



Harmonic Maps with Free Boundary from Degenerating Bordered Riemann Surfaces

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Abstract

We study the blow-up analysis and qualitative behavior for a sequence of harmonic maps with free boundary from degenerating bordered Riemann surfaces with uniformly bounded energy. With the help of Pohozaev type constants associated to harmonic maps defined on degenerating collars, including vertical boundary collars and horizontal boundary collars, we establish a generalized energy identity.

Keywords Harmonic map · Degenerating bordered Riemann surface · Free boundary · Blow-up

Mathematics Subject Classification 58E15 · 35J50 · 35R35

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1 Introduction

Let (Σ, c) be a compact bordered Riemann surface with smooth boundary $\partial\Sigma$ and with complex structure c . Let (N, g) be a compact Riemannian manifold. Let $K \subset N$ be a k -dimensional closed submanifold with $1 \leq k \leq n$ and denote

$$C(K) = \left\{ u \in C^2(\Sigma, N); u(\partial\Sigma) \subset K \right\}.$$

Let h be a Riemannian metric on Σ which is compatible with the complex structure c . A critical point of the energy functional

$$E(u) = \int_{\Sigma} |\nabla u|^2 \, \text{dvol}_h \quad (1.1)$$

over the space $C(K)$ is called a *harmonic map with free boundary* on K .

By Nash's embedding theorem, we embed (N, g) isometrically into some Euclidean space \mathbb{R}^N . Then the Euler–Lagrange equation of the functional (1.1) is

$$\Delta_h u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of $N \subset \mathbb{R}^N$ and Δ_h is the Laplace–Beltrami operator on M which is defined as follows

$$\Delta_h := -\frac{1}{\sqrt{h}} \frac{\partial}{\partial x^\beta} \left(\sqrt{h} h^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \right).$$

Moreover, for $1 \leq k \leq n-1$, u has free boundary $u(\partial\Sigma)$ on K , namely, the following holds

$$u(x) \in K, \quad du(x)(\vec{n}) \perp T_{u(x)}K, \quad \forall x \in \partial\Sigma, \quad (1.2)$$

where \vec{n} is the outward unit normal vector on $\partial\Sigma$ and \perp means orthogonal. For $k = n$, u satisfies a homogeneous Neumann boundary condition on K , that is

$$u(x) \in K, \quad du(x)(\vec{n}) = 0, \quad \forall x \in \partial\Sigma. \quad (1.3)$$

The tension field $\tau(u)$ of a map u is defined by

$$\tau(u) = -\Delta_h u + A(u)(\nabla u, \nabla u). \quad (1.4)$$

Thus, if $u : (\Sigma, h) \rightarrow (N, g)$ is a harmonic map with free boundary on K then it satisfies the Euler–Lagrangian equation

$$\begin{cases} \tau(u) = 0 & \text{in } \Sigma; \\ du(\mathbf{n}) \perp T_u K & \text{on } \partial\Sigma, \end{cases} \quad (1.5)$$

where $\tau(u)$ is the tension field of u . It is well known that both the energy functional (1.1) and the harmonic map system (1.5) are conformally invariant, hence they are independent of the choice of the metric h in the same conformal class $c = [h]$.

Now suppose (Σ_n, c_n) is a sequence of bordered Riemann surface of the same topological type (g, k) such that

$$\tilde{g} = 2g + k - 1 > 1, \tag{1.6}$$

here g is the genus of Σ_n , k is the number of components of $\partial \Sigma_n$ and \tilde{g} is the genus of the complex double Σ_n^c of Σ_n (see Sect. 2). Let $u_n : \Sigma_n \rightarrow N$ be a sequence of harmonic maps with free boundary on the submanifold $K \subset N$, whose energy is uniformly bounded

$$E(u_n) \leq \Lambda < \infty.$$

When the domain surface is fixed, namely $\Sigma_n = \Sigma$, it was shown in [10] that the sequence u_n converges to a limit harmonic map modulo finitely many bubbles, i.e. harmonic spheres or harmonic disks with free boundary on K . Moreover, the energy identity and the no neck property hold, extending the classical case of a closed domain surface developed in [4,8,17,19] etc. and the case of harmonic functions [13].

If we allow the conformal structure c_n on the domain surface to vary. Then, the conformal structure might degenerate and in such a case, the limit surface is a nodal bordered Riemann surface. When the Riemann surface (Σ_n, c_n) is closed, by the uniformization theorem, there is a hyperbolic metric h_n in this conformal class c_n . It is well-known that the degeneration of conformal structures (Σ_n, h_n, c_n) is obtained by shrinking finitely many simple closed geodesics γ_n (for simplicity of notation, we assume there is only one such geodesic) to a point, which is known as the node. Moreover, there is a collar area near γ_n which is conformal to a long standard cylinder

$$P_n = [-T_n, T_n] \times S^1.$$

In [23,25], the asymptotic behaviour of the maps in the limit process was successfully characterized by associating to each P_n a quantity, called *Pohozaev type constant*, defined by the Hopf differential of the map. More precisely, by integrating the Hopf differential on a slice of P_n , one gets a constant

$$\alpha_n := \int_{\{t\} \times S^1} \left(|\partial_t u_n|^2 - |\partial_\theta u_n|^2 \right) d\theta, \quad \text{for } P_n,$$

then the limit energy and length of u_n on P_n can be expressed in terms of α_n and T_n . This result can be considered as a 2nd order extension of Gromov’s compactness theorem for J-holomorphic curves [5]. Inspired by ideas from [23,25] (which was extended to the case of Dirac-harmonic maps in [24]), the notion of Pohozaev type constant were also introduced in other types of two-dimensional geometric variational problems, for instance, super-Liouville type equations on surfaces with conical singularities [9,11]. Such kind of quantity on one hand measures the extent to which the Pohozaev type

identity fails over a punctured disc and on the other hand provides a characterization of the singular behavior of a solution at an isolated singularity.

In the present paper, we shall investigate the situation that the underlying Riemann surface (Σ_n, c_n) is bordered and in particular, the degeneration of (Σ_n, c_n) occurs at the boundary. In fact, for each (Σ_n, c_n) , we can construct a double cover $(\tilde{\Sigma}_n, \tilde{c}_n)$, which is closed. Then the degeneration of $(\tilde{\Sigma}_n, \tilde{c}_n)$ is exactly the same as in the closed case described above. We will see that, if the degeneration happens at the boundary, then there is a collar area which is conformal to the following two types of half cylinders:
 Type I: a vertical half cylinder

$$Q_n^+ = [-T_n, T_n] \times [0, \pi],$$

Type II: a horizontal half cylinder

$$P_n^+ = [0, T_n] \times S^1,$$

to which we can associate a similar quantity defined in terms of the Hopf differential of the map.

