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Isoperimetric Inequality on a Metric Measure Space and Lipschitz Order with an Additive Error

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Abstract
In this paper, we obtain the stability of isoperimetric inequalities with respect to the concentrate topology. The concentration topology is weaker than the □-topology which is like the weak topology. As an application, we obtain isoperimetric inequalities on the non-discrete n-dimensional l^1-cube and l^1-torus by taking the limits of isoperimetric inequalities of discrete l^1-cubes and l^1-torus. The method of this paper builds on by introducing an ε-relaxed (iso-)Lipschitz order.

Keywords Metric measure space · Lipschitz order · 1-measurement · Isoperimetric inequality · l^1-Minkowski distance · Torus with l^1-Minkowski metric · The concentration of measure phenomenon · Concentration topology · Observable distance · Observable diameter

Mathematics Subject Classification Primary 53C23 · Secondary 53C20

1 Introduction

Isoperimetric inequalities are simple and interesting geometric inequalities that have been studied for a long time. Exact solutions are known in some basic spaces, such as the n-dimensional Euclidean space and the n-dimensional sphere. A more detailed list is seen in [8, Appendix H]. Isoperimetric inequalities on metric measure spaces with lower Ricci curvature bounds is studied in [7].

Gromov [12] introduced the Lipschitz order relation on the space of all metric measure spaces and developed a rich theory. In this paper, we focus on applications of the Lipschitz order relation to isoperimetric inequalities. This is a relatively new

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approach to isoperimetric inequalities. Gromov claimed Corollary 3.21 in Sect. 3 which states that an isoperimetric inequality on a continuous metric measure space is represented by using the Lipschitz order under some assumptions. The main results of this paper are Theorems 1.7 and 1.8 that appear later. Theorem 1.7 is a generalization of the Gromov's claim. Theorem 1.8 express a stability of isoperimetric inequalities with respect to the concentrate topology. The concentration topology is defined by the observable distance d_{conc} (Definition 2.21) and weaker than the $\square$-topology (Definition 2.15) which is like the weak topology of measures. These theorems make it possible to treat discrete and continuous spaces in the same way. One of the most important applications is to obtain an isoperimetric inequality on a continuous space by the limit operation of discrete spaces. For example, we obtain the following sharp isoperimetric inequality on the continuous n-dimensional hypercube $[0, 1]^n$ with the l^1-metric d_{l^1} and the uniform measure $\mathcal{L}^n$.

Theorem 1.1 *For any closed subset $\Omega \subset [0, 1]^n$ with $\mathcal{L}^n(\Omega) > 0$, we take the metric ball $B_\Omega \subset [0, 1]^n$ centered at the origin with $\mathcal{L}^n(B_\Omega) = \mathcal{L}^n(\Omega)$. Then we have*

$$\mathcal{L}^n(U_r(\Omega)) \geq \mathcal{L}^n(U_r(B_\Omega))$$

for any $r > 0$, where $U_r(A) := \{x \in [0, 1]^n \mid d_{l^1}(x, A) < r\}$ is the open r-neighborhood of a subset $A \subset [0, 1]^n$.

Similarly, we obtain the following sharp isoperimetric inequality of the l^1-torus T^n by using Corollary 6 in [4]. The l^1-torus T^n is the n-fold l^1-product of the one-dimensional unit sphere S^1 equipped with the uniform measure.

Theorem 1.2 *For any closed subset $\Omega \subset T^n$ with $m_{T^n}(\Omega) > 0$, we take a metric ball B_Ω of T^n with $m_{T^n}(B_\Omega) = m_{T^n}(\Omega)$. Then we have*

$$m_{T^n}(U_r(\Omega)) \geq m_{T^n}(U_r(B_\Omega))$$

for any $r > 0$, where $U_r(A)$ is the open r-neighborhood of a subset $A \subset T^n$ with respect to the l^1-metric.

There are only few spaces satisfying the inequalities in Theorems 1.1 and 1.2 because these are required to hold for any $r > 0$. As a former result, Lévy's isoperimetric inequality (Theorem 2.11) also holds for any $r > 0$. It is a sharp isoperimetric inequality on a unit sphere in an Euclidean space. The isoperimetric inequality on the n-dimensional standard Gaussian space [6,23] is also of the same type.

A usual isoperimetric inequality is given in the case that $r > 0$ is small. There are many variants of Theorem 1.1 and 1.2, if these are not required to hold for any $r > 0$. In [22], the isoperimetric profiles on a l^p-ball in the Euclidean space $\mathbb{R}^n$ is calculated, where $1 \leq p \leq 2$. We note that Theorem 1.1 is the sharp isoperimetric inequality on a l^∞-ball in the Banach space $\mathbb{R}^n$ equipped with the l^1-metric. In [15], a quantitative isoperimetric inequality on the Banach space $\mathbb{R}^2$ equipped with the l^1-metric is given. We remark that the boundary measure is given by the l^1-length in [15] but it is given by Minkowski content with respect to the l^1-metric in Theorem 1.1. In [3], the Cheeger constants on product spaces of metric measure spaces are calculated, where product spaces is equipped with the l^2-product metric. In [13], the concentration function on the unit sphere in a uniformly convex vector space is studied. In Sect. 4.3, we calculate the concentration function on the hypercubes $[0, 1]^n$ equipped with the l^1-metric. In [24], an isoperimetric inequality on the product space equipped with the Talagrand's convex distance is studied. That is a generalization of an isoperimetric inequality on the n-dimensional Hamming cube. The Wulff shapes may be related with Theorem 1.1 but it is nontrivial that there exists an energy which gives Minkowski content with respect to the l^1-metric. The Wulff shape in $\mathbb{R}^n$ is studied in [25].

In this paper, we deal with two key concepts, called "ICL condition" and "iso-dominant". Roughly speaking, "ICL condition" means that an isoperimetric inequality holds for any $r > 0$. The concept "iso-dominant" also means isoperimetric inequality but it is defined by using the (iso-)Lipschitz order. Theorem 1.5 below means that the two concepts are equivalent to each other in some assumptions. However, these assumptions are incompatible with non-continuous spaces. Therefore, we introduce ε-relaxed notions (Definitions 1.3 and 1.6) of those concepts. Theorem 1.7, which appears later, expresses their equivalence. The iso-dominant is kept under taking a limit, as is stated in Theorem 1.8 below.

First, we formulate the form of isoperimetric inequalities. On a general metric measure space, we consider a Lévy type isoperimetric inequality. Namely, we consider that the open r-neighborhood in isoperimetric inequalities for any $r > 0$. Let (X, d_X) be a complete separable metric space with a Borel probability measure m_X. We call such a triple (X, d_X, m_X) an *mm-space* (which is an abbreviation of a metric measure space). If we say that X is an mm-space, the metric and the measure are respectively indicated by d_X and m_X.

Definition 1.3 (Isoperimetric comparison condition of Lévy type; cf. [20]) We say that an mm-space X satisfies the *isoperimetric comparison condition of Lévy type* $\mathrm{ICL}_\varepsilon(\nu)$ for a Borel probability measure ν on $\mathbb{R}$ and a real number $\varepsilon \geq 0$ if we have $V(b) \leq m_X(B_{b-a+\varepsilon}(A))$ for any $a, b \in \operatorname{supp}\nu$ with $a \leq b$ and for any Borel subset $A \subset X$ with $m_X(A) > 0$ and $V(a) \leq m_X(A)$, where $V(t) := \nu((-\infty, t])$ is the cumulative distribution function of ν. We abbreviate $\mathrm{ICL}_0(\nu)$ as $\mathrm{ICL}(\nu)$.

We remark that Definition 1.3 is only defined in the case $\varepsilon = 0$ in [20]. The 1-*measurement* of an mm-space X is defined as

$$\mathcal{M}(X; 1) := \{\varphi_* m_X \mid \varphi : X \to \mathbb{R} \text{ is } 1-\text{Lipschitz}\},$$

where $\varphi_* m_X$ is the push-forward measure of m_X by φ and a 1-*Lipschitz function* is a Lipschitz continuous function with Lipschitz constant less than or equal to one. We denote by $\mathcal{P}(\mathbb{R})$ the set of all Borel probability measures on $\mathbb{R}$ and we see $\mathcal{M}(X; 1) \subset \mathcal{P}(\mathbb{R})$. In the case that $\nu \in \mathcal{M}(X; 1)$, the $\mathrm{ICL}(\nu)$ condition for X means a sharp isoperimetric inequality on X. In fact, if X satisfies $\mathrm{ICL}(\varphi_* m_X)$ for some 1-Lipschitz function $\varphi : X \to \mathbb{R}$, then we have

$$\begin{aligned} m_X(B_r(\Omega)) &\geq m_X(\varphi^{-1}(B_r((-\infty, t]))) \\ &\geq m_X(B_r(\varphi^{-1}((-\infty, t]))) \end{aligned}$$

for any $t \in \operatorname{supp}(\varphi_* m_X)$ and any $r > 0$ with $t + r \in \operatorname{supp}(\varphi_* m_X)$, where a Borel subset $\Omega \subset X$ with $m_X(\Omega) > 0$ satisfies $m_X(\Omega) \geq m_X(\varphi^{-1}((-\infty, t]))$. This means that the subset $\varphi^{-1}((-\infty, t]) \subset X$ is an extremal set for any $t \in \operatorname{supp}(\varphi_* m_X)$. Lévy's isoperimetric inequality is paraphrased as $S^n(1)$ satisfies $\mathrm{ICL}(\xi_* m_{S^n(1)})$, where $\xi : S^n(1) \to \mathbb{R}$ is the distance function from one point. The set of iso-mm-isomorphism class of $\mathcal{P}(\mathbb{R})$ has an order relation called the iso(perimetrically)-Lipschitz order (see Definitions 3.2, 3.3 and Proposition 3.4).

