ and a ball $B = B(x_B, r_B) \subset \mathbb{G}$ such that

(i) $a = L^M b$;
(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, 1, \cdots, M$;
(iii) $\|L^k b\|_{L^2(G)} \le r_B^{2(M-k)} \mu(B)^{1/2-1/p}$, $k = 0, 1, \cdots, M$.

The atomic Hardy space $H^p_{L,\mathrm{at},M}(G)$ is then defined to be set of all $f \in \mathcal{S}'(G)$ of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

with convergence in $\mathcal{S}'(G)$, where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^p$ and each a_j is a (p, M, L)-atom. Finally, the quasi-norm of $f \in H^p_{L,\mathrm{at},M}(G)$ is given by

$$\|f\|_{H^p_{L,\mathrm{at},M}(G)} := \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j, \text{ each } a_j \text{ is a } (p, M, L)\text{-atom} \right\}.$$

In [16, 23, 28], the atomic decomposition of the Hardy spaces $H^p_{\mathcal{L}}(X)$ associated with an abstract nonnegative self-adjoint operator $\mathcal{L}$ on a doubling metric measure space X was established. Recall that $H^p_{\mathcal{L}}(X)$ are defined via the Lusin function (2.2). The result in [16, 23, 28] together with Proposition 2.5 implies the following result.

Proposition 2.11 *Let $p \in (0, 1]$ and $M > \frac{Q(2-p)}{4p}$. Then $H^p(G) = H^p_{L,\mathrm{at},M}(G)$ with equivalent (quasi-)norms.*

2.5 A Complex Interpolation Theorem for $\dot{F}^s_{p,q}(G)$

We record a complex interpolation theorem for homogeneous Triebel-Lizorkin spaces on G, which will be needed in the proof of Theorem 1.2.

Proposition 2.12 (see [6,Proposition 3.18]). *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 < \infty$, and $0 < \theta < 1$. If*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then

$$\left[\dot{F}^{s_0}_{p_0,q_0}(G), \dot{F}^{s_1}_{p_1,q_1}(G)\right]_\theta = \dot{F}^s_{p,q}(G),$$

where $[\cdot,\cdot]_\theta$ stands for the complex interpolation brackets.

3 Proof of Theorem 1.1

Our proof will rely on the following weighted L^2 estimate due to A. Sikora.

Lemma 3.1 *([37]) Let G be a stratified Lie group and let L be the sub-Laplacian on G. Then for any $s > 0$, there exists a constant C such that*

$$\int_G |K_{F(L)}(x)|^2 (1+|x|)^{2s} d\mu(x) \le C\|F\|^2_{L^2_s(\mathbb{R})}$$

for all Borel functions $F : [0,\infty) \to \mathbb{R}$ with the property $\operatorname{supp} F \subset [1/4, 4]$, *where $K_{F(L)}$ is the convolution kernel of the operator $F(L)$, and $L^2_s(\mathbb{R})$ denotes the Sobolev space with norm given by $\|F\|_{L^2_s(\mathbb{R})} = \left\|(I - d^2/d\lambda^2)^{s/2} F\right\|_{L^2(\mathbb{R})}$.*

In the rest of this section, we always assume $0 < p < 1$. We note that, for any $\delta > 0$, by the spectral multiplier theorems for $H^p(G)$ (see, e.g., [20, 31]), $(I+L)^{-\delta}$ is bounded on $H^p(G)$. Hence it suffices to consider the critical case $\beta/\alpha = Q(1/p - 1/2)$.

Let φ be a partition of unity (see Sect. 2 for definition). By Propositions 2.9 and 2.11, it suffices to show that there exists $C > 0$ such that for any (p, M, L)-atom a,

$$\left\|\mathcal{G}_{L,\varphi}\big(\widetilde{m}_{\alpha,\beta,t}(L)a\big)\right\|_{L^p(G)} \le C(1+|t|)^{Q(1/p-1/2)},$$

where $\mathcal{G}_{L,\varphi}$ is the Littlewood–Paley operator defined by

$$\mathcal{G}_{L,\varphi} f := \left(\sum_{\ell\in\mathbb{Z}} |\varphi(2^{-2\ell}L)f|^2\right)^{1/2}.$$

Choose a positive integer M such that

$$2M > \max\{Q/p - Q/2,\ Q/p - Q + 1\}. \tag{3.1}$$

Let a be an arbitrary (p, M, L)-atom associated with some ball $B = B(x_B, r_B)$, and let b be the corresponding function such that $a = L^M b$. Define P by setting

$$P(\lambda) = \sum_{k=1}^{M} (-1)^{k+1} C_k^M e^{-k\lambda}, \quad \lambda \in \mathbb{R}_+.$$

