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Large Perturbations of a Magnetic System with Stein–Weiss Convolution Nonlinearity

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Abstract
In this work we consider a non-local magnetic system with a Stein–Weiss convolution potential. Using variational methods, we study the existence of solutions for the weighted non-local magnetic system and we establish the existence of solutions in the case of large perturbations of the linear absorption term. In addition, we provide new variants of the Brézis–Lieb lemma (Proc Am Math Soc 88:486–490, 1983) with a Stein–Weiss convolution reaction for the non-local magnetic system.

Keywords Nonlinear Stein–Weiss type convolution system · Magnetic field · Variational methods

Mathematics Subject Classification 35J50 (Primary) · 35A15, 35B38 (Secondary)

1 Introduction

The linear Schrödinger equation is a basic tool of quantum mechanics, which provides a description of particle dynamics in a non-relativistic environment. The nonlinear Schrödinger equation appears in different physical theories, for example, see Meystre [22] and Mills [23]. In particular, we are interested in the interaction between the particles, and so we study in this paper the following weighted non-local magnetic system

$$\begin{cases} -(\nabla + iA)^2 u + (\lambda V(x) + 1)u = \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \right) |u|^{p-2}u, \ \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{P_λ}$$

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where $N \geq 2$, $\lambda > 0$ is a real parameter, $\mu \in (0, N)$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $p \in \left(\frac{2N-\mu-2\alpha}{N}, \frac{2N-\mu-2\alpha}{N-2}\right)$ if $N > 2$, $p \in \left(\frac{4-\mu-2\alpha}{2}, +\infty\right)$ if $N = 2$, i is the imaginary unit. The magnetic potential $A : \mathbb{R}^N \mapsto \mathbb{R}^N$ is in $L^2_{\text{loc}}(\mathbb{R}^N)$ and the scalar potential $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a nonnegative continuous function which can vanish somewhere.

We now assume that $Z : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function which satisfies the following hypotheses:

- there exist two positive constants m_0 and m_1 such that

$$\lambda V(x) + Z(x) \geq m_0 \text{ and } |Z(x)| \leq m_1 \text{ for all } x \in \mathbb{R}^N,\ \lambda > 0.$$

In problem (P_λ), if we replace $\lambda V(x) + 1$ with $\lambda V(x) + Z(x)$ and adjust the workspace accordingly, our method is still valid.

Our purpose is to qualitatively analyze the solutions of the weighted non-local magnetic systems with a Stein–Weiss convolution term in whole space. Because of the appearance of magnetic field A, problem (P_λ) cannot be transformed into a pure real-valued problem, so we should deal with a complex-valued problem directly, which brings more new difficulties to our problem by using variational method. On the other hand, the interaction between the Stein–Weiss convolution term and the magnetic field potential makes it necessary to apply or establish new estimates to overcome new interesting challenges.

In the physical case $N = 3$, $A = 0$, $V = 0$, $\alpha = 0$, $\mu = 1$ and $p = 2$, problem (P_λ) reduces to the following *Choquard-Pekar equation*

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} \mathrm{d}y\right) u \text{ in } \mathbb{R}^3, \tag{1}$$

which goes back to the description of a polaron at rest in Quantum Field Theory by Pekar [26] in 1954 and was used to describe an electron trapped in its own hole, as a certain approximation to Hartree-Fock Theory of one component plasma (see Lieb [18]). Eq. (1) was also proposed by Penrose (see [27]) in his discussion on the self-gravitational collapse of a quantum mechanical wave-function. In this context it is also known as the *nonlinear Schrödinger–Newton equation*.

In addition, from a mathematical point of view, Eq. (1) and its generalizations have been extensively studied. In his paper [18], Lieb studied the existence and uniqueness, up to translations, of the ground state to Eq. (1). Via the critical point theory, in [20] Lions proved the existence of a sequence of radially symmetric solutions. Since the non-local term in (1) is invariant under translation, we are able to get easily the existence result by using the Mountain Pass Theorem; see Ackermann [1] for example. For a general case, Ackermann in [1] applied a new method to obtain the existence of infinitely many geometrically distinct weak solutions. For the recent relevant contributions included in the papers are by Alves and Yang [2], Ding et al.[10], Du and Yang [11], Ghimenti and Van Schaftingen [15], Ma and Zhao [21], Moroz and Van Schaftingen [24,25], Wei and Winter [29], and their references. In all the papers mentioned above, the authors proved the existence of solutions by variational method. This method works well due to a Hardy–Littlewood–Sobolev-type inequality in Lieb and Loss [19].

Many authors have studied the problems involving deepening potential well and no magnetic field (i.e., $A = 0$). In [9] Ding and Tanaka considered the local Schrödinger equations with deepening potential well

$$-\Delta u + (\lambda V(x) + Z(x))u = u^p, \ u > 0 \text{ in } \mathbb{R}^N, \tag{2}$$

where $\lambda > 0$, V, Z are suitable continuous functions satisfying some conditions, $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and $1 < p < +\infty$ if $N = 1, \ 2$. For $\lambda > 0$ sufficiently large, the authors proved the existence of multi-bump solutions. For the critical growth case, in [3] Alves et al. introduced some new parameter in (2) and then they established the existence and multiplicity of positive solutions when $N \geq 3$. Very recently, Alves et al. in [5] investigated the existence of multibump solutions for the following non-local equation

$$-\Delta u + (\lambda V(x) + 1)u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|^\mu} \mathrm{d}y \right) |u|^{p-2}u \text{ in } \mathbb{R}^3,$$

here $\mu \in (0, 3)$, $2 < p < 6 - \mu$ and the nonnegative continuous function V has a potential well set. Here we would like to mention the recent work of Filippucci and Ghergu, in [14] they obtained the existence and the asymptotic profile of singular solutions for coercive quasilinear elliptic inequalities with nonlocal terms. Therefore, one of the motivations of this paper is derived from the above results.

Another motivation of this paper comes from several works on magnetic Laplace equations in recent years. For example, in [6] Arioli and Szulkin considered the existence of solutions of the semilinear stationary Schrödinger equation in the presence of a magnetic field:

$$(-i\nabla + A)^2 u + V(x)u = g(x, |u|)u \text{ in } \mathbb{R}^N,$$

where $u : \mathbb{R}^N \mapsto \mathbb{C}$, $N \geq 2$, $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a scalar (or electric) potential, $A : \mathbb{R}^N \mapsto \mathbb{R}^N$ is a vector (or magnetic) potential and g is a nonlinear local term. To prove Theorem 1.3 of [6], they imposed more assumptions on the potentials V, A and the nonlinearity g. For the reader's convenience, we list some of these hypotheses:

(H_1) $V \in L^\infty(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ and $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$;
(H_2) V, g and curl A (in the sense of distributions) are 1-periodic in x_j, $j = 1, 2, \ldots, N$;
(H_3) $0 \notin \sigma(-\Delta_A + V)$, where $\Delta_A = -(-i\nabla + A)^2$.

In general A is not periodic, therefore the operator $\nabla_A = \nabla + iA$ is not translation invariant. However, from hypotheses (H_1) and (H_2), we can define a different "translation" to guarantee some invariants, for instance, see Arioli and Szulkin [6] and Zhang et al. [33]. More recently, in [8] Cingolani et al. studied the following nonlinear magnetic Choquard equation

$$(-i\nabla + A)^2 u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu} \mathrm{d}y \right) |u|^{p-2}u \text{ in } \mathbb{R}^N,$$

where $A : \mathbb{R}^N \mapsto \mathbb{R}$ is a C^1-vector potential, $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a bounded continuous scalar potential with $\inf_{\mathbb{R}^N} V > 0$ with $N \geq 3$, $\mu \in (0, N)$ and $p \in \left(2 - \frac{\mu}{N}, \frac{2N-\mu}{N-2}\right)$. In order to overcome the lack of compactness, they assumed that both A and V scalar potential have certain symmetries. More precisely,

$$A(gx) = gA(x) \text{ and } V(gx) = V(x) \text{ for all } g \in G,\ x \in \mathbb{R}^N, \tag{3}$$

where G is a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$. Therefore, (3) plays an important role in the proofs of [8]. Regarding other related results, we refer to Alves et al. [4], Esteban and Lions [13], Ji and Rădulescu [16,17] and the references therein.

To the best of our knowledge, the first results dealing with the semilinear elliptic equation with Stein–Weiss convolution appear in [12], in which subcritical case and critical cases were studied. Moreover, a system of Schrödinger equations with Stein–Weiss type convolution part was considered in [32], there the authors studied the regularity and symmetry of the nontrivial solutions.

