The Journal of Geometric Analysis (2022) 32:139
https://doi.org/10.1007/s12220-022-00880-9

Non-degeneracy of Positive Solutions for Fractional Kirchhoff Problems: High Dimensional Cases

Zhipeng Yang[1,2]

Received: 4 November 2021 / Accepted: 13 January 2022 / Published online: 8 February 2022

Abstract
In this paper, we establish the nondegeneracy of positive solutions to the fractional Kirchhoff problem

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+u=u^p,\quad \text{in } \mathbb{R}^N,$$

where $a, b > 0$, $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$ and $(-\Delta)^s$ is the fractional Laplacian. In particular, we prove that uniqueness breaks down for dimensions $N > 4s$, i.e., we show that there exist two non-degenerate positive solutions which seem to be completely different from the result of the fractional Schrödinger equation or the low dimensional fractional Kirchhoff equation. As one application, combining this nondegeneracy result and Lyapunov-Schmidt reduction method, we can derive the existence of solutions to the singularly perturbation problems.

Keywords Fractional Kirchhoff equations · Nondegeneracy · Lyapunov–Schmidt reduction

Mathematics Subject Classification 35R11 · 35A15 · 47G20

1 Introduction and Main Results

In this paper, we are concerned with the following fractional Kirchhoff problem

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+u=u^p,\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

✉ Zhipeng Yang
yangzhipeng326@163.com

1 Department of Mathematics, Yunnan Normal University, Kunming 650500, China

2 Mathematical Institute, Georg-August-University of Göttingen, 37073 Göttingen, Germany

where $a, b > 0$, $(-\Delta)^s$ is the pseudo-differential operator defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform, and p satisfies

$$1 < p < 2_s^* - 1 = \begin{cases} \frac{N+2s}{N-2s}, & 0 < s < \frac{N}{2}, \\ +\infty, & s \geq \frac{N}{2}, \end{cases}$$

where 2_s^* is the standard fractional Sobolev critical exponent. Recently, Rădulescu and Yang [41] established uniqueness and nondegeneracy for positive solutions to (1.1) for $\frac{N}{4} < s < 1$. Then in this paper, we will consider the high dimensional cases, i.e. $N \geq 4s$. We also refer to [26, 40, 44, 45] for critical cases and single/multi-peak solutions in this direction.

If $s = 1$, Eq. (1.1) reduces to the well known Kirchhoff type problem, which and their variants have been studied extensively in the literature. The equation that goes under the name of Kirchhoff equation was proposed in [28] as a model for the transverse oscillation of a stretched string in the form

$$\rho h \partial_{tt}^2 u - \left(p_0 + \frac{\mathcal{E}h}{2L}\int_0^L |\partial_x u|^2 \,\mathrm{d}x\right)\partial_{xx}^2 u = 0, \tag{1.2}$$

for $t \geq 0$ and $0 < x < L$, where $u = u(t, x)$ is the lateral displacement at time t and at position x, $\mathcal{E}$ is the Young modulus, ρ is the mass density, h is the cross section area, L the length of the string, p_0 is the initial stress tension. Problem (1.2) and its variants have been studied extensively in the literature. Bernstein obtains the global stability result in [10], which has been generalized to arbitrary dimension $N \geq 1$ by Pohožaev in [37]. We also point out that such problems may describe a process of some biological systems dependent on the average of itself, such as the density of population (see e.g. [9]). Many interesting work on Kirchhoff equations can be found in [15, 27, 33, 43] and the references therein. We also refer to [38] for a recent survey of the results connected to this model.

On the other hand, the interest in generalizing the model introduced by Kirchhoff to the fractional case does not arise only for mathematical purposes. In fact, following the ideas of [11] and the concept of fractional perimeter, Fiscella and Valdinoci proposed in [20] an equation describing the behaviour of a string constrained at the extrema in which appears the fractional length of the rope. Recently, problem similar to (1.1) has been extensively investigated by many authors using different techniques and producing several relevant results (see, e.g. [1–4, 6, 8, 23–25, 34–36, 42]).

Besides, if $b = 0$ in (1.1), then we are led immediately to the following fractional Schrödinger equation

$$a(-\Delta)^s u + u = u^p, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

This equation is related to the standing wave solutions of the time-independent fractional Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = h^{2s}(-\Delta)^s\psi + V(x)\psi - f(x,|\psi|), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.4}$$

where h is the Plank constant and $V(x)$ is a potential function. Eq. (1.4) was introduced by Laskin [29, 30] as a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy process. For $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the natural norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x + \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

From [17], we have

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}(u)|^2 \mathrm{d}\xi = \frac{1}{2}C(N,s)\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y,$$

and the fractional Gagliardo–Nirenberg–Sobolve inequality

$$\int_{\mathbb{R}^N} |u|^{p+1}\mathrm{d}x \le \mathcal{S}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\Big)^{\frac{N(p-1)}{4s}} \Big(\int_{\mathbb{R}^N} |u|^2\mathrm{d}x\Big)^{\frac{p-1}{4s}(2s-N)+1}, \tag{1.5}$$

where $\mathcal{S} > 0$ is the best constant. It follows from (1.5) that

$$J(u) = \frac{\mathcal{S}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)^{\frac{N(p-1)}{4s}} \Big(\int_{\mathbb{R}^N} |u|^2\mathrm{d}x\Big)^{\frac{p-1}{4s}(2s-N)+1}}{\int_{\mathbb{R}^N} |u|^{p+1}\mathrm{d}x} > 0. \tag{1.6}$$

Since the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator, one can not apply directly the usual techniques dealing with the classical Laplacian operator. Therefore, some ideas are proposed recently. In [12], Caffarelli and Silvestre expressed the operator $(-\Delta)^s$ on $\mathbb{R}^N$ as a generalized elliptic BVP with local differential operators defined on the upper half-space $\mathbb{R}_+^{N+1} = \{(t,x) : t > 0, x \in \mathbb{R}^N\}$. By means of Lyapunov–Schmidt reduction, concentration phenomenon of solutions was considered independently in [13, 16]. For more interesting results concerning with the existence, multiplicity and concentration of solutions for the fractional Laplacian equation, we refer reader to [5, 17, 18] and the references therein.