Since $1 \equiv (1 - e^{-r_B^2\lambda})^M + P(r_B^2\lambda)$, by the spectral theory, we have

$$I = (I - e^{-r_B^2 L})^M + P(r_B^2 L).$$

Using the above identity, we write

$$\begin{aligned}
\big\|\mathcal{G}_{L,\varphi}\big(\widetilde{m}_{\alpha,\beta,t}(L)a\big)\big\|_{L^p}^p &\le \Big\|\mathcal{G}_{L,\varphi}\big[(I - e^{-r_B^2 L})^M \widetilde{m}_{\alpha,\beta,t}(L)a\big]\Big\|_{L^p}^p \\
&\quad + \Big\|\mathcal{G}_{L,\varphi}\big[(r_B^2 L)^M P(r_B^2 L)\widetilde{m}_{\alpha,\beta,t}(L) r_B^{-2M} b\big]\Big\|_{L^p}^p \\
&=: \Lambda_1 + \Lambda_2.
\end{aligned}$$

Therefore it suffices to show that

$$\Lambda_1 + \Lambda_2 \lesssim (1 + |t|)^{Q(1-p/2)}. \tag{3.2}$$

3.1 Estimate of Λ_1

Set $B_t = (1 + |t|)B = B(x_B, (1 + |t|)r_B)$, and split Λ_1 into

$$\begin{aligned}
\Lambda_1 &= \Big\|\mathcal{G}_{L,\varphi}\big[(I - e^{-r_B^2 L})^M \widetilde{m}_{\alpha,\beta,t}(L)a\big]\Big\|_{L^p(4\gamma B_t)}^p \\
&\quad + \Big\|\mathcal{G}_{L,\varphi}\big[(I - e^{-r_B^2 L})^M \widetilde{m}_{\alpha,\beta,t}(L)a\big]\Big\|_{L^p(G\setminus 4\gamma B_t)}^p \\
&=: \Lambda_{11} + \Lambda_{12}.
\end{aligned}$$

here, γ is the constant from (1.4), and $4\gamma B_t$ denotes the ball with the same center as B_t and with radius 4γ times of that of B_t. By Hölder's inequality, the L^2 boundedness of $\mathcal{G}_{L,\varphi}$ and the properties of atoms, we have

$$
\begin{aligned}
\Lambda_{11} &\lesssim \mu\big((1+|t|)B\big)^{1-p/2}\big\|\mathcal{G}_{L,\varphi}(I-e^{-r_B^2 L})^M \widetilde{m}_{\alpha,\beta,t}(L)a\big\|_{L^2(4B_t)}^p \\
&\lesssim \mu\big((1+|t|)B\big)^{1-p/2}\|a\|_{L^2}^p \\
&\lesssim \mu\big((1+|t|)B\big)^{1-p/2}\mu(B)^{p/2-1} \\
&= (1+|t|)^{Q(1-p/2)}.
\end{aligned}
\tag{3.3}
$$

Now we estimate Λ_{12}. Setting

$$
F_{\ell,r_B}(\lambda) := \varphi(2^{-2\ell}\lambda)(1-e^{-r_B^2\lambda})^M \widetilde{m}_{\alpha,\beta,t}(\lambda),
$$

we have by Minkowski's inequality

$$
\begin{aligned}
\Lambda_{12} &= \left\| \left(\sum_{\ell\in\mathbb{Z}} \big|F_{\ell,r_B}(L)a\big|^2 \right)^{1/2} \right\|_{L^p(G\setminus 4\gamma B_t)}^p = \left\| \left(\sum_{\ell\in\mathbb{Z}} \big|F_{\ell,r_B}(L)a\big|^2 \right)^{p/2} \right\|_{L^1(G\setminus 4\gamma B_t)} \\
&\le \left\| \sum_{\ell\in\mathbb{Z}} \big|F_{\ell,r_B}(L)a\big|^p \right\|_{L^1(G\setminus 4\gamma B_t)} \le \sum_{\ell\in\mathbb{Z}} \big\|F_{\ell,r_B}(L)a\big\|_{L^p(G\setminus 4\gamma B_t)}^p \\
&\lesssim \sum_{(\alpha-1)\ell\ge\ell_0} \big\|F_{\ell,r_B}(L)a\big\|_{L^p(G\setminus 4\gamma B_t)}^p + \sum_{(\alpha-1)\ell<\ell_0} \big\|F_{\ell,r_B}(L)a\big\|_{L^p(G\setminus 4\gamma B_t)}^p \\
&=: \Lambda_{121} + \Lambda_{122},
\end{aligned}
\tag{3.4}
$$

where ℓ_0 is the (unique) integer such that $2^{\ell_0} \le r_B < 2^{\ell_0+1}$.