According to the comments above, it is quite natural to consider problem (P_λ). In the present paper, we are interested in studying the existence of the solutions for problem (P_λ). Henceforth, we show that if the parameter $\lambda > 0$ is sufficiently large, problem (P_λ) has a nontrivial solution under suitable assumptions on V. Precisely, we require that

- (V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ with $V(x) \geq 0$;
- (V_2) there exists $M_0 > 0$ such that meas $\left(\{x \in \mathbb{R}^N : V(x) \leq M_0\}\right) < +\infty$, where "meas" denotes the Lebesgue's measure;
- (V_3) $\Omega := \text{int } V^{-1}(0)$ is a non-empty set.

In the present work our main result is

Theorem 1 *Let $N \geq 2$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $p \in \left(\frac{2N-\mu-2\alpha}{N}, \frac{2N-\mu-2\alpha}{N-2}\right)$ if $N > 2$ and $p \in \left(\frac{4-\mu-2\alpha}{2}, +\infty\right)$ if $N = 2$. Assume that (V_1)–(V_3) are retained. Then there exists $\lambda^* > 0$ such that, for any $\lambda \geq \lambda^*$ problem (P_λ) admits a nontrivial solution.*

As far as we know, this paper is the first attempt to study the non-local magnetic problem including the Stein–Weiss convolution term.

Since we do not assume that both A and V have some periodicities or symmetries, we are unable to draw a similar conclusion with [6] and [8] directly. In addition, even if both A and V have these properties, due to the appearance of the non-local Stein-Weiss convolution term, we cannot directly obtain the existence of ground state solutions of problem (P_λ) by using Mountain Pass Theorem and Lions' vanishing-nonvanishing arguments. Therefore, the main difficulty in this paper lies in the lack of compactness.

Theorem 1 will be proved by adopting variational methods and making full use of some estimates. On the other hand, as we will see later, it is worth mentioning that, because we are dealing with different problems, in which the functions are complex-valued and the nonlinearity is a non-local Stein-Weiss convolution term, it is necessary to carefully analyze some estimates.

Notations

- $B_r(x)$ denotes the open ball or open disk centered at $x \in \mathbb{R}^N$ $(N \geq 2)$ with radius $r > 0$ and $B_r^c(x)$ denotes the complement of $B_r(x)$ in $\mathbb{R}^N$.
- The usual norm of $L^q(\mathbb{R}^N, \mathbb{R})$ is denoted by $|\cdot|_q$, $q \geq 1$.

2 Variational Setting and Preliminary Results

In order to obtain the existence of solutions to problem (P_λ) by using variational method, we outline the variational framework in this section and give some preliminary results.

For $u : \mathbb{R}^N \mapsto \mathbb{C}$, by ∇_A we denote

$$\nabla_A u := (\nabla + iA)u.$$

Also, we introduce the following Hilbert space

$$H_A^1(\mathbb{R}^N, \mathbb{C}) := \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N, \mathbb{R})\right\}$$

equipped with the scalar product

$$\langle u, \varphi \rangle := \text{Re} \int_{\mathbb{R}^N} \left(\nabla_A u \overline{\nabla_A \varphi} + u\overline{\varphi}\right) \mathrm{d}x \text{ for all } u,\ \varphi \in H_A^1(\mathbb{R}^N, \mathbb{C}),$$

where “Re" and the bar represent the real part of a complex number and the complex conjugation, respectively. By $\|\cdot\|_A$ we denote the norm induced by this inner product.

Let $U \subseteq \mathbb{R}^N$ be an open set. Now, we define

$$H_A^1(U, \mathbb{C}) := \left\{u \in L^2(U, \mathbb{C}) : |\nabla_A u| \in L^2(U, \mathbb{R})\right\}$$

and

$$\|u\|_{H_A^1(U)} = \left(\int_U \left(|\nabla_A u|^2 + |u|^2\right) dx\right)^{\frac{1}{2}}.$$

Moreover, for any fixed $\lambda \geq 0$, let us define the following Hilbert space

$$X_\lambda := \left\{u \in H_A^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x < +\infty\right\},$$

with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}}.$$

So, we can easily see that $X_\lambda \subseteq H^1_A(\mathbb{R}^N, \mathbb{C})$ for any $\lambda \geq 0$. In our argument, it is necessary for the norm $\|\cdot\|_\lambda$ to depend on λ. Hence, we mainly use $\|\cdot\|_\lambda$ in the sequel.

It is worth pointing out that the following well-known *diamagnetic inequality* (see Lieb and Loss [19, Theorem 7.21]):

$$|\nabla |u(x)|| \leq |\nabla_A u(x)| \text{ for all } u \in H^1_A(\mathbb{R}^N, \mathbb{C}) \text{ and for a.e. } x \in \mathbb{R}^N. \tag{4}$$

Thus, from relation (4), we know that, for any $\lambda \geq 0$,

$$\begin{aligned} &\text{if } u \in X_\lambda \Longrightarrow |u| \in H^1(\mathbb{R}^N, \mathbb{R}), \\ \Rightarrow\ & X_\lambda \text{ is continuously embedded in } L^q(\mathbb{R}^N, \mathbb{C}) \text{ for all} \\ & q \in \left[2, \frac{2N}{N-2}\right], N > 2 \text{ (resp. } q \in [2, +\infty),\ N = 2), \\ & \text{and compactly embedded in } L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{C}) \text{ for all} \\ & q \in \left[1, \frac{2N}{N-2}\right),\ N > 2 \text{ (resp. } q \in [1, +\infty),\ N = 2). \end{aligned}$$

Next, we give the Stein–Weiss inequality [28] which plays an important role in the present work.

Proposition 2 *Let* $1 < r,\ s < +\infty$, $0 < \mu < N$, $\alpha + \beta \geq 0$, $0 < \alpha + \beta + \mu \leq N$, $f \in L^r(\mathbb{R}^N, \mathbb{R})$ *and* $g \in L^s(\mathbb{R}^N, \mathbb{R})$. *Then there exists a sharp constant* $C_{(r,s,\alpha,\beta,\mu)}$ *such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} \mathrm{d}x\mathrm{d}y \right| \leq C_{(r,s,\alpha,\beta,\mu)} |f|_r |g|_s,$$

where

$$\frac{1}{r} + \frac{1}{s} + \frac{\alpha + \beta + \mu}{N} = 2$$

and

$$1 - \frac{1}{r} - \frac{\mu}{N} < \frac{\alpha}{N} < 1 - \frac{1}{r},$$

and $C_{(r,s,\alpha,\beta,\mu)}$ *is independent of* f *and* g. *In addition, for all* $g \in L^s(\mathbb{R}^N, \mathbb{R})$, *it holds*

$$\left| \int_{\mathbb{R}^N} \frac{g(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} \mathrm{d}y \right|_t \leq C_{(t,\mu,\alpha,\beta)} |g|_s,$$

where t verifies

$$1+\frac{1}{t}=\frac{1}{s}+\frac{\alpha+\beta+\mu}{N} \text{ and } \frac{\alpha}{N}<\frac{1}{t}<\frac{\alpha+\mu}{N}.$$

Clearly, problem (P_λ) possesses a variational structure: for

$$p\in\left(\frac{2N-\mu-2\alpha}{N},\frac{2N-\mu-2\alpha}{N-2}\right)\ (N\geq 3,\ \alpha\geq 0)$$

and

$$p\in\left(\frac{4-\mu-2\alpha}{2},+\infty\right)\ (N=2,\ \alpha\geq 0),$$

the critical points of the functional $\mathcal{E}_\lambda\in C^1(X_\lambda,\mathbb{R})$ defined for all $u\in X_\lambda$ by

$$\begin{aligned}\mathcal{E}_\lambda(u):=&\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla_A u|^2+(\lambda V(x)+1)|u|^2\right)\mathrm{d}x\\&-\frac{1}{2p}\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u|^p\mathrm{d}x\end{aligned}$$

are weak solutions of problem (P_λ). By Proposition 2 and the Sobolev embeddings, we see that the functional $\mathcal{E}_\lambda$ is well-defined and it holds

$$\begin{aligned}\mathcal{E}_\lambda'(u)\varphi=&\operatorname{Re}\int_{\mathbb{R}^N}\left(\nabla_A u\overline{\nabla_A\varphi}+(\lambda V(x)+1)u\overline{\varphi}\right)\mathrm{d}x\\&-\operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u|^{p-2}u\overline{\varphi}\mathrm{d}x\end{aligned}$$

for all $u,\varphi\in X_\lambda$.