Gromov defined an iso-dominant using the iso-Lipschitz order and claimed that an iso-dominant recollects the isoperimetric inequality [11].

Definition 1.4 (Iso-dominant [11]) We call a Borel probability measure on $\mathbb{R}$ an *iso-dominant* of an mm-space X if it is an upper bound of $\mathcal{M}(X; 1)$ with respect to the iso-Lipschitz order $\succ'$. That means $\nu \succ' \mu$ for all $\mu \in \mathcal{M}(X; 1)$.

We have the following relation between an iso-dominant and ICL.

Theorem 1.5 *([20]) Let X be an mm-space and ν a Borel probability measure on $\mathbb{R}$ with connected support. Assume that the cumulative distribution function V of ν is continuous. Then, X satisfies* $\mathrm{ICL}(\nu)$ *if and only if ν is an iso-dominant of X.*

Gromov claim Corollary 3.21 in Sect. 3 without the proof in Section 9 in [11]. It is a variant of Theorem 1.5. We focus on the continuity of V in Theorem 1.5. Without the continuity of V, we find the following counterexample of Theorem 1.5. We put $[k] := \{0, \ldots, k-1\}$ and consider the n-dimensional discrete cube $[k]^n$ equipped with the l^1-metric and the uniform measure, say $m_{[k]^n}$. Then, $[k]^n$ satisfies $\mathrm{ICL}((d_0)_* m_{[k]^n})$, where d_0 is the distance function from the origin [5]. Since the cumulative distribution function of $(d_0)_* m_{[k]^n}$ is not continuous, we are not able to apply Theorem 1.5 with $[k]^n$ as an mm-space X. In fact, $(d_0)_* m_{[k]^n}$ is not an iso-dominant of $[k]^n$. However, we regard $(d_0)_* m_{[k]^n}$ as an iso-dominant of $[k]^n$ if we allow an error. This is one of our motivations of introducing the iso-Lipschitz order with an error (see Definition 3.23).

The iso-Lipschitz order $\succ'_{(s,t)}$ with error (s, t) satisfies some beneficial properties such as Theorems 3.26, 3.28, and 3.31 in Sect. 3.2. Now, we define the iso-dominant with an error by using the iso-Lipschitz order with an error.

Definition 1.6 [ε-iso-dominant] Let $\varepsilon \geq 0$ be a real number. We call a Borel probability measure ν on $\mathbb{R}$ an *ε-iso-dominant* of an mm-space X if we have $\nu \succ'_{(\varepsilon,0)} \mu$ for all $\mu \in \mathcal{M}(X; 1)$.

We have the following Theorem 1.7, which explains the relation between ε-iso-dominants and $\mathrm{ICL}_\varepsilon(\nu)$. Theorem 1.7 is a generalization of Theorem 1.5.

Theorem 1.7 *Let X be an mm-space and ν a Borel probability measure on $\mathbb{R}$, and let $\varepsilon \geq 0$. We define*

$$\Delta(\operatorname{supp} \nu) := \sup\{\, \delta^-(\operatorname{supp} \nu; a) \mid a \in \operatorname{supp} \nu \setminus \{\inf \operatorname{supp} \nu\} \,\},$$

where $\delta^-(\operatorname{supp} \nu; a) := \inf\{\, t > 0 \mid a - t \in \operatorname{supp} \nu \,\}$. Then we have the following (1) and (2).

(1) *If* $\inf \operatorname{supp} \nu > -\infty$*, we assume* $\nu(\{\inf \operatorname{supp} \nu\}) \le m_X(\{x\})$ *for any* $x \in \operatorname{supp} m_X$*. Then,* ν *is an* $(\varepsilon + \Delta(\operatorname{supp} \nu))$*-iso-dominant of* X *if* X *satisfies* $\mathrm{ICL}_\varepsilon(\nu)$.

(2) *Assume that* $\operatorname{supp} \nu$ *is connected or* $\nu(\{x\}) > 0$ *for any* $x \in \operatorname{supp} \nu$*. Then,* X *satisfies* $\mathrm{ICL}_{2\varepsilon}(\nu)$ *if* ν *is an* ε*-iso-dominant of* X.

Theorem 1.7 implies that $(d_0)_* m_{[k]^n}$ is a 1-iso-dominant of the l^1-discrete hypercube $[k]^n$ since $[k]^n$ satisfies $\mathrm{ICL}((d_0)_* m_{[k]^n})$ (See Example 3.38). It is important to note that we cannot eliminate the term $\Delta(\operatorname{supp} \nu)$ from (1). This is because $(d_0)_* m_{[k]^n}$ is not an 0-iso-dominant of $[k]^n$, even though $[k]^n$ satisfies $\mathrm{ICL}_0((d_0)_* m_{[k]^n})$.

We will show that the condition that ν is an ε-iso-dominant of X is stable under the convergence with respect to the Prokhorov metric d_{P} and the observable distance function d_{conc}. This property enables us to obtain the isoperimetric inequality of a continuous space by using a discretization. The following Theorem 1.8 is one of the main theorems of this paper and represents the stability of ε-iso-dominant.

Theorem 1.8 *Let* X *and* X_n, $n = 1, 2, \ldots,$ *be mm-spaces, let* ν *and* ν_n, $n = 1, 2, \ldots,$ *be Borel probability measures on* $\mathbb{R}$*, and let* ε_n, $n = 1, 2, \ldots$ *be non-negative real numbers. We assume that* $\{X_n\}_n$ d_{conc}*-converges to* X *and* $\{\nu_n\}_n$ *weakly converges to* ν*, and* $\{\varepsilon_n\}_n$ *converges to a real number* ε *as* $n \to \infty$ *and that* ν_n *is an* ε_n*-iso-dominant of* X_n *for any positive integer* n*. Then,* ν *is an* ε*-iso-dominant of* X.

We remark that the distance d_{conc} gives the concentration topology. It is weaker than the $\square$-topology which is like the weak topology of measures.

Now, we obtain a sharp isoperimetric inequality of the n-dimensional continuous l^1-hypercube $[0, 1]^n$. The following Theorem 1.9 is one of the applications of Theorem 1.8. The proof of Theorem 1.8 is in Sect. 4.

Theorem 1.9 *The measure* $(d_0)_* m_{[0,1]^n}$ *is the greatest element of* $\mathcal{M}([0, 1]^n; 1)$ *with respect to the iso-Lipschitz order* $\succ'$*, where* d_0 *is the distance function from the origin.*

By Theorems 1.5 and 1.9, the l^1-hypercube $[0, 1]^n$ satisfies $\mathrm{ICL}((d_0)_* \, m_{[0,1]^n})$. This implies Theorem 1.1.

Similarly, we obtain the following Theorem 1.10 by using Corollary 6 in [4].

Theorem 1.10 *The measure* $\xi_* m_{T^n}$ *is the greatest element of* $\mathcal{M}(T^n; 1)$ *with respect to the iso-Lipschitz order* $\succ'$*, where* ξ *is the distance function from one point.*

By Theorems 1.5 and 1.10, the l^1-torus T^n satisfies $\mathrm{ICL}(\xi_* m_{T^n})$. This yields Theorem 1.2.

Obtaining sharp isometric inequalities using a similar method requires constraints on spaces. If the 1-measurement $\mathcal{M}(M; 1)$ of an compact Riemannian homogeneous space M has the greatest element ν, then M is only a round sphere [19]. Furthermore, a necessary condition for the existence of the maximum of the 1-measurement is given in Theorem 1.9 in [20].

If the 1-measurement $\mathcal{M}(X; 1)$ of an mm-space X has the greatest element ν, then we obtain the precise value of the observable diameter $\mathrm{ObsDiam}(X; -\kappa)$ of X (Definition 2.9) because we have

$$\mathrm{ObsDiam}(X; -\kappa) = \mathrm{diam}(\nu; 1 - \kappa) \quad \text{for any } \kappa \in (0, 1].$$

Hence, we obtain the value of $\mathrm{ObsDiam}([0,1]^n; -\kappa)$ and $\mathrm{ObsDiam}(T^n; -\kappa)$ for any $\kappa \in (0,1]$. As former results, the n-dimensional unit sphere is known to be an mm-space whose 1-measurement has the greatest element (see §9 in [11]). The n-dimensional standard Gaussian space is also such an mm-space by an isoperimetric inequality [6,23]. In Sect. 4.2, we calculate the observable diameters of some spaces as one of the applications of the iso-Lipschitz order with an additive error.