Note that for every ℓ with $(\alpha-1)\ell \ge \ell_0$, we have

$$
X = B\big(x_B, 2^{(\alpha-1)\ell}(1+|t|)\big) \bigcup \left(\bigcup_{j\ge(\alpha-1)\ell-\ell_0} S_j(B_t) \right), \tag{3.5}
$$

where $S_0(B_t) := B_t$ and $S_j(B_t) := 2^j B_t \setminus 2^{j-1} B_t$ for $j \in \mathbb{N}$. Let j_0 be the (unique) integer such that

$$
2\gamma \le 2^{j_0-1} < 4\gamma. \tag{3.6}
$$

Then $j_0 \ge 2$ and

$$
X\setminus 4\gamma B_t \subset X\setminus 2^{j_0-1}B_t = \bigcup_{j\ge j_0} S_j(B_t). \tag{3.7}
$$

Combining (3.5) and (3.7) we have

$$X\backslash 4\gamma B_t = X \cap (X\backslash 4\gamma B_t) \subseteq B\big(x_B, 2^{(\alpha-1)\ell}(1+|t|)\big) \bigcup\Bigg(\bigcup_{j\geq \max\{(\alpha-1)\ell-\ell_0,\ j_0\}} S_j(B_t)\Bigg).$$

It follows that

$$\begin{aligned}\Lambda_{121} &\leq \sum_{(\alpha-1)\ell\geq\ell_0} \big\|F_{\ell,r_B}(L)a\big\|_{L^p(B(x_B,2^{(\alpha-1)\ell}(1+|t|)))} \\ &\quad+ \sum_{(\alpha-1)\ell\geq\ell_0}\ \sum_{j\geq \max\{(\alpha-1)\ell-\ell_0,\ j_0\}} \big\|F_{\ell,r_B}(L)a\big\|^p_{L^p(S_j(B_t))} \\ &=: \Lambda_{1211} + \Lambda_{1212},\end{aligned}$$

The estimate of Λ_{1211} is easy. Indeed, by Hölder's inequality and the properties of atoms, we have

$$\begin{aligned}\Lambda_{1211} &\lesssim \sum_{(\alpha-1)\ell\geq\ell_0} \mu\big(B(x_B, 2^{(\alpha-1)\ell}(1+|t|))\big)^{1-p/2}\big\|F_{\ell,r_B}(L)a\big\|^p_{L^2(B(x_B,2^{(\alpha-1)}\ell(1+|t|)))} \\ &\lesssim \sum_{(\alpha-1)\ell\geq\ell_0} \mu\big(B(x_B, 2^{(\alpha-1)\ell}(1+|t|))\big)^{1-p/2}\|F_{\ell,r_B}\|^p_{L^\infty(\mathbb{R}_+)}\|a\|^p_{L^2(G)} \\ &\lesssim \sum_{(\alpha-1)\ell\geq\ell_0} \mu\big(B(x_B, 2^{(\alpha-1)\ell}(1+|t|))\big)^{1-p/2}\mu(B)^{p/2-1}\big\|F_{\ell,r_B}\big\|^p_{L^\infty(\mathbb{R}_+)} \\ &\sim (1+|t|)^{Q(1-p/2)} \sum_{(\alpha-1)\ell\geq\ell_0} (2^\ell r_B)^{-Q(1-p/2)}2^{\ell\alpha Q(1-p/2)}\big\|F_{\ell,r_B}\big\|^p_{L^\infty(\mathbb{R}_+)}.\end{aligned} \tag{3.8}$$

Since

$$\big\|F_{\ell,r_B}\big\|_{L^\infty(\mathbb{R}_+)} \lesssim \min\{1, (2^\ell r_B)^{2M}\}2^{-\ell\beta} = \min\{1, (2^\ell r_B)^{2M}\}2^{-\ell\alpha Q(1/p-1/2)}, \tag{3.9}$$

it follows from (3.8) and (3.1) that

$$\begin{aligned}\Lambda_{1211} &\lesssim (1+|t|)^{Q(1-p/2)} \sum_{(\alpha-1)\ell\geq\ell_0} (2^\ell r_B)^{-(Q/p-Q/2)p} \min\{1, (2^\ell r_B)^{2Mp}\} \\ &\lesssim (1+|t|)^{Q(1-p/2)}.\end{aligned}$$