Before the end of this section, we shall prove the following useful result, which will be used frequently in the sequel.

Lemma 3 *Let $p>1$ and define $\mathcal{A}:\mathbb{C}^N\mapsto\mathbb{C}^N$ by $\mathcal{A}(z):=|z|^{p-2}z$, $z:=(z_1,z_2,\ldots,z_N)\in\mathbb{C}^N$. Then*

(i) *if $p\geq 2$, for each fixed $\varepsilon>0$, there exists some $C_\varepsilon>0$ such that*

$$|\mathcal{A}(a+b)-\mathcal{A}(a)|\leq\varepsilon|a|^{p-1}+C_\varepsilon|b|^{p-1} \text{ for all } a,\ b\in\mathbb{C}^N;$$

(ii) *if $1<p<2$, it holds*

$$\ell:=\sup_{a,b\in\mathbb{C}^N,\ b\neq 0}\frac{|\mathcal{A}(a+b)-\mathcal{A}(a)|}{|b|^{p-1}}<+\infty.$$

Proof (i) We first consider the case $p = 2$. In this case, we can easily see that for any $\varepsilon > 0$, there exists $C_\varepsilon = 1 > 0$ such that

$$|\mathcal{A}(a+b) - \mathcal{A}(a)| = |b| \leq \varepsilon|a| + |b| \text{ for all } a,\ b \in \mathbb{C}^N.$$

Next, we show the case $p > 2$. Let us define the following functions:

$$\mathcal{A}_j(z) := |z|^{p-2} z_j,\ z \in \mathbb{C}^N,\ z_j \in \mathbb{C},\ j = 1, 2, \ldots, N$$

and

$$g_j(t) := \mathcal{A}_j(a + tb),\ a,\ b \in \mathbb{C}^N,\ t \in \mathbb{R},\ j = 1, 2, \ldots, N.$$

So, we have, for each fixed j ($j = 1, 2, \ldots, N$) and for all $a, b \in \mathbb{C}^N$,

$$\begin{aligned}
&|\mathcal{A}_j(a+b) - \mathcal{A}_j(a)| \\
&\quad = |g_j(1) - g_j(0)| \\
&\quad = |g_j'(\theta)| \text{ for some } \theta \in (0,1) \\
&\quad = |(p-2)|a+\theta b|^{p-4} \operatorname{Re}\left(\overline{b}(a+\theta b)\right)(a_j + b_j) + |a+\theta b|^{p-2} b_j| \\
&\quad \leq (p-2)|a+\theta b|^{p-4} |a+\theta b||b||a_j + \theta b_j| + |a+\theta b|^{p-2}|b_j| \\
&\quad \leq (p-1)(|a|+|b|)^{p-2}|b| \\
&\quad \leq (p-1)2^{p-2}|a|^{p-2}|b| + (p-1)2^{p-2}|b|^{p-1} \text{ (since } p > 2) \\
&\quad \leq \frac{1}{N}\varepsilon|a|^{p-1} + \hat{C}_\varepsilon|b|^{p-1} \text{ (use Young inequality)},
\end{aligned}$$

where

$$\hat{C}_\varepsilon = 2^{p-2}\left(\frac{1}{N}\varepsilon\right)^{2-p} 2^{(p-2)^2}(p-2)^{p-2} + (p-1)2^{p-2},\ p > 2.$$

Combining the cases $p = 2$ and $p > 2$, we infer that, for all $p \geq 2$, for each fixed $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$|\mathcal{A}(a+b) - \mathcal{A}(a)| \leq \varepsilon|a|^{p-1} + C_\varepsilon|b|^{p-1} \text{ for all } a,\ b \in \mathbb{C}^N,$$

where

$$C_\varepsilon = \begin{cases} N\hat{C}_\varepsilon, & \text{if } p > 2, \\ 1 & \text{if } p = 2. \end{cases}$$

(ii) Now, we deal with the case $1 < p < 2$. To this end, we define the function $G : \mathbb{C}^N \times \mathbb{C}^N \mapsto \mathbb{R}$ as follows

$$G(a,b) := \frac{|\mathcal{A}(a+b) - \mathcal{A}(a)|}{|b|^{p-1}} \text{ for all } a,\ b \in \mathbb{C}^N.$$

Clearly, $G(a, tb) = G\left(\frac{a}{t}, b\right)$ for all $t \in \mathbb{R} \setminus \{0\}$. Thus, we have

$$\ell = \sup_{a \in \mathbb{C}^N,\ |b|=1} G(a, b).$$

We observe that

$$\ell_1 := \sup_{|a| \leq 2,\ |b|=1} G(a, b) < +\infty \text{ (by the continuity).}$$

It remains to prove that

$$\ell_2 := \sup_{|a|>2,\ |b|=1} G(a, b) < +\infty.$$

Suppose that $|b| = 1$, $|a| > 2$ and $t \in [0, 1]$. Then, we see that

$$|a + tb| \geq |a| - |b| > 1.$$

So, we have, for any fixed j $(j = 1, 2, \ldots, N)$,

$$\begin{aligned}
&\left| |a + b|^{p-2}(a_j + b_j) - |a|^{p-2} a_j \right| \\
&= \left| \int_0^1 \left((p-2)|a + tb|^{p-4} \operatorname{Re}(a + tb)\overline{b}(a_j + tb_j) + |a + tb|^{p-2} b_j \right) dt \right| \\
&\leq \int_0^1 (2 - p)|a + tb|^{p-2}|b| dt + \int_0^1 |a + tb|^{p-2}|b| dt \\
&\leq 3 - p,
\end{aligned}$$

which implies that $\ell_2 \leq (3 - p)N < +\infty$.

Hence, we have $\ell < +\infty$. This proof is now complete. □

3 The $(PS)_c$ Condition for the Functional $\mathcal{E}_\lambda$

In this section, working with the $(PS)_{c_\lambda}$ sequence of the functional $\mathcal{E}_\lambda$, we will show that, for given $d > 0$ independent of λ and then for $\lambda > 0$ sufficiently large, the $(PS)_{c_\lambda}$ sequence of the energy (Euler) functional $\mathcal{E}_\lambda$ satisfy the $(PS)_{c_\lambda}$ condition at the level $0 \leq c_\lambda < d$, where c_λ and d will be defined later.

Now, we prove that the energy (Euler) functional $\mathcal{E}_\lambda$ verifies the Mountain Pass Geometry (see Willem [30]).

Lemma 4 *For each fixed $\lambda > 0$, the functional $\mathcal{E}_\lambda$ has the following properties*:

(a) *there are $\varrho > 0$, $\rho > 0$ such that $\mathcal{E}_\lambda(u) \geq \varrho$ with $\|u\|_\lambda = \rho$*;
(b) *there is some element $e \in X_\lambda$ with $\|e\|_\lambda > \rho$ such that $\mathcal{E}_\lambda(e) < 0$.*

Proof (a) Notice that $p \in \left(\frac{2N-\mu-2\alpha}{N}, \frac{2N-\mu-2\alpha}{N-2}\right)$ $(N \geq 3,\ \alpha \geq 0)$ and $p \in \left(\frac{4-\mu-2\alpha}{2}, +\infty\right)$ $(N = 2,\ \alpha \geq 0)$. From Proposition 2 and the Sobolev embedding inequalities, it follows that

$$\left|\frac{1}{2p}\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u|^p\mathrm{d}x\right| \leq \hat{C}\|u\|_\lambda^{2p},$$

where $\hat{C} := \hat{C}_{(p,\alpha,\mu)}$ is some positive constant.

By using the inequality above and reviewing the definition of the functional $\mathcal{E}_\lambda$, we get

$$\begin{aligned}\mathcal{E}_\lambda(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \hat{C}\|u\|_\lambda^{2p}\\ &= \frac{1}{2}\|u\|_\lambda^2\left(1-2\hat{C}\|u\|_\lambda^{2p-2}\right).\end{aligned}$$

Set $\rho := 2^{\frac{2}{2-2p}}(\hat{C})^{\frac{1}{2-2p}} > 0$. Then we have

$$\begin{aligned}\mathcal{E}_\lambda(u) &\geq \frac{1}{2}\rho^2\left(1-2\hat{C}\rho^{2p-2}\right)\\ &= \frac{1}{4}\rho^2 =: \varrho > 0 \text{ for all } \|u\|_\lambda = \rho.\end{aligned}$$

This proves (a).