Uniqueness of ground states of nonlocal equations similar to Eq. (1.3) is of fundamental importance in the stability and blow-up analysis for solitary wave solutions of nonlinear dispersive equations, for example, of the generalized Benjamin–Ono equation. In contrast to the classical limiting case when $s = 1$, in which standard ODE techniques are applicable, uniqueness of ground state solutions to Eq. (1.3) is a really difficult problem. In the case that $s = \frac{1}{2}$ and $N = 1$, Amick and Toland [7], they obtained the uniqueness result for solitary waves of the Benjamin–Ono equation. After that, Lenzmann [31] obtained the uniqueness of ground states for the pseudorelativistic Hartree equation in 3-dimension. In [21], Frankand and Lenzmann extends the results in [7] to the case that $s \in (0, 1)$ and $N = 1$ with completely new methods. For the high dimensional case, Fall and Valdinoci [19] established the uniqueness and nondegeneracy of ground state solutions of (1.3) when $s \in (0, 1)$ is sufficiently close to 1 and p is subcritical. In their striking paper [22], Frank, Lenzmann and Silvestre solved the problem completely, and they showed that the ground state solutions of (1.3) is unique for arbitrary space dimensions $N \geq 1$ and all admissible and subcritical exponents $p > 0$. Moreover, they also established the nondegeneracy of ground state solutions. We summarize their main results as follows.

Proposition 1.1 *Let $N \geq 1$, $0 < s < 1$ and $1 < p < 2_s^* - 1$. Then the following holds.*

- (i) *there exists a minimizer $Q \in H^s(\mathbb{R}^N)$ for $J(u)$, which can be chose a nonnegative function that solves Eq. (1.3);*
- (ii) *there exist some $x_0 \in \mathbb{R}^N$ such that $Q(\cdot - x_0)$ is radial, positive and strictly decreasing in $r = |x - x_0|$. Moreover, the function Q belongs to $C^\infty(\mathbb{R}^N) \cap H^{2s+1}(\mathbb{R}^N)$ and it satisfies*

$$\frac{C_1}{1 + |x|^{N+2s}} \leq Q(x) \leq \frac{C_2}{1 + |x|^{N+2s}}, \quad \forall\, x \in \mathbb{R}^N,$$

  with some constants $C_2 \geq C_1 > 0$;
- (iii) *Q is a unique solution of (1.3) up to translation.*

Proposition 1.2 *Let $N \geq 1$, $0 < s < 1$, $1 < p < 2_s^* - 1$ and c be a positive constant. Suppose that $Q \in H^s(\mathbb{R}^N)$ is a ground state solution of*

$$c(-\Delta)^s Q + Q = |Q|^p \quad in \quad \mathbb{R}^N \tag{1.7}$$

and T_+ denotes the corresponding linearized operator given by

$$T_+ = c(-\Delta)^s + 1 - p|Q|^{p-1}.$$

Then the following holds.

- (i) *Q is nondegenerate, i.e., $\ker T_+ = span\{\partial_{x_1} Q, \partial_{x_2} Q, \cdots, \partial_{x_N} Q\}$;*
- (ii) *the restriction of T_+ on $L^2_{rad}(\mathbb{R}^N)$ is one-to-one and thus it has an inverse T_+^{-1} acting on $L^2_{rad}(\mathbb{R}^N)$;*
- (iii) *$T_+ Q = -(p-1)Q^p$ and $T_+ R = -2sQ$, where $R = \frac{2s}{p-1} Q + x \cdot (-\Delta)^{\frac{s}{2}} Q$.*

From the viewpoint of calculus of variation, the fractional Kirchhoff problem (1.1) is much more complex and difficult than the classical fractional Laplacian Eq. (1.3) as the appearance of the term $b\big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\big)(-\Delta)^s u$, which is of order four. So a fundamental task for the study of problem (1.1) is to make clear the effects of this non-local term. The only one uniqueness and non-degeneracy result which we know for the solution of problem (1.1) is proved in [41] for the case $\frac{N}{4} < s < 1$, and [14, 32] for the case $s = 1$. As in [41], let U be a ground state positive solution of (1.1) and set

$$\mathcal{E}_0 = a + b\|(-\Delta)^{\frac{s}{2}}U\|_2^2 \quad \text{and} \quad \tilde{U}(x) = U(\mathcal{E}_0^{\frac{1}{2s}}x).$$

Then, it is easy to check that $\tilde{U}$ is a positive solution of (1.3) and a minimizer of $J(u)$. Therefore, from the uniqueness result for positive solutions of problem (1.3), we know that any solution $U(x)$ of problem (1.1) with $a, b > 0$ has the following form

$$U(x) = Q(\mathcal{E}_0^{-\frac{1}{2s}}x).$$

Consequently, the solvability of the problem (1.1) is simply equivalent to the solvability of the following algebraic equation in $(0, +\infty)$,

$$f(\mathcal{E}) = \mathcal{E} - a - bm^{\frac{2}{p-1}+\frac{2s-N}{2s}}\|(-\Delta)^{\frac{s}{2}}Q\|_2^2\mathcal{E}^{\frac{N-2s}{2s}} = 0, \quad \mathcal{E} \in (a, +\infty).$$

This observation makes the question of uniqueness and multiplicity for solutions to problem (1.1) very simple. Therefore, our main focus of the present paper is non-degeneracy property for positive solutions of problem (1.1). The main results of this paper are collected in the following results.

Theorem 1.1 *Assume that $a, b > 0$ and $1 < p < 2_s^* - 1$. Then the following statements are true:*

- (i) *If $1 < N < 4s$, then problem (1.1) has exactly one solution;*
- (ii) *If $N = 4s$, then problem (1.1) is solvable if and only if $b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2 < 1$, and in this case problem (1.1) has exactly one solution;*
- (iii) *If $N > 4s$, then problem (1.1) is solvable if and only if*

$$b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2 \leq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$

Furthermore, problem (1.1) has exactly one solution when the equality holds, and has exactly two solutions for the other case.

Moreover, define the solution by U, then there exist some $x_0 \in \mathbb{R}^N$ such that $U(\cdot - x_0)$ is radial, positive and strictly decreasing in $r = |x - x_0|$. Moreover, the function U belongs to $C^\infty(\mathbb{R}^N) \cap H^{2s+1}(\mathbb{R}^N)$ and it satisfies

$$\frac{C_1}{1+|x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1+|x|^{N+2s}}, \quad \forall\, x \in \mathbb{R}^N,$$

with some constants $C_2 \geq C_1 > 0$;

Theorem 1.2 *Suppose that $a, b > 0$. Then any positive solution $U(x)$ of problem (1.1) is non-degenerate if one of the following conditions holds:*

- $1 \leq N \leq 4s$;
- $N > 4s$ *and* $b\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 \neq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$.

By Theorem 1.2, it is now possible that we apply Lyapunov–Schmidt reduction to study the perturbed fractional Kirchhoff equation.

$$\left(\varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)(-\Delta)^s u + V(x)u = u^p, \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a bounded continuous function. We want to look for solutions of (1.8) in the Sobolev space $H^s(\mathbb{R}^N)$ for sufficiently small ε, which named semiclassical solutions. We also call such derived solutions as concentrating solutions since they will concentrate at certain point of the potential function V. Moreover, it is expected that this approach can deal with problem (1.8) for all $1 < p < 2_s^* - 1$, in a unified way. To state our following results, let introduce some notations that will be used throughout the paper. For $\varepsilon > 0$ and $y = (y_1, y_2, \cdots y_N) \in \mathbb{R}^N$, write

$$U_{\varepsilon,y}(x) = U\left(\frac{x-y}{\varepsilon}\right), \quad x \in \mathbb{R}^N.$$

Assume that $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(V_1) V is a bounded continuous function with $\inf_{x\in\mathbb{R}^N} V > 0$;

(V_2) There exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$V(x_0) < V(x) \quad \text{for } 0 < |x - x_0| < r_0,$$

and $V \in C^\alpha\left(\bar{B}_{r_0}(x_0)\right)$ for some $0 < \alpha < \frac{N+4s}{2}$. That is, V is of α-th order Hölder continuity around x_0.