More precisely, in this paper, we shall define the Pohozaev type constant associated to u_n over the above two types of half cylinders (see Lemma 3.1) by

$$\alpha_n := \begin{cases} \int_{\{t\} \times [0, \pi]} (|\partial_t u_n|^2 - |\partial_\theta u_n|^2) d\theta, & \text{for } Q_n^+; \\ \int_{\{t\} \times S^1} (|\partial_t u_n|^2 - |\partial_\theta u_n|^2) d\theta, & \text{for } P_n^+. \end{cases}$$

Combined with the case of interior degeneration studied in [25], now we state the following:

Theorem 1.1 *Let $u_n : (\Sigma_n, h_n, c_n) \rightarrow (N, g)$ be a sequence of harmonic maps with free boundary and with uniformly bounded energy $E(u_n, \Sigma_n) \leq \Lambda < \infty$, where (Σ_n, h_n, c_n) is a sequence of compact hyperbolic Riemann surfaces with smooth boundary $\partial \Sigma_n$ and of genus $\tilde{g} > 1$ (see (1.6)), degenerating to a hyperbolic Riemann surface (Σ, h, c) by the following three ways:*

- *interior degeneration: collapsing finitely many pairwise-disjoint interior simple closed geodesics $\{\gamma_{1n}^j, j = 1, \dots, p_1\} \subset \Sigma_n \setminus \partial \Sigma_n$;*
- *Type I boundary degeneration: collapsing finitely many pairwise-disjoint simple geodesics $\{\gamma_{2n}^j, j = 1, \dots, p_2\}$, where each geodesic γ_{2n}^j , connects two points on the boundary $\partial \Sigma_n$;*
- *Type II boundary degeneration: collapsing finitely many pairwise-disjoint boundary simple closed geodesics $\{\gamma_{3n}^j, j = 1, \dots, p_3\} \subset \partial \Sigma_n$.*

For each $j = 1, 2, \dots, p_i, i = 1, 2, 3$, the geodesic γ_{in}^j degenerates into a pair of punctures $(\varepsilon_i^{j,1}, \varepsilon_i^{j,2})$. Here we use the convention that $p_i = 0$ for some $i = 1, 2, 3$ means that such type of degeneration does not occur.

Denote the h_n -length of γ_{in}^j by l_{in}^j . Then, after passing to a subsequence, there exist finitely many blow-up points $\{x_1, \dots, x_I\}$ which are away from the punctures $(\varepsilon_i^{j,1}, \varepsilon_i^{j,2}), i = 1, 2, 3; j = 1, \dots, p_i$, and finitely many harmonic maps:

- $u : (\bar{\Sigma}, \bar{c}) \rightarrow N$ with free boundary $u(\partial\bar{\Sigma})$ on K , where $(\bar{\Sigma}, \bar{c})$ is the normalization of (Σ, c) ¹
- $\tilde{v}_m^l : S^2 \rightarrow N, l = 0, 1, 2, \dots, \tilde{L}_m$, near the blow-up point $x_m, m = 1, 2, \dots, I$;
- $\tilde{w}_m^k : D_1(0) \rightarrow N$ with free boundary $\tilde{w}_m^k(\partial D_1(0))$ on $K, k = 0, 1, 2, \dots, \tilde{K}_m$, near the blow-up point $x_m, m = 1, 2, \dots, I$;
- $\tilde{L}_m + \tilde{K}_m \geq 1, m = 1, 2, \dots, I$;
- $v_i^{j,l} : S^2 \rightarrow N, l = 0, 1, 2, \dots, L_i^j$, near the puncture $(\varepsilon_i^{j,1}, \varepsilon_i^{j,2}), i = 1, 2, 3; j = 1, \dots, p_i$;
- $w_i^{j,k} : D_1(0) \rightarrow N$ with free boundary $w_i^{j,k}(\partial D_1(0))$ on $K, k = 0, 1, 2, \dots, K_i^j$, near the puncture $(\varepsilon_i^{j,1}, \varepsilon_i^{j,2}), i = 2, 3; j = 1, \dots, p_i$;

such that, pulling back the hyperbolic metrics h_n and the compatible complex structures c_n by suitable diffeomorphisms $\Sigma \rightarrow \Sigma_n \setminus \cup_{i=1}^3 \cup_{j=1}^{p_i} \gamma_{in}^j$, passing to a subsequence, u_n converges to u in $C_{loc}^\infty(\Sigma \setminus \{x_1, \dots, x_I\})$ and the following generalized energy identity holds

$$\begin{aligned} \lim_{n \rightarrow \infty} E(u_n) &= E(u) + \sum_{m=1}^I \sum_{l=0}^{\tilde{L}_m} E(\tilde{v}_m^l) + \sum_{m=1}^I \sum_{k=0}^{\tilde{K}_m} E(\tilde{w}_m^k) \\ &\quad + \sum_{i=1}^3 \sum_{j=1}^{p_i} \sum_{l=0}^{L_i^j} E(v_i^{j,l}) + \sum_{i=2}^3 \sum_{j=1}^{p_i} \sum_{k=0}^{K_i^j} E(w_i^{j,k}) \\ &\quad + \sum_{j=1}^{p_1} \lim_{n \rightarrow \infty} \frac{2\pi^2}{l_{1n}^j} \alpha_{1n}^j + \sum_{i=2}^3 \sum_{j=1}^{p_i} \lim_{n \rightarrow \infty} \frac{\pi^2}{l_{in}^j} \alpha_{in}^j, \end{aligned}$$

where α_{in}^j is the Pohozaev type constant defined on the collar area near $\gamma_{in}^j, i = 1, 2, 3, j = 1, 2, \dots, p_i$.

To prove Theorem 1.1, we shall first study the boundary degeneration of bordered Riemann surfaces. We will describe the two types of boundary degeneration both in the language of algebraic geometry and hyperbolic geometry, see Sect. 2. Secondly, we shall study the asymptotic behavior for a sequence of harmonic maps from two kinds of long half cylinders corresponding to two kinds of boundary degenerations. The analysis on the half long cylinder of type I boundary degeneration is similar to the interior degeneration [2,23,25] and the classical blow-up theory of harmonic maps near an interior energy concentration point [4,19], near a boundary (free boundary)

¹ Here, the normalization means adding some points to the nodes.

energy concentration point [10]. In this paper, we shall focus on the type II boundary degeneration. The idea is to extend the maps across the boundary, which are in general no longer harmonic maps. However, combining the structure of extended maps's equations with the Pohozaev type constant, we can still estimate the energy concentration on the cylinder.

Remark 1.2 Theorem 1.1 can be compared with the compactness results of (twisted) holomorphic curves with Lagrangian boundary conditions (cf. [15,22]), which can be viewed as a 1st order analog of the case of harmonic maps with free boundary investigated in this paper.

Finally, we remark that in the case of $\tilde{g} = 1$, namely the cylinder case, one can apply the arguments in Sect. 3 to conclude that similar results as in Theorem 1.1 hold.

The rest of the paper is organized as follows. In Sect. 2, we describe the geometric structure of nodal bordered Riemann surface and the two types of boundary nodes. In Sect. 3, we develop some analytic properties of harmonic maps from two types of long half cylinders and then prove our main Theorem 1.1.

2 Geometric Structure of Nodal Bordered Riemann Surface

In this section, we collect some facts about nodal bordered Riemann surfaces and the two types of boundary nodes as well as the corresponding two types of boundary collar regions. For more details, we refer to e.g. [12,15].

2.1 Bordered Riemann Surface

A smooth bordered Riemann surface Σ is said to be of type (g, k) if Σ is topologically a sphere attached with g handles and k disks removed. It is topologically equivalent to a compact surface of genus g with k punctures.