2 Preliminaries

In this section, we present some basics of mm-spaces. We refer to [12,21] for more details about the contents of this section.

2.1 Some Basics of mm-spaces

Definition 2.1 (mm-space) Let (X, d_X) be a complete separable metric space and m_X a Borel probability measure on X. We call such a triple (X, d_X, m_X) an *mm-space*. We sometimes say that X is an mm-space, for which the metric and measure of X are respectively indicated by d_X and m_X. We put $tX := (X, td_X, m_X)$ for $t > 0$. Since an mm-space is equipped with a probability measure, it is nonempty.

We denote the Borel σ-algebra over X by $\mathcal{B}_X$. For any point $x \in X$, any two subsets $A, B \subset X$ and any real number $r \geq 0$, we define

$$\begin{aligned} d_X(x, A) &:= \inf\{\, d_X(x, y) \mid y \in A \,\}, \\ d_X(A, B) &:= \inf\{\, d_X(x, y) \mid x \in A,\ y \in B \,\}, \\ U_r(A) &:= \{\, y \in X \mid d_X(y, A) < r \,\}, \\ B_r(A) &:= \{\, y \in X \mid d_X(y, A) \leq r \,\}, \end{aligned}$$

where $\inf \emptyset := \infty$. We remark that $U_r(\emptyset) = B_r(\emptyset) = \emptyset$ for any real number $r \geq 0$. The *diameter of* A is defined by $\mathrm{diam} A := \sup_{x,y \in A} d_X(x, y)$ for $A \neq \emptyset$ and $\mathrm{diam}\emptyset := 0$.

Let Y be a topological space and let $p : X \to Y$ be a measurable map from a measure space (X, m_X) to a Borel space $(Y, \mathcal{B}_Y)$. *The push-forward* $p_* m_X$ *of* m_X *by the map* p is defined as $p_* m_X(A) := m_X(p^{-1}(A))$ for any $A \in \mathcal{B}_Y$.

Definition 2.2 (support) Let (X, d_X) be a metric space and m_X a Borel measure on X. We define the *support* $\mathrm{supp}\, m_X$ of m_X by

$$\mathrm{supp}\, m_X := \{x \in X \mid m_X(U_r(x)) > 0 \text{ for any } r > 0\}.$$

Proposition 2.3 *Let (X, d_X) be a metric space and m_X a Borel measure on X. Let Y be a separable metric space. Let $f : X \to Y$ be a continuous map. Then we have*

$$\mathrm{supp}\, f_* m_X = \overline{f(\mathrm{supp}\, m_X)}.$$

Proof Since

$$\begin{aligned} f_* m_X(Y \setminus \overline{f(\operatorname{supp} m_X)}) &= m_X(X \setminus f^{-1}(\overline{f(\operatorname{supp} m_X)})) \\ &\le m_X(Y \setminus \operatorname{supp} m_X) = 0, \end{aligned}$$

we have $\operatorname{supp} f_* m_X \subset \overline{f(\operatorname{supp} m_X)}$ because Y is separable.

Next, let us prove

$$f(\operatorname{supp} m_X) \subset \operatorname{supp} f_* m_X. \tag{2.1}$$

Take any $y \in f(\operatorname{supp} m_X)$. There exists $x \in \operatorname{supp} m_X$ such that $y = f(x)$. Take any positive real number $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that

$$U_\delta(x) \subset f^{-1}(U_\varepsilon(y)).$$

Then we have

$$f_* m_X(U_\varepsilon(y)) \ge m_X(U_\delta(x)) > 0$$

and obtain (2.1). Because $\operatorname{supp} f_* m_X$ is closed, we have

$$\overline{f(\operatorname{supp} m_X)} \subset \operatorname{supp} f_* m_X.$$

This completes the proof. □

Definition 2.4 (mm-isomorphism) Two mm-spaces X and Y are said to be *mm-isomorphic* if there exists an isometry $f : \operatorname{supp} m_X \to \operatorname{supp} m_Y$ such that $f_* m_X = m_Y$, where $\operatorname{supp} m_X$ is the support of m_X. Such an isometry f is called an *mm-isomorphism*. The mm-isomorphism relation is an equivalence relation on the set of mm-spaces. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.

Definition 2.5 (Lipschitz order) Let X and Y be two mm-spaces. We say that X *dominates* Y and write $Y \prec X$ if there exists a 1-Lipschitz map $f : X \to Y$ satisfying

$$f_* m_X = m_Y.$$

We call the relation $\prec$ on $\mathcal{X}$ the *Lipschitz order*.

Proposition 2.6 *(Proposition 2.11 in [21]) The Lipschitz order $\prec$ is a partial order relation on $\mathcal{X}$.*

Definition 2.7 (Transport plan) Let μ and ν be two Borel probability measures on $\mathbb{R}$. We say that a Borel probability measure on $\mathbb{R}^2$ is a transport plan between μ and ν if we have $(\mathrm{pr}_1)_* \pi = \mu$ and $(\mathrm{pr}_2)_* \pi = \nu$, where pr_1 and pr_2 is the first and second projection respectively. We denote by $\Pi(\mu, \nu)$ the set of transport plans between μ and ν.

2.2 Observable Diameter and Partial Diameter

Observable diameter is one of the most important invariants among all invariants for mm-spaces. We remark that the 1-measurement appears in the definition of the observable diameter.

Definition 2.8 (Partial diameter) Let X be an mm-space and let $\alpha \in [0, 1]$ be a real number. We define the *α-partial diameter* $\mathrm{diam}(X; \alpha)$ *of* X as

$$\mathrm{diam}(X; \alpha) := \inf\{ \mathrm{diam} A \mid m_X(A) \geq \alpha,\ A \in \mathcal{B}_X \}.$$

For any Borel probability measure μ on $\mathbb{R}$, we set

$$\mathrm{diam}(\mu; \alpha) := \mathrm{diam}((\mathbb{R}, |\cdot|, \mu); \alpha).$$

Definition 2.9 (Observable diameter) Let X be an mm-space. For any real number $\kappa \in [0, 1]$, we define the *κ-observable diameter* $\mathrm{ObsDiam}(X; -\kappa)$ *of* X as

$$\mathrm{ObsDiam}(X; -\kappa) := \sup_{\mu \in \mathcal{M}(X;1)} \mathrm{diam}(\mu; 1 - \kappa).$$

Proposition 2.10 *(Proposition 2.18 in [21]) Let X and Y be two mm-spaces and $\kappa \in [0, 1]$ a real number. If $Y \prec X$, then we obtain*

$$\mathrm{diam}(Y; 1 - \kappa) \leq \mathrm{diam}(X; 1 - \kappa),$$
$$\mathrm{ObsDiam}(Y; -\kappa) \leq \mathrm{ObsDiam}(X; -\kappa).$$

2.3 Lévy's Isoperimetric Inequality

Let $S^n(r)$ be the n-dimensional sphere of radius $r > 0$ centered at the origin in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let the distance $d_{S^n(r)}(x, y)$ between two points x and y in $S^n(r)$ be the geodesic distance, and let the measure $m_{S^n(r)}$ on $S^n(r)$ be the Riemannian volume measure on $S^n(r)$ normalized as $m_{S^n(r)}(S^n(r)) = 1$. Then, $(S^n(r),\ d_{S^n(r)},\ m_{S^n(r)})$ is an mm-space.

Theorem 2.11 *(Lévy's isoperimetric inequality [10,16]) For any nonempty closed subset $\Omega \subset S^n(1)$, we take a metric ball B_Ω of $S^n(1)$ with $m_{S^n(1)}(B_\Omega) = m_{S^n(1)}(\Omega)$. Then we have*

$$m_{S^n(1)}(U_r(\Omega)) \geq m_{S^n(1)}(U_r(B_\Omega))$$

for any $r > 0$.

2.4 Box Distance and Observable Distance

In this section, we briefly describe the box distance function and the observable distance function.

Definition 2.12 (Parameter) Let $I := [0, 1)$ and let $\mathcal{L}^1$ be the Lebesgue measure on I. Let X be a topological space equipped with a Borel probability measure m_X. A map $\varphi : I \to X$ is called a *parameter of* X if φ is a Borel measurable map such that

$$\varphi_* \mathcal{L}^1 = m_X.$$

Lemma 2.13 *(Lemma 4.2 in [21]) Any mm-space has a parameter.*

Definition 2.14 (Pseudo-metric) A *pseudo-metric* ρ *on a set* S is defined to be a function $\rho : S \times S \to [0, \infty)$ satisfying

(1) $\rho(x, x) = 0$,
(2) $\rho(y, x) = \rho(x, y)$,
(3) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$

for any $x, y, z \in S$

If ρ is a metric, $\rho(x, y) = 0$ implies $x = y$ for any two points $x, y \in S$. However, a pseudo-metric does not necessarily satisfy this condition.