The assumption (V_1) allows us to introduce the inner products

$$\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^N} \left(\varepsilon^{2s} a(-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v + V(x)uv\right) \mathrm{d}x,$$

for $u, v \in H^s(\mathbb{R}^N)$. We also write

$$H_\varepsilon = \left\{u \in H^s(\mathbb{R}^N) : \|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{\frac{1}{2}} < \infty\right\}.$$

Now we state the existence result as follows.

Theorem 1.3 *Let $a, b > 0$, $1 < p < 2_s^* - 1$ and V satisfies (V_1) and (V_2). Assume that $N = 4s$ and $b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} Q|^2 \mathrm{d}x < 1$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.8) has a solution u_ε of the form*

$$u_\varepsilon = U\left(\frac{x - y_\varepsilon}{\varepsilon}\right) + \varphi_\varepsilon$$

with $\varphi_\varepsilon \in H_\varepsilon$, satisfying

$$y_\varepsilon \to x_0,$$
$$\|\varphi_\varepsilon\|_\varepsilon = o\left(\varepsilon^{\frac{N}{2}}\right)$$

as $\varepsilon \to 0$.

Theorem 1.4 *Let $a, b > 0$, $1 < p < 2_s^* - 1$ and V satisfies (V_1) and (V_2). Assume that $N > 4s$ and $b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} Q|^2 dx < \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$. Let $U_i (i = 1, 2)$ be two positive solutions of problem (1.1). Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.8) has two solutions $u_\varepsilon^i(x) (i = 1, 2)$ of the form*

$$u_\varepsilon^i(x) = U_i\left(\frac{x - y_\varepsilon}{\varepsilon}\right) + \varphi_\varepsilon^i(x),$$

with $\varphi_\varepsilon \in H_\varepsilon$, satisfying

$$y_\varepsilon^i \to x_0,$$
$$\left\|\varphi_\varepsilon^i\right\|_\varepsilon = o\left(\varepsilon^{\frac{N}{2}}\right)$$

as $\varepsilon \to 0$.

This paper is organized as follows. We complete the proof of Theorem 1.1 in Sect. 2 and prove Theorem 1.2 in Sect. 3. In Sect. 3, we present some basic results and explain the strategy of the proof of Theorems 1.3 and 1.4.

Notation. Throughout this paper, we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^N$, $B_R(x)$ denotes the ball of radius R centered at x;
- $\|\cdot\|_q$ denotes the usual norm of the space $L^q(\mathbb{R}^N)$, $1 \le q \le \infty$;
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$;
- C or $C_i (i = 1, 2, \cdots)$ are some positive constants may change from line to line.

2 Proof of Theorem 1.1

In this section, we analyze the existence of solutions for the following fractional Kirchhoff problem

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+u=u^p, & \text{in } \mathbb{R}^N,\\ u(x)>0, & \text{in } \mathbb{R}^N,\\ u(x)\in H^s(\mathbb{R}^N). \end{cases} \tag{2.1}$$

As mentioned in the introduction, we know that any solution to (2.1) has the following form

$$U(x)=Q\left(\mathcal{E}_0^{-\frac{1}{2s}}x-x_0\right).$$

and

$$\mathcal{E}_0=a+b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2\mathcal{E}_0^{\frac{N-2s}{2s}},$$

where Q being the unique positive radial solution to the following problem

$$\begin{cases} (-\Delta)^s Q+Q=Q^p, & \text{in } \mathbb{R}^N,\\ Q(x)>0, & \text{in } \mathbb{R}^N,\\ Q(x)\in H^s(\mathbb{R}^N). \end{cases} \tag{2.2}$$

Let Q be the uniquely positive solution of (2.2) and also a minimizer of $J(u)$. Consider the equation

$$f(\mathcal{E})=\mathcal{E}-a-b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2\mathcal{E}^{\frac{N-2s}{2s}}=0,\quad \mathcal{E}\in(a,+\infty). \tag{2.3}$$

Therefore, to find solution $U(x)$ of (2.1), it suffices to find positive solutions of the above algebraic Eq. (2.3).

Case 1 $\mathbf{1<N<4s}$: In this case, we have $\frac{N-2s}{2s}<1$, which implies that $\lim\limits_{\mathcal{E}\to+\infty}f(\mathcal{E})=+\infty$. Moreover, one has $f(a)<0$. Consequently, there exists unique $\mathcal{E}_0>a$ such that $f(\mathcal{E}_0)=0$, which means that (2.1) has a unique solution.

Case 2 $\mathbf{N=4s}$: In this case, (2.3) becomes

$$\mathcal{E}-a-b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2\mathcal{E}=0, \tag{2.4}$$

which means that this equation has a unique positive solution

$$\mathcal{E}_0=\frac{a}{1-b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2}, \tag{2.5}$$

if and only if $b < \frac{1}{\|(-\Delta)^{\frac{s}{2}} Q\|_2^2}$.

Case 3 N > 4s : A simple computation implies that

$$f'(\mathcal{E}) = 1 - \frac{N-2s}{2s} b\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 \mathcal{E}^{\frac{N-4s}{2s}}, \tag{2.6}$$

which means that $f(\mathcal{E})$ has a unique maximum point

$$\mathcal{E}_0 = \left(\frac{2s}{(N-2s)\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 b} \right)^{\frac{2s}{N-4s}} > 0, \tag{2.7}$$

and the maximum of $f(\mathcal{E})$ is

$$f(\mathcal{E}_0) = (\frac{N-4s}{N-2s}) \left(\frac{2s}{(N-2s)\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 b} \right)^{\frac{2s}{N-4s}} - a. \tag{2.8}$$

It is easy to see that $f(\mathcal{E}_0) \geq 0$ implies

$$b\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 \leq \frac{2sa^{\frac{4s-N}{2s}} (N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}. \tag{2.9}$$

Since $f''(\mathcal{E}) < 0$ in $(0, +\infty)$ due to $N > 4s$, we know that $f(\mathcal{E})$ is concave in $(0, +\infty)$. Noting further that $f(0) = -a < 0$ and $\lim_{\mathcal{E}\to+\infty} f(\mathcal{E}) = -\infty$, a sufficient and necessary condition for the solvability of Eq. (2.3) in $(0, +\infty)$ is $f(\mathcal{E}_0) \geq 0$. Hence, Eq. (2.3) has a solution in $(0, +\infty)$ if and only if inequality (2.9) holds. Furthermore, we have

(*i*) If $b\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 = \frac{2sa^{\frac{4s-N}{2s}} (N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, then Eq. (2.3) has exactly one positive solution $\mathcal{E}_0$ defined by (2.7);

(*ii*) If $b\|(-\Delta)^{\frac{s}{2}} Q\|_2^2 < \frac{2sa^{\frac{4s-N}{2s}} (N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, then Eq. (2.3) has exactly two positive solutions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\mathcal{E}_1 \in (0, \mathcal{E}_0)$ and $\mathcal{E}_2 \in (\mathcal{E}_0, +\infty)$.