For any bordered Riemann surface Σ , there exists a compact double cover surface Σ^c with an anti-holomorphic involution map $\sigma : \Sigma^c \rightarrow \Sigma^c$ such that $\Sigma = \Sigma^c / \sigma$, and $\partial\Sigma$ is just the fixed point set of σ . Given a type (g, k) smooth bordered Riemann surface Σ , the genus of its complex double Σ^c is $\tilde{g} = 2g + k - 1$. In fact, we have

Theorem 2.1 *Let Σ be a bordered Riemann surface. There exists a double cover $\pi : \Sigma^c \rightarrow \Sigma$ of Σ by a compact Riemann surface Σ^c and an anti-holomorphic involution $\sigma : \Sigma^c \rightarrow \Sigma^c$ such that $\pi \circ \sigma = \pi$. There is a holomorphic embedding $\iota : \Sigma \rightarrow \Sigma^c$ such that $\pi \circ \iota$ is the identity map. The triple (Σ^c, π, σ) is unique up to isomorphism.*

Proof See [1, Theorem 1.6.1] and also [21]. □

Actually, the double cover Σ^c is decided only by the differentiable structure of Σ , which can be constructed as follows. First, take a maximal atlas $\mathcal{U} = \{U_i \mid i \in I\}$ of Σ , such that any compatible atlas of Σ is contained in \mathcal{U} . Then, the coordinate charts in \mathcal{U} can be divided into two classes according to the orientation. Now let $S = \sqcup_{i \in I} U_i$ be the disjoint union. For each point in $U_i \cap U_j$, we identify the corresponding points

in U_i and U_j if their transition map is orientation-preserving. In this way, we obtain the orientable covering Σ^o .

If $\partial\Sigma \neq \emptyset$, then for all $p \in \partial\Sigma$, we identify the corresponding two points in Σ^o to get the complex double Σ^c , which is closed. The covering $\pi : \Sigma^o \rightarrow \Sigma$ induces a mapping $\pi : \Sigma^c \rightarrow \Sigma$ which is a ramified double covering of Σ . It is a local homeomorphism at all points $p \in \Sigma^c$ for which $\pi(p) \notin \partial\Sigma$. At points lying over the boundary of Σ , the projection is a folding similar to the mapping $x + iy \rightarrow x + i|y|$ at the real axis.

If Σ is orientable, then Σ^o is a trivial double covering of Σ , which is disconnected and consists of two copies of Σ with opposite orientations. Then, Σ^c is obtained by simply gluing the boundary of the two components.

2.2 Nodes

A nodal bordered Riemann surface is of type (g, k) if it is a degeneration of a smooth bordered Riemann surface of the same type.

Let Σ be a (smooth) bordered Riemann surface of type (g, k) , then the following are equivalent:

- Σ is stable, i.e., $Aut(\Sigma)$ is finite
- Its complex double Σ^c is stable
- The genus $\tilde{g} = 2g + k - 1$ of Σ^c is greater than one
- The Euler characteristic $\chi(\Sigma) = 2 - 2g - k$ is negative

There are three types of nodes for a nodal bordered Riemann surface. Let (z, w) be the coordinate on \mathbb{C}^2 and $\sigma(z, w) = (\bar{z}, \bar{w})$ be the complex conjugation. A node on a bordered Riemann surface is a singularity which is locally isomorphic to one of the following:

- (a) Interior node: $(0, 0) \in \{zw = 0\}$
- (b) Type I boundary node: $(0, 0) \in \{z^2 - w^2 = 0\}/\sigma$
- (c) Type II boundary node: $(0, 0) \in \{z^2 + w^2 = 0\}/\sigma$

2.3 Interior Nodes

Let U be a neighborhood of $(0, 0) \in \mathbb{C}^2$ and $\Sigma = \{(z, w) \in U \mid zw = 0\}$. Then, Σ has two components $\Sigma_1 = \{(z, w) \in U \mid z = 0\}$ and $\Sigma_2 = \{(z, w) \in U \mid w = 0\}$, which are attached at the nodal point $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$. Note that the metric near the node on each component is flat.

2.4 Type I Boundary Nodes

Let $\Sigma = \{(z, w) \in U \mid z^2 - w^2 = 0\}$. Then Σ has two components $\Sigma_1 = \{(z, w) \in U \mid z = w\}$ and $\Sigma_2 = \{(z, w) \in U \mid z = -w\}$, which are attached at the point $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$. The map σ gives an involution ($\sigma^2 = id$) on each component Σ_i . The fixed point set of σ is $F = \{(z, w) \in \mathbb{C}^2 \mid Im(z) = Im(w) = 0\}$. The boundary of Σ_1/σ is just the fixed point set of the involution in Σ_1 , i.e. $\gamma_1 = F \cap \Sigma_1 = \{z = w \in \mathbb{R}^1\}$,

which is a 1-dimensional curve. Similarly, the boundary of Σ_2/σ is the curve $\gamma_2 = F \cap \Sigma_2 = \{z = -w \in \mathbb{R}^1\}$. Thus the boundary curves γ_1 and γ_2 intersects at the node $(0, 0)$.

By sending (z, w) to $(z - w, z + w)$, it is clear that the surface Σ here is isomorphic to the one in the interior node case. Indeed, Σ is the complex double of Σ/σ . Thus by using the involution map σ , the picture of Type 1 boundary node is actually a (*vertical*) half of the interior node case.

2.5 Type II Boundary Nodes

Let $\Sigma = \{(z, w) \in U | z^2 + w^2 = 0\}$. Then, Σ has two components $\Sigma_1 = \{(z, w) \in U | z = iw\}$ and $\Sigma_2 = \{(z, w) \in U | z = -iw\}$, which are attached at the point $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$. The map σ gives an anti-holomorphic involution between Σ_1 and Σ_2 . The only fixed point of σ on Σ is the node $(0, 0)$. Thus the boundary of Σ/σ is simply the node itself.

By sending (z, w) to $(z - iw, z + iw)$, it is clear that the surface Σ here is also isomorphic to the one in the interior node case. Indeed, Σ is the complex double of Σ/σ . Thus by using the symmetry map σ , the picture of Type 2 boundary node is actually a (*horizontal*) half of the interior node case.

2.6 Hyperbolic Geometry Picture Near Nodes

Suppose Σ is a bordered surface of general type (g, k) . Let $(\tilde{\Sigma}, \pi, \sigma)$ be the corresponding complex double which is a closed surface with genus $\tilde{g} = 2g + k - 1 > 1$. Then by uniformization theorem, for each complex structure c on Σ , there exists a (unique) hyperbolic metric h on Σ . It is easy to see that h and c can be extended to its complex double $\tilde{\Sigma}$ such that \tilde{h} and \tilde{c} is symmetric w.r.t. the (anti-holomorphic) involution σ . It follows that the boundary components are geodesics in this metric. See for example [16] for a proof of the uniformization theorem of bordered surfaces and a construction of the hyperbolic metric by a flow method.

Now suppose $\Sigma_n = (\Sigma, h_n, c_n)$ is a sequence of degenerating hyperbolic surfaces which converges to a hyperbolic surface $\Sigma_\infty = (\Sigma_\infty, h, c)$, with finitely many nodes $\{y_1, \dots, y_m; z_1, \dots, z_l\}$ where y_i 's are on the boundary and z_j 's are interior nodes. Then the complex double $(\tilde{\Sigma}, \tilde{h}, \tilde{c})$ converges to the complex double $\tilde{\Sigma}_\infty$ of Σ_∞ , with nodes $\{y_1, \dots, y_m; z_1, \dots, z_l, z'_1, \dots, z'_l\}$ where $z'_j = \sigma(z_j) \in \Sigma'_\infty$. By classical results e.g. [3], $\tilde{\Sigma}_\infty$ is obtained by pinching $m + 2l$ pairwise-disjoint Jordan curves to the nodes.

