The Journal of Geometric Analysis (2022) 32:7
https://doi.org/10.1007/s12220-021-00800-3

Two Flow Approaches to the Loewner–Nirenberg Problem on Manifolds

Gang Li[1]

Received: 6 May 2021 / Accepted: 25 September 2021 / Published online: 2 December 2021

Abstract

We introduce two flow approaches to the Loewner–Nirenberg problem on compact Riemannian manifolds (M^n, g) with boundary and establish the convergence of the corresponding Cauchy–Dirichlet problems to the solution of the Loewner–Nirenberg problem. In particular, when the initial data u_0 is a subsolution to (1.1), the convergence holds for both the direct flow (1.3)–(1.5) and the Yamabe flow (1.6). Moreover, when the background metric satisfies $R_g \geq 0$, the convergence holds for any positive initial data $u_0 \in C^{2,\alpha}(M)$ for the direct flow; while for the case the first eigenvalue $\lambda_1 < 0$ for the Dirichlet problem of the conformal Laplacian L_g, the convergence holds for $u_0 > v_0$ where v_0 is the largest solution to the homogeneous Dirichlet boundary value problem of (1.1) and $v_0 > 0$ in M°. We also give an equivalent description between the existence of a metric of positive scalar curvature in the conformal class of (M, g) and $\inf_{u \in C^1(M)-\{0\}} Q(u) > -\infty$ when (M, g) is smooth, provided that the positive mass theorem holds, where Q is the energy functional (see (3.2)) of the second type Escobar–Yamabe problem.

Keywords Conformal flow · Initial-boundary value problem · Loewner–Nirenberg problem · Blowing up at boundary · Monotonicity

Mathematics Subject Classification Primary 53C44; Secondary 35K55 · 35R01 · 53C21

Research partially supported by the National Natural Science Foundation of China No. 11701326 and the Young Scholars Program of Shandong University 2018WLJH85.

✉ Gang Li
runxing3@gmail.com

1 Department of Mathematics, Shandong University, Jinan, Shandong Province, China

1 Introduction

In the well-known paper [34], Loewner and Nirenberg studied the blowing up boundary value problem

$$\Delta u = \frac{1}{4}n(n-2)u^{\frac{n+2}{n-2}}, \text{ in } \Omega,$$
$$u(x) \to \infty, \text{ as } x \to \partial\Omega,$$

with Ω a bounded domain of C^2 in $\mathbb{R}^n$. They proved that there exists a unique positive solution u to this problem, and there exists a constant $C > 0$ depending on the domain Ω such that

$$|\text{dist}(x, \partial\Omega)^{\frac{n-2}{2}} u - 1| \leq C\text{dist}(x, \partial\Omega)$$

near the boundary, where $\text{dist}(x, \partial\Omega)$ is the distance of x to $\partial\Omega$. This is equivalent to seeking the conformal metric $g = u^{\frac{4}{n-2}}\delta$ with δ the Euclidean metric on Ω has constant scalar curvature $R_g = -n(n-1)$.

In [3] and [4], Aviles and McOwen generalized the Loewner–Nirenberg problem to compact Riemannian manifolds (M, g) with boundary. Denote M° to be the interior of M. In particular, they considered the blowing up Dirichlet boundary value problem

$$\frac{4(n-1)}{n-2}\Delta u - R_g u - n(n-1)u^{\frac{n+2}{n-2}} = 0, \text{ in } M^\circ, \tag{1.1}$$
$$u(p) \to \infty, \text{ as } p \to \partial M. \tag{1.2}$$

We call (1.1)–(1.2) the Loewner–Nirenberg problem on (M, g). Using classical variational method they obtained a sequence of solutions to (1.1) with enlarging Dirichlet boundary data that go to infinity, and using maximum principle and an integral type weak Harnack inequality, they obtained the existence of the unique solution u to (1.1)–(1.2) and analyzed the asymptotic behavior of u near the boundary. For regularity of the Loewner–Nirenberg metric in the conformal class of a smooth compact manifold (M^n, g) with boundary, an nice expansion of the solution near the boundary is given in [1,38]. Recently, Xumin Jiang and Qing Han developed a type of weighted Schauder estimates near the boundary, see [25] (see also [30]), which fits in this expansion well, and for expansion of the solution near the boundary for manifolds with corners on the boundary see [27] and for more references on this topic one is referred to [26].

In this article, we derive two flow approaches to the Loewner–Nirenberg problem. Indeed, we introduce the Cauchy–Dirichlet problems to a direct scalar curvature flow (see (1.3)–(1.5)) and the Yamabe flow (1.6) on a compact Riemannian manifold (M, g) with boundary.

Let (M^n, g) be a compact Riemannian manifold with boundary. Let M° be the interior of M. Define the function spaces

$C^{k+\alpha,m+\beta}_{loc}(\partial M\times[0,+\infty))=\{u\in C^{k+\alpha,m+\beta}(\partial M\times[0,T]$ for any $T>0\}$, and
$C^{k+\alpha,m+\beta}_{loc}(M\times[0,+\infty))=\{u\in C^{k+\alpha,m+\beta}(M\times[0,T])$ for any $T>0\}$.

Consider the Cauchy–Dirichlet problem

$$u_t=\frac{4(n-1)}{n-2}\Delta u-R_g u-n(n-1)u^{\frac{n+2}{n-2}},\quad \text{in } M\times[0,+\infty), \tag{1.3}$$

$$u(p,0)=u_0(p),\quad p\in M, \tag{1.4}$$

$$u(q,t)=\phi(q,t),\quad q\in\partial M, \tag{1.5}$$

with $\phi\to\infty$ as $t\to\infty$, which is called *the direct flow* in this paper. For the direct flow, we first derive the long time existence of the flow for general initial data, see Lemma 2.4. If the boundary data $\phi\to\infty$ uniformly of certain speed (2.4) as $t\to\infty$, we obtain an asymptotic blowing up lower bound estimates near the boundary, see Lemma 2.3. Together with the interior upper bound estimates (see Lemma 2.2, in comparison with the local upper bound estimates in [4]) and the Harnack inequality, we obtain the uniform upper and lower bound of u on any given compact subset of M°. The convergence of the flow is the main part of the discussion. We have the following theorem for the case $R_g\geq 0$.

Theorem 1.1 *Let* (M,g) *be a smooth compact manifold with boundary such that* $R_g\geq 0$. *Assume two positive functions* $u_0\in C^{2,\alpha}(M)$ *and* $\phi\in C^{2,\alpha}_{loc}(\partial M\times[0,+\infty))$ *satisfy the compatible condition* (2.1) *on* $\partial M\times\{0\}$. *Moreover, assume* $\phi_t\geq 0$ *for* $t\geq 0$, ϕ *satisfies* (2.4) *as* $t\to\infty$ *and* $\liminf_{t\to\infty}\inf_{\partial M}\phi=\infty$. *Then there exists a unique solution* u *to the Cauchy–Dirichlet problem* (1.3)–(1.5), *converging in* $C^2_{loc}(M^\circ)$ *to a solution* u_∞ *to the Loewner–Nirenberg problem* (1.1)–(1.2) *as* $t\to+\infty$. *Moreover, there exists a constant* $C>0$ *such that*

$$\frac{1}{C}\Big(x+\big(\inf_{p\in\partial M}\phi(p,t)\big)^{\frac{2}{2-n}}\Big)^{\frac{2-n}{2}}-C\leq u\leq Cx^{\frac{2-n}{2}}$$

near the boundary ∂M, *where* x *is the distance function to the boundary.*

To achieve that, when the initial data is a subsolution to (1.1) and ϕ is increasing in a certain speed to infinity as $t\to\infty$, we first show that the solution is increasing and converges to the solution of the Loewner–Nirenberg problem, see Lemma 2.6 to Proposition 2.5. In particular, $u(\cdot,t)$ is a sub-solution to (1.1) for each $t\geq 0$. Then for general positive initial data, we can first solve a Cauchy–Dirichlet problem with smaller initial data which is a subsolution to (1.1), and hence the solution u_1 is increasing and converges to the solution of the Loewner–Nirenberg problem; and then we use u_1 to give the lower bound of the flow with general initial data by maximum principle, and we use Hamilton's technique in [24] to derive the convergence of the flow. We provide examples of the compact manifolds with boundary with positive scalar curvature in Sect. 3, and based on the positive mass theorem, we obtain that on a smooth compact manifold (M,g) with boundary, there exists a positive scalar curvature metric in the conformal class if and only if $\inf_{u\in C^1(M)-\{0\}}Q(u)>-\infty$, where Q (see (3.2)) is the energy functional of the second type Escobar–Yamabe problem (3.1), see Theorem 3.2.

For a general conformal class $(M, [g])$, we have

Theorem 1.2 *Let (M, g) be a compact Riemannian manifold with boundary of $C^{4,\alpha}$. Let $u_0 \in C^{2,\alpha}(M)$ be any positive function such that $u_0 > v_0$ on M, where v_0 is the largest solution to he homogeneous Dirichlet boundary value problem of the Yamabe equation (A.1)–(A.2). Then there exists a direct flow $g(t)$ starting from $g_0 = u_0^{\frac{4}{n-2}} g$ and converges in $C^2_{loc}(M^\circ)$ to $g_\infty = u_{LN}^{\frac{4}{n-2}} g$, where u_{LN} the solution to the Loewner–Nirenberg problem (1.1)–(1.2).*

Notice that $v_0 > 0$ in M° when $\lambda_1(L_g) < 0$ (see Appendix A), where $\lambda_1(L_g)$ is the first eigenvalue of the Dirichlet boundary value problem of the conformal Laplacian operator $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$; while $v_0 = 0$ by maximum principle when $R_h \geq 0$ for some conformal metric $h \in [g]$. When (M, g) is smooth and the positive mass theorem holds, if $\lambda_1(L_g) > 0$, there exists $h \in [g]$ such that $R_h > 0$, see Sect. 3. To prove Theorem 1.2, we first take a conformal metric g with $R_g = -n(n-1)$ as the background metric and show that the flow is monotone and converges to the Loewner–Nirenberg metric when we choose the initial data $u_0 = 1$ and increasing boundary data in a certain speed; while for $u_0 > v_0$, we construct a compatible boundary data increasing in $t \geq 0$ ina certain speed, and then use a solution to the Cauchy–Dirichlet problem (1.3)–(1.5) with smaller boundary data and smaller initial data which is a solution to (1.1) to give a lower bound, and we use Hamilton's technique in [24] again for the upper bound control to conclude the convergence of the flow.

Yamabe flow is studied as an alternative approach to the Yamabe problem on closed manifolds, see [8,9,15,49,55], etc. It is also used in the study of general prescribed scalar curvature equation, see [12,53], etc. For Yamabe flow on manifolds with boundary on the Neumann type boundary problems posed by Escobar, see [10,11], etc. On complete non-compact Riemannian manifolds, there are also many works on the long time existence and convergence of the Yamabe flow, see [14,37], etc. On compact manifolds with incomplete edge singularities Yamabe flow is studied in [5], etc. see also [48,50] and the references there for Yamabe flow on incomplete manifolds. In [47], an instantaneously complete Yamabe flow was derived in the conformal class of the hyperbolic space. In [48], the author studied the relationship between the dimension of the singular sub-manifold and instantaneous completeness and incompleteness of the flow.

Notice that the Yamabe flow is conformally covariant and it makes no difference which metric in the conformal class is chosen as the background metric of the flow. We take a conformal metric g with $R_g = -n(n-1)$ in the conformal class $(M, [g])$ as the background metric. Now we introduce the Cauchy–Dirichlet problem of the Yamabe flow

$$
\begin{aligned}
&(u^{\frac{n+2}{n-2}})_t = \frac{(n-1)(n+2)}{n-2}\left(\Delta_g u + \frac{n(n-2)}{4}(u - u^{\frac{n+2}{n-2}})\right), \\
&u(q, 0) = u_0(q), \ \ q \in M, \\
&u(q, t) = \phi(q, t), \ \ (q, t) \in \partial M \times [0, +\infty),
\end{aligned}
\tag{1.6}
$$

where $\phi \to \infty$ as $t \to \infty$, to get a natural connection between a compact metric g on M with the complete Loewner–Nirenberg metric in the conformal class. For the convergence, we have that

Theorem 1.3 *Let (M^n, g) be a compact Riemannian manifold with boundary of $C^{4,\alpha}$ and $R_g = -n(n-1)$. Let $u_0 \in C^{4,\alpha}(M)$ with $u_0 \geq 1$ be a subsolution of the equation (1.1) and satisfies*

$$L_g(\mu) \geq 0 \tag{1.7}$$

at the points $q \in \partial M$ such that $\mu = 0$, where μ and $L_g(\mu)$ are defined in (4.5) and (4.4). Assume $\phi \in C_{loc}^{4+\alpha, 2+\frac{\alpha}{2}}(M \times [0, \infty))$ satisfies the compatible condition (4.3)–(4.4) and $\phi_t \geq 0$ on $\partial M \times [0, \infty)$. Moreover, assume $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$ and ϕ satisfies (4.7)–(4.8) as $t \to \infty$. Then there exists a unique positive solution u to (1.6) on $M \times [0, +\infty)$ with $u \in C_{loc}^{4+\alpha, 2+\frac{\alpha}{2}}(M \times [0, \infty))$, and $u \to u_{LN}$ in $C^4_{loc}(M^\circ)$ as $t \to \infty$, where u_{LN} is the solution to the Loewner–Nirenberg problem (1.1)–(1.2).