Definition 2.15 (Box distance) For two pseudo-metrics ρ_1 and ρ_2 on $I := [0, 1)$, we define $\square(\rho_1, \rho_2)$ to be the infimum of $\varepsilon \ge 0$ such that there exists a Borel subset $I_0 \subset I$ satisfying

(1) $|\rho_1(s, t) - \rho_2(s, t)| \le \varepsilon$ for any $s, t \in I_0$,
(2) $\mathcal{L}^1(I_0) \ge 1 - \varepsilon$.

We define the *box distance* $\square(X, Y)$ *between two mm-spaces* X *and* Y to be the infimum of $\square(\varphi^* d_X, \psi^* d_Y)$, where $\varphi : I \to X$ and $\psi : I \to Y$ run over all parameters of X and Y, respectively, and where $\varphi^* d_X(s, t) := d_X(\varphi(s), \varphi(t))$ for $s, t \in I$.

Theorem 2.16 *(Theorem 4.10 in [21]) The function* $\square$ *is a metric on the set* $\mathcal{X}$ *of mm-isomorphism classes of mm-spaces.*

Definition 2.17 (Prokhorov metric) The Prokhorov metric d_P is defined by

$$d_P(\mu, \nu) := \inf\{\, \varepsilon > 0 \mid \mu(U_\varepsilon(A)) \ge \nu(A) - \varepsilon \text{ for any Borel set } A \subset X \,\}$$

for any two Borel probability measures μ and ν on a metric space X.

Proposition 2.18 *(Proposition 4.12 in [21]) For any two Borel probability measures* μ *and* ν *on a complete separable metric space* X*, we have*

$$\square((X, \mu), (X, \nu)) \le 2 d_P(\mu, \nu).$$

Definition 2.19 (Ky Fan metric) Let (X, μ) be a measure space. For two μ-measurable maps $f, g : X \to \mathbb{R}$, we define the *Ky Fan metric* $d_{KF} = d_{KF}^{\mu}$ by

$$d_{KF}^{\mu}(f, g) := \inf\{\, \varepsilon \ge 0 \mid \mu(\{\, t \in I \mid |f(t) - g(t)| > \varepsilon \,\}) \le \varepsilon \,\}.$$

Lemma 2.20 *(Lemma 1.26 in [21]) Let (X, μ) be a measure space. For two μ-measurable maps $f, g : X \to \mathbb{R}$, we have*

$$d_{\mathrm{P}}(f_*\mu, g_*\mu) \le d_{\mathrm{KF}}^{\mu}(f, g).$$

Definition 2.21 (Observable distance) For a parameter φ of an mm-space X, we define

$$\mathcal{L}ip_1(X) := \{ f : X \to \mathbb{R} \mid \text{1-Lipschitz} \}$$

and

$$\varphi^* \mathcal{L}ip_1(X) := \{ f \circ \varphi \mid f \in \mathcal{L}ip_1(X) \}.$$

The Hausdorff distance function $d_{\mathrm{H}}^{\mathrm{KF}}$ is defined by

$$d_{\mathrm{H}}^{\mathrm{KF}}(A, B) := \inf\{\varepsilon > 0 \mid A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}$$

for two subsets A and B of Borel measurable functions from I, where the open ε-neighborhood of A is defined by

$$U_\varepsilon(A) := \{g : I \to \mathbb{R} \mid d_{\mathrm{KF}}^{\mathcal{L}^1}(A, g) < \varepsilon\}.$$

We define the *observable distance* d_{conc} between two mm-spaces X and Y by

$$d_{\mathrm{conc}}(X, Y) := \inf_{\varphi, \psi} d_{\mathrm{H}}^{\mathrm{KF}}(\varphi^* \mathcal{L}ip_1(X), \psi^* \mathcal{L}ip_1(Y))$$

where $\varphi : I := [0, 1) \to X$ and $\psi : I \to Y$ are two parameters of X and Y respectively.

Theorem 2.22 *(Theorem 5.13 in [21]) The function d_{conc} is a metric on $\mathcal{X}$.*

Proposition 2.23 *(Proposition 5.5 in [21]) For two mm-spaces X and Y, we have $d_{\mathrm{conc}}(X, Y) \le \square(X, Y)$.*

3 Isoperimetric Comparison Condition

3.1 Isoperimetric Comparison Condition Without an Error

In this subsection, we investigate the relation between iso-dominant and isoperimetric comparison condition. We refer to [20] for more details about the contents of this subsection. The main aim of this subsection is to introduce the following Theorems 3.5 and 3.7 which are extensions of Theorem 1.5. In a continuous space, isoperimetric problem is represented in terms of an isoperimetric profile. Let X be an mm-space. The *boundary measure* of a Borel set $A \subset X$ is defined to be

$$m_X^+(A) := \limsup_{\varepsilon \to 0+} \frac{m_X(U_\varepsilon(A)) - m_X(A)}{\varepsilon}.$$

Set

$$\operatorname{Im} m_X := \{\, m_X(A) \mid A \subset X \text{ is Borel} \,\}.$$

The *isoperimetric profile* $I_X : \operatorname{Im} m_X \to [0, +\infty)$ of X is defined by

$$I_X(v) := \inf\{\, m_X^+(A) \mid A \subset X : \text{Borel},\ m_X(A) = v \,\}$$

for $v \in \operatorname{Im} m_X$. The following isoperimetric comparison condition is a generalization of an isoperimetric inequality. It is a derivative of the ICL condition.

Definition 3.1 (Isoperimetric comparison condition [20]) We say that X satisfies the *isoperimetric comparison condition* $\mathrm{IC}(\nu)$ for a Borel probability measure ν on $\mathbb{R}$ if

$$I_X \circ V \geq V' \quad \mathcal{L}^1\text{-a.e. on } V^{-1}(\operatorname{Im} m_X),$$

where V means the cumulative distribution function of ν and $\mathcal{L}^1$ the one-dimensional Lebesgue measure on $\mathbb{R}$.

We define the following iso-Lipschitz order in order to define an iso-dominant (See Definition 1.4).

Definition 3.2 (Iso-Lipschitz order [11, §9]) Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$. We say that μ *iso-dominates* ν if there exists a monotonically non-decreasing 1-Lipschitz function f : $\operatorname{supp}\mu \to \operatorname{supp}\nu$ such that $f_*\mu = \nu$, where $\operatorname{supp}\mu$ is the support of μ. We write $\mu \succ' \nu$ if μ iso-dominates ν.

Definition 3.3 (Iso-mm-isomorphism) Two Borel probability measures μ and ν on $\mathbb{R}$ are said to be *iso-mm-isomorphic* if there exists a real number c such that $(\mathrm{id}_{\mathbb{R}} + c)_*\mu = \nu$, where $\mathrm{id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$. The iso-mm-isomorphism relation is an equivalence relation on the set of Borel probability measures on $\mathbb{R}$.

In the definition of iso-dominant (Definition 1.4), an upper bound of a 1-measurement appears. An upper bound is defined on a partially ordered set. The following Proposition 3.4 asserts that a 1-measurement is a partially ordered set.

Proposition 3.4 *The iso-Lipschitz order is a partial order on the set of iso-mm-isomorphism class of Borel probability measures on* $\mathbb{R}$.

The purpose of this subsection is to introduce the following Theorem 3.5 which is an extension of Theorem 1.5 because one of the aims of this paper is to prove Theorem 1.7 which is a generalization of Theorem 1.5.

Denote by $\mathcal{V}$ the set of Borel probability measures on $\mathbb{R}$ absolutely continuous with respect to the one-dimensional Lebesgue measure $\mathcal{L}^1$ and with connected support.

An mm-space X is said to be *essentially connected* if we have $m_X^+(A) > 0$ for any closed set $A \subset X$ with $0 < m_X(A) < 1$.

Theorem 3.5 *([20]) Let X be an essentially connected mm-space and let $\nu \in \mathcal{V}$. Then the following* (1), (2)*, and* (3) *are all equivalent.*

(1) The measure ν is an iso-dominant of X.
(2) The space X satisfies $\mathrm{ICL}(\nu)$.
(3) The space X satisfies $\mathrm{IC}(\nu)$.

Gromov claim the following Corollary 3.21 without the proof in Sect. 9 in [11]. It is a variant of Theorem 3.5. We prepare to state it and extend Theorem 3.5 to prove it.

Denote $\mathcal{F}(X)$ by the set of all closed subsets of X. Put $m_X(\mathcal{F}(X)) := \{ m_X(A) \mid A \in \mathcal{F}(X) \}$. The *isoperimetric profile with respect to closed subsets* $I_X^{\mathrm{cl}} : m_X(\mathcal{F}(X)) \to [0, +\infty)$ of X is defined by

$$I_X^{\mathrm{cl}}(v) := \inf\{ m_X^+(A) \mid A \subset X : \text{closed},\ m_X(A) = v \}$$

for $v \in m_X(\mathcal{F}(X))$.