Up to now, we have proved Theorem 1.1. Next, we analyze the asymptotic behavior of solution obtained above as $b \to 0$. In the case $1 < N \leq 4s$, if we denote by $\mathcal{E}_0$ the unique positive solution to Eq. (2.3), we have $\lim_{b\to 0} b\mathcal{E}_0 = 0$. It infers from this that the following conclusion holds.

Theorem 2.1 *Assume that* $1 < N \leq 4s$. *Let* $U_b(x)$ *be the unique solution to problem* (2.1). *Then* $\lim_{b\to 0} U_b(x) = Q(x)$ *in point wise.*

In the case $N > 4s$, if $b\|(-\Delta)^{\frac{s}{2}}Q\|_2^2 < \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, Eq. (2.1) has exactly two solutions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that

$$0 < \mathcal{E}_1 < \mathcal{E}_0 \text{ and } \mathcal{E}_0 < \mathcal{E}_2 < +\infty, \text{ where } \mathcal{E}_0 = \left(\frac{2s}{(N-2s)\|(-\Delta)^{\frac{s}{2}}Q\|_2^2 b}\right)^{\frac{2s}{N-4s}}. \tag{2.10}$$

Correspondingly, problem (2.1) has exactly two solutions

$$U_b^1(x) = Q\left(\mathcal{E}_1^{-\frac{1}{2s}}x\right) \quad \text{and} \quad U_b^2(x) = Q\left(\mathcal{E}_2^{-\frac{1}{2s}}x\right).$$

From (2.10), we can see that

$$\lim_{b\to 0} b\mathcal{E}_2 \geq \lim_{b\to 0} b\mathcal{E}_0 = +\infty.$$

Hence,

$$\lim_{b\to 0} U_b^2(x) = Q(0), \quad \forall x \in \mathbb{R}^N.$$

By a similar analysis, we have $\lim_{b\to 0} b\mathcal{E}_1 = 0$, and the following conclusion is true.

Theorem 2.2 *Suppose that $N > 4s$. Then*

$$\lim_{b\to 0} U_b^1(x) = Q(x) \quad \textit{and} \quad \lim_{b\to 0} U_b^2(x) = Q(0), \quad \forall x \in \mathbb{R}^N.$$

3 Nondegeneracy Results

In this section we prove the nondegeneracy results of Theorem 1.2. For positive constants a, b, we define the differential operator L as

$$L(u) = \left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)(-\Delta)^s u + u - |u|^{p-1}u,$$

for any $u \in H^s(\mathbb{R}^N)$ in the weak sense. The linearized operator $\mathcal{L}_+$ of L at U is defined as

$$\mathcal{L}_+(\varphi) = \left.\frac{dL(U + t\varphi)}{dt}\right|_{t=0}, \quad \forall \varphi \in H^s(\mathbb{R}^N).$$

It is easy to see that for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\begin{aligned}\mathcal{L}_+(\varphi) &= \Big(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 \mathrm{d}x\Big)(-\Delta)^s \varphi + \varphi - pU^{p-1}\varphi + 2b \\ &\quad \Big(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x\Big)(-\Delta)^s U \\ &= T_+(\varphi) + L_2(\varphi)(-\Delta)^s U,\end{aligned}$$

acting on $L^2(\mathbb{R}^N)$ with domain $D(L)$, where

$$T_+(\varphi) = c(-\Delta)^s \varphi + \varphi - pU^{p-1}\varphi,$$

with $c = a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 \mathrm{d}x$ and

$$L_2(\varphi) = 2b\Big(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x\Big).$$

We also denote by $\mathrm{Ker}(L)$ the kernel space of a linear operator L, that is

$$\mathrm{Ker}(L) = \{\varphi \in D(L) : L(\varphi) = 0\}.$$

Definition 3.1 Let $U \in H^s(\mathbb{R}^N)$ be a solution to $L(u) = 0$. We say that U is non-degenerate if $\mathrm{Ker}\,(\mathcal{L}_+) = \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$.

In the sequel, we always use $U(x)$ to denote a positive solution to the equation $L(u) = 0$ in $H^s(\mathbb{R}^N)$. We divide the proof of Theorem (1.2) into the following series of lemmas.

Lemma 3.1 $\mathrm{Ker}(T_+) = \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$.

Proof Since $U(x)$ is a positive solution to the equation $L(u) = 0$, $U(x)$ satisfies

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 dx\right)(-\Delta)^s U + U - U^p = 0, \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

For any fixed $i \in \{1, 2, \ldots, N\}$, taking partial derivative with respect to x_i on both sides of the above Eq. (3.1), we obtain

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 dx\right)(-\Delta)^s \frac{\partial U}{\partial x_i} + \frac{\partial U}{\partial x_i} - pU^{p-1}\frac{\partial U}{\partial x_i} = 0, \quad \text{in } \mathbb{R}^N.$$

This implies that $T_+\left(\frac{\partial U}{\partial x_i}\right) = 0$ for any fixed $i \in \{1, 2, \ldots, N\}$. Therefore,

$$\mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\} \subseteq \mathrm{Ker}(T_+).$$

On the other hand, for any $\varphi \in \mathrm{Ker}(T_+)$, from the definition of $\mathrm{Ker}(T_+)$, we have

$$c(-\Delta)^s\varphi + \varphi - pU^{p-1}\varphi = 0. \tag{3.2}$$

Let $x = c^{\frac{1}{2s}}y$, $\hat{\varphi}(y) = \varphi(c^{\frac{1}{2s}}y) = \varphi(x)$ and $Q(y) = U(c^{\frac{1}{2s}}y) = U(x)$. Then Eq. (3.2) becomes

$$(-\Delta)^s\hat{\varphi}(y) + \hat{\varphi}(y) - pQ^{p-1}(y)\hat{\varphi}(y) = 0. \tag{3.3}$$

Noting that $Q(y)$ is a solution to Eqs.(1.3), (3.3) implies that $\hat{\varphi}(y) \in \mathrm{Ker}\ (T_+)$. Therefore, it follows from Proposition 1.2 that there are real numbers $a_i (i \in \{1, 2, \ldots, N\})$ such that

$$\hat{\varphi}(y) = \sum_{i=1}^{N} a_i \frac{\partial Q}{\partial y_i}.$$

Since $\frac{\partial Q}{\partial y_i} = c^{\frac{1}{2s}}\frac{\partial U}{\partial x_i}$, we have

$$\varphi(x) = \hat{\varphi}(y) = \sum_{i=1}^{N} a_i \frac{\partial Q}{\partial y_i} = \sum_{i=1}^{N} a_i c^{\frac{1}{2s}} \frac{\partial U}{\partial x_i}.$$

This implies that $\varphi \in \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$. From the arbitrariness of φ, we have $\mathrm{Ker}(T_+) \subseteq \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$. Thus, $\mathrm{Ker}(T_+) = \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$.