To estimate Λ_{1212}, first note that by Hölder's inequality and the properties of atoms,

$$\Lambda_{1212} \lesssim \sum_{(\alpha-1)\ell\geq\ell_0}\ \sum_{j\geq \max\{(\alpha-1)\ell-\ell_0,\ j_0\}} \big[2^j(1+|t|)\big]^{Q(1-p/2)}\big\|F_{\ell,r_B}(L)\big\|^p_{L^2(B)\to L^2(S_j(B_t))}. \tag{3.10}$$

We then claim that for any $s > Q/2$ and $j \geq j_0$, there holds

$$\begin{aligned}&\|F_{\ell,r_B}(L)\|_{L^2(B)\to L^2(S_j(B_t))}\\&\qquad\lesssim 2^{-js}(2^\ell r_B)^{-(s-Q/2)}\min\{1,(2^\ell r_B)^{2M}\}\max\{1,2^{-\ell\beta}2^{\ell\alpha s}\}.\end{aligned}\tag{3.11}$$

Let us prove the claim. For any $h \in L^2(B)$, by Minkowski's and Hölder's inequalities, we have

$$\begin{aligned}\left\|F_{\ell,r_B}(L)h\right\|_{L^2(S_j(B_t))} &= \left\|x \mapsto \int_B K_{F_{\ell,r_B}(L)}(y^{-1}x)h(y)d\mu(y)\right\|_{L^2(S_j(B_t))}\\&\leq \int_B \left\|x\mapsto K_{F_{\ell,r_B}(L)}(y^{-1}x)\right\|_{L^2(S_j(B_t))}|h(y)|d\mu(y)\\&\leq \sup_{y\in B}\left\|x\mapsto K_{F_{\ell,r_B}(L)}(y^{-1}x)\right\|_{L^2(S_j(B_t))}\int_B |h(y)|d\mu(y)\\&\leq \sup_{y\in B}\left\|x\mapsto K_{F_{\ell,r_B}(L)}(y^{-1}x)\right\|_{L^2(S_j(B_t))}\mu(B)^{1/2}\|h\|_{L^2(B)},\end{aligned}$$

where $K_{F_{\ell,r_B}(L)}$ is the convolution kernel of the operator $F_{r_B,\ell}(L)$. Hence

$$\left\|F_{\ell,r_B}(L)\right\|_{L^2(B)\to L^2(S_j(B_t))} \leq \sup_{y\in B}\left\|x\mapsto K_{F_{\ell,r_B}(L)}(y^{-1}x)\right\|_{L^2(S_j(B_t))}\mu(B)^{1/2}.\tag{3.12}$$

Note that for any $y \in B$ and $x \in S_j(B_t)$ with $j \geq j_0$, we have $|y^{-1}x| \sim 2^j(1+|t|)r_B$. Indeed, on the one hand, by (1.4), we have $|y^{-1}x| \leq \gamma(|x|+|y|) \leq \gamma\big[2^j(1+|t|)r_B + r_B\big] \lesssim 2^j(1+|t|)r_B$; on the other hand, by (1.4) and (3.6), we have $|y^{-1}x| \geq \gamma^{-1}|y| - |x| \geq \gamma^{-1}2^{j_0-1}r_B - r_B \geq 2r_B - r_B = r_B$. Thus, for every $y \in B$, applying Lemma 3.1,

$$\begin{aligned}&\left\|x\mapsto K_{F_{\ell,r_B}(L)}(y^{-1}x)\right\|^2_{L^2(S_j(B_t))}\\&= \int_{S_j(B_t)}\left|K_{F_{\ell,r_B}(L)}(y^{-1}x)\right|^2 d\mu(x)\\&\sim \big[2^j(1+|t|)r_B\big]^{-2s}\int_{S_j(B_t)}|y^{-1}x|^{2s}\left|K_{F_{\ell,r_B}(L)}(y^{-1}x)\right|^2 d\mu(x)\\&\leq \big[2^j(1+|t|)r_B\big]^{-2s}\int_G |x|^{2s}\left|K_{F_{\ell,r_B}(L)}(x)\right|^2 d\mu(x)\\&= \big[2^j(1+|t|)r_B\big]^{-2s}\int_G |x|^{2s}\left|2^{\ell Q}K_{\widetilde{F}_{\ell,r_B}(L)}(2^\ell x)\right|^2 d\mu(x)\\&= \big[2^j(1+|t|)r_B\big]^{-2s}2^{\ell Q}2^{-2\ell s}\int_G |x|^{2s}\left|K_{\widetilde{F}_{\ell,r_B}(L)}(x)\right|^2 d\mu(x)\\&\lesssim \big[2^j(1+|t|)r_B\big]^{-2s}2^{\ell Q}2^{-2\ell s}\left\|\widetilde{F}_{\ell,r_B}\right\|^2_{L^2_s(\mathbb{R})},\end{aligned}\tag{3.13}$$