Notice that the condition (1.7) is to guarantee the $C^{4+\alpha, 2+\frac{\alpha}{2}}$ regularity of the solution u at $\partial M \times \{0\}$ and the condition $\phi_t \geq 0$ for $t \geq 0$, and it holds automatically if u_0 is a strict subsolution to (1.1) in a neighborhood of ∂M, or u_0 is a solution to (1.1) in a neighborhood of ∂M. It is clear that $\phi = \log(t)$, t, t^2, e^t and te^t satisfy (4.7). The strategy of the proof of this theorem is similar as that of Theorem 1.2, but more is involved because of the nonlinear term on the left hand side of the Yamabe flow equation. Notice that we do not show the convergence of the Yamabe flow for general initial data, because when using Hamilton's technique in [24] to get the upper bound of u, we are not able to show that $\limsup_{t\to\infty} \sup_{M^\circ}(u - u_{LN}) \leq 0$, although $\sup_{M^\circ}(u(\cdot, t) - u_{LN}(\cdot))$ is decreasing in t, where u_{LN} is the solution to the Loewner–Nirenberg problem (1.1)–(1.2).

The Cauchy-Dirchlet problem of the Yamabe flow is generalized to the Cauchy-Dirchlet problem of σ_k-Ricci flow in [31], i.e., a flow approach to the generalized Loewner–Nirenberg problem of the fully nonlinear equations in [22] and [23], where the fact the sub-solution property is preserving along the flow also plays an important role for the convergence of the flow.

It would be interesting to know whether the direct flow converges to the solution to Loewner–Nirenberg problem when $u_0 < v_0$ somewhere in M° when $\lambda_1(L_g) < 0$.

2 A Direct Flow

Let (M^n, g) be a smooth compact Riemannian manifold with boundary ∂M, with $g \in C^{4,\alpha}$. We consider the direct flow, i.e. the Cauchy–Dirichlet problem (1.3)–(1.5) with $\lim_{t\to+\infty} \phi(q, t) = +\infty$, uniformly. To guarantee the solution u is in $C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T_0])$ for some $T_0 > 0$ and $0 < \alpha < 1$, we need the compatible condition

$$\begin{aligned} u_0(p) &= \phi(p, 0), \\ \phi_t(p, 0) &= \frac{4(n-1)}{n-2}\Delta u_0(p) - R_g u_0(p) - n(n-1) u_0(p)^{\frac{n+2}{n-2}} \quad \text{for } p \in \partial M, \end{aligned} \tag{2.1}$$

with $u_0 \in C^{2,\alpha}(M)$ and $\phi \in C^{2,\alpha}(\partial M \times [0,T])$ for all $T > 0$. Moreover, in order that $u \in C^{4+\alpha,2+\frac{\alpha}{2}}(M \times [0,T_0])$, we need the additional condition

$$\phi_{tt}(p,0) = L(v)(p) \quad \text{for } p \in \partial M, \tag{2.2}$$

with $u_0 \in C^{4,\alpha}(M)$ and $\phi \in C^{4,\alpha}(\partial M \times [0,T])$ for all $T > 0$, where

$$v = \frac{4(n-1)}{n-2}\Delta u_0 - R_g u_0 - n(n-1)u_0^{\frac{n+2}{n-2}}$$

on M and L is a linear operator such that

$$L(\varphi) = \frac{4(n-1)}{n-2}\Delta\varphi - R_g\varphi - \frac{n(n-1)(n+2)}{n-2}u_0^{\frac{4}{n-2}}\varphi$$

for any $\varphi \in C^2(M)$.

For later use, we now present a well-known weak Harnack inequality for parabolic inequalities, see in [32] or Peter Li's online lecture notes on geometric analysis for instance.

Lemma 2.1 *Assume $\bar{B}_r(p)$ is a closed geodesic r-ball in a complete Riemannian manifold (M,g) with $r \le 1$. Let $u \in C^{2,1}(M \times [0,T_1])$ be a positive function such that*

$$u_t \le \frac{4(n-1)}{(n-2)}\Delta u + C_0 u,$$

for some constant $C_0 > 0$. Assume that there exists a constant $C_S > 0$, such that we have the Sobolev inequality

$$\frac{1}{|B_r(p)|}\int_{B_r(p)} |\nabla\phi|^2 \ge C_S r^{-2}\Big(\frac{1}{|B_r(p)|}\int_{B_r(p)} \phi^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}},$$

for each $\phi \in C_0^1(B_r(p))$. Then for $\theta \in (0,1)$ and $T \in (T_0,T_1)$, there exists a constant $C_1 > 0$ depending on C_0, such that

$$\begin{aligned}
&\sup_{q \in B_{\theta r}(p),\, t \in [T,T_1]} |u(q)| \\
&\le C_1 C_S^{-\frac{n-2}{4}}\left((1-\theta)^{-2}r^{-2} + (T-T_0)^{-1}\right)^{\frac{n^2-4}{4n}} r^{\frac{n-2}{2}}(T_1-T_0)^{\frac{n-2}{2n}} \\
&\quad \times \Big(\frac{1}{|T_1-T_0| \times |B_r(p)|}\int_{T_0}^{T_1}\int_{B_r(p)} |u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}}.
\end{aligned}$$

Remark. For a complete Riemannian manifold (M^n,g), if $Ric_g \ge -k(n-1)g$ with some constant $k > 0$, and assume that there exists a constant $C_v > 0$ such that for a geodesic ball $B_r \subseteq M$, the volume ratio satisfies

$$\frac{|B_r|}{|B_r^0|} \geq C_v, \tag{2.3}$$

where $|B_r^0|$ is the volume of a geodesic ball of radius r in the space form of sectional curvature $-k$, then the Sobolev inequality holds on B_r with C_S depending on k and C_v. For a compact manifold (M, g) with boundary of $C^{2,\alpha}$, such constants are uniform for each geodesic ball in the interior. Denote M° the interior of M, for each $p \in M^\circ$, we can also use classical parabolic theory in Euclean domain to get the weak Harnack inequality. Indeed, let $r = \frac{1}{2}\min\{r_0, \operatorname{dist}(p, \partial M)\}$ with r_0 the injectivity radius at p. In $B_{2r}(p)$, using the geodesic normal coordinates, we can write the equality as an parabolic inequality in Euclidean domain $B_r(0)$, and apply the weak Harnack theorem in [33] to get the same control.

We start with a parabolic analog to Aviles-McOwen's local upper bound estimates in [4] on the solutions to (1.3).

Lemma 2.2 *Assume $u > 0$ is a solution to (1.3) on $M \times [0, T_1)$ with $T_1 > \epsilon_0$ for some $2 > \epsilon_0 > 0$. There exists a uniform constant $C_3 > 0$ depending on $\epsilon_0 > 0$ but independent of T_1 and r, such that for each closed geodesic ball $\bar{B}_{2r}(p) \subseteq M^\circ$, we have*

$$u(q, t) \leq C_3 r^{-\frac{(n-2)}{2}},$$

for $q \in \bar{B}_r(p)$ with $r \leq \min\{1, \sqrt{\epsilon_0}\}$, and $T_1 > t \geq \frac{\epsilon_0}{2}$.

Proof Pick up a function $\varphi(q, t) = \xi(q)\eta(t) \in C^2(M \times [0, T_1))$ to be determined later. Let $\xi \geq 0$ be a cut off function with compact support $B_{2r}(p)$, so that $\xi(q) = 1$ for $q \in B_r(p)$, $0 \leq \xi(q) \leq 1$ for $q \in B_{2r}(p)$ and there exists a constant $C > 0$ independent of $p \in M^\circ$ and r such that $|\nabla \xi| \leq C r^{-1}$ in $B_{2r}(p)$. Multiply $u\varphi^\alpha$ on both sides of (1.3) and do integration on M, we have

$$\int_M u u_t \varphi^\alpha dV_g = \int_M \left[\frac{4(n-1)}{n-2} u \Delta u \varphi^\alpha - R_g u^2 \varphi^\alpha - n(n-1) u^{\frac{2n}{n-2}} \varphi^\alpha\right] dV_g$$

Integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_M u^2 \varphi^\alpha dV_g\Big) - \frac{\alpha}{2}\int_M u^2 \varphi^{\alpha-1}\varphi_t dV_g \\
&= \int_M \Big[-\frac{4(n-1)}{n-2}|\nabla u|^2 \varphi^\alpha - \frac{4(n-1)}{n-2}\alpha u \varphi^{\alpha-1}\nabla u \cdot \nabla \varphi \\
&\quad - R_g u^2 \varphi^\alpha - n(n-1) u^{\frac{2n}{n-2}} \varphi^\alpha\Big] dV_g \\
&\leq \int_M \Big[u^2\Big(\frac{(n-1)}{n-2}\alpha^2 \varphi^{\alpha-2}|\nabla \varphi|^2 - R_g \varphi^\alpha\Big) - n(n-1) u^{\frac{2n}{n-2}} \varphi^\alpha\Big] dV_g,
\end{aligned}$$

where the last inequality is by Cauchy inequality. Therefore,

$$\frac{1}{2}\frac{d}{dt}\Big(\int_M u^2\varphi^\alpha dV_g\Big) \le \int_M \Big[\Big(\frac{\alpha}{2}\varphi^{\alpha-1}\varphi_t + \frac{(n-1)}{n-2}\alpha^2\varphi^{\alpha-2}|\nabla\varphi|^2 - R_g\varphi^\alpha\Big)u^2 - n(n-1)u^{\frac{2n}{n-2}}\varphi^\alpha\Big]dV_g.$$

Now for any T such that $T_1 > T \ge \frac{\epsilon_0}{2}$, we assume $1 \ge \eta \ge 0$ with $\eta(t) = 0$ for $t \le T - \frac{r^2}{4}$, $\eta = 1$ for $t \ge T - \frac{r^2}{8}$ and $|\eta'(t)| \le \frac{12}{r^2}$ for $t > 0$. Integrate the above inequality on $t \in [T - \frac{r^2}{4}, T]$ to have

$$\begin{aligned}
0 &\le \frac{1}{2}\Big(\int_M u^2\varphi^\alpha dV_g\Big)\Big|_{t=T}\\
&\le \int_{T-\frac{r^2}{4}}^T \int_M \Big[\Big(\frac{\alpha}{2}\varphi^{\alpha-1}\varphi_t + \frac{(n-1)}{n-2}\alpha^2\varphi^{\alpha-2}|\nabla\varphi|^2 - R_g\varphi^\alpha\Big)u^2 - n(n-1)u^{\frac{2n}{n-2}}\varphi^\alpha\Big]dV_g dt.
\end{aligned}$$

Therefore, by Hölder inequality,

$$\begin{aligned}
&n(n-1)\int_{T-\frac{r^2}{4}}^T \int_M u^{\frac{2n}{n-2}}\varphi^\alpha dV_g dt\\
&\le \int_{T-\frac{r^2}{4}}^T \int_M \Big[\Big(\frac{\alpha}{2}\varphi^{\alpha-1}\varphi_t + \frac{(n-1)}{n-2}\alpha^2\varphi^{\alpha-2}|\nabla\varphi|^2 - R_g\varphi^\alpha\Big)u^2\Big]dV_g dt\\
&\le \Big[\int_{T-\frac{r^2}{4}}^T \int_M u^{\frac{2n}{n-2}}\varphi^\alpha dV_g dt\Big]^{\frac{n-2}{n}}\\
&\quad \Big[\int_{T-\frac{r^2}{4}}^T \int_M [\Big(\frac{\alpha}{2}\varphi\varphi_t + \frac{(n-1)}{n-2}\alpha^2|\nabla\varphi|^2 - R_g\varphi^2\Big)\varphi^{\alpha-2-\frac{n-2}{n}\alpha}]^{\frac{n}{2}} dV_g dt\Big]^{\frac{2}{n}}\\
&= \Big[\int_{T-\frac{r^2}{4}}^T \int_M u^{\frac{2n}{n-2}}\varphi^\alpha dV_g dt\Big]^{\frac{n-2}{n}}\\
&\quad \Big[\int_{T-\frac{r^2}{4}}^T \int_M \Big[\Big(\frac{\alpha}{2}\varphi\varphi_t + \frac{(n-1)}{n-2}\alpha^2|\nabla\varphi|^2 - R_g\varphi^2\Big)\varphi^{-2+\frac{2}{n}\alpha}\Big]^{\frac{n}{2}} dV_g dt\Big]^{\frac{2}{n}}.
\end{aligned}$$

Taking $\alpha = n$, we have

$$\begin{aligned}
&\Big(\int_{T-\frac{r^2}{8}}^T \int_{B_r(p)} u^{\frac{2n}{n-2}} dV_g dt\Big)^{\frac{2}{n}}\\
&\le \left(\int_{T-\frac{r^2}{4}}^T \int_M u^{\frac{2n}{n-2}}\varphi^n dV_g dt\right)^{\frac{2}{n}}\\
&\le \frac{1}{n(n-1)}\Big[\int_{T-\frac{r^2}{4}}^T \int_M \Big(\frac{n}{2}\varphi\varphi_t + \frac{(n-1)}{n-2}n^2|\nabla\varphi|^2 - R_g\varphi^2\Big)^{\frac{n}{2}} dV_g dt\Big]^{\frac{2}{n}},
\end{aligned}$$

where the right hand side is bounded from above independent of u, p, T and r. Recall that

$$u_t \leq \frac{4(n-1)}{(n-2)}\Delta_g u - R_g u$$

on $M \times [0, T_1)$. Hence on $B_r(p) \times [T - \frac{r^2}{4}, T]$, combining with the weak Harnack type inequality on $B_{2r}(q) \times [T - \frac{r^2}{4}, T]$ with $\theta = \frac{1}{2}$ and by the choice of φ, we have

$$\sup_{q \in B_r(p), T - \frac{r^2}{8} \leq t \leq T} u(q, t) \leq C_3 r^{-\frac{n-2}{2}},$$

for $T_1 > T \geq \frac{\epsilon_0}{2}$, with a constant $C_3 = C_3(\epsilon_0) > 0$ independent of $p \in M^\circ$, u, T_1 and r. In particular, for any compact subset in M° and $t \geq \frac{\epsilon_0}{2}$, u has a uniform upper bound independent of t. □

For the Cauchy–Dirichlet problem (1.3)–(1.5) with $\phi \to \infty$ uniformly on ∂M, we now estimate the asymptotic behavior of the solution u near the boundary as $t \to \infty$.