□

Since $\frac{\partial U}{\partial x_i}$ is non-radially symmetric, we have the following corollary:

Corollary 3.1 *T_+ is invertible on $L^2_{rad}(\mathbb{R}^N)$.*

Lemma 3.2 *Let $U(x)$ be a positive solution to the equation $L(u) = 0$ in $H^s(\mathbb{R}^N)$. Then $L_2\left(\frac{\partial U}{\partial x_i}\right) = 0$ for $i \in \{1, 2, \ldots, N\}$.*

Proof From the definition of L_2, and U is the solution of the equation

$$c(-\Delta)^s U + U = U^p.$$

We have

$$L_2\left(\frac{\partial U}{\partial x_i}\right) = -2b\int_{\mathbb{R}^N} \frac{\partial U}{\partial x_i}(-\Delta)^s U \mathrm{d}x.$$

Therefore,

$$L_2\left(\frac{\partial U}{\partial x_i}\right) = \frac{-2b}{c}\int_{\mathbb{R}^N} \left(U^p - U\right)\frac{\partial U}{\partial x_i}\mathrm{d}x = \frac{-2b}{c}\int_{\mathbb{R}^N} \frac{\partial\left(\frac{1}{p+1}U^{p+1} - \frac{1}{2}U\right)}{\partial x_i}\mathrm{d}x.$$

Since, for any fixed i, up to a translation, the function $\frac{\partial\left(\frac{1}{p+1}U^{p+1}-\frac{1}{2}U\right)}{\partial x_i}$ is odd in variable x_i, it is easy to see that

$$\int_{\mathbb{R}^N} \frac{\partial\left(\frac{1}{p+1}U^{p+1}-\frac{1}{2}U\right)}{\partial x_i}\mathrm{d}x = 0.$$

Therefore, $L_2\left(\frac{\partial U}{\partial x_i}\right) = 0.$ □

Lemma 3.3 *Let $U(x)$ be a positive solution to the equation $L(u) = 0$ in $H^s(\mathbb{R}^N)$. If $N > 4s$ and*

$$\frac{(N-2s)b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2\mathrm{d}x}{2sc} = 1,$$

then

$$b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}Q|^2\mathrm{d}x = \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}},$$

where $Q \in H^s(\mathbb{R}^N)$ is the unique positive solution to the equation $L_0(u) = 0$.

Proof Noting that $c = a + b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2\mathrm{d}x$, the assumption

$$\frac{(N-2s)b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2\mathrm{d}x}{2sc} = 1,$$

implies

$$b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2dx = \frac{2sa}{N-4s} \text{ and } c = \frac{(N-2s)a}{N-4s}.$$

Since $U(x) \in H^s(\mathbb{R}^N)$ is a positive solution to the equation $L(u) = 0$, we know that $U(x)$ has the following form

$$U(x) = Q\left(c^{-\frac{1}{2s}}x\right),$$

with $Q(x) \in H^s(\mathbb{R}^N)$ being the unique positive solution to the Eq. (1.3). Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2dx &= c^{\frac{N-2s}{2s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}Q|^2\mathrm{d}x\\ &= \left(\frac{(N-2s)a}{N-4s}\right)^{\frac{N-2s}{2s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}Q|^2\mathrm{d}x.\end{aligned}$$

Therefore, we have

$$b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}Q|^2\mathrm{d}x=\frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$

This completes the proof. □

Lemma 3.4 *Let $U(x)$ be a positive solution to the equation $L(u)=0$ in $H^s(\mathbb{R}^N)$. Suppose that*

$$1<N\leq 4s,$$

or

$$N>4s \quad and \quad b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}Q|^2\mathrm{d}x\neq\frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$

Then

$$\mathrm{Ker}\,(\mathcal{L}_+)\bigcap L^2_{rad}(\mathbb{R}^N)=\{0\},$$

Proof Assume that $v\in H^s(\mathbb{R}^N)\cap L^2_{rad}(\mathbb{R}^N)$ belongs to $\ker\mathcal{L}_+$. Then we have

$$\begin{aligned}&\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2\mathrm{d}x\Big)(-\Delta)^s v+v-pU^{p-1}v\\&=-2b\Big(\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}U(-\Delta)^{\frac{s}{2}}v\mathrm{d}x\Big)(-\Delta)^sU.\end{aligned}\tag{3.4}$$

Let $c=a+b\|(-\Delta)^{\frac{s}{2}}U\|_2^2$. Recall that U is a ground state solution of (1.1). It follows from above that c is a constant independent of U under the assumptions of Theorem 1.1. Hence, U solves (1.3) with $c=a+b\|(-\Delta)^{\frac{s}{2}}U\|_2^2$. We then can rewrite (3.4) as

$$T_+v=-2b\Big(\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}U(-\Delta)^{\frac{s}{2}}v\mathrm{d}x\Big)(-\Delta)^sU=-\frac{2b\sigma_v}{c}(-U+U^p),\tag{3.5}$$

where

$$\sigma_v=\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}U(-\Delta)^{\frac{s}{2}}v\mathrm{d}x.$$

By applying Proposition 1.2, we conclude that

$$v=-\frac{2b\sigma_v}{c}T_+^{-1}(-U+U^p)=-\frac{b\sigma_v}{sc}\psi,\tag{3.6}$$

where $\psi = x \cdot \nabla U$. Multiplying (3.6) by $(-\Delta)^s U$ and integrating over $\mathbb{R}^N$, we see that

$$\int_{\mathbb{R}^N} v(-\Delta)^s U \mathrm{d}x = -\frac{b\sigma_v}{sc} \int_{\mathbb{R}^N} \psi(-\Delta)^s U \mathrm{d}x. \tag{3.7}$$

Note that

$$\int_{\mathbb{R}^N} v(-\Delta)^s U \mathrm{d}x = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U (-\Delta)^{\frac{s}{2}} v \mathrm{d}x, \tag{3.8}$$

and

$$\int_{\mathbb{R}^N} \psi(-\Delta)^s U \mathrm{d}x = \frac{2s-N}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 \mathrm{d}x, \tag{3.9}$$