For simplicity, we first assume there is only one node z . If z is an interior nodes, i.e. $z \notin \partial \Sigma_\infty = \Sigma_\infty^\sigma$, then there exists a collar area $A_n \subset \Sigma_n$ near a closed geodesic $\gamma_n \subset \Sigma_n$, which is isomorphic to a hyperbolic cylinder $P_n = [-T_n, T_n] \times S^1$ with $T_n \rightarrow \infty$ with metric (cf. [18,25])

$$ds^2 = \left(\frac{l_n}{2\pi \cos\left(\frac{l_n t}{2\pi}\right)} \right)^2 (dt^2 + d\theta^2),$$

where l_n is the length of γ_n satisfying

$$T_n = \frac{2\pi}{l_n} \left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{l_n}{2}\right)\right) \right).$$

Moreover, $\Sigma_n \setminus A_n$ converges to $\Sigma_\infty \setminus \{z\}$ smoothly. The local geometry near the node z on each component of Σ_∞ is a standard hyperbolic cusp with metric $(\frac{|dz|}{|z| \ln|z|})^2$.

If $y \in \partial\Sigma_\infty$ is a boundary node, then there also exists a collar area $\tilde{A}_n \subset \tilde{\Sigma}_n$ which lies in the complex double, near a closed geodesic $\tilde{\gamma}_n \subset \tilde{\Sigma}_n$ and isomorphic to a hyperbolic cylinder $\tilde{P}_n = [-T_n, T_n] \times S^1$. Note that the cylinder \tilde{P}_n is the δ -thin part of $\tilde{\Sigma}_n$, and hence is symmetric w.r.t. the involution σ , i.e. $\sigma^*ds^2 = ds^2$. However, there are only two possible involutions on the hyperbolic cylinder \tilde{P}_n which is anti-holomorphic as follows. (The antipodal map $\theta \rightarrow \pi + \theta$ is holomorphic.)

The first one corresponds to the symmetry of \tilde{P}_n w.r.t. the horizontal lines

$$\xi_n = \{(t, \theta) \mid \theta = 0, \pi\} \subset \tilde{P}_n.$$

Namely, the involution $\sigma : \tilde{P}_n \rightarrow \tilde{P}_n$ maps (t, θ) to $(t, 2\pi - \theta)$. In this case, ξ_n lies in the boundary $\partial\Sigma_n$. Let $Q_n^+ = [-T_n, T_n] \times [0, \pi]$ be the vertical half of \tilde{P}_n in Σ_n . Then, $\Sigma_n \setminus Q_n^+$ converges to $\Sigma_\infty \setminus \{y\}$ smoothly. In hyperbolic geometry, the node is obtained by shrinking a geodesic $\gamma_n^+ = \{0\} \times [0, \pi]$ which connects two points at the boundary. This is exactly the type I node described in the algebraic language above.

The second one corresponds to the symmetry of \tilde{P}_n w.r.t. the vertical middle circle

$$\tilde{\gamma}_n = \{(t, \theta) \mid t = 0\} \subset \tilde{P}_n.$$

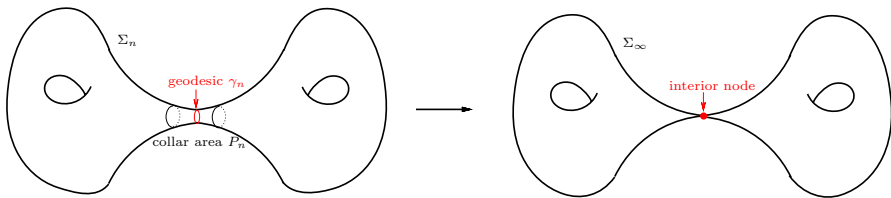


Fig. 1 Interior node (closed surface with $g = 2$)

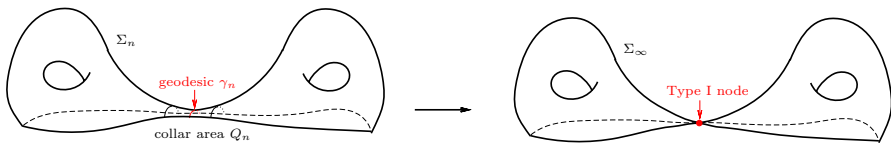


Fig. 2 Type I node (bordered surface with $g = 2$ and $k = 1$)

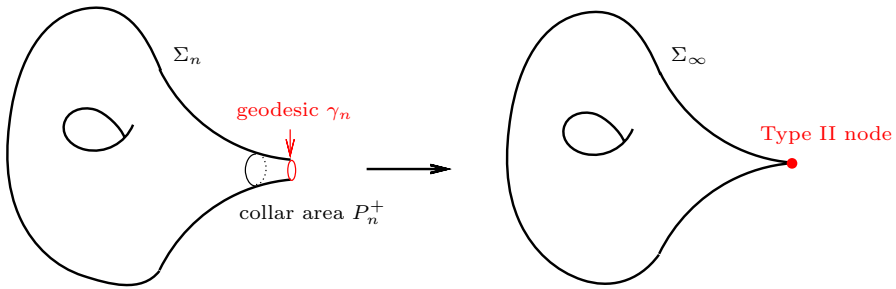


Fig. 3 Type II node (bordered surface with $g = 1$ and $k = 1$)

Namely, the involution $\sigma : \tilde{P}_n \rightarrow \tilde{P}_n$ maps (t, θ) to $(-t, \theta)$. In this case, $\tilde{\gamma}_n$ is fixed by σ and hence belongs to the boundary $\partial \Sigma_n$. Let $P_n^+ = [0, T_n] \times S^1$ be the horizontal half of \tilde{P}_n in Σ_n . Then, $\Sigma_n \setminus P_n^+$ converges to $\Sigma_\infty \setminus \{y\}$ smoothly. In hyperbolic geometry, the node is obtained by shrinking a boundary curve $\gamma_n = \tilde{\gamma}_n$ (also a closed geodesic). This is exactly the type II node described in the algebraic language.

See Figs. 1, 2, and 3 for a simple illustration of the different types of nodes.

3 Harmonic Maps from Half Cylinders and Proof of Theorem 1.1

In this section, we develop the blow-up analysis for harmonic maps from long half cylinders and then complete the proof of Theorem 1.1.

Let u_n be a sequence of harmonic maps defined on the vertical half cylinder

$$Q_n^+ = [-T_n, T_n] \times [0, \pi]$$

with free boundary $u_n([-T_n, T_n] \times \{0, \pi\})$ on K and with uniformly bounded energy

$$E(u_n, Q_n^+) \leq \Lambda$$

or defined on the horizontal half cylinder

$$P_n^+ = [0, T_n] \times S^1$$

with free boundary $u_n(\{0\} \times S^1)$ on K and with uniformly bounded energy

$$E(u_n, P_n^+) \leq \Lambda,$$

where

$$[-T_n, T_n] \times \{0, \pi\} = [-T_n, T_n] \times \{0\} \cup [-T_n, T_n] \times \{\pi\}.$$

For a harmonic map u from a cylinder $[-T, T] \times S^1$, using the fact that the Hopf differential $\phi(u)(dt + id\theta)^2$ is holomorphic, where

$$\phi(u) = |\partial_t u|^2 - |\partial_\theta u|^2 - 2i \partial_t u \cdot \partial_\theta u,$$

it was observed in [25] that the integration

$$\int_{\{t\} \times S^1} \phi(u) d\theta$$

is independent of t , which is a complex constant. Here, for harmonic maps from half cylinders, by integrating by parts, we can also define a quantity independent of t , which plays an important role to characterize the asymptotic properties of the maps in the limit process.

Lemma 3.1 *If u is a harmonic map defined on Q_n^+ with free boundary $u([-T_n, T_n] \times \{0, \pi\})$ on K , then*

$$\int_{\{t\} \times [0, \pi]} (|\partial_t u|^2 - |\partial_\theta u|^2) d\theta$$

is independent of t , where $\partial_t u = \frac{\partial u}{\partial t}$ and $\partial_\theta u = \frac{\partial u}{\partial \theta}$.