(b) Choose $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}$ with supp $(\varphi) \subset \Omega$. We observe that

$$\begin{aligned}\mathcal{E}_\lambda(t\varphi) &= \frac{t^2}{2}\|\varphi\|_\lambda^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|\varphi(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|\varphi|^p\mathrm{d}x\\ &= \frac{t^2}{2}\|\varphi\|^2_{H_A^1(\mathrm{supp}\,(\varphi))} - \frac{t^{2p}}{2p}\int_{\mathrm{supp}\,(\varphi)}\frac{1}{|x|^\alpha}\left(\int_{\mathrm{supp}\,(\varphi)}\frac{|\varphi(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|\varphi|^p\mathrm{d}x\\ &\to -\infty \text{ as } t \to -\infty,\ \text{ since } 2p > 2.\end{aligned}$$

The proof is now complete. □

Let us define

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{E}_\lambda(\gamma(t))$$

and

$$\Gamma := \{\gamma \in C([0,1], X_\lambda) : \gamma(0) = 0,\ \gamma(1) = e\},$$

where e is given in Lemma 4.

Since supp $(\varphi) \subset \Omega$, we can easily see that there exists some constant $d > 0$, independent of $\lambda > 0$, such that $\max_{t>0} \mathcal{E}_\lambda(t\varphi) < d$. So, we deduce that $c_\lambda < d$ for all $\lambda > 0$.

Lemma 5 *Assume that $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is a $(PS)_c$ sequence of the functional $\mathcal{E}_\lambda$ at the level c. Then the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is bounded, and moreover $c \geq 0$.*

Proof Let $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ be a $(PS)_c$ sequence, that is,

$$\mathcal{E}_\lambda(u_n) \to c \text{ and } \mathcal{E}'_\lambda(u_n) \to 0 \text{ as } n \to \infty.$$

Therefore, for n sufficiently large, it follows that

$$\begin{aligned} c + 1 + \|u_n\|_\lambda &\geq \mathcal{E}_\lambda(u_n) - \frac{1}{2p}\mathcal{E}'_\lambda(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_\lambda^2, \end{aligned}$$

which yields the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$, and so $c \geq 0$. Thus, we complete the proof of the lemma. □

Corollary 6 *Assume that $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is a $(PS)_0$ sequence of the functional $\mathcal{E}_\lambda$ at the level 0. Then $u_n \to 0$ in X_λ as $n \to \infty$.*

The next three lemmas are variants of the Brézis–Lieb Lemma [7] for the Stein–Weiss type convolution term, which seem to be new and of independent interest. The results we get here will be useful to everyone working in this direction.

Lemma 7 *Assume that $N \geq 2$, $\alpha \geq 0$, $0 < \mu < N$, $2\alpha + \mu \leq N$ and $1 \leq p \leq \dfrac{2N-\mu-2\alpha}{N-2}$ if $N \geq 3$ (resp. $1 \leq p < +\infty$ if $N = 2$) are fulfilled. If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{\frac{2Np}{2N-\mu-2\alpha}}(\mathbb{R}^N, \mathbb{C})$ is a bounded sequence with $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then we have the following property:*

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u_n|^p\mathrm{d}x - \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|v_n|^p\mathrm{d}x \\ \to \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u|^p\mathrm{d}x, \end{aligned}$$

as $n \to \infty$, where $v_n := u_n - u$.

Proof Since the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{\frac{2Np}{2N-\mu-2\alpha}}(\mathbb{R}^N, \mathbb{C})$ is bounded and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, from Proposition 5.4.7. of Willem [31], it follows that

$$|v_n|^p \xrightarrow{w} 0 \text{ in } L^{\frac{2N}{2N-\mu-2\alpha}}(\mathbb{R}^N, \mathbb{R}) \text{ as } n \to \infty.$$

On the one hand, arguing as in the proof of the Brézis-Lieb Lemma [7], we can deduce that

$$|u_n|^p - |v_n|^p - |u|^p \to 0 \text{ in } L^{\frac{2N}{2N-\mu-2\alpha}}(\mathbb{R}^N, \mathbb{R}) \text{ as } n \to \infty. \tag{5}$$

In addition, we can use relation (5) and Proposition 2 to infer that

$$\int_{\mathbb{R}^N} \frac{|u_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y - \int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \to \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y$$
$$\text{in } L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N, \mathbb{R}) \text{ as } n \to \infty. \tag{6}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |u_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |v_n|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \left(|u_n|^p - |v_n|^p \right) \mathrm{d}x \\
&\quad + 2 \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |v_n|^p \mathrm{d}x.
\end{aligned}$$

Using the above information, we can easily get the desired result. This proof is now complete. □

Lemma 8 *Assume that* $N \geq 2$, $\alpha \geq 0$, $0 < \mu < N$, $2\alpha + \mu \leq N$, $p > 1$ *and* $\frac{2N-\mu-2\alpha}{N} \leq p < \frac{2N-\mu-2\alpha}{N-2}$ *if* $N \geq 3$ $\left(\textit{resp.}\ \frac{4-\mu-2\alpha}{2} \leq p < +\infty \textit{ if } N = 2 \right)$ *are fulfilled. Let* $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ *be such that* $u_n \xrightarrow{w} u$ *in* X_λ *as* $n \to \infty$. *Set* $v_n := u_n - v_n$. *Then, passing to a subsequence, for any* $\varphi \in X_\lambda$ *such that* $\|\varphi\|_\lambda \leq 1$ *it holds*

$$\sup_{\|\varphi\|_\lambda \leq 1} \left| (\Phi'(u_n) - \Phi'(v_n) - \Phi'(u))\varphi \right| = o_n(1), \ \textit{as } n \to \infty,$$

that is,

$$\Phi'(u_n) - \Phi'(v_n) - \Phi'(u) = o_n(1) \textit{ in } X_\lambda^* \textit{ as } n \to \infty,$$

where

$$\Phi(u) := \frac{1}{2p} \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |u|^p \mathrm{d}x$$

and

$$\Phi'(u)\varphi := \mathrm{Re} \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |u|^{p-2} u \overline{\varphi} \mathrm{d}x.$$

Proof To prove Lemma 8, we will discuss the following four cases respectively:

(i) if $2\alpha + \mu \leq \min\{N, 4\}$ and $2\alpha + \mu < N$,

$$\Rightarrow \frac{2N-\mu-2\alpha}{N} \leq p < \frac{2N-\mu-2\alpha}{N-2} \text{ (resp. } +\infty,\ N=2);$$

(ii) if $2\alpha + \mu = N \leq 4$,

$$\Rightarrow 1 < p < \frac{2N-\mu-2\alpha}{N-2} \text{ (resp. } +\infty,\ N=2);$$

(iii) if $4 < 2\alpha + \mu < N$,

$$\Rightarrow \frac{2N-\mu-2\alpha}{N} \leq p < \frac{2N-\mu-2\alpha}{N-2} < 2;$$

(iv) if $4 < 2\alpha + \mu = N$,

$$\Rightarrow 1 < p < \frac{2N-\mu-2\alpha}{N-2} < 2.$$

Next, we only consider the case (i) and leave the other cases to the reader. In the sequel, we show the following limit:

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left||u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u\right|^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}}\mathrm{d}x = 0. \tag{7}$$

Firstly, we deal with the situation

$$2 \leq p < \frac{2N-\mu-2\alpha}{N-2} \text{ (resp. } +\infty,\ N=2).$$

Applying (i) of Lemma 3, we know that for each fixed $\varepsilon > 0$, there is some positive constant $C_\varepsilon > 0$ such that

$$\left||u_n|^{p-2}u_n - |v_n|^{p-2}v_n\right| \leq \varepsilon|v_n|^{p-1} + C_\varepsilon|u|^{p-1}.$$

Now, we introduce the function $H_{\varepsilon,n} : \mathbb{R}^N \mapsto \mathbb{R}^+$ defined by

$$H_{\varepsilon,n}(x) := \max\left\{\left||u_n(x)|^{p-2}u_n(x) - |v_n(x)|^{p-2}v_n(x) - |u(x)|^{p-2}u(x)\right| - \varepsilon|v_n(x)|^{p-1}, 0\right\}.$$

Obviously, $H_{\varepsilon,n}(x) \to 0$ a.e. $\mathbb{R}^N$ as $n \to \infty$ (up to a subsequence) and

$$0 \le H_{\varepsilon,n} \le \widehat{C}|u|^{p-1} \in L^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}}(\mathbb{R}^N, \mathbb{R}),$$

where $\widehat{C} > 0$ is some constant. Therefore, we use the Dominated Convergence Theorem to derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} H_{\varepsilon,n}^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} \,\mathrm{d}x = 0.$$