(see e.g. [39]). We then conclude from (3.7)-(3.9) that

$$\sigma_v = -\frac{b(2s-N)\sigma_v}{2sc} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 \mathrm{d}x = -\frac{(c-a)(2s-N)}{2sc} \sigma_v.$$

It follows from Lemma 3.3 that

$$1 + \frac{(2s-N)b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 \mathrm{d}x}{2sc} \neq 0,$$

provided that $1 < N \leq 4s$, or $N > 4s$ and $b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} Q|^2 \mathrm{d}x \neq \frac{2sa^{\frac{4s-N}{2s}} (N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$. Therefore, under this assumption, we have $v \equiv 0$. This completes the proof. □

Proof of Theorem 1.2 Let $U(x) \in H^s(\mathbb{R}^N)$ be a positive solution to the equation $L(u) = 0$. For any $i \in \{1, 2, \ldots, N\}$, by Lemmas 3.1 and 3.2, we have

$$\mathcal{L}_+ \left(\frac{\partial U}{\partial x_i} \right) = T_+ \left(\frac{\partial U}{\partial x_i} \right) + L_2 \left(\frac{\partial U}{\partial x_i} \right) (-\Delta)^s U = 0.$$

This implies that

$$\operatorname{span} \left\{ \frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N} \right\} \subseteq \operatorname{Ker} (\mathcal{L}_+).$$

On the other hand, for any $\varphi(x) \in \operatorname{Ker} (\mathcal{L}_+)$, we have

$$T_+(\varphi) = -L_2(\varphi)(-\Delta)^s U. \tag{3.10}$$

To prove $\varphi \in \text{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$, it follows from Corollary 3.1 that there exists a unique radial function $\psi_1(r) \in L^2_{\text{rad}}(\mathbb{R}^N)$ such that

$$T_+(\psi_1) = -L_2(\varphi)(-\Delta)^s U. \tag{3.11}$$

Set $W = \psi(x) - \psi_1(r)$. Then, from (3.10) and (3.11), we have $T_+(W) = 0$. Therefore, it follows from Lemma 3.1 that there are some real numbers a_i such that

$$W = \sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i}.$$

This implies that any solution $\psi(x)$ to the Eq. (3.10) has the following form

$$\psi(x) = \psi_1(r) + \sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i}.$$

Since $\varphi(x)$ is a solution to (3.10), we conclude that

$$\varphi(x) = \psi_1(r) + \sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i} \tag{3.12}$$

for some real numbers a_i. Noting that $\varphi(x)$ and $\frac{\partial U}{\partial x_i}$ are in Ker $(\mathcal{L}_+)$, we can conclude from (3.12) that $\mathcal{L}_+(\psi_1(r)) = 0$. That is $\psi_1(r) \in \text{Ker}(\mathcal{L}_+)$. Hence, it follows from Lemma 3.4 that $\psi_1(r) \equiv 0$. Now, from (3.12), we have

$$\varphi(x) = \sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i},$$

for some real numbers a_i. This implies that $\varphi \in \text{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$. From the arbitrariness of φ, we see that Ker $(\mathcal{L}_+) \subseteq \text{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$.

In conclusion, we have Ker $(\mathcal{L}_+) = \text{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots, \frac{\partial U}{\partial x_N}\right\}$. That is, $U(x)$ is non-degenerate. This completes the proof of Theorem 1.2. □

4 The Lyapunov–Schmidt Reduction

As mentioned in the Introduction, non-degeneracy property of positive solutions for the limit problem in entire space can be used to construct concentrated solutions for singularly perturbed problems. Here, we take the following problem as an example:

$$\begin{cases} \left(\varepsilon^{2s}a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)(-\Delta)^s u + V(x)u = u^p, & \text{in } \mathbb{R}^N \\ 0 < u(x) \in H^s(\mathbb{R}^N). \end{cases} \tag{4.1}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(V_1) V is a bounded continuous function with $\inf\limits_{x \in \mathbb{R}^N} V > 0$;

(V_2) There exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$V(x_0) < V(x) \quad \text{for } 0 < |x - x_0| < r_0,$$

and $V \in C^\alpha\left(\bar{B}_{r_0}(x_0)\right)$ for some $0 < \alpha < \frac{N+4s}{2}$. That is, V is of α-th order Hölder continuity around x_0.

It is known that every solution to Eq. (4.1) is a critical point of the energy functional $I_\varepsilon : H_\varepsilon \to \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b\varepsilon^{4s-N}}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x,$$

for $u \in H_\varepsilon$. It is standard to verify that $I_\varepsilon \in C^2(H_\varepsilon)$. So we are left to find a critical point of I_ε. Since the procedure of Lyapunov–Schmidt reduction is the same as in [41], we just state some Lemmas and explain the strategy of the proof. Readers interested in the full proof shall refer to [41].

4.1 Finite Dimensional Reduction

We will restrict our argument to the existence of a critical point of I_ε that concentrates, as ε small enough. For $\delta, \eta > 0$, fixing $y \in B_\delta(x_0)$, we define

$$M_{\varepsilon,\eta} = \left\{(y, \varphi) : y \in B_\delta(x_0), \varphi \in E_{\varepsilon,y}\right\},$$

where we denote $E_{\varepsilon,y}$ by

$$E_{\varepsilon,y} := \left\{\varphi \in H_\varepsilon : \left\langle \frac{\partial U_{\varepsilon,y^i}}{\partial y^i}, \varphi \right\rangle_\varepsilon = 0, i = 1, \ldots, N\right\}.$$

We are looking for a critical point of the form

$$u_\varepsilon = U_{\varepsilon,y} + \varphi_\varepsilon.$$

For this we introduce a new functional $J_\varepsilon : M_{\varepsilon,\eta} \to \mathbb{R}$ defined by

$$J_\varepsilon(y, \varphi) = I_\varepsilon\left(U_{\varepsilon,y} + \varphi\right), \quad \varphi \in E_{\varepsilon,y}.$$

In fact, we divide the proof of Theorem 1.3 and 1.4 into two steps:

Step 1 for each ε, δ sufficiently small and for each $y \in B_\delta(x_0)$, we will find a critical point $\varphi_{\varepsilon,y}$ for $J_\varepsilon(y,\cdot)$ (the function $y \mapsto \varphi_{\varepsilon,y}$ also belongs to the class C^1 (H_ε));

Step 2 for each ε, δ sufficiently small, we will find a critical point y_ε for the function $j_\varepsilon : B_\delta(x_0) \to \mathbb{R}$ induced by

$$y \mapsto j_\varepsilon(y) \equiv J\left(y, \varphi_{\varepsilon,y}\right). \tag{4.2}$$

That is, we will find a critical point y_ε in the interior of $B_\delta(x_0)$.