Lemma 2.3 *Assume that the positive function $u \in C^{2,1}(M \times [0, \infty))$ is a solution to the Cauchy–Dirichlet problem (1.3)–(1.5) with $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$. Moreover, assume ϕ satisfies that*

$$\begin{aligned} &\phi^{\frac{n}{2-n}} \frac{d\phi}{dt} \to 0, \quad \phi^{\frac{n-1}{2-n}} |\nabla_g \phi| \to 0, \\ &\phi^{\frac{n}{2-n}} |\nabla_g^2 \phi| \to 0, \quad \textit{uniformly on } \partial M, \quad \textit{as } t \to +\infty. \end{aligned} \tag{2.4}$$

Then there exist constants $C > 0$, $t_1 > 0$ large and $x_1 > 0$ small such that

$$u \geq \frac{1}{C}\big(x + (\inf_{p \in \partial M} \phi(p, t))^{\frac{2}{2-n}}\big)^{\frac{2-n}{2}} - C,$$

for $x \leq x_1$ and $t \geq t_1$, where $x = x(p)$ is the distance function to the boundary.

Proof Recall that there exists a conformal metric h with $g = f^{\frac{4}{n-2}} h$ so that $R_h > 0$ in a neighborhood V of the boundary ∂M, where $f \in C^{2,\alpha}(M)$ is a positive function, see in Lemma 3.1 in [1]. Indeed, for that one can use the formula of the scalar curvature under conformal change and do Taylor expansion of f in the direction of x in a small neighborhood of the boundary and just take f to be a quadratic polynomial of x with coefficients functions on ∂M. Let x be the distance function to the boundary ∂M under h. Take $x_1 > 0$ small so that the boundary neighborhood $U = \{0 \leq x \leq x_1\}$ lies in V and hence $R_h > 0$, and U is diffeomorphic to $\partial M \times [0, x_1]$ under the exponential map $F : \partial M \times [0, x_1] \to U$, where $F(q, x) = \mathrm{Exp}_q^h(x) \in U$ is on the geodesic starting from $q \in \partial M$ in (V, h) in the inner normal direction of ∂M of distance x to q. We define a function $\varphi \in C^{2,\alpha}_{loc}(U \times [0, +\infty))$ such that

$$\varphi(\mathrm{Exp}_q^h(x),t)=c\left[(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{2-n}{2}}-(x_1+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{2-n}{2}}\right] \tag{2.5}$$

for $(q,x,t)\in\partial M\times[0,x_1]\times[0,+\infty)$, where $c>0$ is a small constant to be determined. Let $\tilde{u}=fu$. Then we have

$$\begin{aligned}
&\tilde{u}_t=f^{-\frac{4}{n-2}}\left[\frac{4(n-1)}{n-2}\Delta_h\tilde{u}-R_h\tilde{u}-n(n-1)\tilde{u}^{\frac{n+2}{n-2}}\right],\ \text{ in } M\times[0,+\infty),\\
&\tilde{u}(p,0)=f(p)u_0(p),\ \ p\in M,\\
&\tilde{u}(q,t)=f(q)\phi(q,t),\ \ q\in\partial M.
\end{aligned}$$

Note that h has the orthogonal decomposition $h=dx^2+h_x$ on U with $h_0=h\big|_{\partial M}$. Then by direct calculation, one has

$$\begin{aligned}
\Delta_h\varphi&=\partial_x^2\varphi+\frac{1}{2}H_x\partial_x\varphi+\Delta_{h_x}\varphi,\\
\varphi_t&=cf^{\frac{2}{n-2}}[(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}}-(x_1+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}}]\phi^{\frac{n}{2-n}}\frac{d\phi}{dt}\\
\partial_x\varphi&=\frac{2-n}{2}c(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}},\quad \partial_x^2\varphi=\frac{n(n-2)}{4}c(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n-2}{2}},\\
|\nabla_{h_x}\varphi|&\le c(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}}(f\phi)^{\frac{n}{2-n}}|\nabla_{h_x}(f\phi)|,\\
|\nabla_{h_x}^2\varphi|&\le c(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}}\Big[(f\phi)^{\frac{n}{2-n}}|\nabla_{h_x}^2(f\phi)|+\frac{n}{n-2}(f\phi)^{\frac{2n-2}{2-n}}|\nabla_{h_x}(f\phi)|^2\\
&\quad+\frac{n}{n-2}(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{-1}(f\phi)^{\frac{2n}{2-n}}|\nabla_{h_x}(f\phi)|^2\Big],
\end{aligned} \tag{2.6}$$

where H_s is the mean curvature of the hypersurface $\Sigma_s=\{x=s\}$. Recall that $\phi\to\infty$ uniformly on ∂M as $t\to\infty$. By the assumption (2.4), we have that there exists $t_1\ge 0$ such that for $t\ge t_1$,

$$\varphi_t\le f^{-\frac{4}{n-2}}\left[\frac{4(n-1)}{(n-2)}\Delta_h\varphi-R_h\varphi-n(n-1)\varphi^{\frac{n+2}{n-2}}\right]$$

on U, since all other terms are lower order terms as $t\to\infty$ in comparison with the terms

$$\begin{aligned}
&\frac{4(n-1)}{(n-2)}\Delta_h\varphi-n(n-1)\varphi^{\frac{n+2}{n-2}}\\
&\quad=n(n-1)c(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n-2}{2}}(1+o(1))\\
&\qquad-n(n-1)c^{\frac{n+2}{n-2}}[(x+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{2-n}{2}}-(x_1+(f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{2-n}{2}}]^{\frac{n+2}{n-2}},
\end{aligned} \tag{2.7}$$

with $c>0$ small and the term $o(1)\to 0$ uniformly on U as $t\to+\infty$. Let $v=\tilde{u}-\varphi$. Therefore,

$$v_t \geq f^{\frac{-4}{n-2}} \left[\frac{4(n-1)}{(n-2)} \Delta_h v - \left(R_h + n(n-1)\xi(p,t) \right) v \right]$$

on $U \times [t_1, +\infty)$, where $\xi(p,t) = \frac{u^{\frac{n+2}{n-2}} - \varphi^{\frac{n+2}{n-2}}}{u-\varphi}$ when $u \neq \varphi$, and $\xi = \frac{n+2}{n-2} u^{\frac{4}{n-2}}$ otherwise, and hence, $\xi(p,t) > 0$ on $U \times [t_1, +\infty)$. Since $\tilde{u}$ is continuous on $U \times \{t = t_1\}$, we choose $c > 0$ small so that $\tilde{u} > \varphi$ on $U \times \{t = t_1\}$. On the other hand, $\varphi = 0 < \tilde{u}$ on $\{x = x_1\} \times [t_1, \infty)$ and by definition $\varphi \leq \tilde{u}$ on ∂M and hence, $v \geq 0$ on $\partial U \times [t_1, +\infty) \bigcup U \times \{t_1\}$. Recall that $R_h > 0$ on U, and hence by maximum principle,

$$v \geq 0$$

on $U \times [t_1, +\infty)$. Therefore,

$$u \geq f^{-1} \varphi$$

on $U \times [t_1, +\infty)$. Combining this lower bound estimate of u near the boundary with the Harnack inequality (2.9) based on the uniform interior upper bound estimates in Lemma 2.2, and also using finite cover of geodesic balls, on any compact sub-domain of M° we obtain uniform positive lower bound of u for $t \in [t_1, +\infty)$. □

Now we give the long time existence of the flow.

Lemma 2.4 *There exists a unique solution* $u \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, \infty))$ *to the Cauchy–Dirichlet boundary* (1.3)–(1.5) *with positive functions* $u_0 \in C^{2,\alpha}(M)$ *and* $\phi \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(\partial M \times [0, \infty))$ *satisfying* (2.1)*. Moreover, if in addition,* $u_0 \in C^{4,\alpha}(M)$ *and* $\phi \in C_{loc}^{4+\alpha, 2+\frac{\alpha}{2}}(\partial M \times [0, \infty))$ *satisfy* (2.2)*, then* $u \in C_{loc}^{4+\alpha, 2+\frac{\alpha}{2}}(M \times [0, \infty))$*. Moreover, if* $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$ *and* ϕ *satisfies* (2.4) *as* $t \to \infty$*, then there exists constants* $C > 0$*,* $t_2 > 0$ *large and* $x_1 > 0$ *small such that*

$$u \geq \frac{1}{C} \left(x + \left(\inf_{p \in \partial M} \phi(p,t) \right)^{\frac{2}{2-n}} \right)^{\frac{2-n}{2}} - C, \tag{2.8}$$

for $x \leq x_1$ *and* $t \geq t_2$*, where* $x = x(p)$ *is the distance function to the boundary, and hence, for each compact subset* $F \subseteq M^\circ$*, there exists* $C = C(F) > 0$ *such that*

$$u(q,t) \geq C$$

for each $(q,t) \in F \times [0, \infty)$*. Moreover, for each compact subset* $F \subseteq M^\circ$*, there exists* $m = m(F) > 0$ *independent of* u_0 *and* ϕ *such that*

$$\liminf_{t\to\infty} \inf_{q \in F} u(q,t) \geq m,$$

provided that $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$.

Proof Since (1.3) is parabolic, by the compatible condition and regularity of the positive functions u_0 and ϕ, standard implicit function theorem yields the existence of a unique positive solution u to (1.3)–(1.5) on $M \times [0, T_0)$ for some $T_0 > 0$ with u satisfying $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(M \times [0, T])$ for each $0 < T < T_0$ when $u_0 \in C^{2,\alpha}(M)$ and $\phi \in C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(M \times [0, \infty))$ satisfying (2.1); moreover, $u \in C^{4+\alpha,2+\frac{\alpha}{2}}(M \times [0, T])$ for each $0 < T < T_0$ when $u_0 \in C^{4,\alpha}(M)$ and $\phi \in C^{4+\alpha,2+\frac{\alpha}{2}}$ satisfy (2.2) in addition. We assume T_0 is the maximum time for the existence of the positive solution u on $[0, T_0)$. By maximum principle, we have the upper bound

$$\sup_{q\in M,\, t\in[0,T)} u \le \max\left\{\sup_{q\in\partial M,\, t\in[0,T)} \phi,\ \sup_{q\in M} u_0(q),\ \Big(\frac{1}{n(n-1)}\max\{-\inf_{q\in M} R_g(q), 0\}\Big)^{\frac{n-2}{4}}\right\},$$

for any $0 < T < T_0$, which is uniformly bounded on $M \times [0, T_0)$. Then standard Schauder theory derives that there exists $C = C(T_0) > 0$ such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(M\times[0,T])} \le C$$

in the $C^{2+\alpha,1+\frac{\alpha}{2}}$ setting Cauchy–Dirichlet problem, for each $0 < T < T_0$; while

$$\|u\|_{C^{4+\alpha,2+\frac{\alpha}{2}}(M\times[0,T])} \le C$$

in the $C^{4+\alpha,2+\frac{\alpha}{2}}$ setting Cauchy–Dirichlet problem, for each $0 < T < T_0$. To establish the lower bound estimates, for any $q \in M^\circ$, assume $0 < r < \frac{\mathrm{dist}(q,\partial M)}{2}$ and $r < 1$. Then by the Harnack inequality, there exists a constant $C > 0$ depending on $B_{2r}(q)$ but independent of T such that

$$\sup_{B_r(q)\times[T-\frac{3r^2}{4},T-\frac{r^2}{2}]} u \le C \inf_{B_r(q)\times[T-\frac{r^2}{4},T]} u \tag{2.9}$$

for each $r^2 < T < T_0$. By Lemma 2.2, $|R_g + n(n-1)u^{\frac{4}{n-2}}|$ is uniformly bounded in $B_{2r}(q)$ (independent of T_0) and hence, the constant C in (2.9) is independent of T_0. Recall that $\phi > 0$ on $\partial M \times [0, \infty)$. Therefore, u can be extended to a positive solution of $C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(M \times [0, \infty))$ (resp. of $C_{loc}^{4+\alpha,2+\frac{\alpha}{2}}(M \times [0, \infty))$).