where

$$\widetilde{F}_{\ell,r_B}(\lambda) := F_{\ell,r_B}(2^{2\ell}\lambda) = \varphi(\lambda)\big(1-e^{-(2^\ell r_B)^2\lambda}\big)^M \widetilde{m}_{\alpha,\beta,t}(2^{2\ell}\lambda).$$

Let $\psi \in C_0^\infty(0,\infty)$ such that supp $\psi \subset [1/8, 8]$ and $\psi(\lambda) = 1$ for $\lambda \in [1/4, 4]$. Then $\psi(\lambda)\varphi(\lambda) = \varphi(\lambda)$, and hence by the algebra property of $L^2_s(\mathbb{R})$,

$$\begin{aligned}\big\|\widetilde{F}_{\ell,r_B}\big\|_{L^2_s(\mathbb{R})} &= \big\|\lambda \mapsto \varphi(\lambda)\big(1-e^{-(2^\ell r_B)^2\lambda}\big)^M \widetilde{m}_{\alpha,\beta,t}(2^{2\ell}\lambda)\big\|_{L^2_s(\mathbb{R})}\\ &\le \big\|\lambda \mapsto \psi(\lambda)\big(1-e^{-(2^\ell r_B)^2\lambda}\big)^M\big\|_{L^2_s(\mathbb{R})}\big\|\lambda \mapsto \varphi(\lambda)\widetilde{m}_{\alpha,\beta,t}(2^{2\ell}\lambda)\big\|_{L^2_s(\mathbb{R})}\\ &\lesssim \min\{1, (2^\ell r_B)^{2M}\}(1+|t|)^s \max\{1, 2^{-\ell\beta}2^{\ell\alpha s}\}.\end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) and (3.14) we have

$$\begin{aligned}&\big\|F_{\ell,r_B}(L)\big\|_{L^2(B)\to L^2(S_j(B_t))}\\ &\quad\lesssim \big[2^j(1+|t|)r_B\big]^{-s}2^{\ell Q/2}2^{-\ell s}\min\{1, (2^\ell r_B)^{2M}\}(1+|t|)^s\max\{1, 2^{-\ell\beta}2^{\ell\alpha s}\}\mu(B)^{1/2}\\ &\quad= 2^{-js}(2^\ell r_B)^{-(s-Q/2)}\min\{1, (2^\ell r_B)^{2M}\}\max\{1, 2^{-\ell\beta}2^{\ell\alpha s}\},\end{aligned}$$

as claimed.

From (3.10) and (3.11), it follows that

$$\begin{aligned}\Lambda_{1212} \lesssim \sum_{(\alpha-1)\ell\ge\ell_0}\ \sum_{j\ge\max\{(\alpha-1)\ell-\ell_0,\ j_0\}} &[2^j(1+|t|)]^{Q(1-p/2)}2^{-jsp}(2^\ell r_B)^{-(s-Q/2)p}\\ &\times\min\{1, (2^\ell r_B)^{2Mp}\}\max\{1, 2^{\ell\alpha(s-\beta/\alpha)p}\}.\end{aligned} \tag{3.15}$$

Letting $s = \beta/\alpha + \varepsilon = Q(1/p - 1/2) + \varepsilon$ for some $\varepsilon \in (0, 1)$, we rewrite (3.15) as

$$\begin{aligned}\Lambda_{1212} &\lesssim \sum_{(\alpha-1)\ell\ge\ell_0}\ \sum_{j\ge\max\{(\alpha-1)\ell-\ell_0,\ j_0\}} 2^{-jp\varepsilon}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q+\varepsilon)p}\\ &\quad\times\min\{1, (2^\ell r_B)^{2Mp}\}\max\{1, 2^{\ell\alpha\varepsilon p}\}\\ &= \sum_{\substack{(\alpha-1)\ell\ge\ell_0\\ \ell<0}}\ \sum_{j\ge\max\{(\alpha-1)\ell-\ell_0,\ j_0\}}\cdots + \sum_{\substack{(\alpha-1)\ell\ge\ell_0\\ \ell\ge0}}\ \sum_{j\ge\max\{(\alpha-1)\ell-\ell_0,\ j_0\}}\cdots\\ &=: \Lambda_{12121} + \Lambda_{12122}.\end{aligned} \tag{3.16}$$