If u is a harmonic map defined on P_n^+ with free boundary $u_n(\{0\} \times S^1)$ on K , then

$$\int_{\{t\} \times S^1} (|\partial_t u|^2 - |\partial_\theta u|^2) d\theta$$

is independent of t .

Proof We shall only prove for first case, since the second case is similar and easier.

Since u is a harmonic map defined on Q_n , denoting $Q_{t_1 t_2} = [t_1, t_2] \times [0, \pi]$, then we have

$$\begin{aligned} 0 &= \int_{Q_{t_1 t_2}} \Delta u \cdot \partial_t u d\theta dt \\ &= \int_{Q_{t_1 t_2}} (\partial_t^2 u + \partial_\theta^2 u) \cdot \partial_t u d\theta dt \\ &= \frac{1}{2} \int_{Q_{t_1 t_2}} \frac{\partial}{\partial t} (|\partial_t u|^2 - |\partial_\theta u|^2) d\theta dt + \int_{t_1}^{t_2} \partial_t u \cdot \partial_\theta u|_0^\pi dt \\ &= \frac{1}{2} \left(\int_{\{t_2\} \times [0, \pi]} (|\partial_t u|^2 - |\partial_\theta u|^2) d\theta - \int_{\{t_1\} \times [0, \pi]} (|\partial_t u|^2 - |\partial_\theta u|^2) d\theta \right), \end{aligned}$$

where we used the free boundary condition, i.e.

$$\partial_t u \cdot \partial_\theta u|_{[-T_n, T_n] \times \{0, \pi\}} = 0.$$

Then, the conclusions of lemma follow immediately. □

By above lemma, we shall give the following definition of Pohozaev-type constant:

Definition 3.1 The constant

$$\alpha_n := \int_{\{t\} \times [0, \pi]} (|\partial_t u_n|^2 - |\partial_\theta u_n|^2) d\theta \quad \text{for } Q_n^+ \tag{3.1}$$

or

$$\alpha_n := \int_{\{t\} \times S^1} (|\partial_t u_n|^2 - |\partial_\theta u_n|^2) d\theta \quad \text{for } P_n^+ \tag{3.2}$$

is called the Pohozaev-type constant for u_n on Q_n^+ or P_n^+ , respectively.

In this paper, we want to study the energy concentration of u_n on Q_n^+ and P_n^+ . First, by the classical blow-up theory of harmonic maps near an interior energy concentration point [4], near a (free) boundary energy concentration point [10] and near the interior degenerating area [25], it is not hard to prove the following:

Theorem 3.2 $u_n : P_n^+ \rightarrow N$ be a sequence of harmonic maps with free boundary $u_n(\{0\} \times S^1)$ on K and with uniformly bounded energy

$$E(u_n; P_n^+) \leq \Lambda,$$

where P_n^+ is a cylinder with standard flat metric $ds^2 = dt^2 + d\theta^2$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$.

Then, there exist a finite harmonic spheres $v^i : S^2 \rightarrow N$, $i = 1, \dots, I$ and harmonic disks $w^j : D_1(0) \rightarrow N$, $j = 1, \dots, J$ with free boundary $w^j(\partial D_1(0))$ on K such that, after passing to a subsequence, there holds

$$\lim_{n \rightarrow \infty} E(u_n, P_n^+) = \sum_{i=1}^I E(v^i) + \sum_{j=1}^J E(w^j) + \lim_{n \rightarrow \infty} 2\alpha_n T_n,$$

where α_n is defined by (3.2).

Proof With the help of [4,10], one can refer to a similar proof in [2,25]. We omit the details here. □

Next, we will focus on the case of Q_n^+ . We first consider a simpler case by assuming that there is no energy concentration points in Q_n^+ , i.e.

$$\lim_{n \rightarrow \infty} \sup_{-T_n \leq t \leq T_n - 1} E(u_n, [t, t + 1] \times [0, \pi]) = 0. \tag{3.3}$$

By the small energy regularity theory of harmonic maps in the interior case [19] and the free boundary case [10], we have

Lemma 3.3 *Let $u_n : Q_n^+ \rightarrow N$ be a sequence of harmonic maps with free boundary $u_n([-T_n, T_n] \times \{0, \pi\})$ on K . Assuming (3.3) holds, then for any $\epsilon > 0$, there hold*

$$\sup_{-T_n+1 \leq t \leq T_n-1} \|\nabla u_n\|_{W^{1,2}([t-\frac{1}{2}, t+\frac{1}{2}] \times [0, \pi])} \leq C(K, N) \sup_{-T_n+1 \leq t \leq T_n-1} \|\nabla u_n\|_{L^2([t-1, t+1] \times [0, \pi])} \tag{3.4}$$

and

$$\begin{aligned} \sup_{-T_n+1 \leq t \leq T_n-1} \|u_n\|_{Osc([t-\frac{1}{2}, t+\frac{1}{2}] \times [0, \pi])} &\leq C(K, N) \\ \sup_{-T_n+1 \leq t \leq T_n-1} \|\nabla u_n\|_{L^2([t-1, t+1] \times [0, \pi])} &\leq C(K, N)\epsilon \end{aligned} \tag{3.5}$$

when n is big enough, where

$$\|u_n\|_{Osc(\Omega)} := \sup_{x, y \in \Omega} |u_n(x) - u_n(y)|.$$

In particular, the free boundary condition tells us that the image of u_n is contained in a small tubular neighborhood of K in N , i.e.

$$u_n([-T_n + 1, T_n - 1] \times [0, \pi]) \subset K_{C(N)\epsilon},$$

where $K_{C(N)\epsilon}$ denotes the $C(N)\epsilon$ -tubular neighborhood of K in N .

Denote by K_{δ_0} the δ_0 -tubular neighborhood of K in N . Taking $\delta_0 > 0$ small enough, then for any $y \in K_{\delta_0}$, there exists a unique projection $y' \in K$. Set $\bar{y} = \exp_{y'}\{-\exp_{y'}^{-1}y\}$. Thus, we may define an involution σ , i.e. $\sigma^2 = Id$ as in [6,7,20] by

$$\sigma(y) = \bar{y} \text{ for } y \in K_{\delta_0}.$$

Then, it is easy to check that the linear operator $D\sigma : TN|_{K_{\delta_0}} \rightarrow TN|_{K_{\delta_0}}$ satisfies $D\sigma(V) = V$ for $V \in TK$ and $D\sigma(\xi) = -\xi$ for $\xi \in T^\perp K$.

By Lemma 3.3, we can define an extension of u_n to $Q_n := [-T_n + 1, T_n - 1] \times S^1$ that

$$\widehat{u}_n(t, \theta) = \begin{cases} u_n(t, \theta), & \text{if } (t, \theta) \in [-T_n + 1, T_n - 1] \times [0, \pi]; \\ \sigma(u_n(\rho(t, \theta))), & \text{if } (t, \theta) \in [-T_n + 1, T_n - 1] \times [\pi, 2\pi], \end{cases} \tag{3.6}$$

where $\rho(t, \theta) := (t, 2\pi - \theta)$.

Now, we derive the equation for the extended map \widehat{u}_n . One can see that it is no longer a harmonic map, but it satisfies the following property. See also Proposition 3.3 in [10].