If we obtain a Borel set $A_0 \subset X$ such that $I_X(m_X(A_0)) = m_X^+(A_0)$, an isoperimetric inequality on X is represented by

$$m_X^+(A_0) \leq m_X^+(A)$$

for any $A \subset X$ with $m_X(A) = m_X(A_0)$. The following Definition 3.6 is a variant of Definition 3.1.

Definition 3.6 (Isoperimetric comparison condition with respect to closed subsets) We say that X satisfies the *isoperimetric comparison condition with respect to closed subsets* $\mathrm{IC}^{\mathrm{cl}}(\nu)$ for a Borel probability measure ν on $\mathbb{R}$ if

$$I_X^{\mathrm{cl}} \circ V \geq V' \quad \mathcal{L}^1\text{-a.e. on } V^{-1}(m_X(\mathcal{F}(X))),$$

where V denotes the cumulative distribution function of ν.

The following Theorem 3.7 is an extension of Theorem 3.5.

Theorem 3.7 *Let X be an essentially connected mm-space and let $\nu \in \mathcal{V}$. Then the following* (1), (2), (3)*, and* (4) *are all equivalent.*

(1) The measure ν is an iso-dominant of X.
(2) The space X satisfies $\mathrm{ICL}(\nu)$.
(3) The space X satisfies $\mathrm{IC}(\nu)$.
(4) The space X satisfies $\mathrm{IC}^{\mathrm{cl}}(\nu)$.

Let us prove Theorem 3.7. By Theorem 3.5, it is satisfied to prove that (3) implies (4) and that (4) implies (2). The following Proposition 3.8 means that (3) implies (4). The following Theorem 3.15 means that (4) implies (2).

Proposition 3.8 *Let X be an mm-space and ν a Borel probability measure on $\mathbb{R}$. If X satisfies* $\mathrm{IC}(\nu)$*, then X satisfies* $\mathrm{IC}^{\mathrm{cl}}(\nu)$.

Proof This follows from $I_X \leq I_X^{\mathrm{cl}}$ on $m_X(\mathcal{F}(X))$. □

To prove the following Theorem 3.15, we define the following Definition 3.9 and we prepare some lemmas. These lemmas are also used in the proof of Theorem 1.7.

Definition 3.9 (Generalized inverse function) For a monotonically non-decreasing and right-continuous function $F : \mathbb{R} \to [0, 1]$ with

$$\lim_{t\to-\infty} F(t) = 0 \text{ and } \lim_{t\to+\infty} F(t) = 1,$$

we define a generalized inverse function $\tilde{F} : [0, 1] \to \mathbb{R}$ by

$$\tilde{F}(s) := \begin{cases} \inf\{\, t \in \mathbb{R} \mid s \leq F(t)\,\} & \text{if } s \in (0, 1), \\ 0 & \text{if } s = 0 \text{ or } s = 1 \end{cases}$$

for $s \in [0, 1]$.

Lemma 3.10 *For any F as in Definition 3.9, $\tilde{F}$ takes finite values and $\tilde{F}|_{(0,1)}$ is non-decreasing.*

Proof Fix any real number $s \in (0, 1)$ and define $A := \{\, t \in \mathbb{R} \mid s \leq F(t)\,\}$. The set A is nonempty because $\lim_{t\to\infty} F(t) = 1$. Since $\lim_{t\to-\infty} F(t) = 0$, there exists $t_0 \in \mathbb{R}$ such that $F(t_0) < s$. For any element $t \in A$, we have $F(t_0) < s \leq F(t)$. Since F is non-decreasing, the inequality $t_0 < t$ follows. This implies that t_0 is a lower bound of A. Hence, $\tilde{F}(s)$ takes finite values. The function $\tilde{F}$ is a non-decreasing function on $(0, 1)$ because we have $\{\, t \in \mathbb{R} \mid s' \leq F(t)\,\} \supset \{\, t \in \mathbb{R} \mid s \leq F(t)\,\}$ for any $0 < s' \leq s < 1$. This completes the proof. □

Lemma 3.11 *(cf. [19]) For any F as in Definition 3.9, we have the following* (1), (2)*, and* (3).

(1) $F \circ \tilde{F}(s) \geq s$ *for any real number s with* $0 \leq s < 1$.
(2) $\tilde{F} \circ F(t) \leq t$ *for any real number t with* $0 < F(t) < 1$.
(3) $\tilde{F}^{-1}((-\infty, t\,]) \setminus \{0, 1\} = (0, F(t)] \setminus \{1\}$ *for any real number t.*

Proof First we prove (1). If $s = 0$, we have (1) because $\mathrm{Im}F \subset [0, 1]$. Fix a real number $s \in (0, 1)$ and define $A := \{\, t \in \mathbb{R} \mid s \leq F(t)\,\} \neq \emptyset$. By the definition of infimum, we have

$$F(t') \geq \inf_{t\in A} F(t)$$

for any $t' \in A$. For any $t' > \inf A$, we have $t' \in A$ because F is non-decreasing. By this, we have

$$\lim_{t'\to\inf A+0} F(t') \geq \inf_{t\in A} F(t).$$

By the right continuity of F, we obtain

$$F(\inf A) \geq \inf_{t \in A} F(t).$$

Therefore, we have

$$\begin{aligned} F(\tilde{F}(s)) &= F(\inf A) \\ &\geq \inf_{t \in A} F(t) \\ &= \inf\{\, F(t) \mid s \leq F(t) \,\} \\ &\geq s. \end{aligned}$$

Next we prove (2). We take any real number $t \in \mathbb{R}$ with $0 < F(t) < 1$, then we have

$$\tilde{F}(F(t)) = \inf\{\, t' \in \mathbb{R} \mid F(t') \geq F(t) \,\} \leq t.$$

Last we prove (3). Take any real number $s \in \tilde{F}^{-1}((-\infty, t]) \setminus \{0, 1\}$. It follows from $\tilde{F}(s) \leq t$ and the non-decreasing property of F that $F \circ \tilde{F}(s) \leq F(t)$. This implies that $s \leq F(t)$ by (1) and we have $s \in (0, F(t)] \setminus \{1\}$. Conversely, take any real number $s \in (0, F(t)] \setminus \{1\}$. Since $s \leq F(t)$, we obtain $\tilde{F}(s) \leq t$ by the definition of $\tilde{F}(s)$. Hence $s \in \tilde{F}^{-1}((-\infty, t]) \setminus \{0, 1\}$. This completes the proof. □

Remark 3.12 The generalized inverse function $\tilde{F}$ of a function F is a Borel measurable function. In fact, $\tilde{F}|_{(0,1)}$ is monotonically non-decreasing.

Lemma 3.13 *Let μ be a Borel probability measure on $\mathbb{R}$ with cumulative distribution function F. Then we have*

$$\mu = \tilde{F}_* \mathcal{L}^1|_{[\,0,1\,]},$$

where $\mathcal{L}^1|_{[\,0,1\,]}$ is the one-dimensional Lebesgue measure on $[\,0, 1\,]$.

Proof By Lemma 3.11(3), we have

$$\begin{aligned} \tilde{F}_* \mathcal{L}^1|_{[\,0,1\,]}((\,-\infty, t\,]) &= \mathcal{L}^1|_{[\,0,1\,]}(\tilde{F}^{-1}((\,-\infty, t\,]) \setminus \{0, 1\}) \\ &= \mathcal{L}^1|_{[\,0,1\,]}((\,0, F(t)\,] \setminus \{1\}) \\ &= F(t) = \mu((\,-\infty, t\,]) \end{aligned}$$

for any $t > 0$. This completes the proof. □

Lemma 3.14 *(Lemma 3.13 in [20]) Let $g : \mathbb{R} \to \mathbb{R}$ be a monotonically non-decreasing function, $f : \mathbb{R} \to [\,0, +\infty\,)$ a Borel measurable function, and $A \subset \mathbb{R}$ a Borel set. Then we have*

$$\int_{g^{-1}(A)} (f \circ g) \cdot g' \, d\mathcal{L}^1 \leq \int_A f \, d\mathcal{L}^1.$$

Theorem 3.15 *Let X be an essentially connected mm-space and $\nu \in \mathcal{V}$. If X satisfies* $\mathrm{IC}^{\mathrm{cl}}(\nu)$*, then X satisfies* $\mathrm{ICL}(\nu)$.