It is standard to verify that $\left(y_\varepsilon, \varphi_{\varepsilon,y_\varepsilon}\right)$ is a critical point of J_ε for ε sufficiently small by the chain rule. This gives a solution $u_\varepsilon = U_{\varepsilon,y_\varepsilon} + \varphi_{\varepsilon,y_\varepsilon}$ to Eq. (4.1) for ε sufficiently small in virtue of the following lemma.

Lemma 4.1 *There exist $\varepsilon_0, \eta_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\eta \in (0, \eta_0]$, and $(y, \varphi) \in M_{\varepsilon,y}$ the following are equivalent:*

(i) $u_\varepsilon = U_{\varepsilon,y_\varepsilon} + \varphi_{\varepsilon,y_\varepsilon}$ is a critical point of I_ε in H_ε.
(i) (y, φ) is a critical point of J_ε.

Now, in order to realize **Step 1**, we expand $J_\varepsilon(y,\cdot)$ near $\varphi = 0$ for each fixed y as follows:

$$J_\varepsilon(y,\varphi) = J_\varepsilon(y,0) + l_\varepsilon(\varphi) + \frac{1}{2}\langle \mathcal{L}_\varepsilon \varphi, \varphi\rangle + R_\varepsilon(\varphi),$$

where $J_\varepsilon(y,0) = I_\varepsilon\left(U_{\varepsilon,y}\right)$, and l_ε, $\mathcal{L}_\varepsilon$ and R_ε are defined for $\varphi, \psi \in H_\varepsilon$ as follows:

$$\begin{aligned} l_\varepsilon(\varphi) &= \left\langle I_\varepsilon'\left(U_{\varepsilon,y}\right), \varphi\right\rangle \\ &= \left\langle U_{\varepsilon,y}, \varphi\right\rangle_\varepsilon + b\varepsilon^{4s-N}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y}\right|^2 \mathrm{d}x\right)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y} \\ &\quad \cdot (-\Delta)^{\frac{s}{2}}\varphi \mathrm{d}x - \int_{\mathbb{R}^N} U_{\varepsilon,y}^p \varphi \mathrm{d}x, \end{aligned} \tag{4.3}$$

and $\mathcal{L}_\varepsilon : L^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right)$ is the bilinear form around $U_{\varepsilon,y}$ defined by

$$\begin{aligned} \langle \mathcal{L}_\varepsilon \varphi, \psi\rangle &= \left\langle I_\varepsilon''\left(U_{\varepsilon,y}\right)[\varphi], \psi\right\rangle \\ &= \langle \varphi, \psi\rangle_\varepsilon + b\varepsilon^{4s-N}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y}\right|^2 \mathrm{d}x\right)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\varphi \cdot (-\Delta)^{\frac{s}{2}}\psi \mathrm{d}x \\ &\quad + 2\varepsilon^{4s-N} b\left(\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{s}{2}}\varphi \mathrm{d}x\right) \\ &\quad \left(\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{s}{2}}\psi \mathrm{d}x\right) - p\int_{\mathbb{R}^N} U_{\varepsilon,y}^{p-1}\varphi\psi \mathrm{d}x, \end{aligned}$$

and R_ε denotes the second order reminder term given by

$$R_\varepsilon(\varphi) = J_\varepsilon(y,\varphi) - J_\varepsilon(y,0) - l_\varepsilon(\varphi) - \frac{1}{2}\left\langle \mathcal{L}_\varepsilon\varphi, \varphi\right\rangle. \tag{4.4}$$

We remark that R_ε belongs to $C^2(H_\varepsilon)$ since so is every term in the right hand side of (4.4).

Lemma 4.2 *Assume that V satisfies (V_1) and (V_2). Then, there exists a constant $C>0$, independent of ε, such that for any $y \in B_1(0)$, there holds*

$$|l_\varepsilon(\varphi)| \le C\varepsilon^{\frac{N}{2}}\left(\varepsilon^\alpha + (|V(y) - V(x_0)|)\right)\|\varphi\|_\varepsilon,$$

for $\varphi \in H_\varepsilon$. Here α denotes the order of the Hölder continuity of V in $B_{r_0}(0)$.

Lemma 4.3 *There exists a constant $C>0$, independent of ε and b, such that for $i \in \{0,1,2\}$, there hold*

$$\left\|R_\varepsilon^{(i)}(\varphi)\right\| \le C\varepsilon^{-\frac{N(p-1)}{2}}\|\varphi\|_\varepsilon^{p+1-i} + C(b+1)\varepsilon^{-\frac{N}{2}}\left(1+\varepsilon^{-\frac{N}{2}}\|\varphi\|_\varepsilon\right)\|\varphi\|_\varepsilon^{N-i},$$

for all $\varphi \in H_\varepsilon$.

Lemma 4.4 *Assume that V satisfies $(V1)$ and $(V2)$. Then, for $\varepsilon>0$ sufficiently small, there is a small constant $\tau>0$ and $C>0$ such that,*

$$I_\varepsilon\left(U_{\varepsilon,y}\right) = A\varepsilon^N + B\varepsilon^N\left((V(y) - V(x_0))\right) + O\varepsilon^{N+\alpha},$$

where

$$A = \tfrac{1}{2}\int_{\mathbb{R}^N}\left(a|(-\Delta)^{\frac{s}{2}}U|^2 + U^2\right)\mathrm{d}x + \tfrac{b}{4}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U|^2\mathrm{d}x\right)^2 \\ -\tfrac{1}{p+1}\int_{\mathbb{R}^N}U^{p+1}\mathrm{d}x,$$

and

$$B = \frac{1}{2}\int_{\mathbb{R}^N}U^2\mathrm{d}x.$$

In this subsection we complete **Step 1** for the Lyapunov–Schmidt reduction method as in Sect. 4. We first consider the operator $\mathcal{L}_\varepsilon$,

$$\begin{aligned}\langle \mathcal{L}_\varepsilon \varphi, \psi\rangle = \langle \varphi, \psi\rangle_\varepsilon + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} U_{\varepsilon,y}\right|^2 \mathrm{d}x \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi \cdot (-\Delta)^{\frac{s}{2}} \psi \mathrm{d}x \\ + 2\varepsilon^{4s-N} b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x\right) \\ \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{s}{2}} \psi \mathrm{d}x\right) \\ - p \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p-1} \varphi \psi \mathrm{d}x,\end{aligned}$$

for $\varphi, \psi \in H_\varepsilon$. The following result shows that $\mathcal{L}_\varepsilon$ is invertible when restricted on $E_{\varepsilon,y}$

Lemma 4.5 *There exist $\varepsilon_1 > 0, \delta_1 > 0$ and $\rho > 0$ sufficiently small, such that for every $\varepsilon \in (0, \varepsilon_1), \delta \in (0, \delta_1)$, there holds*

$$\|\mathcal{L}_\varepsilon \varphi\|_\varepsilon \geq \rho \|\varphi\|_\varepsilon, \quad \forall \varphi \in E_{\varepsilon,y},$$

uniformly with respect to $y \in B_\delta(x_0)$.