In addition, according to the definition of $H_{\varepsilon,n}$, we have

$$\Big||u_n(x)|^{p-2}u_n(x) - |v_n(x)|^{p-2}v_n(x) - |u(x)|^{p-2}u(x)\Big| \le H_{\varepsilon,n} + \varepsilon|v_n(x)|^{p-1},$$

hence, we conclude that

$$\begin{aligned} &\Big||u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u\Big|^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} \\ &\le \widetilde{C}\left(H_{\varepsilon,n}^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} + \varepsilon^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}}|v_n|^{\frac{2Np}{2N-\mu-2\alpha}}\right), \end{aligned}$$

where $\widetilde{C}$ is a positive constant. So, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big||u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u\Big|^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} \,\mathrm{d}x \\ &\le \widetilde{C}\varepsilon^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}}|v_n|^{\frac{2Np}{2N-\mu-2\alpha}}_{\frac{2Np}{2N-\mu-2\alpha}} \\ &\le \overline{C}\varepsilon^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}}, \end{aligned}$$

where $\overline{C}$ is a positive constant. Using the arbitrariness of $\varepsilon > 0$, we see that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big||u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u\Big|^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} \,\mathrm{d}x \le 0.$$

In this case the proof of relation (7) is now complete.

Now, assume that

$$p > 1 \text{ and } \frac{2N-\mu-2\alpha}{N} \le p < 2 < \frac{2N-\mu-2\alpha}{N-2} \text{ (resp. } +\infty,\ N=2).$$

From (ii) of Lemma 3, it follows that

$$\sup_{x\in\mathbb{R}^N,\ u(x)\ne 0} \left|\frac{|u_n(x)|^{p-2}u_n(x) - |v_n(x)|^{p-2}v_n(x)}{|u(x)|^{p-1}}\right| < +\infty.$$

Using the Dominated Convergence Theorem, we can finish the proof of relation (7) under this situation.

Combining the above two situations, we now complete the proof of relation (7). As with the above-mentioned proof, we can also show the other situations, so we omit the details.

Since $u_n \xrightarrow{w} u$ in X_λ as $n \to \infty$, we see that the set $\{u_n, v_n, u\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is bounded, that is, there is some constant $C_1 > 0$ such that

$$\|u_n\|_\lambda,\ \|v_n\|_\lambda,\ \|u\|_\lambda \le C_1 \text{ for all } n \in \mathbb{N}. \tag{8}$$

So, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left||u_n|^{p-2}u_n - |v_n|^{p-2}v_n\right|^{\frac{2N}{2N-\mu-2\alpha}} |\overline{\varphi}|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x \\
&\le 2^{\frac{2\alpha+\mu}{2N-\mu-2\alpha}} \int_{\mathbb{R}^N} \left(|u_n|^{\frac{2N(p-1)}{2N-\mu-2\alpha}} + |v_n|^{\frac{2N(p-1)}{2N-\mu-2\alpha}}\right) |\varphi|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x \\
&\left(\text{since } \frac{2N}{2N-\mu-2\alpha} > 1\right) \\
&\le 2^{\frac{2\alpha+\mu}{2N-\mu-2\alpha}} \left(|u_n|^{\frac{2N(p-1)}{2N-\mu-2\alpha}}_{\frac{2Np}{2N-\mu-2\alpha}} + |v_n|^{\frac{2N(p-1)}{2N-\mu-2\alpha}}_{\frac{2Np}{2N-\mu-2\alpha}}\right) |\varphi|^{\frac{2N}{2N-\mu-2\alpha}}_{\frac{2Np}{2N-\mu-2\alpha}} \\
&\text{(use the Hölder inequality)} \\
&\le C_2 \|\varphi\|_\lambda^{\frac{2N}{2N-\mu-2\alpha}} \text{ (recall that the Sobolev embedding and use (8))} \\
&\text{for some constant } C_2 > 0.
\end{aligned} \tag{9}$$

In the same fashion as in the proof of relation (9), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left||v_n|^{p-2}v_n\overline{\varphi}\right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x,\ \int_{\mathbb{R}^N} \left||u|^{p-2}u\overline{\varphi}\right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x \le C_3\|\varphi\|_\lambda^{\frac{2N}{2N-\mu-2\alpha}} \\
&\text{for some constant } C_3 > 0.
\end{aligned} \tag{10}$$

Now, we define the following notations:

$$\begin{aligned}
I_n^1 :=& \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n\right) \overline{\varphi}\mathrm{d}x \\
&- \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) |u|^{p-2}u\overline{\varphi}\mathrm{d}x \\
I_n^2 :=& \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) |v_n|^{p-2}v_n\overline{\varphi}\mathrm{d}x \\
I_n^3 :=& \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n\right) \overline{\varphi}\mathrm{d}x.
\end{aligned}$$

So, for $n \in \mathbb{N}$ large enough, we deduce that

$$
\begin{aligned}
|I_n^1| &\le \left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |v_n(y)|^p - |u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \right. \\
&\quad \left. \times \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n \right) \overline{\varphi} \mathrm{d}x \right| \\
&\quad + \left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u \right) \overline{\varphi} \mathrm{d}x \right| \\
&\le \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{\left||u_n(y)|^p - |v_n(y)|^p - |u(y)|^p\right|}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \\
&\quad \times \left| \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n \right) \overline{\varphi} \right| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \left| \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u \right) \overline{\varphi} \right| \mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{\left||u_n(y)|^p - |v_n(y)|^p - |u(y)|^p\right|}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right)^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x \right)^{\frac{2\alpha+\mu}{2N}} \\
&\quad \times \left(\int_{\mathbb{R}^N} \left| \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n \right) \overline{\varphi} \right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x \right)^{\frac{2N-\mu-2\alpha}{2N}} \\
&\quad \text{(use the Hölder inequality)} \\
&\quad + |u|^p_{\frac{2Np}{2N-\mu-2\alpha}} \left(\int_{\mathbb{R}^N} \left| \left(|u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u \right) \overline{\varphi} \right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x \right)^{\frac{2N-\mu-2\alpha}{2N}} \\
&\quad \text{(invoke Proposition 2)} \\
&\le C_2^{\frac{2N-\mu-2\alpha}{2N}} \varepsilon \|\varphi\|_\lambda \text{ (by (6), (9))} \\
&\quad + C_4 \left(\int_{\mathbb{R}^N} \left| |u_n|^{p-2}u_n - |v_n|^{p-2}v_n - |u|^{p-2}u \right|^{\frac{2Np}{(2N-\mu-2\alpha)(p-1)}} \mathrm{d}x \right)^{\frac{(2N-\mu-2\alpha)(p-1)}{2Np}} \\
&\quad \times \|\varphi\|_\lambda \text{ for some constant } C_4 > 0 \text{ (by the Hölder inequality, the Sobolev} \\
&\quad \text{embedding, (8))} \\
&\le C_5 \varepsilon \|\varphi\|_\lambda \text{ (by (7)).}
\end{aligned} \tag{11}
$$

Concerning $u \in X_\lambda$, together with Proposition 2 and the Sobolev embedding, we see that

$$
\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x < +\infty
$$

and

$$
\int_{\mathbb{R}^N} |u|^{\frac{2Np}{2N-\mu-2\alpha}} \mathrm{d}x < +\infty.
$$

From the above inequalities, we can deduce that, for any $\varepsilon > 0$, there exists $R := R(\varepsilon) > 0$ such that

$$\left(\int_{B_R^c(0)} \left|\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y\right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x\right)^{\frac{2\alpha+\mu}{2N}} < -\left(\int_{B_R^c(0)} |u|^{\frac{2Np}{2N-\mu-2\alpha}} \mathrm{d}x\right)^{\frac{(2N-\mu-2\alpha)(p-1)}{2Np}} + \varepsilon. \tag{12}$$

Moreover, using Proposition 2 and (8), we conclude that

$$\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y\right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x,\ \int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y\right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x < C_6 \tag{13}$$

for some positive constant $C_6 > 0$.