Proposition 3.4 *Let $u \in W^{2,p}([T_1, T_2] \times [0, \pi], N)$, $1 \leq p \leq \infty$, be a map with free boundary $u([T_1, T_2] \times \{0, \pi\})$ on K . Let $u([T_1, T_2] \times [0, \pi]) \subset K_{\delta_0}$, then the extended map \widehat{u} defined by 3.6 satisfying $\widehat{u} \in W^{2,p}([T_1, T_2] \times S^1)$ and*

$$\Delta \widehat{u} + \Upsilon_{\widehat{u}}(\nabla \widehat{u}, \nabla \widehat{u}) = 0 \text{ in } [T_1, T_2] \times S^1, \tag{3.7}$$

where $\Upsilon_{\widehat{u}}(\cdot, \cdot)$ is a bounded bilinear form defined by

$$\Upsilon_{\widehat{u}}(\cdot, \cdot) = \begin{cases} A(\widehat{u})(\cdot, \cdot) \text{ in } [T_1, T_2] \times (0, \pi), \\ A(\widehat{u})(\cdot, \cdot) - D^2\sigma|_{\sigma(\widehat{u})}(D\sigma|_{\widehat{u}} \circ \cdot, D\sigma|_{\widehat{u}} \circ \cdot) \text{ in } [T_1, T_2] \times (\pi, 2\pi); \end{cases}$$

satisfying

$$|\Upsilon_{\widehat{u}}(\nabla \widehat{u}, \nabla \widehat{u})| \leq C(K, N)|\nabla \widehat{u}|^2.$$

Proof According to the properties of $D\sigma$, it is easy to see that $\widehat{u} \in W^{2,p}([T_1, T_2] \times S^1)$ since $u \in W^{2,p}([T_1, T_2] \times [0, \pi], N)$ and satisfies free boundary condition. Next, we derive the equation for \widehat{u} .

Firstly, for $(t, \theta) \in (T_1, T_2) \times (\pi, 2\pi)$, we have

$$\Delta \widehat{u} + A(\widehat{u})(\nabla \widehat{u}, \nabla \widehat{u}) = \nabla_{\widehat{d}\widehat{u}(e_\beta)}^{*\widehat{u}TN}(\widehat{d}\widehat{u}(e_\beta)),$$

where $\nabla^{*\widehat{u}TN}$ is the covariant derivative on the pull back bundle.

Computing directly in local normal coordinates $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ of target manifold N , we get

$$\begin{aligned} \nabla_{\widehat{d}\widehat{u}(e_\beta)}^{*\widehat{u}TN}(\widehat{d}\widehat{u}(e_\beta)) &= \nabla_{\widehat{d}\widehat{u}(e_\beta)}^{*\widehat{u}TN} \left(\frac{\partial \sigma^j}{\partial y^i} du^i \circ d\rho(e_\beta) \frac{\partial}{\partial y^j} \right) \\ &= \frac{\partial^2 \sigma^j}{\partial y^k \partial y^i} du^i \circ d\rho(e_\beta) du^k \circ d\rho(e_\beta) \frac{\partial}{\partial y^j} + \frac{\partial \sigma^j}{\partial y^i} \Delta u^i \frac{\partial}{\partial y^j} \\ &= \frac{\partial^2 \sigma^j}{\partial y^k \partial y^i} du^i \circ d\rho(e_\beta) du^k \circ d\rho(e_\beta) \frac{\partial}{\partial y^j} \\ &= D^2\sigma|_{\sigma(\widehat{u})}(D\sigma|_{\widehat{u}} \circ \nabla \widehat{u}, D\sigma|_{\widehat{u}} \circ \nabla \widehat{u}), \end{aligned}$$

where the last second equality follows from the fact that u is a harmonic map which is equivalent to say that $\Delta u^i = 0$, $i = 1, \dots, n$ in local normal coordinate system $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$.

Combining this with the fact that \widehat{u} is a harmonic map in $[T_1, T_2] \times [0, \pi]$, the conclusions of the proposition follows immediately. \square

Now, we estimate the energy of u_n on Q_n^+ when there is no energy concentration. We have

Theorem 3.5 *Let N be a compact Riemannian manifold and $K \subset N$ is a smooth submanifold. Let $u_n : Q_n^+ \rightarrow N$ be a sequence of harmonic maps with free boundary $u_n([-T_n, T_n] \times \{0, \pi\})$ on K and with uniformly bounded energy*

$$E(u_n; Q_n^+) \leq \Lambda,$$

where Q_n^+ is a cylinder with standard flat metric $ds^2 = dt^2 + d\theta^2$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$.

Suppose there is no energy concentration for u_n , i.e. (3.3) holds, then we have

$$\lim_{n \rightarrow \infty} E(u_n; Q_n^+) = \lim_{n \rightarrow \infty} 2T_n \alpha_n,$$

where α_n is defined by (3.1).

Proof Since (3.3) holds, by Lemma 3.3, we know that for any $\epsilon > 0$, there holds

$$u_n([-T_n + 1, T_n - 1] \times [0, \pi]) \subset K_{C(N)\epsilon},$$

when n is big enough. Taking $\epsilon > 0$ small such that $C(N)\epsilon \leq \delta_0$, then we can use the definition (3.6) to extend u_n to \widehat{u}_n , which is defined on $[-T_n + 1, T_n - 1] \times S^1$ and satisfies Eq. (3.7).

Setting

$$\widehat{u}_n^*(t) := \frac{1}{2\pi} \int_0^{2\pi} \widehat{u}_n(t, \theta) d\theta, \quad t \in [-T_n + 1, T_n - 1],$$

then, by (3.3) and Lemma 3.3, we have

$$\begin{aligned} \|\widehat{u}_n - \widehat{u}_n^*\|_{L^\infty([-T_n+1, T_n-1] \times S^1)} &\leq \sup_{t \in [-T_n+1, T_n-1]} \|\widehat{u}_n\|_{Osc(\{t\} \times S^1)} \\ &\leq C(K, N) \sup_{t \in [-T_n+1, T_n-1]} \|u_n\|_{Osc(\{t\} \times [0, \pi])} \\ &\leq C(K, N)\epsilon, \end{aligned} \tag{3.8}$$

when n is big enough.

Multiplying the Eq. (3.7) by $\widehat{u}_n - \widehat{u}_n^*$ and integrating by parts, we get

$$\begin{aligned} &\int_{[-T_n+1, T_n-1] \times S^1} \Upsilon_{\widehat{u}_n}(\nabla \widehat{u}_n, \nabla \widehat{u}_n)(\widehat{u}_n - \widehat{u}_n^*) dt d\theta \\ &= \int_{[-T_n+1, T_n-1] \times S^1} -\Delta \widehat{u}_n (\widehat{u}_n - \widehat{u}_n^*) dt d\theta \\ &= \int_{[-T_n+1, T_n-1] \times S^1} \nabla \widehat{u}_n \nabla (\widehat{u}_n - \widehat{u}_n^*) dt d\theta - \int_{\{T_n-1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta \\ &\quad + \int_{\{-T_n+1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta. \end{aligned} \tag{3.9}$$