Assume that $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$ and ϕ satisfies (2.4) as $t \to \infty$. Then by Lemma 2.3 we have that there exists $x_1 > 0$, $C > 0$ and $t_2 > 0$ such that (2.8) holds in $\{0 \le x \le x_1\}$ for $t \ge t_2$. Using this uniform lower bound estimate near the boundary, combining with the above Harnack inequality and a finite geodesic ball cover of the path connecting to $\{0 \le x \le x_1\}$, we have that for each compact subset $F \subseteq M^\circ$, there exists $C = C(F) > 0$ such that

$$u(q, t) \ge C$$

for $(q, t) \in F \times [0, \infty)$, and hence by Lemma 2.2, u is uniformly bounded from above and below by positive constants in $F \times [0, \infty)$.

On any compact subset $F \subseteq M^\circ$, since the upper bound of u obtained in Lemma 2.2 is independent of u_0 and ϕ, and so is $|R_g + n(n-1)u^{\frac{4}{n-2}}|$ and hence the constant C in the Harnack inequality is independent of u_0 and ϕ. Therefore, by Lemma 2.3 there exists $m > 0$ depending on F but independent of u_0 and ϕ such that

$$\liminf_{t\to\infty} \inf_{q\in F} u(q,t) \geq m.$$

□

Now we consider the convergence of the flow. To warm up, we first consider the case $R_g \geq 0$.

Proposition 2.5 *Let (M^n, g) be a compact manifold with $g \in C^{4,\alpha}$ up to the boundary such that $R_g \geq 0$. Assume $u_0 \in C^{4,\alpha}(M)$ and $\phi \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(M \times [0,\infty))$ satisfy (2.1) and (2.2), and also $\phi_t \geq 0$ on $M \times [0,\infty)$, $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$ and ϕ satisfies (2.4) as $t \to \infty$. Moreover, assume u_0 is a positive subsolution to (1.1) and*

$$L(v)(p) \geq 0 \tag{2.10}$$

for any $p \in \partial M$ such that $v(p) = 0$, where L and v are defined in (2.2). Then we have that the Cauchy–Dirichlet problem (1.3)–(1.5) has a unique global solution u, which converges in $C^4_{loc}(M^\circ)$ to the unique solution u_∞ to the Loewner–Nirenberg problem.

Remark that for a sub-solution u_0 of (1.1), which is a strict sub-solution in a neighborhood of ∂M, the condition (2.10) disappears automatically. For instance, if $\varphi > 0$ is a subsolution of (1.1), then $\epsilon\varphi$ is a strict sub-solution for any constant $0 < \epsilon < 1$. A solution $\varphi > 0$ to (1.1) automatically satisfies (2.10). Notice that the functions $\log(t)$, t and e^t all satisfy (2.4). Also, if u_0 is a solution to (1.1) in a neighborhood of ∂M, (2.10) holds automatically. To prove this proposition, we need two lemmas.

Lemma 2.6 *Let (M, g) be a smooth compact manifold with boundary such that $R_g \geq 0$. Assume $\phi \in C^4(\partial M \times [0,+\infty))$ is a positive function with $\phi_t \geq 0$, and $u_0 \in C^4(M)$ is a positive function which is a subsolution to (1.1) on M. Assume that a positive function $u \in C^{4,2}(M \times [0,\infty))$ is a solution to the Cauchy–Dirichlet problem (1.3)–(1.5). Then u satisfies that $u_t \geq 0$ in $M^\circ \times (0,+\infty)$. That's to say, for any $t > 0$, $u(\cdot,t)$ is a subsolution to the Yamabe equation (1.1).*

Proof The proof is an application of the maximum principle to the equation satisfied by u_t. Indeed, denote $v = u_t$. It is clear that $v \in C^{2,1}(M \times [0,+\infty))$. We take derivative of t on both sides of (1.3) and (1.5) to obtain the Cauchy–Dirichlet problem

$$v_t = \frac{4(n-1)}{n-2}\Delta v - R_g v - \frac{n(n-1)(n+2)}{n-2}u^{\frac{4}{n-2}}v, \quad \text{in } M \times [0,+\infty), \tag{2.11}$$

$$v(p,0) = u_t(p,0) = \frac{4(n-1)}{n-2}\Delta u_0(p) - R_g u_0(p) - n(n-1)u_0(p)^{\frac{n+2}{n-2}} \geq 0, \quad p \in M,$$

$$v(q,t) = \phi_t(q,t) \geq 0, \quad q \in \partial M.$$

Therefore, by maximum principle, $v = u_t \geq 0$ on $M \times [0, +\infty)$. Moreover, if there exists a constant $t_1 > 0$ such that $\phi_t > 0$ for $t > t_1$, then by strong maximum principle, we have $v = u_t > 0$ in $M^\circ \times (t_1, +\infty)$. □

Proposition 2.5 is a direct consequence of the following lemma.

Lemma 2.7 *Let (M, g), u_0 and ϕ be as in Proposition 2.5. Then the unique solution u to the Cauchy–Dirichlet problem (1.3)–(1.5) converges in $C^4_{loc}(M^\circ)$ to the solution u_∞ of the Loewner–Nirenberg problem (1.1)–(1.2) as $t \to +\infty$. Moreover, there exists a constant $C > 0$ such that $\frac{1}{C}x^{\frac{2-n}{2}} \leq u_\infty \leq Cx^{\frac{2-n}{2}}$ near the boundary ∂M, where x is the distance function to the boundary.*

Proof By Lemma 2.4, the exists a unique positive solution $u \in C^{4+\alpha, 2+\frac{\alpha}{2}}_{loc}(M \times [0, \infty))$ to (1.3)–(1.5). By Lemma 2.6, $u_t \geq 0$ on $M \times [0, \infty)$. Since we have the upper and lower bound estimates on u on any compact sub-domain of M° and u is increasing pointwisely on M° as t increases, we have that u converges pointwisely in M° as $t \to +\infty$. By the Harnack inequality for $v = u_t$ in the parabolic equation (2.11), we have that $u_t \to 0$ locally uniformly and hence u converges locally uniformly in M° to a positive function u_∞. Using classical parabolic estimates, we have that $u(\cdot, t) \to u_\infty$ in $C^4_{loc}(M)$ as $t \to +\infty$. Therefore, u_∞ is a solution to (1.1) in M°. By the upper bound and lower bound estimates of u near the boundary ∂M in Lemma 2.2 and Lemma 2.3, we obtain the estimate of u_∞ near the boundary as stated in the lemma, and hence u_∞ is the unique solution to the Loewner–Nirenberg problem. This completes the proof of the lemma and Proposition 2.5. □

Proof of Theorem 1.1 By Lemma 2.4, there exists a unique positive solution $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}_{loc}(M \times [0, \infty))$ to (1.3)–(1.5).

By Lemma A.1, for any $\epsilon > 0$, there exists a unique positive solution $w \in C^{4,\alpha}(M)$ to the Dirichlet boundary value problem

$$\begin{aligned} &\frac{4(n-1)}{n-2}\Delta w - R_g w - n(n-1)w^{\frac{n+2}{n-2}} = 0,\\ &w\big|_{\partial M} = \epsilon. \end{aligned}$$

By maximum principle, $w \leq \epsilon$.

Let $\epsilon > 0$ be a small constant so that $\epsilon < \frac{1}{10n} \inf_M u_0$. It is easy to choose a boundary data $\tilde{\phi}(q, t) \in C^{4+\alpha, 2+\frac{\alpha}{2}}_{loc}(\partial M \times [0, +\infty))$ such that $\tilde{\phi} = \epsilon$ and $\tilde{\phi}_t = \tilde{\phi}_{tt} = 0$ on $\partial M \times \{0\}$, moreover, $\tilde{\phi}_t \geq 0$ and $\tilde{\phi} \leq \phi$ on $\partial M \times [0, +\infty)$. Also, we require that $\tilde{\phi} \to \infty$ and $\tilde{\phi}$ satisfies (2.4) as $t \to \infty$. Therefore, by Proposition 2.5 there exists a unique solution $\tilde{u}$ to the problem

$$\begin{aligned} &\tilde{u}_t = \frac{4(n-1)}{n-2}\Delta \tilde{u} - R_g \tilde{u} - n(n-1)\tilde{u}^{\frac{n+2}{n-2}}, \quad \text{in } M \times [0, +\infty),\\ &\tilde{u}(p, 0) = w, \quad p \in M,\\ &\tilde{u}(q, t) = \tilde{\phi}(q, t), \quad q \in \partial M, \end{aligned}$$

such that $\tilde{u}$ is increasing in t, and converges to the Loewner–Nirenberg solution u_∞. Let $v = u - \tilde{u}$. Then v satisfies

$$\begin{aligned} v_t &= \frac{4(n-1)}{n-2}\Delta v - (R_g + n(n-1)\zeta)v, \quad \text{in } M \times [0, +\infty), \\ v(p, 0) &= u_0(p) - w(p) > 0, \;\; p \in M, \\ v(q, t) &= \phi(q, t) - \tilde{\phi}(q, t) \geq 0, \;\; q \in \partial M, \end{aligned}$$

where $\zeta = \frac{u^{\frac{n+2}{n-2}} - \tilde{u}^{\frac{n+2}{n-2}}}{u - \tilde{u}}$ for $u \neq \tilde{u}$, and $\zeta = \frac{n+2}{n-2}u^{\frac{4}{n-2}}$ otherwise. In particular, $\zeta > 0$. By maximum principle, $v \geq 0$ on $M \times [0, +\infty)$.

Now we establish the upper bound of u. Let $\xi(p, t) = u(p, t) - u_\infty(p)$, with u_∞ the Loewner–Nirenberg solution. Therefore, $\xi(p, t) \to -\infty$ as $p \to \partial M$ for each $t > 0$. Hence, for any $t > 0$, there exists $p_t \in M^\circ$ such that $\xi(p_t, t) = \sup_{p \in M^\circ} \xi(p, t)$, and hence $\Delta_g \xi(p_t, t) \leq 0$. Therefore, we have

$$\begin{aligned} \xi_t &= \frac{4(n-1)}{n-2}\Delta\xi - (R_g + n(n-1)\mu(x, t))\xi \\ &\leq -(R_g + n(n-1)\mu)\xi, \end{aligned} \tag{2.12}$$

at the point (p_t, t) for $t > 0$, where $\mu(x, t) = \frac{u^{\frac{n+2}{n-2}} - u_\infty^{\frac{n+2}{n-2}}}{u - u_\infty}$ for $u \neq u_\infty$, and $\mu = \frac{n+2}{n-2}u_\infty^{\frac{4}{n-2}}$ otherwise. In particular, $\mu > 0$. By maximum principle for the equation (2.12) satisfied by ξ, if $\xi(p_{t_1}, t_1) \leq 0$ for some $t_1 > 0$, then $\xi(p, t) \leq 0$ for all $t > t_1$. Therefore, there are two possibilities: one is that there exists $t_2 > 0$ so that for $t > t_2$ we have $\xi \leq 0$ on M°; the second case is that $\xi(p_t, t) > 0$ for $t \in [0, +\infty)$. For the second case, let $\eta(t) = \sup_{p \in M^\circ} \xi(p, t)$. By the inequality in (2.12), $\eta(t)$ is decreasing since $\eta(t) > 0$. Now we use the discussion as in [24] to show that $\limsup_{t \to +\infty} \eta(t) \leq 0$.