For the term Λ_{12111}, since $2M \ge Q/p - Q + 1 > Q/p - Q + \varepsilon$ (see (3.1)), we have

$$\begin{aligned}\Lambda_{12121} &\lesssim \sum_{\substack{(\alpha-1)\ell\ge\ell_0\\ \ell<0}} 2^{-j_0p\varepsilon}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q+\varepsilon)p}\min\{1, (2^\ell r_B)^{2Mp}\}\\ &\lesssim (1+|t|)^{Q(1-p/2)}.\end{aligned}$$

For the term Λ_{12112}, using that $r_B \sim 2^{\ell_0}$, we have

$$\begin{aligned}
\Lambda_{12122} &\lesssim \sum_{\substack{(\alpha-1)\ell\geq\ell_0\\ \ell\geq 0}} 2^{-[(\alpha-1)\ell-\ell_0]p\varepsilon}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q+\varepsilon)p}\min\{1,(2^\ell r_B)^{2Mp}\}2^{\ell\alpha\varepsilon p}\\
&\sim \sum_{\substack{(\alpha-1)\ell\geq\ell_0\\ \ell\geq 0}} (1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q)p}\min\{1,(2^\ell r_B)^{2Mp}\}\\
&\lesssim (1+|t|)^{Q(1-p/2)}.
\end{aligned}$$

Thus we have proved $\Lambda_{1212} \lesssim (1+|t|)^{Q(1-p/2)}$.

Combining the estimates of Λ_{1211} and Λ_{1212}, we obtain

$$\Lambda_{121} \lesssim (1+|t|)^{Q(1-p/2)}.$$

We now estimate Λ_{122}. Indeed, by (3.7) we have

$$\Lambda_{122} = \sum_{(\alpha-1)\ell<\ell_0} \left\|F_{\ell,r_B}(L)a\right\|^p_{L^p(G\backslash 4\gamma B_t)} \leq \sum_{(\alpha-1)\ell<\ell_0}\sum_{j\geq j_0}\left\|F_{\ell,r_B}(L)a\right\|^p_{L^p(S_j(B_t))}.$$

An argument similar to that used in the estimate of Λ_{1212} [see (3.16)] yields

$$\begin{aligned}
\Lambda_{122} &\lesssim \sum_{(\alpha-1)\ell<\ell_0}\sum_{j\geq j_0} 2^{-jp\varepsilon}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q+\varepsilon)p}\\
&\qquad\qquad \times \min\{1,(2^\ell r_B)^{2Mp}\}\max\{1,2^{\ell\alpha\varepsilon p}\}\\
&\lesssim \sum_{(\alpha-1)\ell<\ell_0}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q/p-Q+\varepsilon)p}\\
&\qquad\qquad \times \min\{1,(2^\ell r_B)^{2Mp}\}\max\{1,2^{\ell\alpha\varepsilon p}\}\\
&= \sum_{\substack{(\alpha-1)\ell<\ell_0\\ \ell<0}}\cdots + \sum_{\substack{(\alpha-1)\ell<\ell_0\\ \ell\geq 0}}\cdots\\
&=: \Lambda_{1221}+\Lambda_{1222}.
\end{aligned}$$

The estimate of Λ_{1221} is easy. Indeed, since $2M > Q/p - p + \varepsilon$, we have

$$\begin{aligned}
\Lambda_{1221} &\leq \sum_{\substack{(\alpha-1)\ell\leq 0\\ \ell<0}}(1+|t|)^{Q(1-p/2)}(2^\ell r_B)^{-(Q-Q/p+\varepsilon)p}\min\{1,(2^\ell r_B)^{2Mp}\}\\
&\lesssim (1+|t|)^{Q(1-p/2)}.
\end{aligned}$$