Proof Setting $E := (\operatorname{supp}\nu)^{\circ}$, we easily see the bijectivity of $V|_E : E \to (0,1)$. We define a function $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(t) := \begin{cases} V'(t) & \text{for any } t \in V^{-1}(m_X(\mathcal{F}(X))) \text{ where } V \text{ is differentiable} \\ & \text{and such that } I_X^{\mathrm{cl}} \circ V(t) \geq V'(t), \\ 0 & \text{otherwise,} \end{cases}$$

for a real number t. We see that $\rho = V'$ $\mathcal{L}^1$-a.e. and that ρ is a density function of ν with respect to $\mathcal{L}^1$. Since $I_X^{\mathrm{cl}} \circ V \geq \rho$ everywhere on $V^{-1}(m_X(\mathcal{F}(X)))$, we have $I_X^{\mathrm{cl}} \geq \rho \circ (V|_E)^{-1}$ on $m_X(\mathcal{F}(X)) \setminus \{0,1\}$. To prove $\mathrm{ICL}(\nu)$, we take two real numbers $a, b \in \operatorname{supp}\nu$ with $a \leq b$ and a closed set $A \subset X$ with $m_X(A) > 0$ and $V(a) \leq m_X(A)$. Note that replacing a Borel set A by a closed set A in the Definition 1.3 is equivalent to the original definition. We may assume $m_X(B_{b-a}(A)) < 1$. Let s be any real number with $0 \leq s \leq b-a$. Remarking $m_X(B_s(A)) \in m_X(\mathcal{F}(X)) \setminus \{0,1\}$, we see

$$m_X^+(B_s(A)) \geq I_X^{\mathrm{cl}}(m_X(B_s(A))) \geq \rho \circ (V|_E)^{-1}(m_X(B_s(A))).$$

Setting $g(s) := m_X(B_s(A))$, we have

$$g'(s) = m_X^+(B_s(A)) \geq \rho \circ (V|_E)^{-1}(g(s)) \quad \mathcal{L}^1\text{-a.e. } s \geq 0$$

and hence

$$1 \leq \frac{g'(s)}{\rho \circ (V|_E)^{-1}(g(s))} \leq +\infty \quad \mathcal{L}^1\text{-a.e } s \in [0,+\infty),$$

where we remark that $g'(s) > 0$ by essential connectedness of X. Since $g(0) = m_X(A)$, we have

$$\begin{aligned} (V|_E)^{-1} \circ g(0) &= (V|_E)^{-1}(m_X(A)) \\ &\geq (V|_E)^{-1}(V(a)) = a \end{aligned}$$

if $V(a) > 0$. If $V(a) = 0$, we have $a = \inf \operatorname{supp}\nu \leq (V|_E)^{-1} \circ g(0)$ since $(V|_E)^{-1} \circ g(0) \in E = (\operatorname{supp}\nu)^{\circ}$. By Lemmas 3.14 and 3.13,

$$\begin{aligned} b - a &\leq \int_{[0,b-a]} g'(s) \cdot \left(\rho \circ (V|_E)^{-1}(g(s))\right)^{-1} ds \\ &\leq \int_{g^{-1}(g([0,b-a]))} g'(s) \cdot \left(\rho \circ (V|_E)^{-1}(g(s))\right)^{-1} ds \\ &\leq \int_{g([0,b-a])} \frac{d\mathcal{L}^1}{\rho \circ (V|_E)^{-1}} \end{aligned}$$

$$
\begin{aligned}
&= \int_{(V|_E)^{-1}\circ g([0,b-a])} \frac{1}{\rho}\, d((V|_E)^{-1}_* \mathcal{L}^1) \\
&= \int_{(V|_E)^{-1}\circ g([0,b-a])} \frac{1}{\rho}\, d\nu \\
&\leq \int_{(V|_E)^{-1}\circ g([0,b-a])} d\mathcal{L}^1 \\
&\leq \mathcal{L}^1([(V|_E)^{-1}\circ g(0), (V|_E)^{-1}\circ g(b-a)]) \\
&= (V|_E)^{-1}\circ g(b-a) - (V|_E)^{-1}\circ g(0) \\
&\leq (V|_E)^{-1}\circ g(b-a) - a,
\end{aligned}
$$

which implies

$$
V(b) \leq g(b-a) = m_X(B_{b-a}(A)).
$$

This completes the proof. □

This completes the proof of Theorem 3.7. At the end of this subsection, let us prove Corollary 3.21 by Theorem 3.7. We prepare some definitions and propositions to prove Corollary 3.21.

Proposition 3.16 *([20]) Let X and Y be mm-spaces such that X dominates Y. Then we have*

$$
m_Y(\mathcal{F}(Y)) \subset m_X(\mathcal{F}(X)) \quad \text{and} \quad I_X^{\mathrm{cl}} \leq I_Y^{\mathrm{cl}} \text{ on } m_Y(\mathcal{F}(Y)).
$$

In particular, if X satisfies $\mathrm{IC}^{\mathrm{cl}}(\nu)$ for a Borel probability measure ν on $\mathbb{R}$, then Y also satisfies $\mathrm{IC}^{\mathrm{cl}}(\nu)$.

Proof Since X dominates Y, there is a 1-Lipschitz map $f : X \to Y$ such that $f_* m_X = m_Y$. For any closed set $A \subset Y$, we see $f^{-1}(B_\varepsilon(A)) \supset B_\varepsilon(f^{-1}(A))$ by the 1-Lipschitz continuity of f, and hence

$$
\begin{aligned}
m_Y^+(A) &= \limsup_{\varepsilon \to +0} \frac{m_Y(B_\varepsilon(A)) - m_Y(A)}{\varepsilon} \\
&\geq \limsup_{\varepsilon \to +0} \frac{m_X(B_\varepsilon(f^{-1}(A))) - m_X(f^{-1}(A))}{\varepsilon} \\
&= m_X^+(f^{-1}(A)),
\end{aligned}
$$

which implies that, for any $v \in m_Y(\mathcal{F}(Y))$,

$$
I_Y^{\mathrm{cl}}(v) = \inf_{m_Y(A)=v} m_Y^+(A) \geq \inf_{m_X(f^{-1}(A))=v} m_X^+(f^{-1}(A)) \geq I_X^{\mathrm{cl}}(v).
$$

The rest is easy. This completes the proof. □

Definition 3.17 (Dominant [11, §9]) We call a Borel probability measure ν on $\mathbb{R}$ a *dominant* of an mm-space X if ν is an upper bound of $\mathcal{M}(X;1)$ with respect to the Lipschitz order $\succ$. That means $(\mathbb{R}, |\cdot|, \nu) \succ (\mathbb{R}, |\cdot|, \mu)$ for all $\mu \in \mathcal{M}(X;1)$. The Lipschitz order $\succ$ is defined in Definition 2.5.

Using Proposition 3.16, we prove the following Proposition 3.18.

Proposition 3.18 *(Gromov [11, §9]) If ν is a dominant of an mm-space X, then*

$$m_X(\mathcal{F}(X)) \subset \nu(\mathcal{F}(\mathbb{R})) \quad \textit{and} \quad I_\nu^{\mathrm{cl}} \le I_X^{\mathrm{cl}} \textit{ on } m_X(\mathcal{F}(X)),$$

where I_ν^{cl} is the isoperimetric profile with respect to closed subsets of $(\mathbb{R}, \nu)$.

Proof We take any real number $v \in m_X(\mathcal{F}(X))$ and fix it. If $v = 0$, then it is obvious that $v \in \nu(\mathcal{F}(\mathbb{R}))$ and $I_\nu^{\mathrm{cl}}(v) = 0 = I_X^{\mathrm{cl}}(v)$. Assume $v > 0$. For any $\varepsilon > 0$ there is a closed set $A \subset X$ such that $m_X(A) = v$ and $m_X^+(A) < I_X^{\mathrm{cl}}(v) + \varepsilon$. Note that A is nonempty because $v > 0$. Define a function $f : X \to \mathbb{R}$ by $f(x) := d_X(x, A)$. Then f is 1-Lipschitz continuous. Since $f_* m_X((-\infty, 0\,]) = m_X(A) = v$, we have

$$I_{f_* m_X}^{\mathrm{cl}}(v) \le (f_* m_X)^+((-\infty, 0]) = m_X^+(A) < I_X^{\mathrm{cl}}(v) + \varepsilon.$$

Since ν dominates $f_* m_X$, Proposition 3.16 implies that $v \in \nu(\mathcal{F}(\mathbb{R}))$ and $I_\nu^{\mathrm{cl}}(v) \le I_{f_* m_X}^{\mathrm{cl}}(v)$. We therefore have $I_\nu^{\mathrm{cl}}(v) < I_X^{\mathrm{cl}}(v) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $I_\nu^{\mathrm{cl}}(v) \le I_X^{\mathrm{cl}}(v)$. This completes the proof. □

Definition 3.19 [Iso-simpleness [11, §9]] A Borel probability measure ν on $\mathbb{R}$ is said to be *iso-simple* if $\nu \in \mathcal{V}$ and if

$$I_\nu \circ V = V' \quad \mathcal{L}^1\text{-a.e.}$$

Remark 3.20 For any Borel probability measure ν on $\mathbb{R}$, we always observe $I_\nu \circ V \le V'$ $\mathcal{L}^1$-a.e. In fact, we have

$$V'(t) = \nu^+((-\infty, t]) \ge \inf_{\nu(A) = V(t)} \nu^+(A) = I_\nu \circ V(t)$$

$\mathcal{L}^1$-a.e. t.

Gromov [11]*§9 stated the following corollary without proof.