Denote $(\frac{d\eta}{dt})_+(t) = \limsup_{\tau \searrow 0} \dfrac{\eta(t+\tau) - \eta(t)}{\tau}$ and define the set

$$S(t) = \{p \in M^\circ \big| \, p \text{ is a maximum point of } \xi \text{ on } M^\circ \times \{t\}\}.$$

Then $S(t)$ is compact for any $t \geq 0$. By Lemma 3.5 in [24],

$$\left(\frac{d\eta}{dt}\right)_+(t) \leq \sup\{\xi_t(p_t, t) \big| \, p_t \in S(t)\},$$

since $\eta(t) > 0$. By the mean value theorem we have that $\frac{u^{\frac{n+2}{n-2}} - u_\infty^{\frac{n+2}{n-2}}}{u - u_\infty} \geq \frac{n+2}{n-2}(\inf_{q \in M} u_\infty)^{\frac{4}{n-2}}$. Therefore, for the second case,

$$\left(\frac{d\eta}{dt}\right)_+(t) \leq -\left(R_g + \frac{n(n-1)(n+2)}{n-2}\left(\inf_{q \in M} u_\infty\right)^{\frac{4}{n-2}}\right)\eta(t),$$

for $t \geq 0$. By integration or a contradiction argument we can easily obtain that $\limsup_{t\to+\infty} \eta(t) \leq 0$. In summary, for both cases we obtain the upper bound control that

$$\limsup_{t\to+\infty} \sup_{q\in M^\circ} (u(q,t) - u_\infty(q)) \leq 0.$$

Based on the upper bound and lower bound control on u, we have that $\lim_{t\to+\infty} u(q,t) \to u_\infty(q)$ locally uniformly on M° as $t \to +\infty$. By the standard interior parabolic estimates, we obtain that the convergence is in $C^2_{loc}(M^\circ)$ sense. □

Let (M, g) be a general compact Riemannian manifold of $C^{4,\alpha}$ with boundary. By the discussion in Appendix A, there exists a conformal metric $h \in [g]$ of $C^{4,\alpha}$ such that $R_h = -n(n-1)$. We still denote it as g and choose it as the background metric of our flow, and hence the Cauchy–Dirichlet problem (1.3)–(1.5) becomes

$$\begin{aligned} &u_t = \frac{4(n-1)}{n-2}\Delta_g u + n(n-1)\big(u - u^{\frac{n+2}{n-2}}\big), \quad \text{in } M \times [0,+\infty),\\ &u(p,0) = u_0(p), \quad p \in M,\\ &u(q,t) = \phi(q,t), \quad q \in \partial M, \end{aligned} \tag{2.13}$$

with $\phi > 0$ and $u_0 > 0$ compatible, and $\phi \to +\infty$ uniformly as $t \to +\infty$. Recall that we have a unique positive solution $u \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(M \times [0,+\infty))$ to the Cauchy–Dirichlet boundary value problem with uniform upper and lower bound estimates on any compact sub-domain in M° for $t \in [0,+\infty)$, and the upper bound and lower bound asymptotic behavior estimates near the boundary as $t \to +\infty$. We now consider the convergence. To insure the asymptotic behavior near the boundary as $t \to +\infty$, we always assume the boundary data ϕ satisfies (2.4) for t large.

We start with the special initial data $u_0 = 1$.

Lemma 2.8 *Assume (M, g) is a compact Riemannian manifold with boundary of $C^{4,\alpha}$ and $R_g = -n(n-1)$. Let $u > 0$ be the solution to the Cauchy–Dirichlet problem*

$$\begin{aligned} &u_t = \frac{4(n-1)}{n-2}\Delta_g u + n(n-1)\big(u - u^{\frac{n+2}{n-2}}\big), \quad \text{in } M \times [0,+\infty),\\ &u(p,0) = 1, \quad p \in M,\\ &u(q,t) = \phi(q,t), \quad q \in \partial M, \end{aligned} \tag{2.14}$$

with a positive function $\phi \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(\partial M \times [0,+\infty))$ such that $\phi(q,0) = 1$, $\phi_t(q,0) = \phi_{tt}(q,0) = 0$ for $q \in \partial M$, $\phi_t \geq 0$ on $\partial M \times [0,+\infty)$ and $\liminf_{t\to\infty} \inf_{\partial M} \phi = \infty$. Assume ϕ satisfies (2.4). Then $u \to u_\infty$ in $C^4_{loc}(M^\circ)$ as $t \to +\infty$, where u_∞ is the solution to the Loewner–Nirenberg problem (1.1)–(1.2).

Proof By Lemma 2.4, we have that the exists a positive solution $u \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(M \times [0,\infty))$ to the Cauchy–Dirichlet problem (2.14). By maximum principle, we have that

$u \geq 1$ in $M \times [0, +\infty)$ and $u > 1$ for t large. As in Lemma 2.6, the solution u to (2.14) is increasing. Indeed, let $v = u_t$, and then v satisfies

$$
\begin{aligned}
&v_t = \frac{4(n-1)}{n-2}\Delta_g v + n(n-1)\big(1 - \frac{n+2}{n-2}u^{\frac{4}{n-2}}\big)v, \quad \text{in } M \times [0, +\infty),\\
&v = 0, \ \ p \in M,\\
&v(q,t) = \phi_t(q,t) \geq 0, \ \ q \in \partial M.
\end{aligned}
$$

By maximum principle, $v \geq 0$ for $t \geq 0$ and $v > 0$ for t large. Therefore, u is increasing in t. By the uniform upper bound estimates on any compact sub-domain of M°, we have that u converges pointwisely to $u_\infty > 0$ locally in M°. Using the Harnack inequality on v we have that u converges to u_∞ locally uniformly in M° as $t \to +\infty$. By the standard parabolic estimates, the convergence is in $C^4_{loc}(M^\circ)$. By Lemma 2.2 and Lemma 2.3, there exists $C > 0$ such that

$$
\frac{1}{C}x^{-\frac{n-2}{2}} \leq u_\infty \leq Cx^{-\frac{n-2}{2}},
$$

in a neighborhood of ∂M, where x is the distance function to ∂M. □

Lemma 2.9 *Assume (M, g) is a compact Riemannian manifold with boundary of $C^{4,\alpha}$ and $R_g = -n(n-1)$. For any $u_0 \in C^{2,\alpha}(M)$ such that $u_0 > 1$ and a positive function $\phi \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(M, [0, +\infty))$ satisfying the compatible condition (2.1) with u_0, $\phi_t \geq 0$ for $t \geq 0$ and satisfying (2.4) as $t \to \infty$, there exists a unique positive solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(M \times [0, +\infty))$ to (2.13), and u converges to u_∞ in $C^2_{loc}(M^\circ)$ as $t \to +\infty$, where u_∞ is the solution to the Loewner–Nirenberg problem (1.1)–(1.2).*

Proof Remark that for any $u_0 \in C^{2,\alpha}(M)$ such that $u_0 > 1$, there always exists such a function ϕ in the lemma. By Lemma 2.4, we have that the exists a positive solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(M \times [0, \infty))$ to the Cauchy–Dirichlet problem (2.14). By maximum principle, we have that $u \geq 1$ on $M \times [0, \infty)$.

Now we pick up a function $\tilde{\phi} \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(\partial M \times [0, \infty))$ such that $\tilde{\phi}(q, 0) = 1$, $\tilde{\phi}_t(q, 0) = \tilde{\phi}_{tt}(q, 0) = 0$ on ∂M, moreover, $\tilde{\phi}_t \geq 0$ and $\tilde{\phi} \leq \phi$ on $\partial M \times [0, \infty)$, and $\tilde{\phi}$ satisfies (2.4) as $t \to \infty$. By Lemma 2.8, for the initial data $\tilde{u}_0 = 1$ and the boundary data $\tilde{\phi}$, there exists a unique positive solution $\tilde{u} \in C^{4+\alpha,2+\frac{\alpha}{2}}_{loc}(M \times [0, \infty))$ to the Cauchy–Dirichlet problem (2.14), and $\tilde{u}$ converges to u_∞ in $C^4_{loc}(M^\circ)$ as $t \to \infty$, where u_∞ is the solution of the Loewner–Nirenberg problem (1.1)–(1.2). By maximum principle, we have that $\tilde{u} \geq 1$ in $M \times [0, +\infty)$.

Now let $\xi = u - \tilde{u}$. Then ξ satisfies

$$
\begin{aligned}
&\xi_t = \frac{4(n-1)}{n-2}\Delta_g \xi + n(n-1)\zeta\xi, \quad \text{in } M \times [0, +\infty),\\
&\xi(p, 0) = u_0(p) - 1 > 0, \ \ p \in M,\\
&\xi(q,t) = \phi(q,t) - \tilde{\phi}(q,t) \geq 0, \ \ q \in \partial M,
\end{aligned}
$$

where $\zeta = \frac{\left((\tilde{u}-\tilde{u}^{\frac{n+2}{n-2}})-(u-u^{\frac{n+2}{n-2}})\right)}{\tilde{u}-u}$ when $\tilde{u} \neq u$, and $\zeta = 1 - \frac{n+2}{n-2}u^{\frac{4}{n-2}}$ otherwise, and hence $\zeta < 0$. By maximum principle, $u \geq \tilde{u}$ in $M \times [0, +\infty)$.

The upper bound estimates is the same as the case $R_g \geq 0$. Notice that $u_\infty > 1$ by maximum principle of the equation (A.4). Let $\eta = u - u_\infty$. Then $\eta(q,t) \to -\infty$ as $q \to \partial M$ for any $t > 0$. Notice that η satisfies the equation

$$\eta_t = \frac{4(n-1)}{n-2}\Delta_g \eta + n(n-1)\beta\eta, \quad \text{in } M \times [0, +\infty),$$

where $\beta = \frac{(u-u^{\frac{n+2}{n-2}})-(u_\infty-u_\infty^{\frac{n+2}{n-2}})}{u-u_\infty}$ when $u \neq u_\infty$ and $\beta = 1 - \frac{n+2}{n-2}u^{\frac{4}{n-2}}$ otherwise, and hence, $\beta < 0$. Therefore, by the same argument as the case $R_g \geq 0$, we have that $\limsup_{t\to+\infty} \sup_{q\in M^\circ} \eta(q,t) \leq 0$. Therefore, $u(\cdot,t) \to u_\infty(\cdot)$ locally uniformly on M° as $t \to \infty$. By standard interior estimates for the parabolic equation, we have that $\tilde{u} \to u_\infty$ in $C^2_{loc}(M^\circ)$ as $t \to +\infty$. □

Proof of Theorem 1.2 By the argument in Appendix A, there exists a positive solution $\bar{u} \in C^{2,\alpha}(M)$ to (1.1) such that $v_0 < \bar{u} < u_0$ on M. Let $\bar{g} = \bar{u}^{\frac{4}{n-2}}g$ and hence $R_{\bar{g}} = -n(n-1)$. Let $\eta = \frac{u_0}{\bar{u}}$. Then by Lemma 2.9, we have that there exists a positive function $\phi \in C^{2,\alpha}_{loc}(\partial M \times [0, +\infty))$ such that to the flow

$$\begin{aligned} u_t &= \frac{4(n-1)}{n-2}\Delta_{\bar{g}} u + n(n-1)\left(u - u^{\frac{n+2}{n-2}}\right), \quad \text{in } M \times [0, +\infty),\\ u(p,0) &= \eta(p), \quad p \in M,\\ u(q,t) &= \phi(q,t), \quad q \in \partial M, \end{aligned}$$

there exists a unique positive solution u, which converges to the Loewner–Nirenberg solution u_∞ in the conformal class in $C^2_{loc}(M^\circ)$. That is to say, there exists a direct flow starting from $g_0 = u_0^{\frac{4}{n-2}}g$ and converging to the Loewner–Nirenberg metric. Notice that if the first Dirichlet eigenvalue satisfies $\lambda_1(L_g) \leq 0$, by Lemma 3.1, it is equivalent to say that $\inf_{u\in C^1(M)} Q(u) = -\infty$ when g is smooth, where the energy $Q(u)$ is defined in (3.2). □

3 Positive Scalar Curvature Metrics on Compact Manifolds with Boundary

In this section, we present examples for conformal classes on compact manifold with boundary that admit positive scalar curvature metrics. Let M be a compact smooth manifold with boundary ∂M. It is well-known that there is not topological obstruction for the existence of positive scalar curvature metrics on M. For any given smooth Riemannian metric g on M, we can extend (M, g) to a smooth closed manifold (N, h). Then by the well known theorems by Kazdan and Warner, Theorem 3.3 in [28] and Theorem 6.2 in [29], we have that there exists a diffeomorphism $F : N \to N$ and a smooth positive function $u \in N$, such that the scalar curvature of $u^{\frac{4}{n-2}} F^*h$ is positive on M where F^*h is the pull back metric of h by the map F. Notice that F^*h is not necessarily in the conformal class $[g]$ on M. Another interesting result obtained by Shi-Wang-Wei (Theorem 1.1 in [51]) recently, answering a question by Gromov, states that, any smooth Riemannian metric h on ∂M can be extended to a smooth Riemannian metric g with positive scalar curvature on M.

Now we consider the existence of positive scalar curvature metrics in a conformal class on a compact manifold with boundary.

Let (M^n, h) be a smooth compact Riemannian manifold with boundary ∂M of dimension $n \geq 4$. Let g be a complete metric equipped on the interior M° such that $x^2 g$ extends in $C^{k,\alpha}$ up to the boundary, where x is the distance function to the boundary under the metric h, then we call (M°, g) conformally compact of $C^{k,\alpha}$. If moreover, g is Einstein, we call (M°, g) conformally compact Einstein (CCE). In the well-known paper [42], Qing shows that for a conformally compact Einstein manifold which has conformal infinity of positive Yamabe constant, there exists a conformal compactification $\bar{g}$ with positive scalar curvature. Definitely, the CCE metric g is the Loewner–Nirenberg metric in the conformal class of $\bar{g}$, and hence the flow here connects these two metrics with the starting metric $\bar{g}$ and end at g as $t \to +\infty$.