To estimate the term Λ_{1222}, note that if $(\alpha-1)\ell < \ell_0$ then $2^{(\alpha-1)\ell} r_B^{-1} \sim 2^{(\alpha-1)\ell} 2^{-\ell_0} < 1$. Hence

$$\begin{aligned}
\Lambda_{1222} &\lesssim \sum_{\substack{(\alpha-1)\ell<\ell_0 \\ \ell\geq 0}} (1+|t|)^{Q(1-p/2)} (2^\ell r_B)^{-(Q/p-Q+\varepsilon)p} \min\{1, (2^\ell r_B)^{2Mp}\} 2^{\ell\alpha\varepsilon p} \\
&\lesssim \sum_{\substack{(\alpha-1)\ell<\ell_0 \\ \ell\geq 0}} (1+|t|)^{Q(1-p/2)} \big[2^{(\alpha-1)\ell} r_B^{-1}\big]^{-\varepsilon p} (2^\ell r_B)^{-(Q/p-Q+\varepsilon)p} \min\{1, (2^\ell r_B)^{2Mp}\} 2^{\ell\alpha\varepsilon p} \\
&= \sum_{\substack{(\alpha-1)\ell<\ell_0 \\ \ell\geq 0}} (1+|t|)^{Q(1-p/2)} (2^\ell r_B)^{-(Q/p-Q)p} \min\{1, (2^\ell r_B)^{2Mp}\} \\
&\lesssim (1+|t|)^{Q(1-p/2)}.
\end{aligned}$$

Therefore we have proved

$$\Lambda_{122} \lesssim (1+|t|)^{Q(1-p/2)}.$$

Collecting the estimates for Λ_{121} and Λ_{122} we have

$$\Lambda_{12} \lesssim 2^{-kp\epsilon} (1+|t|)^{Q(1-p/2)},$$

which along with (3.3) yields

$$\Lambda_1 \lesssim (1+|t|)^{Q(1-p/2)}.$$

3.2 Estimate of Λ_2

The term Λ_2 can be handled by an argument analogous to that we used in the estimate of Λ_1. Indeed, setting

$$G_{\ell,r_B}(\lambda) := \varphi(2^{-2\ell})(r_B^2\lambda)^M P(r_B^2\lambda)\widetilde{m}_{\alpha,\beta,t}(\lambda),$$

we have

$$\|G_{\ell,r_B}\|_{L^\infty(\mathbb{R}_+)} \lesssim \min\{(2^\ell r_B)^{-2M}, (2^\ell r_B)^{2M}\} 2^{-\ell\alpha Q(1/p-1/2)}.$$

Also, analogously to (3.14) we have

$$\big\|\widetilde{G}_{\ell,r_B}\big\|_{L^2_s(\mathbb{R})} \lesssim \min\{(2^\ell r_B)^{-2M}, (2^\ell r_B)^{2M}\}(1+|t|)^s \max\{1, 2^{-\ell\beta} 2^{\ell\alpha s}\},$$

where $\widetilde{G}_{\ell,r_B}(\lambda) := G_{\ell,r_B}(2^{2\ell}\lambda)$. We note that the function $r_B^{-2M} b$ has similar properties as the atom a; more precisely,

$$\operatorname{supp}(r_B^{-2M} b) \subset B \quad \text{and} \quad \big\|r_B^{-2M} b\big\|_{L^2(G)} \leq \mu(B)^{1/2-1/p}.$$

Using these facts and Lemma 3.1, and argue similarly as in estimate of Λ, we obtain

$$\Lambda_2 \lesssim (1+|t|)^{Q(1-p/2)}.$$

The proof of Theorem 1.1 is thus complete.

4 Proof of Theorem 1.2

We first prove the assertion of Theorem 1.2 in the case $0 < p < 1$. Indeed, we set

$$\widetilde{\psi}(\lambda) := \psi(\lambda)\lambda^{-\beta/2}(1+\lambda)^{\beta/2}, \quad \lambda \in \mathbb{R}_+,$$

so that

$$m_{\alpha,\beta,t}(L) = \widetilde{\psi}(L)\widetilde{m}_{\alpha,\beta,t}(L). \tag{4.1}$$

Obviously, the multiplier function $\widetilde{\psi}$ satisfies the Mihlin-Hörmander condition

$$\left|\frac{d^k}{d\lambda^k}\widetilde{\psi}(\lambda)\right| \le C_k\lambda^{-k}, \quad \forall\lambda > 0,\ k \in \mathbb{N}_0.$$