Then, for $n \in \mathbb{N}$ large enough, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) |v_n|^{p-2} v_n \overline{\varphi} \mathrm{d}x\right| \\
&\leq \left|\int_{B_R^c(0)} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) |v_n|^{p-2} v_n \overline{\varphi} \mathrm{d}x\right| \\
&\quad + \left|\int_{B_R(0)} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y\right) |v_n|^{p-2} v_n \overline{\varphi} \mathrm{d}x\right| \\
&\leq \left(\int_{B_R^c(0)} \left|\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y\right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x\right)^{\frac{2\alpha+\mu}{2N}} \\
&\quad \times \left(\int_{\mathbb{R}^N} \left||v_n|^{p-2} v_n \overline{\varphi}\right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x\right)^{\frac{2N-\mu-2\alpha}{2N}} \\
&\quad + \left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y\right|^{\frac{2N}{2\alpha+\mu}} \mathrm{d}x\right)^{\frac{2\alpha+\mu}{2N}} \\
&\quad \times \left(\int_{B_R(0)} \left||v_n|^{p-2} v_n \overline{\varphi}\right|^{\frac{2N}{2N-\mu-2\alpha}} \mathrm{d}x\right)^{\frac{2N-\mu-2\alpha}{2N}}
\end{aligned}$$

(by the Hölder inequality)

$$\begin{aligned}
&\leq C_3^{\frac{2N-\mu-2\alpha}{2N}} \varepsilon \|\varphi\|_\lambda \\
&\quad + C_6 \left(\int_{B_R(0)} |v_n|^{\frac{2Np}{2N-\mu-2\alpha}} \mathrm{d}x\right)^{\frac{(2N-\mu-2\alpha)(p-1)}{2Np}} |\varphi|_{\frac{2Np}{2N-\mu-2\alpha}} \quad \text{(see (10), (12), (13))}
\end{aligned}$$

$$\leq C_3^{\frac{2N-\mu-2\alpha}{2N}} \varepsilon \|\varphi\|_\lambda + C_7 \varepsilon \|\varphi\|_\lambda \text{ for some constant } C_7 > 0$$

(by the local Sobolev compact embedding and the Sobolev continuous embedding). (14)

From relation (14), we infer that

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |v_n|^{p-2} v_n \overline{\varphi} \mathrm{d}x \right| \leq \widehat{C}_7 \varepsilon \|\varphi\|_\lambda. \tag{15}$$

Next, we show that the following inequality

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x-y|^\mu |y|^\alpha} \mathrm{d}y \right) |u|^{p-2} u \overline{\varphi} \mathrm{d}x \right| \leq C_8 \|\varphi\|_\lambda \tag{16}$$

holds true for some constant $C_8 > 0$.

In fact, for any $\varepsilon > 0$, we can find some $K_1 > 0$ and $R_0 := R_0(\varepsilon) > \max\{1, R\}$ [see (12)] such that

$$\limsup_{n\to\infty} \int_{B_{R_0}(0)} |v_n|^p \left(\int_{B_{R_0}(0)} \frac{|u(x)|^{p-1}|\varphi|}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}x \right) \mathrm{d}y \leq K_1 \varepsilon \|\varphi\|_\lambda, \tag{17}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p \left(\int_{\mathbb{R}^N \setminus B_{R_0}(0)} \frac{|u(x)|^{p-1}|\varphi|}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}x \right) \mathrm{d}y \leq K_1 \varepsilon \|\varphi\|_\lambda. \tag{18}$$

In order to prove relation (16), it remains to consider the following term:

$$J_n := \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |v_n|^p \left(\int_{B_{R_0}(0)} \frac{|u(x)|^{p-1}|\varphi(x)|}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}x \right) \mathrm{d}y.$$

For this reason, we will discuss it in two parts.

$(*)$ If $|u(x)|^{p-1}|\varphi(x)| = 0$ a.e. on $B_{R_0}(0)$. Consequently, for any $\varepsilon > 0$, we obtain

$$J_n \leq K_2 \varepsilon \|\varphi\|_\lambda \text{ for some constant } K_2 > 0.$$

$(**)$ If meas $\left(\left\{x \in B_{R_0}(0) : |u(x)|^{p-1}|\varphi(x)| > 0\right\}\right) > 0$. So, we have

$$\int_{B_{R_0}(0)} \left| |u|^{p-1}|\varphi| \right|^{\frac{6N}{6N-\mu-2\alpha}} \mathrm{d}x > 0.$$

Moreover, we have

$$\int_{B_{R_0}(0)} \left| |u|^{p-1}|\varphi| \right|^{\frac{6N}{6N-\mu-2\alpha}} \mathrm{d}x \leq K_3 \|\varphi\|_\lambda^{\frac{6N}{6N-\mu-2\alpha}} \text{meas} \left(B_{R_0}(0)\right)^{\frac{2\mu+4\alpha}{6N-\mu-2\alpha}}.$$

Set

$$d_\varepsilon := \left(\frac{\left| |u|^{p-1}|\varphi| \right|_{L^{6N/(6N-\mu-2\alpha)}(B_{R_0}(0))}}{\varepsilon \|\varphi\|_\lambda} \right)^{\frac{3}{\mu}}$$

and

$$\widehat{R}_0 := R_0 + \varepsilon^{-\frac{3}{\mu}} K_3^{\frac{6N-\mu-2\alpha}{2N\mu}} \operatorname{meas}(B_{R_0}(0))^{\frac{6\mu+12\alpha}{(6N-\mu-2\alpha)\mu}}$$

In the case $(**)$, we can use the above relations, Proposition 2 and the local Sobolev compactness and the continuous embedding to conclude that, for $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
J_n &= \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{\mathbb{R}^N \setminus B_{R_0}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&= \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{\mathbb{R}^N \setminus B_{R_0+d_\varepsilon}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&\quad + \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{B_{R_0+d_\varepsilon}(0) \setminus B_{R_0}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&\leq \frac{1}{R_0^{\frac{2\alpha}{3}}} \frac{\varepsilon \|\varphi\|_\lambda}{\left| |u|^{p-1}|\varphi| \right|_{L^{6N/(6N-\mu-2\alpha)}(B_{R_0}(0))}} \\
&\quad \times \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{\mathbb{R}^N \setminus B_{R_0+d_\varepsilon}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^{2\mu/3} |y|^{\alpha/3}} \mathrm{d}y \right) \mathrm{d}x \\
&\quad + \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{B_{R_0+d_\varepsilon}(0) \setminus B_{R_0}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&\leq \frac{1}{R_0^{\frac{2\alpha}{3}}} \frac{\varepsilon \|\varphi\|_\lambda}{\left| |u|^{p-1}|\varphi| \right|_{L^{6N/(6N-\mu-2\alpha)}(B_{R_0}(0))}} \\
&\quad \times \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^{2\mu/3} |y|^{\alpha/3}} \mathrm{d}y \right) \mathrm{d}x \\
&\quad + \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{B_{R_0+d_\varepsilon}(0) \setminus B_{R_0}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&\leq \frac{K_4}{R_0^{\frac{2\alpha}{3}}} \frac{\varepsilon \|\varphi\|_\lambda}{\left| |u|^{p-1}|\varphi| \right|_{L^{6N/(6N-\mu-2\alpha)}(B_{R_0}(0))}} \left| |u|^{p-1}|\varphi| \right|_{L^{6N/(6N-\mu-2\alpha)}(B_{R_0}(0))} \\
&\quad + \int_{B_{R_0}(0)} |u|^{p-1}|\varphi| \left(\int_{B_{\widehat{R}_0}(0) \setminus B_{R_0}(0)} \frac{|v_n(y)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&\leq C_8 \varepsilon \|\varphi\|_\lambda \text{ for some positive constant } K_4 > 0. \qquad (19)
\end{aligned}$$

Combining (17), (18), (19) and (∗), we can easily complete the proof of relation (16).

Now, we estimate $|I_n^2|$ and $|I_n^3|$.