To continue, we now introduce a well-known Neumann type boundary value problem on (M, g) introduced by Escobar ([17], see also [16]), which is sometimes called the second type Escobar–Yamabe problem:

$$
\begin{aligned}
&-\frac{4(n-1)}{(n-2)}\Delta_g u + R_g u = 0, \text{ in } M,\\
&\frac{4(n-1)}{n-2}\frac{\partial}{\partial n_g}u + 2(n-1)H_g u = 2(n-1)u^{\frac{n}{n-2}}, \text{ on } \partial M,
\end{aligned}
\tag{3.1}
$$

where n_g is the unit out-normal vector field on ∂M and H_g is the mean curvature on the boundary. The solution of problem (3.1) is a critical point of the energy functional:

$$
Q(u) = \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2) dV_g + \frac{n-2}{2}\int_{\partial M} H_g u^2 dS}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} dS\right)^{\frac{n-2}{n-1}}}, \tag{3.2}
$$

for any $u \in C^1(M)$, where H_g is the mean curvature of the boundary ∂M on (M, g), dV_g is the volume element of (M, g) and dS is the volume element of $(\partial M, g|_{\partial M})$. Notice that the functional does not always have a finite lower bound in a general conformal class, as pointed by Zhiren Jin, see [19]. For instance, $\inf_{u \in C^1(M) - \{0\}} Q(u) = -\infty$ when (M, g) is obtained by deleting a small geodesic ball on a closed Riemannian manifold with negative scalar curvature. Under the assumption $\inf_{u \in C^1(M)} Q(u) > -\infty$ and the positive mass theorem, the problem has been solved: it was studied in [13, 19, 20, 39, 40] assuming the positive mass theorem or the Weyl tensor $W_g = 0$ on M, and the remaining cases ($n \geq 6$, ∂M is umbilic, $\inf_{u \in C^1(M) - \{0\}} Q(u) > 0$ and the subset on ∂M where the vanishing of the Weyl tensor of g is of certain order is the whole boundary ∂M) were recently solved in [41] without the positive mass theorem.

Recall that by an easy computation, Escobar observed in [19] the following lemma.

Lemma 3.1 (Escobar, [19]) *Assume (M^n, g) is a smooth compact Riemannian manifold with boundary. Let $\lambda_1(L_g)$ be the first eigenvalue of the Dirichlet boundary value problem of the conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$. Then $\inf\limits_{u\in C^1(M)-\{0\}} Q(u) > -\infty$ if and only if $\lambda_1(L_g) > 0$.*

When $R_g \geq 0$, we know that $\lambda_1(L_g) > 0$, and hence $\inf\limits_{u\in C^1(M)-\{0\}} Q(u) > -\infty$. Based on the final solution of the second type of Escobar–Yamabe problem (3.1) in [41], we will use a perturbation argument to show that the other direction holds.

Theorem 3.2 *Let (M, g) be a smooth compact Riemannian manifold with boundary. Assuming either the condition in Theorem 1.1 in [41] holds on (M, g) or the positive mass theorem holds, there exists a positive scalar curvature metric in the conformal class of (M, g) if and only if $\inf\limits_{u\in C^1(M)-\{0\}} Q(u) > -\infty$.*

Proof Necessity is a trivial consequence of Lemma 3.1. Now we show the other direction. **Claim.** If there is a metric $h \in [g]$ such that $R_h = 0$ on M, then there exists a metric $h_1 \in [g]$ such that $R_{h_1} > 0$ on M. This is an application of the implicit function theorem as proof of Theorem 6.11 in [2]. Consider the Dirichlet boundary value problem

$$
\begin{aligned}
&-\frac{4(n-1)}{n-2}\Delta_h u = f u^{\frac{n+2}{n-2}}, \ \text{ in } M,\\
&u = 1, \ \text{ on } \partial M,
\end{aligned}
$$

for any given function $f \in C^\alpha(M)$. We want to obtain a positive solution to the problem. Let $\mathcal{C}_0 = \{u \in C^{2,\alpha}(M) : \ u\big|_{\partial M} = 0\}$. Define the map $F : \mathcal{C}_0 \times C^\alpha(M) \to C^\alpha(M)$ such that

$$F(u, f) = -\frac{4(n-1)}{n-2}\Delta_h(1+u) - f(1+u)^{\frac{n+2}{n-2}}.$$

This is a C^1 map. We take derivative

$$D_u F(v, f) = -\frac{4(n-1)}{n-2}\Delta_h v - \frac{n+2}{n-2} f(1+u)^{\frac{4}{n-2}} v.$$

Recall that $F(0, 0) = 0$ since $R_h = 0$. At $(0, 0)$,

$$D_u F(v, 0) = -\frac{4(n-1)}{n-2}\Delta_h v$$

is invertible. By implicit function theorem, there exists a constant $\epsilon > 0$ small such that for any $f \in C^\alpha(M)$ with $\|f\|_{C^\alpha(M)} \leq \epsilon$, we have that the Dirichlet problem has a unique solution $\tilde{u} = 1 + u$ in the neighborhood of 1 in $C^{2,\alpha}(M)$. Take f to be a small positive constant. We have that $\tilde{u} > 0$ on M. This completes the proof of **Claim**. The solution of Escobar–Yamabe problem (3.1), where positive mass theorem is used for certain cases, tells that when $\inf_{u \in C^1(M)} Q(u) > -\infty$, there exists a metric in the conformal class with zero scalar curvature, and hence by **Claim**, we have that there exists a positive scalar curvature metric in the conformal class. This completes the proof of the Theorem. □

In particular, for manifolds of dimension $3 \le n \le 7$ (see [43–45]), or spin manifolds [54], positive mass theorem holds. Recently, Schoen-Yau [46] presented a proof that positive mass theorem holds true in general dimension.

Remark. For positive mass theorem and the existence theory of solutions to the Escobar–Yamabe problem, for the regularity of the metric, it is enough that the metric is of C^k for some sufficiently large $k > 0$. The above argument shows that for a smooth compact Riemannian manifold (M, g), if $\inf_{u \in C^1(M) - \{0\}} Q(u) = -\infty$, then there exists no conformal metric $g_1 = u^{\frac{4}{n-2}} g$ with $u \in C^{2,\alpha}(M)$ such that $R_{g_1} = 0$.

4 The Yamabe Flow: A Conformally Invariant Flow

We consider the Yamabe flow

$$u_t = (n-1)u^{-\frac{4}{n-2}} \left(\Delta_g u - \frac{n-2}{4(n-1)} (R_g u + n(n-1) u^{\frac{n+2}{n-2}}) \right), \tag{4.1}$$

which can also be written as

$$(u^{\frac{n+2}{n-2}})_t = \frac{(n-1)(n+2)}{n-2} \left(\Delta_g u - \frac{n-2}{4(n-1)} (R_g u + n(n-1) u^{\frac{n+2}{n-2}}) \right). \tag{4.2}$$

The flow is conformally covariant in the sense that, under the conformal change $g = \varphi^{\frac{4}{n-2}} h$, the above equation (4.1) becomes

$$v_t = (n-1)v^{-\frac{4}{n-2}} \left(\Delta_h v - \frac{n-2}{4(n-1)} (R_h v + n(n-1) v^{\frac{n+2}{n-2}}) \right),$$

where $v = u\varphi$. So without loss of generality, let g be the metric on M in the conformal class such that $R_g = -n(n-1)$. We consider the Cauchy–Dirichlet problem (1.6) where $u_0 \in C^{2k+2,\alpha}(M)$ and $\phi(q,t) \in C_{loc}^{2k+2+\alpha, k+1+\frac{\alpha}{2}}(\partial M \times [0, +\infty))$ with $k = 0, 1$ satisfying the $C^{2+\alpha, 1+\frac{\alpha}{2}}$ compatible condition

$$\begin{aligned} &u_0(p) = \phi(p, 0), \\ &\phi(p,0)^{\frac{4}{n-2}} \phi_t(p,0) = (n-1) \left[\Delta_g u_0(p) + \frac{n(n-2)}{4} (u_0(p) - u_0(p)^{\frac{n+2}{n-2}}) \right], \end{aligned} \tag{4.3}$$

for $p \in \partial M$, and in addition the $C^{4+\alpha,2+\frac{\alpha}{2}}$ compatible condition

$$\phi(p,0)^{\frac{4}{n-2}}\phi_{tt}(p,0)+\frac{4}{n-2}\phi(p,0)^{\frac{6-n}{n-2}}(\phi_t(p,0))^2=(n-1)L_g(\mu), \tag{4.4}$$

where the linear operator L_g is defined as

$$L_g(\mu)=\Delta_g\mu+\frac{n(n-2)}{4}\left(1-\frac{n+2}{n-2}u_0(p)^{\frac{4}{n-2}}\right)\mu,$$

and the function μ is defined as

$$\mu(p)=(n-1)u_0(p)^{-\frac{4}{n-2}}\left[\Delta_g u_0(p)+\frac{n(n-2)}{4}(u_0(p)-u_0(p)^{\frac{n+2}{n-2}})\right]. \tag{4.5}$$

Lemma 4.1 *Assume (M^n, g) is a compact Riemannian manifold with boundary of $C^{4,\alpha}$ such that $R_g=-n(n-1)$. Let $u_0 \in C^{2k+2,\alpha}(M)$ be a positive function, and $\phi \in C_{loc}^{2k+2+\alpha,k+1+\frac{\alpha}{2}}(M\times[0,\infty))$ be a positive function satisfying the compatible condition (4.3) for $k=0$ and (4.4) in addition for $k=1$. Then there exists a unique positive solution $u \in C_{loc}^{2k+2+\alpha,k+1+\frac{\alpha}{2}}(M\times[0,\infty))$ to the Cauchy–Dirichlet problem (1.6). Moreover, if $u_0 \geq 1$ and $\phi \geq 1$, then $u \geq 1$.*

Proof Since $u_0 \in C^{2k+2,\alpha}(M)$ is positive, the equation (4.1) is uniform parabolic, and by the compatible condition on u_0 and ϕ, there exists $T>0$ such that a positive solution u on $M\times[0,T)$ such that $u \in C^{2k+2+\alpha,k+1+\frac{\alpha}{2}}(M\times[0,T_1])$ for any $T_1<T$ and $k=0,\ 1$ respectively. Now for any $0<T_1<T$, by maximum principle,

$$u \geq \min\{1,\ \inf_M u_0,\ \inf_{M\times[0,T_1])}\phi\}$$

on $M\times[0,T_1]$. In fact, if there exists $(q,t)\in M^\circ\times(0,T_1]$, such that $0<u(q,t)=\inf_{M\times[0,T_1]}u<1$, then we have $(u^{\frac{n+2}{n-2}})_t(q,t)\leq 0$, $\Delta_g u(q,t)\geq 0$, $(u-u^{\frac{n+2}{n-2}})(q,t)>0$, contradicting with the equation. Also, by similar argument, we have that

$$u \leq \max\{1,\ \sup_M u_0,\ \sup_{M\times[0,T_1]}\phi\},$$

on $M\times[0,T_1]$. Therefore, by the standard a prior $C^{2k+2+\alpha,1+k+\frac{\alpha}{2}}$ estimates of parabolic equations, the solution $u>0$ can be extended on $M\times[0,+\infty)$ with $u\in C^{2k+2+\alpha,k+1+\frac{\alpha}{2}}(M\times[0,T])$ for any $T>0$. □

Lemma 4.2 *Let (M,g), $u_0\in C^{4,\alpha}(M)$ and $\phi\in C_{loc}^{4+\alpha,2+\frac{\alpha}{2}}(M\times[0,\infty))$ be as in Lemma 4.1 with the compatible condition (4.3)–(4.4). In particular, $R_g=-n(n-1)$.*

Moreover, let $u_0 \geq 1$ *be a subsolution to the equation* (1.1). *And moreover, we assume* $u_0 \in C^{4,\alpha}(M)$ *and*

$$L_g(\mu) \geq 0$$

at the points $q \in \partial M$ *such that* $\mu(q) = 0$ *where* L_g *and* μ *are as in* (4.4). *Let a positive function* $\phi \in C_{loc}^{4+\alpha,2+\frac{\alpha}{2}}(M \times [0,\infty))$ *satisfy the compatible condition* (4.3) *and* (4.4). *Also we assume* $\phi_t \geq 0$ *on* $\partial M \times (0,+\infty)$. *Then the solution* u *to* (1.6) *satisfies* $u_t \geq 0$ *on* $M \times [0,+\infty)$. *In particular,* $u(\cdot,t)$ *is a sub-solution to* (1.1) *for each* $t \geq 0$.