Using Hölder’s inequality, we have

$$\begin{aligned}
 & \int_{[-T_n+1, T_n-1] \times S^1} \nabla \widehat{u}_n \nabla (\widehat{u}_n - \widehat{u}_n^*) \, dt \, d\theta \\
 &= \int_{[-T_n+1, T_n-1] \times S^1} \left(|\nabla \widehat{u}_n|^2 - \frac{\partial \widehat{u}_n}{\partial t} \frac{\partial \widehat{u}_n^*}{\partial t} \right) \, dt \, d\theta \\
 &\geq \int_{[-T_n+1, T_n-1] \times S^1} |\nabla \widehat{u}_n|^2 \, dt \, d\theta - \left(\int_{[-T_n+1, T_n-1] \times S^1} \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \, dt \, d\theta \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{[-T_n+1, T_n-1] \times S^1} \left| \frac{\partial \widehat{u}_n^*}{\partial t} \right|^2 \, dt \, d\theta \right)^{\frac{1}{2}} \geq \int_{[-T_n+1, T_n-1] \times S^1} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) \, dt \, d\theta,
 \end{aligned} \tag{3.10}$$

where the last inequality follows from the fact that

$$\begin{aligned}
 \left(\int_{[-T_n+1, T_n-1] \times S^1} \left| \frac{\partial \widehat{u}_n^*}{\partial t} \right|^2 \, dt \, d\theta \right)^{\frac{1}{2}} &= \left(\int_{[-T_n+1, T_n-1] \times S^1} \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \widehat{u}_n}{\partial t} \, d\theta \right|^2 \, dt \, d\theta \right)^{\frac{1}{2}} \\
 &\leq \left(\int_{[-T_n+1, T_n-1] \times S^1} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \, d\theta \, dt \, d\theta \right)^{\frac{1}{2}} \\
 &= \left(\int_{[-T_n+1, T_n-1] \times S^1} \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \, dt \, d\theta \right)^{\frac{1}{2}}.
 \end{aligned}$$

Combining (3.9) and (3.10), we arrived

$$\begin{aligned}
 & \int_{[-T_n+1, T_n-1] \times S^1} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) \, dt \, d\theta \\
 &\leq \int_{[-T_n+1, T_n-1] \times S^1} \Gamma_{\widehat{u}_n}(\nabla \widehat{u}_n, \nabla \widehat{u}_n) (\widehat{u}_n - \widehat{u}_n^*) \, dt \, d\theta \\
 &\quad + \int_{\{T_n-1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) \, d\theta - \int_{\{-T_n+1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) \, d\theta.
 \end{aligned} \tag{3.11}$$

A direct computation yields that

$$\begin{aligned}
 & \int_{-T_n+1}^{T_n-1} \int_{\pi}^{2\pi} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) \, dt \, d\theta \\
 &= \int_{-T_n+1}^{T_n-1} \int_0^{\pi} \left(|D\sigma(u_n) \cdot \nabla u_n|^2 - |D\sigma(u_n) \cdot \frac{\partial u_n}{\partial t}|^2 \right) \, dt \, d\theta.
 \end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}
 |D\sigma(u_n) \cdot \nabla u_n|^2 &= \langle D\sigma \cdot \nabla u_n, D\sigma \cdot \nabla u_n \rangle \\
 &= \langle (D\sigma)^* \cdot D\sigma \cdot \nabla u_n, \nabla u_n \rangle \\
 &= \left\langle \left((D\sigma)^* \cdot D\sigma - Id \right) \cdot \nabla u_n, \nabla u_n \right\rangle + |\nabla u_n|^2,
 \end{aligned}$$

where $(D\sigma)^*$ is the adjoint operator of linear operator $D\sigma : TN|_{K_{\delta_0}} \rightarrow TN|_{K_{\delta_0}}$.

Similarly,

$$\left| D\sigma \cdot \frac{\partial u_n}{\partial t} \right|^2 = \left\langle \left((D\sigma)^* \cdot D\sigma - Id \right) \cdot \frac{\partial u_n}{\partial t}, \frac{\partial u_n}{\partial t} \right\rangle + \left| \frac{\partial u_n}{\partial t} \right|^2.$$

Noting that $(D\sigma)^* \cdot D\sigma|_K = Id$, by the continuity of eigenvalues of $(D\sigma)^* \cdot D\sigma$, we have that for any $\delta' > 0$, there exists a constant $\delta_1 = \delta_1(\delta', K, N)$, such that for any $y \in K_{\delta_1}$ and $\xi \in TN|_{K_{\delta_1}}$, there holds

$$\langle (D\sigma)^*|_y \cdot D\sigma|_y \cdot \xi, \xi \rangle \leq (1 + \delta')|\xi|^2.$$

By Lemma 3.3, for any $\epsilon > 0$, when n is big enough, there holds

$$\|dist(u_n, K)\|_{L^\infty([-T_n+1, T_n-1] \times S^1)} \leq C\epsilon.$$

Thus,

$$\left\langle \left((D\sigma)^*|_{u_n(t,\theta)} \cdot D\sigma|_{u_n(t,\theta)} - Id \right) \cdot \xi, \xi \right\rangle \leq C\epsilon|\xi|^2, \quad (t, \theta) \in [-T_n + 1, T_n - 1] \times S^1,$$

implies that

$$\begin{aligned} \int_{-T_n+1}^{T_n-1} \int_{\pi}^{2\pi} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) dt d\theta &\geq \int_{-T_n+1}^{T_n-1} \int_0^{\pi} \left(|\nabla u_n|^2 - \left| \frac{\partial u_n}{\partial t} \right|^2 \right) dt d\theta \\ &\quad - C\epsilon \int_{-T_n+1}^{T_n-1} \int_0^{\pi} |\nabla u_n|^2 dt d\theta. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\int_{[-T_n+1, T_n-1] \times S^1} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) dt d\theta \\ &= \int_{-T_n+1}^{T_n-1} \int_0^{\pi} \left(|\nabla u_n|^2 - \left| \frac{\partial u_n}{\partial t} \right|^2 \right) dt d\theta + \int_{-T_n+1}^{T_n-1} \int_{\pi}^{2\pi} \left(|\nabla \widehat{u}_n|^2 - \left| \frac{\partial \widehat{u}_n}{\partial t} \right|^2 \right) dt d\theta \\ &\geq 2 \int_{-T_n+1}^{T_n-1} \int_0^{\pi} \left(|\nabla u_n|^2 - \left| \frac{\partial u_n}{\partial t} \right|^2 \right) dt d\theta - C\epsilon \int_{-T_n+1}^{T_n-1} \int_0^{\pi} |\nabla u_n|^2 dt d\theta. \end{aligned}$$

Combining this with (3.11), (3.8) and Lemma 3.3, it yields

$$\begin{aligned} &2 \int_{-T_n+1}^{T_n-1} \int_0^{\pi} \left(|\nabla u_n|^2 - \left| \frac{\partial u_n}{\partial t} \right|^2 \right) dt d\theta \\ &\leq \int_{[-T_n+1, T_n-1] \times S^1} \Gamma_{\widehat{u}_n}(\nabla \widehat{u}_n, \nabla \widehat{u}_n)(\widehat{u}_n - \widehat{u}_n^*) dt d\theta + C\epsilon \int_{-T_n+1}^{T_n-1} \int_0^{\pi} |\nabla u_n|^2 dt d\theta \end{aligned}$$

$$\begin{aligned}
 & + \int_{\{T_n-1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta - \int_{\{-T_n+1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta \\
 \leq & C \epsilon \int_{-T_n+1}^{T_n-1} \int_0^\pi |\nabla u_n|^2 dt d\theta + \int_{\{T_n-1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta \\
 & - \int_{\{-T_n+1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta.
 \end{aligned} \tag{3.13}$$

For the boundary terms, by trace theory and Lemma 3.3, we have

$$\begin{aligned}
 \left| \int_{\{T_n-1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta \right| & \leq \|\widehat{u}_n - \widehat{u}_n^*\|_{L^\infty(\{T_n-1\} \times S^1)} \int_{\{T_n-1\} \times S^1} \left| \frac{\partial \widehat{u}_n}{\partial t} \right| d\theta \\
 & \leq C \|\widehat{u}_n\|_{Osc(\{T_n-1\} \times S^1)} \int_{\{T_n-1\} \times [0, \pi]} \left| \frac{\partial u_n}{\partial t} \right| d\theta \\
 & \leq C \|u_n\|_{Osc(\{T_n-1\} \times [0, \pi])} \left(\|\nabla^2 u_n\|_{L^2(\{T_n-\frac{3}{2}, T_n-\frac{1}{2}\} \times [0, \pi])} \right. \\
 & \quad \left. + \|\nabla u_n\|_{L^2(\{T_n-\frac{3}{2}, T_n-\frac{1}{2}\} \times [0, \pi])} \right) \\
 & \leq C \|u_n\|_{Osc(\{T_n-1\} \times [0, \pi])} \|\nabla u_n\|_{L^2(\{T_n-2, T_n\} \times [0, \pi])} \\
 & \leq C \|\nabla u_n\|_{L^2(\{T_n-2, T_n\} \times [0, \pi])}^2 \leq C \epsilon,
 \end{aligned}$$

where the last inequality follows from (3.3).