Lemma 4.5 implies that by restricting on $E_{\varepsilon,y}$, the quadratic form $\mathcal{L}_\varepsilon : E_{\varepsilon,y} \to E_{\varepsilon,y}$ has a bounded inverse, with $\left\|\mathcal{L}_\varepsilon^{-1}\right\| \leq \rho^{-1}$ uniformly with respect to $y \in B_\delta(x_0)$. This further implies the following reduction map.

Lemma 4.6 *There exist $\varepsilon_0 > 0, \delta_0 > 0$ sufficiently small such that for all $\varepsilon \in (0, \varepsilon_0), \delta \in (0, \delta_0)$, there exists a C^1 map $\varphi_\varepsilon : B_\delta(x_0) \to H_\varepsilon$ with $y \mapsto \varphi_{\varepsilon,y} \in E_{\varepsilon,y}$ satisfying*

$$\left\langle \frac{\partial J_\varepsilon\left(y, \varphi_{\varepsilon,y}\right)}{\partial \varphi}, \psi \right\rangle_\varepsilon = 0, \quad \forall \psi \in E_{\varepsilon,y}.$$

Moreover, there exists a constant $C > 0$ independent of ε small enough and $\kappa \in (0, \frac{\alpha}{2})$ such that

$$\|\varphi_{\varepsilon,y}\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}+\alpha-\kappa} + C\varepsilon^{\frac{N}{2}} \left(V(y) - V(x_0)\right)^{1-\kappa}.$$

4.2 Proof of Theorems 1.3 and 1.4

Let ε_0 and δ_0 be defined as in Lemma 4.6 and let $\varepsilon < \varepsilon_0$. Fix $0 < \delta < \delta_0$. Let $y \mapsto \varphi_{\varepsilon,y}$ for $y \in B_\delta(x_0)$ be the map obtained in Lemma 4.6. As aforementioned in **Step 2**, it is equivalent to find a critical point for the function j_ε defined as in (4.2) by Lemma 4.1. By the Taylor expansion, we have

$$j_\varepsilon(y) = J\left(y, \varphi_{\varepsilon,y}\right) = I_\varepsilon\left(U_{\varepsilon,y}\right) + l_\varepsilon\left(\varphi_{\varepsilon,y}\right) + \frac{1}{2}\left\langle \mathcal{L}_\varepsilon \varphi_{\varepsilon,y}, \varphi_{\varepsilon,y}\right\rangle + R_\varepsilon\left(\varphi_{\varepsilon,y}\right).$$

We analyze the asymptotic behavior of j_ε with respect to ε first.

By Lemmas 4.2, 4.3, 4.4 and 4.6, we have

$$\begin{aligned} j_\varepsilon(y) &= I_\varepsilon\left(U_{\varepsilon,y}\right) + O\left(\|l_\varepsilon\|\,\|\varphi_\varepsilon\| + \|\varphi_\varepsilon\|^2\right) \\ &= A\varepsilon^N + B\varepsilon^N\left(V(y) - V(x_0)\right) + \varepsilon^N\left(\varepsilon^{\alpha-\kappa}\right. \\ &\quad \left. + \left(V(y) - V(x_0)\right)^{1-\kappa}\right)^2 + O\varepsilon^{N+\alpha}. \end{aligned} \tag{4.5}$$

Now consider the minimizing problem

$$j_\varepsilon\left(y_\varepsilon\right) \equiv \inf_{y\in B_\delta(x_0)} j_\varepsilon(y).$$

Assume that j_ε is achieved by some y_ε in $B_\delta(x_0)$. We will prove that y_ε is an interior point of $B_\delta(x_0)$.

To prove the claim, we apply a comparison argument. Let $e \in \mathbb{R}^N$ with $|e| = 1$ and $\eta > 1$. We will choose η later. Let $z_\epsilon = \epsilon^\eta e \in B_\delta(0)$ for a sufficiently large $\eta > 1$. By the above asymptotics formula, we have

$$\begin{aligned} j_\epsilon\left(z_\epsilon\right) &= A\epsilon^N + B\epsilon^N\left(V(z_\epsilon) - V(0)\right) + O\left(\epsilon^{N+\alpha}\right) \\ &\quad + O\left(\epsilon^N\right)\left(\epsilon^{\alpha-\kappa} + \left(V(z_\epsilon) - V(0)\right)^{1-\kappa}\right)^2. \end{aligned}$$

Applying the Hölder continuity of V, we derive that

$$\begin{aligned} j_\epsilon\left(z_\epsilon\right) &= A\epsilon^N + O\left(\epsilon^{N+\alpha\eta}\right) + O\left(\epsilon^{N+\alpha}\right) \\ &\quad + O\left(\epsilon^N\left(\epsilon^{2(\alpha-\tau)} + \epsilon^{2\eta\alpha(1-\kappa)}\right)\right) \\ &= A\epsilon^N + O\left(\epsilon^{N+\alpha}\right). \end{aligned}$$

where $\eta > 1$ is chosen to be sufficiently large accordingly. Note that we also used the fact that $\kappa \ll \alpha/2$. Thus, by using $j\left(y_\epsilon\right) \le j\left(z_\epsilon\right)$ we deduce

$$B\epsilon^N\left(V(y_\epsilon) - V(0)\right) + O\left(\epsilon^N\right)\left(\epsilon^{\alpha-\kappa} + \left(V(y_\epsilon) - V(0)\right)^{1-\kappa}\right)^2 \le O\left(\epsilon^{N+\alpha}\right)$$

That is,

$$B\left(V(y_\epsilon) - V(0)\right) + O(1)\left(\epsilon^{\alpha-\kappa} + \left(V(y_\epsilon) - V(0)\right)^{1-\kappa}\right)^2 \le O\left(\epsilon^\alpha\right). \tag{4.6}$$

If $y_\epsilon \in \partial B_\delta(0)$, then by the assumption (V_2), we have

$$V(y_\epsilon) - V(0) \ge c_0 > 0,$$

for some constant $0 < c_0 \ll 1$ since V is continuous at $x = 0$ and δ is sufficiently small. Thus, by noting that $B > 0$ from Lemma 4.4 and sending $\epsilon \to 0$, we infer from (4.6) that

$$c_0 \leq 0.$$

We reach a contradiction. This proves the claim. Thus y_ϵ is a critical point of j_ϵ in $B_\delta(x_0)$. Then Theorems 1.3 and 1.4 now follows from the claim and Lemma 4.1.

Acknowledgements We would like to thank the anonymous referee for his/her careful readings of our manuscript and the useful comments. This work is partially supported by National Natural Science Foundation of China (No. 11771385) and RTG 2491 "Fourier Analysis and Spectral Theory".

Funding Open Access funding enabled and organized by Projekt DEAL.

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