We first have, for $n \in \mathbb{N}$ large enough,

$$\begin{aligned}
|I_n^2| &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{||u_n(y)|^p-|v_n(y)|^p-|u(y)|^p|}{|x|^\alpha|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)\left||v_n|^{p-2}v_n\overline{\varphi}\right|\mathrm{d}x\\
&\quad+\left|\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|v_n|^{p-2}v_n\overline{\varphi}\mathrm{d}x\right|\\
&\le C_9\left||u_n(y)|^p-|v_n(y)|^p-|u(y)|^p\right|_{\frac{2N}{2N-\mu-2\alpha}}\left||v_n|^{p-2}v_n\overline{\varphi}\right|_{\frac{2N}{2N-\mu-2\alpha}}\\
&\quad+\widehat{C}_7\varepsilon\|\varphi\|_\lambda \text{ for some constant } C_9>0 \text{ (by Proposition 2 and (15))}\\
&\le\left(C_9C_3^{\frac{2N-\mu-2\alpha}{2N}}+\widehat{C}_7\right)\varepsilon\|\varphi\|_\lambda \text{ (see (5) and (10)).} \qquad (20)
\end{aligned}$$

Also, we have

$$\begin{aligned}
|I_n^3| &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)\left|\left(|u_n|^{p-2}u_n-|v_n|^{p-2}v_n-|u|^{p-2}u\right)\overline{\varphi}\right|\mathrm{d}x\\
&\quad+\left|\int_{\mathbb{R}^N}\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|v_n(y)|^p}{|x-y|^\mu|y|^\alpha}\mathrm{d}y\right)|u|^{p-2}u\overline{\varphi}\mathrm{d}x\right|\\
&\le C_{10}\left(\int_{\mathbb{R}^N}\left||u_n|^{p-2}u_n-|v_n|^{p-2}v_n-|u|^{p-2}u\right|^{\frac{2Np}{(2N-\mu-2)(p-1)}}\mathrm{d}x\right)^{\frac{(2N-\mu-2\alpha)(p-1)}{2Np}}\\
&\quad\times\|\varphi\|_\lambda+C_8\varepsilon\|\varphi\|_\lambda \text{ for some constant } C_{10}>0,\\
&\quad\text{(by Proposition 2, the Sobolev embedding, the Hölder inequality, (8) and (16))}\\
&\le (C_8+C_{10})\,\varepsilon\|\varphi\|_\lambda \text{ (see (7)) for } n\in\mathbb{N} \text{ large enough.} \qquad (21)
\end{aligned}$$

Note that

$$\Phi'(u_n)\varphi-\Phi'(v_n)\varphi-\Phi'(u)\varphi=\mathrm{Re}\left(I_n^2+I_n^2+I_n^3\right) \text{ for all } n\in\mathbb{N}.$$

Using the last equality and relations (11), (20) and (21), for $n \in \mathbb{N}$ large enough we conclude that

$$\begin{aligned}
\left|\Phi'(u_n)\varphi-\Phi'(v_n)\varphi-\Phi'(u)\varphi\right| &= \left|\mathrm{Re}\left(I_n^2+I_n^2+I_n^3\right)\right|\\
&\le C_{11}\varepsilon\|\varphi\|_\lambda\\
&\quad \text{for some constant } C_{11}>0,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}\sup_{\|\varphi\|_\lambda\le 1}\left|(\Phi'(u_n)-\Phi'(v_n)-\Phi'(u))\varphi\right|\le 0 \text{ (by the arbitrariness of } \varepsilon),$$

that is,

$$\Phi'(u_n) - \Phi'(v_n) - \Phi'(u) = o_n(1) \text{ in } X_\lambda^* \text{ as } n \to \infty.$$

This proof is now complete. □

Lemma 9 *Let $N \geq 2$, $\alpha \geq 0$, $0 < \mu < N$, $2\alpha + \mu \leq N$ and $\frac{2N-\mu-2\alpha}{N} < p < \frac{2N-\mu-2\alpha}{N-2}$ (resp. $+\infty$, $N = 2$). Assume that $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is a $(PS)_c$ sequence of the functional $\mathcal{E}_\lambda$ at the level $c \geq 0$. Up to a subsequence, there exists some $u \in X_\lambda$ such that $u_n \xrightarrow{w} u$ in X_λ and have the following relations*

$$\mathcal{E}_\lambda(v_n) - \mathcal{E}_\lambda(u_n) + \mathcal{E}_\lambda(u) = o_n(1) \text{ in } X_\lambda \text{ as } n \to \infty, \tag{22}$$

$$\mathcal{E}'_\lambda(v_n) - \mathcal{E}'_\lambda(u_n) + \mathcal{E}'_\lambda(u) = o_n(1) \text{ in } X_\lambda^* \text{ as } n \to \infty, \tag{23}$$

where $v_n := u_n - u$. Moreover, the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a $(PS)_{c-\mathcal{E}_\lambda(u)}$ sequence.

Proof By Lemma 5, we know that the sequence $\{u_n\}$ is bounded in X_λ. So, passing to a subsequence, we may assume that $u_n \xrightarrow{w} u$ in X_λ, $\nabla_A u_n \xrightarrow{w} \nabla_A u$ in $L^2(\mathbb{R}^N, \mathbb{C})^N$ and $u_n(x) \to u(x)$ in $\mathbb{R}^N$ as $n \to \infty$.

Since X_λ is a Hilbert space, together with the fact that $u_n \xrightarrow{w} u$ in X_λ as $n \to \infty$, we see that

$$\|u_n\|_\lambda^2 - \|v_n\|_\lambda^2 - \|u\|_\lambda^2 = o_n(1) \text{ as } n \to \infty. \tag{24}$$

Next, we can argue as in the proof of relation (7) to infer that

$$\int_{\mathbb{R}^N} (\lambda V(x) + 1)\, |u_n - v_n - u|^2 \,\mathrm{d}x = o_n(1) \text{ as } n \to \infty. \tag{25}$$

Finally, we show that

$$\int_{\mathbb{R}^N} |\nabla_A u_n - \nabla_A v_n - \nabla_A u|^2 \,\mathrm{d}x = o_n(1) \text{ as } n \to \infty. \tag{26}$$

For this purpose, we first prove that

$$\nabla_A u_n(x) \to \nabla_A u(x) \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty. \tag{27}$$

Fix $R > 0$ and $\psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ with $\psi(x) = 1$ for $x \in B_R(0)$. Since $\{u_n, u\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is bounded and $\mathcal{E}'_\lambda(u_n) \to 0$ in X_λ^* as $n \to \infty$, we know that

$$\mathcal{E}'_\lambda(u_n)(u_n\psi) = o_n(1) \text{ and } \mathcal{E}'_\lambda(u_n)(u\psi) = o_n(1) \text{ as } n \to \infty.$$

Then, we conclude that

$$\begin{aligned}
&\int_{B_R(0)} |\nabla_A(u_n-u)|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} |\nabla_A(u_n-u)|^2 \psi \mathrm{d}x \\
&= \mathcal{E}_\lambda'(u_n)(u_n\psi) - \mathcal{E}_\lambda'(u_n)(u\psi) - \operatorname{Re}\int_{\mathbb{R}^N} \nabla_A u(\overline{\nabla_A u_n} - \overline{\nabla_A u})\psi \mathrm{d}x \\
&\quad - \operatorname{Re}\int_{\mathbb{R}^N} (\overline{u_n}-\overline{u})\nabla_A u_n \nabla\psi \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}^N} (\lambda V(x)+1)u_n(\overline{u_n}-\overline{u})\psi \mathrm{d}x \\
&\quad - \operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{|x|^\alpha}\left(\frac{|u_n(y)|^p}{|x-y|^\alpha |y|^\alpha}\mathrm{d}y\right)|u_n|^{p-2}u_n(\overline{u_n}-\overline{u})\psi \mathrm{d}x.
\end{aligned}$$

Using the above all information, we can infer that

$$\int_{B_R(0)} |\nabla_A(u_n-u)|^2 \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Since R is arbitrary, we deduce that (27) holds true.

Applying (27) and proceeding as in the proof of relation (7), we can derive that (26) is true.

Therefore, from Lemmas 7 and 8, together with relations (24)–(26), we can get the desired results. This proof is now finished. □

Lemma 10 *Let* $N \geq 2$, $\alpha \geq 0$, $0 < \mu < N$, $2\alpha + \mu \leq N$ *and* $\frac{2N-\mu-2\alpha}{N} < p < \frac{2N-\mu-2\alpha}{N-2}$ *(resp.* $+\infty$, $N=2$*). Assume that* $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ *is a* $(PS)_c$ *sequence of the functional* $\mathcal{E}_\lambda$ *at the level* $c \geq 0$*. Then* $c = 0$*, or there exists* $d_* > 0$*, independent of* λ*, such that* $c \geq d_*$ *for any* $\lambda > 0$.