Hence by the spectral multiplier theorem in [31] we see that $\widetilde{\psi}(L)$ is bounded on $H^p(G)$ for all $0 < p < \infty$. This along with (4.1) and Theorem 1.1 yields that for $0 < p < 1$,

$$\|m_{\alpha,\beta,t} f\|_{H^p(G)} \le C(1+|t|)^{Q(1/p-1/2)}\|f\|_{H^p(G)}, \quad t \in \mathbb{R}, \quad \beta/\alpha \ge Q(1/p-1/2). \tag{4.2}$$

We now use complex interpolation of $\dot{F}^s_{p,q}(G)$ to prove the assertion of Theorem 1.2 in the case $1 \le p < \infty$. Indeed, fix an arbitrary $q \in (0, 1)$. Then (4.2) implies

$$\left\|\psi(L)e^{itL^{\alpha/2}} f\right\|_{H^q(G)} \le C(1+|t|)^{Q(1/q-1/2)}\left\|L^{\beta/2} f\right\|_{H^q(G)}, \quad \beta = \alpha Q(1/q-1/2).$$

In view of Proposition 2.9 and the lifting property of homogeneous Triebel-Lizorkin spaces on G (see [25,Theorem 13]), this is equivalent to that

$$\left\|\psi(L)e^{itL^{\alpha/2}} f\right\|_{\dot{F}^0_{q,2}(G)} \le C(1+|t|)^{Q(1/q-1/2)}\|f\|_{\dot{F}^\beta_{q,2}(G)}, \quad \beta = \alpha Q(1/q-1/2). \tag{4.3}$$

On the other hand, by the spectral theory, we have

$$\left\|\psi(L)e^{itL^{\alpha/2}} f\right\|_{L^2(G)} \le C\|f\|_{L^2(G)},$$

which implies, by Proposition 2.9 and Remark 2.3,

$$\left\|\psi(L)e^{itL^{\alpha/2}} f\right\|_{\dot{F}^0_{2,2}(G)} \le C\|f\|_{\dot{F}^0_{2,2}(G)}. \tag{4.4}$$

Let $p \in (q, 2)$ and set $\theta = \frac{2(p-q)}{p(2-q)} \in (0, 1)$. Then by Proposition 2.12, we have

$$\big[\dot F^0_{q,2}(G), \dot F^0_{2,2}(G)\big]_\theta = \dot F^0_{p,2}(G), \quad \text{and} \quad [\dot F^\beta_{q,2}(G), \dot F^0_{2,2}(G)]_\theta = \dot F^{(1-\theta)\beta}_{p,2}(G).$$

This, along with (4.3), (4.4) and the lifting property of the homogeneous Triebel-Lizorkin spaces on G implies that

$$\begin{aligned}\big\|\psi(L)e^{itL^{\alpha/2}} f\big\|_{\dot F^0_{p,2}(G)} &\le (1+|t|)^{(1-\theta)Q(1/q-1/2)}\|f\|_{\dot F^{(1-\theta)\beta}_{p,2}(G)}\\ &\sim (1+|t|)^{(1-\theta)Q(1/q-1/2)}\big\|L^{(1-\theta)\beta/2} f\big\|_{\dot F^0_{p,2}(G)}\\ &= (1+|t|)^{Q(1/p-1/2)}\big\|L^{\alpha Q(1/p-1/2)/2} f\big\|_{\dot F^0_{p,2}(G)}.\end{aligned}$$

Hence

$$\big\|\psi(L)L^{-\alpha Q(1/p-1/2)/2}e^{itL^{\alpha/2}} f\big\|_{\dot F^0_{p,2}(G)} \le (1+|t|)^{Q(1/p-1/2)}\|f\|_{\dot F^0_{p,2}(G)}, \quad \forall p \in (q, 2).$$

In particular, by Proposition 2.9 and Remark 2.3, this implies

$$\big\|\psi(L)L^{-\alpha Q/4}e^{itL^{\alpha/2}} f\big\|_{H^1(G)} \le (1+|t|)^{Q/2}\|f\|_{H^1(G)}$$

and

$$\big\|\psi(L)L^{-\alpha Q(1/p-1/2)/2}e^{itL^{\alpha/2}} f\big\|_{L^p(G)} \le (1+|t|)^{Q(1/p-1/2)}\|f\|_{L^p(G)}, \quad 1 < p < 2.$$

By duality, we also have

$$\big\|\psi(L)L^{-\alpha Q(1/2-1/p)/2}e^{itL^{\alpha/2}} f\big\|_{L^p(G)} \le (1+|t|)^{Q(1/2-1/p)}\|f\|_{L^p(G)}, \quad 2 < p < \infty.$$

This completes our proof of Theorem 1.2.

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