Similarly,

$$\left| \int_{\{-T_n+1\} \times S^1} \frac{\partial \widehat{u}_n}{\partial t} (\widehat{u}_n - \widehat{u}_n^*) d\theta \right| \leq C \|\nabla u_n\|_{L^2(\{-T_n, -T_n+2\} \times [0, \pi])}^2 \leq C \epsilon.$$

Then by (3.13) and Lemma 3.1, we have

$$\begin{aligned}
 C \epsilon & \geq 2 \int_{-T_n+1}^{T_n-1} \int_0^\pi \left(|\nabla u_n|^2 - \left| \frac{\partial u_n}{\partial t} \right|^2 \right) dt d\theta \\
 & = \left| \int_{-T_n+1}^{T_n-1} \int_0^\pi |\nabla u_n|^2 dt d\theta - \int_{-T_n+1}^{T_n-1} \int_0^\pi \left(\left| \frac{\partial u_n}{\partial t} \right|^2 - \left| \frac{\partial u_n}{\partial \theta} \right|^2 \right) dt d\theta \right| \\
 & = \left| \int_{-T_n+1}^{T_n-1} \int_0^\pi |\nabla u_n|^2 dt d\theta - 2(T_n - 1)\alpha_n \right|,
 \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} E(u_n; [-T_n + 1, T_n - 1] \times [0, \pi]) = \lim_{n \rightarrow \infty} 2T_n \alpha_n.$$

We finished the proof of theorem. □

Next, we will consider the more general case of allowing the phenomenon of energy concentration, i.e. (3.3) does not hold. Combining Theorem 3.5 with [4, 10, 25], we can prove the following theorem.

Theorem 3.6 *Let N be a compact Riemannian manifold and $K \subset N$ is a smooth submanifold. Let $u_n : Q_n^+ \rightarrow N$ be a sequence of harmonic maps with free boundary $u_n([-T_n, T_n] \times \{0, \pi\})$ on K and with uniformly bounded energy*

$$E(u_n; Q_n^+) \leq \Lambda,$$

where Q_n^+ is a cylinder with standard flat metric $ds^2 = dt^2 + d\theta^2$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$.

Then there exist a finite harmonic spheres $v^i : S^2 \rightarrow N, i = 1, \dots, I$ and harmonic disks $w^j : D_1(0) \rightarrow N, j = 1, \dots, J$ with free boundary $w^j(\partial D_1(0))$ on K such that, after passing to a subsequence, there holds

$$\lim_{n \rightarrow \infty} E(u_n, Q_n^+) = \sum_{i=1}^I E(v^i) + \sum_{j=1}^J E(w^j) + \lim_{n \rightarrow \infty} 2\alpha_n T_n,$$

where α_n is defined by (3.1).

Proof Based on the neck analysis scheme in [4], Zhu [25] gives a refined ‘‘bubble domain and neck domain’’ decomposition for a sequence of harmonic maps from a long cylinder. Here, Combining [25] with the blow-up theory for a blow-up sequence of harmonic maps near a (free) boundary point [10], it is not hard to see this kind of decomposition also holds for a sequence of harmonic maps u_n from a long half cylinder Q_n^+ with free boundary $u_n([-T_n, T_n] \times \{0, \pi\})$ on K . We leave the detailed proof to interested readers.

Precisely, we can show that there exists a constant $L > 0$ independent of n and $2L$ sequences $\{a_n^1\}, \{b_n^1\}, \{a_n^2\}, \{b_n^2\}, \dots, \{a_n^L\}, \{b_n^L\}$, such that

$$-T_n \leq a_n^1 \ll b_n^1 \leq a_n^2 \ll b_n^2 \leq \dots \leq a_n^L \ll b_n^L \leq T_n \quad (a_n^i \ll b_n^i \text{ means } \lim_{n \rightarrow \infty} b_n^i - a_n^i = \infty)$$

and $(b_n^i - a_n^i) \ll T_n$, i.e.

$$\lim_{n \rightarrow \infty} \frac{b_n^i - a_n^i}{T_n} = 0, \quad i = 1, \dots, L.$$

Denote

$$\begin{aligned} J_n^j &:= [a_n^j, b_n^j] \times S^1, \quad j = 1, \dots, L, \\ I_n^0 &:= [-T_n, a_n^1] \times S^1, \quad I_n^L := [b_n^L, T_n] \times S^1, \\ I_n^i &:= [b_n^i, a_n^{i+1}] \times S^1, \quad i = 1, \dots, L - 1. \end{aligned}$$

Then, after passing to a subsequence, which we still denote by $\{u_n\}$, there hold

$$(1) \quad \forall i = 0, 1, \dots, L,$$

$$\lim_{n \rightarrow \infty} \sup_{t \in I_n^i} \int_{[t, t+1] \times S^1} |\nabla u_n|^2 = 0.$$

The maps ϕ_n on I_n^j are necks corresponding to collapsing homotopically nontrivial curves.

- (2) $\forall j = 1, \dots, L$, there are finitely many harmonic maps $v^{j,l} : S^2 \rightarrow N, l = 1, \dots, l_j$, and finitely harmonic maps $w^{j,m} : D_1(0) \rightarrow N, m = 1, \dots, m_j$ with free boundary $w^{j,m}(\partial D_1(0))$ on K , such that

$$\lim_{n \rightarrow \infty} E(u_n, J_n^j) = \sum_{l=1}^{l_j} E(v^{j,l}) + \sum_{m=1}^{m_j} E(w^{j,m}). \tag{3.14}$$

By Theorem 3.5, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=0}^L E(u_n; I_n^i) &= \lim_{n \rightarrow \infty} \alpha_n \left(a_n^1 - (-T_n) + T_n - b_n^L + \sum_{i=1}^{L-1} (a_n^{i+1} - b_n^i) \right) \\ &= \lim_{n \rightarrow \infty} \alpha_n \left(2T_n - \sum_{i=1}^L (b_n^i - a_n^i) \right) \\ &= \lim_{n \rightarrow \infty} 2\alpha_n T_n \left(1 - \sum_{i=1}^L \frac{b_n^i - a_n^i}{2T_n} \right) = \lim_{n \rightarrow \infty} 2\alpha_n T_n. \end{aligned}$$

Combining this with (3.14), we proved the conclusion of the theorem. □

At the end of this section, we give the proof of our main Theorem 1.1.

Proof of Theorem 1.1 With the help of the blow-up analysis for a sequence of harmonic maps near an interior blow-up point [4], near a free boundary point [10], and the asymptotic behaviour near an interior node [25], it is easy to see that the conclusions of Theorem 1.1 follow immediately from Theorems 3.6 and 3.2. □

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