Proof Let $v = u_t$, condition on u_0 implies $v(q,0) \geq 0$. By assumption, we have $\phi \geq 1$ on $\partial M \times [0,+\infty)$ and hence, by Lemma 4.1, $u \geq 1$ in $M \times [0,\infty)$. Take derivative of t on both sides of the equation in (1.6), and we have

$$\begin{aligned} &\frac{n+2}{n-2}u^{\frac{4}{n-2}}v_t + \frac{4(n+2)}{(n-2)^2}u^{\frac{6-n}{n-2}}v^2 \\ &\quad = \frac{(n-1)(n+2)}{n-2}\left(\Delta_g v + \frac{n(n-2)}{4}(1 - \frac{n+2}{n-2}u^{\frac{4}{n-2}})v\right). \end{aligned} \tag{4.6}$$

By maximum principle, v can not obtain a negative minimum on $M \times [0,T]$ at a point $(q,t) \in M^\circ \times (0,T]$, for any $T > 0$. Indeed, if otherwise, since $v \geq 0$ on $M \times \{0\} \bigcup \partial M \times [0,\infty)$ and recall that $u \geq 1$, by continuity of v, there exists $t_1 > 0$ and $p \in M^\circ$ such that

$$v(p,t_1) = \inf_{M\times[0,t_1]} v < 0,$$

and $|v(p,t_1)|$ is so small that at the point (p,t_1),

$$-\frac{4(n+2)}{(n-2)^2}u^{\frac{6-n}{n-2}}v^2 + \frac{n(n-1)(n+2)}{4}\left(1 - \frac{n+2}{n-2}u^{\frac{4}{n-2}}\right)v > 0,$$

contradicting with the equation (4.6) and the fact $v_t(p,t_1) \leq 0$ and $\Delta_g v(p,t_1) \geq 0$. Therefore, $v \geq 0$ in $M \times [0,+\infty)$. □

Lemma 4.3 *Let* (M,g), u_0 *and* ϕ *be as in Lemma* 4.2. *Moreover, assume* ϕ *satisfies that there exists a constant* $\beta > 0$ *such that*

$$\phi^{-1}\phi_t \leq \beta \tag{4.7}$$

on $\partial M \times [0,\infty)$, *and*

$$\begin{aligned} &\phi^{\frac{n-1}{2-n}}|\nabla_g\phi| \to 0, \\ &\phi^{\frac{n}{2-n}}|\nabla_g^2\phi| \to 0, \ \textit{uniformly on} \ \partial M, \ \textit{as} \ t \to +\infty. \end{aligned} \tag{4.8}$$

Let x be the distance function to the boundary on (M, g). Let $U = \{0 \le x \le x_1\}$. Then there exist constants $C > 0$, $x_1 > 0$ small and $t_1 > 0$ large such that the solution $u > 0$ to (1.6) satisfies

$$u(q,t) \ge C(x + \phi^{\frac{2}{2-n}})^{\frac{2-n}{2}} - C$$

on $U \times [t_1, +\infty)$.

For instance, take $\phi(t) = e^t,\ t^2 e^t,\ t$, or any other monotone function of polynomial growth for t large.

Proof The proof is similar to Lemma 2.3. Let $f \equiv 1$ and $x_1 > 0$ small be as in Lemma 2.3 such that the exponential map $F = \text{Exp} : \partial M \times [0, x_1] \to U$ is a diffeomorphism, and let the barrier function $\varphi \in C^{2,\alpha}_{loc}(U \times [0,\infty))$ be defined in (2.5). For φ we have the estimates (2.6). In particular,

$$\begin{aligned}
\varphi^{\frac{4}{n-2}}\varphi_t &= \varphi^{\frac{4}{n-2}} c f^{\frac{2}{n-2}}[(x + (f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}} \\
&\quad - (x_1 + (f(q)\phi(q,t))^{\frac{2}{2-n}})^{\frac{-n}{2}}]\phi^{\frac{n}{2-n}}\frac{\partial \phi}{\partial t} \\
&\le c^{\frac{n+2}{n-2}}(x + \phi(q,t)^{\frac{2}{2-n}})^{-\frac{n+4}{2}}\phi^{\frac{n}{2-n}}\frac{\partial \phi}{\partial t} \\
&\le c^{\frac{n+2}{n-2}}(x + \phi(q,t)^{\frac{2}{2-n}})^{-\frac{n+2}{2}}\phi^{-1}\frac{\partial \phi}{\partial t},
\end{aligned}$$

and hence by (2.7) and our condition on ϕ, for the constants $c > 0$ and x_1 small enough and $t_1 > 0$ large enough, we have

$$(\varphi^{\frac{n+2}{n-2}})_t \le \frac{(n-1)(n+2)}{n-2}\left(\Delta_g \varphi + \frac{n(n-2)}{4}(\varphi - \varphi^{\frac{n+2}{n-2}})\right)$$

on $U \times [t_1, \infty)$. Now we take $c > 0$ small enough so that $u > \varphi$ on $U \times \{t_1\}$. By the definition of φ, we also know that $u > \varphi$ on $\partial U \times [0, +\infty)$. Let $v = u - \varphi$. Take difference of this inequality and the equation satisfied by u in (1.6), we have

$$\begin{aligned}
&u^{\frac{4}{n-2}} v_t + (u^{\frac{4}{n-2}} - \varphi^{\frac{4}{n-2}})\varphi_t \\
&\quad \ge (n-1)(\Delta_g v + \frac{n(n-2)}{4}(u - u^{\frac{n+2}{n-2}} - \varphi + \varphi^{\frac{n+2}{n-2}}))
\end{aligned}$$

on $U \times [t_1, \infty)$. Assume that there exists $(q,t) \in U^\circ \times (t_1, +\infty)$ such that $v(q,t) = \inf_{M \times [t_1, t]} v < 0$, and then since $u \ge 1$, we have $\varphi(q,t) > 1$ and hence at the point (q,t), we have

$$\begin{aligned}
&u^{\frac{4}{n-2}} v_t \le 0,\ \ (u^{\frac{4}{n-2}} - \varphi^{\frac{4}{n-2}})\varphi_t \le 0, \\
&\Delta_g v \ge 0,\ \ u - u^{\frac{n+2}{n-2}} - \varphi + \varphi^{\frac{n+2}{n-2}} > 0,
\end{aligned}$$

where $\varphi_t \geq 0$ by the assumption $\phi_t \geq 0$, contradicting with the above inequality. Therefore, $v \geq 0$ on $U \times [t_1, +\infty)$. This completes the proof of the lemma. □

Proof of Theorem 1.3 By Lemma 4.1 and Lemma 4.2, we have that there exists a positive solution u on $M \times [0, +\infty)$ with $u \in C^{4+\alpha, 2+\frac{\alpha}{2}}(M \times [0, T])$ for any $T > 0$, and $u_t \geq 0$, i.e., $u(\cdot, t)$ is a sub-solution to (1.1) for any $t \geq 0$ and also, $u \geq 1$. By maximum principle, $u(q, t) \leq u_{LN}(q)$ for any $(q, t) \in M^\circ \times [0, +\infty)$, where u_{LN} is the solution to the Loewner–Nirenberg problem (1.1)–(1.2). Alternatively, one can use the local super-solution constructed in Lemma 5.2 in [23] to give the local upper bound estimates of u in M°. Indeed, let $x = x(q)$ be the distance function of $q \in M^\circ$ to ∂M. There exists $x_1 > 0$ small such that for $q \in \{0 < x(q) \leq x_1\}$, the injectivity radius $i(q)$ at q is larger than $\frac{x(q)}{2}$. We then define a function $\bar{u}$ on $B_R(q)$ by

$$\bar{u}(p) = \left(\frac{2R}{R^2 - r(p)^2}\right)^{\frac{n-2}{2}} e^{\frac{n-2}{2}(\sqrt{R^2 - r(p)^2 + \epsilon^2} - \epsilon)}$$

for $p \in B_R(q)$ where $R = \frac{x(q)}{2}$, $\epsilon > 0$ is some small constant and $r(p)$ is the distance function from p to q, and hence $\bar{u} \in C^2(B_R(q))$ and $\bar{u} = \infty$ on $\partial B_R(q)$. In fact x_1 and ϵ are chosen small enough as in [23] so that $\bar{u}$ is a super-solution to (1.1) on $B_R(q)$. By maximum principle, $u \leq \bar{u}$ on $B_R(q)$, and hence there exists a uniform constant $C > 0$, such that

$$u(q, t) \leq Cx(q)^{\frac{2-n}{2}}$$

on $\{0 < x(q) \leq x_1\}$ for $t \geq 0$. For any $q \in M - \{0 \leq x \leq x_1\}$, taking $R < \min\{\frac{i(q)}{2}, x_1\}$ and $\epsilon > 0$ small so that $\bar{u}$ is a super-solution to (1.1) on $B_R(q)$, and hence we have that there exists a uniform constant $C > 0$ depending on $B_R(q)$ such that $u(p, t) \leq C$ for $p \in B_{\frac{R}{2}}(q)$ and $t \geq 0$. Now by standard interior Schauder estimates of parabolic equations, we have that for any compact subset $F \subseteq M^\circ$, there exists a uniform constant $C = C(F) > 0$ such that

$$\|u\|_{C^{4+\alpha, 2+\frac{\alpha}{2}}(F \times [T, T+1])} \leq C$$

for any $T \geq 0$. Since u is locally uniformly bounded from above in M° and $u_t \geq 0$ on $M \times [0, \infty)$, by Harnack inequality with respect to the equation (4.6) satisfied by u_t, we have that u converges locally uniformly in M° to a positive function u_∞ on M° as $t \to +\infty$. By interior Schauder estimates of the uniform parabolic equation in (1.6), we have that $u \to u_\infty$ in $C^4_{loc}(M^\circ)$, and hence u_∞ is a solution to (1.1) in M°. By the lower bound estimates near the boundary in Lemma 4.3 and the above upper bound estimates, there exists a constant $C > 0$ such that

$$Cx^{\frac{2-n}{2}} \geq u_\infty \geq \frac{1}{C}x^{\frac{2-n}{2}}$$

in $\{0 < x \leq x_1\}$ for some constant $x_1 > 0$, where x is the distance function to the boundary on (M, g). Therefore, by uniqueness of the solution to (1.1)–(1.2), $u_\infty = u_{LN}$. This completes the proof of the theorem. □

Acknowledgements The author would like to thank Wei Yuan, Xiaoyang Chen, Jian Ge and Professor Yuguang Shi for helpful discussion.

Appendix A

In this section, we show that the homogeneous Dirichlet boundary value problem

$$\frac{4(n-1)}{n-2}\Delta u - R_g u - n(n-1)u^{\frac{n+2}{n-2}} = 0, \quad \text{in } M, \tag{A.1}$$

$$u(p) = 0, \quad \text{for } p \in \partial M. \tag{A.2}$$

admits a nontrivial solution $u \geq 0$ in M when the conformal Laplacian $L = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ has a negative eigenvalue $\lambda_1(L_g) < 0$ for the Dirichlet boundary value problem.

Let (M, g) be a compact Riemannian manifold of $C^{k+2,\alpha}$ with boundary with $k \geq 0$. For any given positive function $\varphi_0 \in C^{k+2,\alpha}(\partial M)$, by the classical variational method (see Lemma A.1), there exists a unique positive solution $u_0 \in C^{k+2,\alpha}(M)$ to the Dirichlet boundary value problem

$$\begin{aligned} &\frac{4(n-1)}{n-2}\Delta_g w - R_g w - n(n-1)w^{\frac{n+2}{n-2}} = 0, \\ &w\big|_{\partial M} = \varphi_0. \end{aligned} \tag{A.3}$$

Claim 1 If u_1 and u_2 are the corresponding solutions to (A.3) with respect to $\varphi_0 = \varphi_1$ and $\varphi_0 = \varphi_2$ for two positive functions $\varphi_1 \leq \varphi_2$ on ∂M, then $u_1 \leq u_2$. We now use maximum principle to prove the claim. Let $g_2 = u_2^{\frac{4}{n-2}} g$, $\phi = \frac{\varphi_1}{\varphi_2}$ and $v = \frac{u_1}{u_2}$. Then v satisfies

$$\begin{aligned} &\frac{4(n-1)}{n-2}\Delta_{g_2} v + n(n-1)(v - v^{\frac{n+2}{n-2}}) = 0, \\ &v\big|_{\partial M} = \phi \leq 1. \end{aligned} \tag{A.4}$$

Then by maximum principle, v could not obtain its maximum point with $\sup\limits_M v > 1$ in M°, and hence Claim 1 is proved.

We present a well-known existence result for the Dirichlet boundary value problem of the Yamabe equation (A.3), see [3,34,35,52]. The proof of which is by a direct variational method, in seek of a minimizer of the corresponding energy functional.

Lemma A.1 *Let (M, g) be a compact Riemannian manifold of $C^{k+2,\alpha}$ with boundary. For any positive function $\varphi_0 \in C^{k+2,\alpha}(\partial M)$ with $k \geq 0$, there exists a unique positive* *solution* $w \in C^{k+2,\alpha}(M)$ *to the Dirichlet boundary value problem* (A.3). *In particular,* $R_{w^{\frac{4}{n-2}} g} = -n(n-1)$.