Corollary 3.21 *(Gromov [11]*§9) Let X be an essentially connected mm-space and ν an iso-simple Borel probability measure on $\mathbb{R}$. Then, we have $I_\nu^{\mathrm{cl}} \le I_X^{\mathrm{cl}}$ on $m_X(\mathcal{F}(X))$ if and only if ν is an iso-dominant of X.*

Proof We assume that ν is an iso-dominant of X. By Proposition 3.18, we have $I_\nu^{\mathrm{cl}} \le I_X^{\mathrm{cl}}$ on $m_X(\mathcal{F}(X))$. Conversely, we assume $I_\nu^{\mathrm{cl}} \le I_X^{\mathrm{cl}}$ on $m_X(\mathcal{F}(X))$. Then we have $I_\nu \circ V \le I_\nu^{\mathrm{cl}} \circ V \le I_X^{\mathrm{cl}} \circ V$ on $V^{-1}(m_X(\mathcal{F}(X)))$. Since ν is iso-simple, we have

$$V' = I_\nu \circ V \le I_X^{\mathrm{cl}} \circ V \quad \mathcal{L}^1\text{-a.e. on } V^{-1}(m_X(\mathcal{F}(X))).$$

This means that X satisfies $\mathrm{IC}^{\mathrm{cl}}(\nu)$. By Theorem 3.7, ν is an iso-dominant of X. This completes the proof. □

3.2 Iso-Lipschitz Order with an Error

In this section, we define the iso-Lipschitz order with an additive error and present some properties. To define the iso-Lipschitz order with an error, we use transport plans (Definition 2.7) and the following iso-deviation.

Definition 3.22 (Iso-deviation) We define the *iso-deviation* $\operatorname{dev}_{\succ}$ of a subset $S \subset \mathbb{R}^2$ by

$$\operatorname{dev}_{\succ} S := \sup\{\, y - y' - \max\{\, x - x', 0\,\} \mid (x, y), (x', y') \in S\,\}$$

if S is nonempty. We set $\operatorname{dev}_{\succ} \emptyset := 0$.

The iso-deviation evaluates the deviation from the monotonically non-decreasing and 1-Lipschitz property. The following iso-Lipschitz order with an error is a generalization of the iso-Lipschitz order (Definition 3.2).

Definition 3.23 (Iso-Lipschitz order $\succ'_{(s,t)}$ with error (s, t)) Let μ and ν be two Borel probability measures on $\mathbb{R}$ and $s, t \geq 0$ two real numbers. We say that μ *iso-dominates* ν *with error* (s, t) and denote $\mu \succ'_{(s,t)} \nu$ if there exists a transport plan $\pi \in \Pi(\mu, \nu)$ and a Borel subset $S \subset \mathbb{R}^2$ such that $\operatorname{dev}_{\succ} S \leq s$ and $1 - \pi(S) \leq t$.

The following Propositions 3.24 and 3.25 are useful properties of the iso-deviation. By Proposition 3.24, we obtain $\operatorname{dev}_{\succ} \overline{S} \leq \varepsilon$ if we check $\operatorname{dev}_{\succ} S \leq \varepsilon$ for any real number $\varepsilon \geq 0$. Proposition 3.25 implies that a subset $S \subset \mathbb{R}^2$ determines a 1-Lipschitz function if we have $\operatorname{dev}_{\succ} S = 0$.

Proposition 3.24 *For a subset $S \subset \mathbb{R}^2$, we have*

$$\operatorname{dev}_{\succ} S = \operatorname{dev}_{\succ} \overline{S}.$$

Proposition 3.25 *Let $S \subset \mathbb{R}^2$. For any two points $(x, y), (x', y') \in S$, we have*

$$|y - y'| - |x - x'| \leq \operatorname{dev}_{\succ} S.$$

Proof Take any $(x, y), (x', y') \in S$. By symmetry, we may assume that $y \geq y'$. Then we have

$$|y - y'| - |x - x'| \leq y - y' - \max\{\, x - x', 0\,\} \leq \operatorname{dev}_{\succ} S.$$

This completes proof. □

Theorem 3.26 *Let μ and ν be two Borel probability measures on $\mathbb{R}$. Then $\mu \succ' \nu$ if and only if $\mu \succ'_{(0,0)} \nu$.*

Proof Assume that $\mu \succ' \nu$. Then, there exists a monotonically non-decreasing 1-Lipschitz function $f : \operatorname{supp}\mu \to \operatorname{supp}\nu$ such that $f_*\mu = \nu$. We put $\pi := (\mathrm{id}_{\mathbb{R}}, f)_*\mu \in \Pi(\mu, \nu)$. Let us prove $\operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$. By Proposition 2.3, we have

$$\operatorname{supp}\pi = \overline{(\mathrm{id}_{\mathbb{R}}, f)(\operatorname{supp}\mu)},$$

which implies that

$$\operatorname{dev}_{\succ} \operatorname{supp}\pi = \operatorname{dev}_{\succ}((\mathrm{id}_{\mathbb{R}}, f)(\operatorname{supp}\mu))$$

by Proposition 3.24. Hence, it suffices to prove $\operatorname{dev}_{\succ}((\mathrm{id}_{\mathbb{R}}, f)(\operatorname{supp}\mu)) = 0$. Take any two points $(x_1, y_1), (x_2, y_2) \in (\mathrm{id}_{\mathbb{R}}, f)(\operatorname{supp}\mu)$. Then, we have $x_1, x_2 \in \operatorname{supp}\mu$ and $y_1 = f(x_1)$, $y_2 = f(x_2)$. In the case that $x_1 \geq x_2$, we have

$$\begin{aligned} y_1 - y_2 - \max\{x_1 - x_2, 0\} &= f(x_1) - f(x_2) - |x_1 - x_2| \\ &\leq |f(x_1) - f(x_2)| - |x_1 - x_2| \leq 0 \end{aligned}$$

because f is 1-Lipschitz. In the case that $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$ since f is monotonically non-decreasing. Then we have

$$y_1 - y_2 - \max\{x_1 - x_2, 0\} = y_1 - y_2 = f(x_1) - f(x_2) \leq 0.$$

Therefore we obtain $\operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$. It follows that $\mu \succ'_{(0,0)} \nu$.

Conversely, assume that $\mu \succ'_{(0,0)} \nu$. Then there exists $\pi \in \Pi(\mu, \nu)$ such that $\operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$. Take any point $x \in \operatorname{supp}\mu$. We now claim that there exists a unique point $y \in \operatorname{supp}\nu$ such that $(x, y) \in \operatorname{supp}\pi$. Let us prove the existence of y. Take any $x \in \operatorname{supp}\mu$. By Proposition 2.3, we have

$$\operatorname{supp}\mu = \operatorname{supp}(\mathrm{pr}_1)_*\pi = \overline{\mathrm{pr}_1(\operatorname{supp}\pi)}.$$

Hence, there exists $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \operatorname{supp}\pi$ such that x_n converges to x. By Proposition 3.25, we have

$$|y_m - y_n| - |x_m - x_n| \leq \operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$$

for any positive integers m and n. This means that $\{y_n\}$ is a Cauchy sequence. Therefore, $\{y_n\}$ converges to some $y \in \mathbb{R}$. Since $\operatorname{supp}\pi$ is closed, we have $(x, y) \in \operatorname{supp}\pi$. In addition, we have

$$y \in \mathrm{pr}_2(\operatorname{supp}\pi) \subset \operatorname{supp}(\mathrm{pr}_2)_*\pi = \operatorname{supp}\nu.$$

The uniqueness of $y \in \operatorname{supp}\nu$ follows from $\operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$ and Proposition 3.25.

Now, we define a function $f : \operatorname{supp}\mu \to \operatorname{supp}\nu$ by $f(x) := y$ for $x \in \operatorname{supp}\mu$, where $y \in \operatorname{supp}\nu$ satisfies $(x, y) \in \operatorname{supp}\pi$. By $\operatorname{dev}_{\succ} \operatorname{supp}\pi = 0$ and Proposition 3.25,

f is a 1-Lipschitz function. Let us prove that f is monotonically non-decreasing. Take any $x, x' \in \operatorname{supp}\mu$ with $x \le x'$. Then we have

$$f(x) - f(x') = f(x) - f(x') - \max\{x - x', 0\} \le \operatorname{dev}_{\succ} \operatorname{supp}\pi = 0.$$

It remains to show $f_*\mu = \nu$. Let us prove

$$\operatorname{supp}\pi = \{(x, f(x)) \mid x \in \operatorname{supp}\mu\}. \tag{3.1}$$

By the definition of f, we have $\operatorname{supp}\pi \supset \{(x, f(x)) \mid x \in \operatorname{supp}\mu\}$. We now check $\operatorname{supp}\pi = \{(x, f(x)) \mid x \in \operatorname{supp}\mu\}$. Take any point $(x, y) \in \operatorname{supp}\pi$. By Proposition (2.3), we have

$$x \in \operatorname{pr}_1(\operatorname{supp}\pi) \subset \operatorname{supp}(\operatorname{pr}_1)_*\pi = \operatorname{supp}\mu.$$

Because f is well-defined, we have $y = f(x)$. Thus we have $(x, y) \in \{(x, f(x)) \mid x \in \operatorname{supp}\mu\}$. Therefore we obtain (3.1).