Proof Assume that $c > 0$. On account of the fact that

$$\frac{2N-\mu-2\alpha}{N} < p < \frac{2N-\mu-2\alpha}{N-2} \; (\text{resp. } +\infty,\; N=2),$$

then we can employ Proposition 2 and the Sobolev embedding to conclude that there exists $\sigma_0 > 0$ such that

$$\mathcal{E}_\lambda'(u_n)u_n \geq \frac{1}{4}\|u_n\|_\lambda^2, \text{ for } \|u_n\|_\lambda < \sigma_0.$$

In addition, since $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence of the functional $\mathcal{E}_\lambda$ at the level $c > 0$, it is easy to check that

$$\limsup_{n\to\infty} \|u_n\|_\lambda \leq \sqrt{\frac{2pc}{p-1}}.$$

So, if $c \in \left(0, \frac{(p-1)\sigma_0^2}{2p}\right)$, for $n \in \mathbb{N}$ large enough it follows that

$$\|u_n\|_\lambda < \sigma_0.$$

Then we can deduce that

$$\lim_{n\to\infty} \|u_n\|_\lambda = 0,$$
$$\Rightarrow\ 0 < c = \lim_{n\to\infty} \mathcal{E}_\lambda(u_n) = \mathcal{E}_\lambda(0) = 0,$$

a contradiction. Thus, $c \geq \frac{(p-1)\sigma_0^2}{2p} := d_* > 0$. The proof is now complete. □

Lemma 11 *Assume that* $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ *is a* $(PS)_c$ *sequence of the functional* $\mathcal{E}_\lambda$ *at the level* $c \geq 0$. *Then there is a positive number* $\sigma_1 > 0$ *independent of* $\lambda > 0$, *such that*

$$\liminf_{n\to\infty} |u_n|^{2p}_{\frac{2Np}{2N-\mu-2\alpha}} \geq c\sigma_1.$$

Proof Since $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is a $(PS)_c$ sequence of the functional $\mathcal{E}_\lambda$ at the level $c \geq 0$, we can use Proposition 2 and the Sobolev embedding to infer that

$$c\sigma_1 := c\frac{2p}{p-1}C_0 \leq \liminf_{n\to\infty} |u_n|^{2p}_{\frac{2Np}{2N-\mu-2\alpha}}$$

for some constant $C_0 > 0$, where C_0 does not depend on λ.

This proves the lemma. □

Lemma 12 *Let* $d > 0$ *be a real number independent of* λ, *and assume that* $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ *is a* $(PS)_c$ *sequence of the functional* $\mathcal{E}_\lambda$ *at the level* $c \in [0, d]$. *For any* $\varepsilon > 0$, *there are some positive constants* $\Lambda = \Lambda(\varepsilon)$ *and* $R = R(d, \varepsilon)$ *such that*

$$\limsup_{n\to\infty} |u_n|^{2p}_{L^{\frac{2Np}{2N-\mu-2\alpha}}(B_R^c(0))} < \varepsilon \text{ for all } \lambda \geq \Lambda.$$

Proof For any fixed $R > 0$, we introduce the following sets:

$$A(R) := \left\{x \in \mathbb{R}^N : |x| > R,\ V(x) > M_0\right\}$$

and

$$B(R) := \left\{x \in \mathbb{R}^N : |x| > R,\ V(x) \leq M_0\right\}.$$

So, for $n \in \mathbb{N}$ large enough we have

$$\begin{aligned}\int_{A(R)} |u_n|^2 \mathrm{d}x &\le \frac{1}{\lambda M_0 + 1} \int_{A(R)} (\lambda V(x) + 1)|u_n|^2 \mathrm{d}x \\ &\le \frac{1}{\lambda M_0 + 1} \|u_n\|_\lambda^2 \\ &\le \frac{1}{\lambda M_0 + 1} \left(\frac{2pc}{p-1} + o_n(1) \right) \\ &\le \frac{1}{\lambda M_0 + 1} \left(\frac{2pd}{p-1} + o_n(1) \right), \\ &\text{where } d \text{ is independent of } \lambda. \end{aligned} \tag{28}$$

Then we deduce form relation (28) that there is $\Lambda > 0$ such that, for all $\lambda \ge \Lambda$,

$$\limsup_{n\to\infty} \int_{A(R)} |u_n|^2 \mathrm{d}x < \frac{\varepsilon}{2}. \tag{29}$$

From the Hölder inequality and the Sobolev embedding, we can find some constant $K > 0$ (which is independent of λ) such that

$$\begin{aligned}&\int_{B(R)} |u_n|^2 \mathrm{d}x \le \left(\frac{2pKd}{p-1} + o_n(1) \right) \text{meas } (B(R))^{\frac{1}{r'}} \\ &\left(\text{where } 1 \le r \le \frac{N}{N-2} \ (N \ge 3) \text{ and } r > 1 \ (N = 2),\ r' = \frac{r}{r-1} \right), \\ \Rightarrow\ &\limsup_{n\to\infty} \int_{B(R)} |u_n|^2 \mathrm{d}x < \frac{\varepsilon}{2} \text{ for some } R \text{ large enough (recall that hypothesis } V). \end{aligned} \tag{30}$$

Combining (29) with (30), we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^2 \mathrm{d}x < \varepsilon \text{ for some } R \text{ large enough.}$$

Finally, we can use the above inequality and the interpolation inequality to conclude the desired result. □

Proposition 13 *Let $d > 0$ be a real number independent of λ. Then there is a $\Lambda = \Lambda(d) > 0$ such that, for all $\lambda \ge \Lambda$ the functional $\mathcal{E}_\lambda$ satisfies the $(PS)_{c_\lambda}$ condition for all $c_\lambda \in [0, d]$.*

Proof Suppose that $\{u_n\}_{n\in\mathbb{N}} \subseteq X_\lambda$ is a $(PS)_{c_\lambda}$ sequence. Going to a subsequence if necessary, we may assume that $u_n \to u \in X_\lambda$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ ($N \ge 2$) and $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$ for all $1 \le q < \dfrac{2N}{N-2}$ (resp. $+\infty$, if $N = 2$) as $n \to \infty$. Following the standard density arguments, we observe that $\mathcal{E}'_\lambda(u) = 0$ and $\mathcal{E}_\lambda(u) \ge 0$.

Let $v_n = u_n - u$. Then we use Lemma 9 to obtain that $\{v_n\}_{n\in n} \subseteq X_\lambda$ is a $(PS)_{c_\lambda - \mathcal{E}_\lambda(u)}$ sequence. Moreover, $0 \leq c_\lambda - \mathcal{E}_\lambda(u) \leq c_\lambda < d$.

Now, we prove that $c_\lambda = \mathcal{E}_\lambda(u)$ if $\lambda > 0$ large enough. Arguing by contradiction, we may assume that $\mathcal{E}_\lambda(u) < c_\lambda$ for some $\lambda > 0$ large enough. From Lemmas 10 and 11, we see that there exists $d_* > 0$ (which is independent of λ) such that

$$c_\lambda - \mathcal{E}_\lambda(u) \geq d_* \text{ and } \liminf_{n\to\infty} |v_n|^{2p}_{\frac{2Np}{2N-\mu-2\alpha}} \geq \sigma_1 d_* > 0. \tag{31}$$

In Lemma 12 we choose $\varepsilon = \dfrac{\sigma_1 d_*}{2} > 0$ and then we know that there are $\Lambda > 0$, $R > 0$ such that, for some $\lambda \geq \Lambda$,

$$\limsup_{n\to\infty} |v_n|^{2p}_{L^{\frac{2Np}{2N-\mu-2\alpha}}(B_R^c(0))} < \frac{\sigma_1 d_*}{2} \tag{32}$$

So, we infer from relations (31) and (32) that

$$\liminf_{n\to\infty} |v_n|^{2p}_{L^{\frac{2Np}{2N-\mu-2\alpha}}(B_R(0))} > \frac{\sigma_1 d_*}{2} > 0.$$

This is impossible. In fact, since $v_n \xrightarrow{w} 0$ in X_λ as $n \to \infty$, then we can use the compact Sobolev embedding $X_\lambda \hookrightarrow L^{\frac{2Np}{2N-\mu-2\alpha}}(B_R(0))$ to obtain

$$\liminf_{n\to\infty} |v_n|^{2p}_{L^{\frac{2Np}{2N-\mu-2\alpha}}(B_R(0))} = 0.$$

Therefore, we arrive at the conclusion that $0 > \dfrac{\sigma_1 d_*}{2} > 0$ is a contradiction.

So, for $\lambda > 0$ large enough we deduce that $c_\lambda = \mathcal{E}_\lambda(u)$ and $\{v_n\}_{n\in N} \subseteq X_\lambda$ is a $(PS)_0$ sequence. Thus, it follows from Corollary 6 that, for $\lambda > 0$ sufficiently large, $v_n \to 0$ in X_λ as $n \to \infty$. We now complete the proof of the proposition. □

Proof of Theorem 1 Using Lemma 4 and Proposition 13, we can complete the proof of Theorem 1. This proves Theorem 1. □

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