By Lemma A.1, we take the metric g such that $R_g = -n(n-1)$ in the conformal class as the background metric. Let $v_j > 0$ be the solution to the Dirichlet boundary value problem (A.4) with $g_2 = g$ and with the boundary data $\phi = \frac{1}{j}$. Then by Claim 1, $\{v_j\}_{j=1}^{\infty}$ is decreasing as j increases. By standard elliptic estimates we have $\{v_j\}_j$ converges in $C^2(M)$ to a non-negative function v_0, i.e., $v_0 = \lim_{j\to\infty} v_j$. Then v_0 is a non-negative solution to the problem

$$\begin{aligned} &\frac{4(n-1)}{n-2}\Delta_g v_0 + n(n-1)v_0 - n(n-1)|v_0|^{\frac{n+2}{n-2}} = 0, \\ &v_0\big|_{\partial M} = 0. \end{aligned} \tag{A.5}$$

We want to show that the limit v_0 of $\{v_j\}_j$ is not zero when the first eigenvalue $\lambda_1(L_g) < 0$ for the Dirichlet problem of the conformal Laplacian $L_g = -(\frac{4(n-1)}{n-2}\Delta_g + n(n-1))$. For the case $\lambda_1(L_g) < 0$, let ϕ_1 be the first eigenfunction with $1 > \phi_1 > 0$ in M°. Recall that the minimizer of the energy

$$E(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|_g^2 + \int_M \frac{n(n-1)}{2}\left(-u^2 + \frac{n-2}{n}|u|^{\frac{2n}{n-2}}\right), \tag{A.6}$$

in the function space

$$S = \{u \in W^{1,2}(M) \,\big|\, u - \varphi_0 \in W_0^{1,2}(M)\} \tag{A.7}$$

is the unique solution to (A.3) when $\varphi_0 > 0$. Here for the homogeneous Dirichlet problem (A.5), just take $\varphi_0 = 0$. Then let $\epsilon > 0$ be small enough, we have that

$$\begin{aligned} E(\epsilon\phi_1) &= \frac{2(n-1)}{n-2}\epsilon^2 \int_M |\nabla \phi_1|_g^2 - \frac{n(n-2)}{4}\phi_1^2 dV_g \\ &\quad + \frac{(n-2)(n-1)}{2}\epsilon^{\frac{2n}{n-2}} \int_M \phi_1^{\frac{2n}{n-2}} dV_g \\ &= \epsilon^2 \left[\frac{2(n-1)}{n-2}\int_M |\nabla \phi_1|_g^2 - \frac{n(n-2)}{4}\phi_1^2 dV_g \right. \\ &\quad \left. + \frac{(n-2)(n-1)}{2}\epsilon^{\frac{4}{n-2}} \int_M \phi_1^{\frac{2n}{n-2}} dV_g\right]. \end{aligned}$$

Since $\int_M |\nabla \phi_1|_g^2 - \frac{n(n-2)}{4}\phi_1^2 dV_g < 0$, there exists $\epsilon > 0$, such that $E(\epsilon\phi_1) < 0$. Therefore,

$$-m \equiv \inf_{v \in W_0^{1,2}(M)} E(v) < 0.$$

Let $\{u_j\}_{j=0}^{\infty}$ be a minimizing sequence of E on the function space S defined in (A.7). Then for j large,

$$\frac{-m}{2} \geq E(u_j) = \frac{2(n-1)}{n-2} \int_M |\nabla u_j|_g^2 + \int_M \frac{n(n-1)}{2} \left(-u_j^2 + \frac{n-2}{n} |u_j|^{\frac{2n}{n-2}} \right),$$

and hence,

$$\int_M \frac{n(n-1)}{2} u_j^2 \geq \frac{m}{2} + \frac{2(n-1)}{n-2} \int_M |\nabla u_j|_g^2 + \int_M \frac{(n-2)(n-1)}{2} |u_j|^{\frac{2n}{n-2}} \geq \frac{m}{2} > 0.$$

Since $u_j \rightharpoonup \bar{v}$ weakly in $W^{1,2}(M)$ sense, by the Sobolev embedding theorem, we have that $u_j \to \bar{v}$ in $L^2(M)$ up to a subsequence. Therefore, $\|\bar{v}\|_{L^2(M)} \geq \sqrt{\frac{m}{2}} > 0$, and it is a weak solution to the problem. It is clear that $|\bar{v}|$ is also a minimizer of E. By the regularity argument in Appendix B in [52], $|\bar{v}| \in C^2(M)$. By Harnack inequality, we have that $|\bar{v}| > 0$ in M° since $\bar{v}$ is not zero, and hence $\bar{v} > 0$ in M°. Therefore, the homogeneous Dirichlet boundary value problem has a non-zero solution $\bar{v} \in C^2(M) \bigcap W_0^{1,2}(M)$ with $\bar{v} > 0$ in M°. By maximum principle, $\bar{v} \leq 1$ in M.

On the other hand, by Claim 1, we have $v_0 = \lim_{j\to\infty} v_j \geq \bar{v}$ on M, and hence $v_0 > 0$ in M°. In particular, v_0 is the largest solution to (A.5).

In summary, for a general compact Riemannian manifold (M, g) of $C^{k+2,\alpha}$ with boundary such that $\lambda_1(L_g) < 0$, where $\lambda_1(L_g)$ is the first eigenvalue of the conformal Laplacian L_g of the Dirichlet boundary value problem, there exists a largest solution v_0 to (A.1)–(A.2) such that $v_0 > 0$ in M°.

By the convergence of v_j to v_0, for any continuous function $u_0 > v_0$ on M, there exists $j > 0$ such that $u_0 > v_j$ on M.

On a smooth compact manifold (M, g) with boundary, if $\lambda_1(L_g) > 0$, then by Lemma 3.1 and Theorem 3.2, there exists a conformal metric $h \in [g]$ such that $R_h \geq 0$. By maximum principle, $v_0 = 0$ in this case. We do not know if v_0 vanishes when $\lambda_1(L_g) = 0$.

References

1. Andersson, L., Chruściel, P., Friedrich, H.: On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einsteins field equations. Commun. Math. Phys. **149**, 587–612 (1992)
2. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer Monograph in Mathematics, Springer, Berlin (1998)
3. Aviles, P., McOwen, R.C.: Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds. Duke Math. J. **56**(2), 395–398 (1988)
4. Aviles, P., McOwen, R.C.: Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds. J. Differ. Geom. **27**, 225–239 (1988)
5. Bahuaud, E., Vertman, B.: Long-time existence of the edge Yamabe flow. J. Math. Soc. Jpn. **71**(2), 651–688 (2019)
6. Besse, A.: Einstein Manifolds. Springer, Berlin (1987)
7. Borel, A.: Some remarks about Lie groups transitive on spheres and tori. Bull. A.M.S. **55**, 580–587 (1949)

8. Brendle, S.: Convergence of the Yamabe flow for arbitrary initial energy. J. Differ. Geom. **69**, 217–278 (2005)
9. Brendle, S.: Convergence of the Yamabe flow in dimension 6 and higher. Invent. Math. **170**(3), 541–576 (2007)
10. Brendle, S.: A generalization of the Yamabe flow for manifolds with boundary. Asian J. Math. **6**, 625–644 (2002)
11. Chen, X., Ho, P.T., Sun, L.: Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant. Ann. Glob. Anal. Geom. **53**(1), 121–150 (2018)
12. Chen, X., Xu, X.: The scalar curvature flow on S^n-perturbation theorem revisited. Invent. Math. **187**(2), 395–506 (2012)
13. Chen, S.Y.S.: Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions, preprint. arXiv:0912.1302
14. Cheng, L., Ma, L.: Yamabe flow and Myers type theorem on a complete manifold. J. Geom. Anal. **24**(1), 246–270 (2014)
15. Chow, B.: The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. Commun. Pure Appl. Math. **45**, 1003–1014 (1992)
16. Cherrier, P.: Problèmes de Neumann non linéaires sur les variétés riemanniennes. J. Funct. Anal. **57**(2), 154–206 (1984)
17. Escobar, J.F.: The Yamabe problem on manifolds with boundary. J. Differ. Geom. **35**(1), 21–84 (1992)
18. Escobar, J.F.: Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. Ann. Math. 2 **136**(1), 1–50 (1992)
19. Escobar, J.F.: Addendum: "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary". Ann. Math. 2 **139**(3), 749–750 (1994)
20. Escobar, J.F.: Conformal metrics with prescribed mean curvature on the boundary. Calc. Var. Partial Differ. Equ. **4**(6), 559–592 (1996)
21. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin (2001)
22. Guan, B.: Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary. Int. Math. Res. Not. IMRN (2008). Art. ID rnn 105
23. Gursky, M., Streets, J., Warren, M.: Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary. Calc. Var. PDE **1–2**, 21–43 (2011)
24. Hamilton, R.: Four-manifolds with positive curvature operator. J. Differ. Geom. **24**, 153–179 (1986)
25. Han, Q., Jiang, X.: Boundary expansions for minimal graphs in the hyperbolic space. arXiv:1412.7608
26. Han, Q., Jiang, X., Shen, W.: The Loewner-Nirenberg problem in cones. arXiv:2012.06799
27. Han, Q., Shen, W.: The Loewner-Nirenberg problem in singular domains. J. Funct. Anal. **279**(6), 108604 (2020)
28. Kazdan, J.L., Warner, F.W.: Scalar curvature and conformal deformation of Riemannian structure. J. Differ. Geom. **10**, 113–134 (1975)
29. Kazdan, J.L., Warner, F.W.: Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. Ann. Math. **100**(2), 317–331 (1975)
30. Kichenassamy, S.: Boundary behavior in the Loewner-Nirenberg problem. J. Funct. Anal. **222**, 98–113 (2005)
31. Li, G.: A flow approach to the generalized Loewner-Nirenberg problem of the σ_k-Ricci equation, preprint
32. Li, P.: Geometric Analysis. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. ISBN: 978-1-107-02064-1 (2012)
33. Lieberman, G.M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge, NJ. ISBN: 981-02-2883-X (1996)
34. Loewner, C., Nirenberg, L.: Partial Differential Equations Invariant Under Conformal or Projective Transformations, Contributions to Analysis, pp. 245–272. Academic Press, New York (1974)
35. Ma, L.: The Yamabe problem with Dirichlet data. C. R. Acad. Sci. Paris Sér. I Math. **320**(6), 709–712 (1995)
36. Ma, L., An, Y.: The maximum principle and the Yamabe flow. In: Chen, H., Rodino, L. (eds.) Partial Differential Equations and Their Applications (Wuhan, 1999), pp. 211–224. World Scientific, River Edge, NJ (1999)
37. Ma, L.: Yamabe flow and metrics of constant scalar curvature on a complete manifold. Calc. Var. Partial Differ. Equ. **58**(1) (2019). Paper No. 30

38. Mazzeo, R.: Regularity for the singular Yamabe problem. Indiana Univ. Math. J. **40**, 1277–1299 (1991)
39. Marques, F.: Existence results for the Yamabe problem on manifolds with boundary. Indiana Univ. Math. J. **54**(6), 1599–1620 (2005)
40. Marques, F.: Conformal deformations to a scalar-flat metrics with constant mean curvature on the boundary. Commun. Anal. Geom. **15**(2), 381–405 (2007)
41. Mayer, M., Ndiaye, C.: Bacycenter technique and the Riemann mapping problem of Cherrier-Escobar. J. Differ. Geom. **107**(3), 519–560 (2017)
42. Qing, J.: On the rigidity for conformally compact Einstein manifolds. Int. Math. Res. Not. **21**, 1141–1153 (2003)
43. Schoen, R., Yau, S.T.: On the proof of positive mass conjecture in general relativity. Commun. Math. Phys. **65**, 45–76 (1979)
44. Schoen, R., Yau, S.T.: Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity. Proc. Nat. Acad. Sci. USA **76**, 1024–1025 (1979)
45. Schoen, R., Yau, S.T.: The energy and the linear momentum of space-times in general relativity. Commun. Math. Phys. **79**, 47–51 (1981)
46. Schoen, R., Yau, S.T.: Positive scalar curvature and minimal hypersurface singularities, preprint. arXiv:1704.05490
47. Schulz, M.B.: Instantaneously complete Yamabe flow on hyperbolic space. Calc. Var. PDE. **58**(6), 1–30 (2019)
48. Schulz, M.B.: Incomplete Yamabe flows and removable singularities. J. Funct. Anal. **278**(11), 108475 (2020)
49. Schwetlick, H., Struwe, M.: Convergence of the Yamabe flow for "large" energies. J. Reine Angew. Math. **562**, 59–100 (2003)
50. Shao, Y.: The Yamabe flow on imcomplete manifolds. J. Evol. Equ. **18**, 1595–1632 (2018)
51. Shi, Y., Wang, W., Wei, G.: Total mean curvature of boundary and nonnegative scalar curvature fill-ins, preprint. arXiv:2007.06756v3
52. Struwe, M.: Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. 4th edn. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 34. Springer, Berlin (2008)
53. Struwe, M.: A flow approach to Nirenberg problem. Duke Math. J. **128**(1), 19–64 (2005)
54. Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. **80**(3), 381–402 (1981)
55. Ye, R.: Global existence and convergence of Yamabe flow. J. Differ. Geom. **39**(1), 35–50 (1994)