By (3.1), we have

$$(A \times B) \cap \operatorname{supp}\pi = \{(A \cap f^{-1}(B)) \times \mathbb{R}\} \cap \operatorname{supp}\pi$$

for any Borel sets A and B of $\mathbb{R}$. Since

$$\begin{aligned}\pi(A \times B) &= \pi((A \cap f^{-1}(B)) \times \mathbb{R}) \\ &= \mu(A \cap f^{-1}(B)) \\ &= (\operatorname{id}_{\mathbb{R}}, f)_*\mu(A \times B),\end{aligned}$$

we have $\pi = (\operatorname{id}_{\mathbb{R}}, f)_*\mu$, which implies $\nu = (\operatorname{pr}_2)_*\pi = f_*\mu$. This completes the proof. □

Proposition 3.27 *Let d_{l^1} be the l^1-distance $d_{l^1}((x, y), (x', y')) := |x - x'| + |y - y'|$ on $\mathbb{R}^2$ and d_H the Hausdorff distance function with respect to d_{l^1}. For any two closed subsets $S, S' \subset \mathbb{R}^2$, we have*

$$|\operatorname{dev}_{\succ} S - \operatorname{dev}_{\succ} S'| \le 2d_H(S, S').$$

Proof Take any real number $\varepsilon > 0$ with $\varepsilon > d_H(S, S')$. We have $S' \subset U_\varepsilon(S)$. Let us prove $\operatorname{dev}_{\succ} U_\varepsilon(S) \le \operatorname{dev}_{\succ} S + 2\varepsilon$. Take a point $(x_i, y_i) \in U_\varepsilon(S)$ for $i = 1, 2$. Then there exists $(x_i', y_i') \in S$ such that $d_{l^1}((x_i, y_i), (x_i', y_i')) < \varepsilon$. Now, we have

$$\begin{aligned}& y_1 - y_2 - \max\{x_1 - x_2, 0\} \\ &\quad = y_1' - y_2' + (y_1 - y_1') + (y_2' - y_2) \\ &\qquad - \max\{x_1' - x_2' + (x_1 - x_1') + (x_2' - x_2), 0\} \\ &\quad \le y_1' - y_2' + |y_1 - y_1'| + |y_2' - y_2|\end{aligned}$$

$$
\begin{aligned}
&\quad - \max\{ x_1' - x_2' - |x_1 - x_1'| - |x_2' - x_2|, 0 \} \\
&\le y_1' - y_2' + |y_1 - y_1'| + |y_2' - y_2| \\
&\quad - (\max\{ x_1' - x_2', 0 \} - |x_1 - x_1'| - |x_2' - x_2|) \\
&\le y_1' - y_2' - \max\{ x_1' - x_2', 0 \} + 2\varepsilon \\
&\le \mathrm{dev}_{\succ} S + 2\varepsilon.
\end{aligned}
$$

Therefore we obtain

$$
\mathrm{dev}_{\succ} S' \le \mathrm{dev}_{\succ} U_\varepsilon(S) \le \mathrm{dev}_{\succ} S + 2\varepsilon.
$$

This implies $\mathrm{dev}_{\succ} S' - \mathrm{dev}_{\succ} S \le 2d_H(S, S')$. By exchanging S for S', we also obtain $\mathrm{dev}_{\succ} S - \mathrm{dev}_{\succ} S' \le 2d_H(S, S')$. □

Theorem 3.28 *Let μ and ν be two Borel probability measures on $\mathbb{R}$ and $s, t \ge 0$. If $\mu \succ'_{(s+\varepsilon, t+\varepsilon)} \nu$ for every $\varepsilon > 0$, then we have $\mu \succ'_{(s,t)} \nu$.*

Proof Suppose that $\mu \succ'_{(s+\frac{1}{n}, t+\frac{1}{n})} \nu$ for any positive integer n. For any positive integer n, there exist $\pi_n \in \Pi(\mu, \nu)$ and a closed subset $S_n \subset \mathbb{R}^2$ such that $\mathrm{dev}_{\succ} S_n \le s + \frac{1}{n}$ and $\pi_n(S_n) \ge 1 - t - \frac{1}{n}$. Due to the weak compactness of $\Pi(\mu, \nu)$, we may assume that π_n converges weakly to some Borel probability measure π by taking a subsequence. By Prokhorov's theorem, for any positive number m, there exists a compact subset $K_m \subset \mathbb{R}^2$ such that $\sup_{n\in\mathbb{N}} \pi_n(K_m^c) \le \frac{1}{m}$ and $\pi(K_m^c) \le \frac{1}{m}$. We may assume that the sequence of $\{K_m\}$ is monotonically non-decreasing with respect to the inclusion relation. Let d_H be the Hausdorff distance function of $(\mathbb{R}^2, d_{l^1})$ and d_H^m the Hausdorff distance function of (K_m, d_{l^1}). Since K_m is compact, $(\mathcal{F}(K_m), d_H^m)$ is also compact, where $\mathcal{F}(K_m)$ is the set of all closed subsets of K_m. By taking a subsequence $\{n_i^{(1)}\}_{i\in\mathbb{N}} \subset \mathbb{N}$, we have $d_H^1(S_{n_i^{(1)}} \cap K_1, S_\infty^1) \to 0$ as $i \to \infty$ for some $S_\infty^1 \in \mathcal{F}(K_1)$, where $\mathbb{N}$ is the set of positive integers. Furthermore, we take some subsequence $\{n_i^{(2)}\}_{i\in\mathbb{N}} \subset \{n_i^{(1)}\}_{i\in\mathbb{N}}$ and we have $d_H^2(S_{n_i^{(2)}} \cap K_2, S_\infty^2) \to 0$ for some $S_\infty^2 \in \mathcal{F}(K_2)$. By repeating this procedure, we take a subsequence $\{n_i^{(m)}\}_{i\in\mathbb{N}} \subset \{n_i^{(m-1)}\}_{i\in\mathbb{N}}$ and we have $d_H^m(S_{n_i^{(m)}} \cap K_m, S_\infty^m) \to 0$ for some $S_\infty^m \in \mathcal{F}(K_m)$. Put $n_i := n_i^{(i)}$. Since the convergence on $(\mathcal{F}(K_m), d_H^m)$ implies the convergence on $(\mathcal{F}(\mathbb{R}), d_H)$, we obtain

$$
d_H(S_{n_i} \cap K_m, S_\infty^m) \to 0 \tag{3.2}
$$

for any positive integer m. Since $\{K_m\}$ is monotonically non-decreasing with respect to the inclusion relation, $\{S_\infty^m\}$ is also monotonically non-decreasing. By Proposition

3.27 and (3.2), we have

$$\begin{aligned} \operatorname{dev}_{\succ} S_{\infty}^{m} &\leq \liminf_{i\to\infty} \operatorname{dev}_{\succ}(S_{n_i} \cap K_m) \\ &\leq \liminf_{i\to\infty} \operatorname{dev}_{\succ}(S_{n_i}) \\ &\leq \liminf_{i\to\infty} \left(s + \frac{1}{n_i}\right) = s \end{aligned} \tag{3.3}$$

Since $\{\pi_{n_i}\}$ converges weakly to π and (3.2), we also have

$$\begin{aligned} \pi(S_{\infty}^{m}) &\geq \limsup_{i\to\infty} \pi_{n_i}(S_{n_i} \cap K_m) \\ &= \limsup_{i\to\infty} (\pi_{n_i}(S_{n_i}) - \pi_{n_i}(S_{n_i} \cap K_m^c)) \\ &\geq \limsup_{i\to\infty} (\pi_{n_i}(S_{n_i}) - \pi_{n_i}(K_m^c)) \\ &\geq \limsup_{i\to\infty} \left(1 - t - \frac{1}{n_i} - \frac{1}{m}\right) = 1 - t - \frac{1}{m} \end{aligned} \tag{3.4}$$

for any positive number m. Now, we put $S := \bigcup_{m=1}^{\infty} S_{\infty}^{m}$. By (3.3), we have

$$\operatorname{dev}_{\succ} S = \sup_{m\in\mathbb{N}} \operatorname{dev}_{\succ} S_{\infty}^{m} \leq s$$

By (3.4), we have

$$\pi(S) = \lim_{m\to\infty} \pi(S_{\infty}^{m}) \geq \lim_{m\to\infty} \left(1 - t - \frac{1}{m}\right) = 1 - t,$$

where we remark that the limit exists because $\{S_{\infty}^{m}\}$ is monotonically non-decreasing. Therefore we obtain $\mu \succ'_{(s,t)} \nu$. This completes the proof. □

The following Theorem 3.31 is a variation of the transitive property. To prove Theorem 3.31, we prepare the following Definition 3.29 and Proposition 3.30.

Definition 3.29 (Subtransport plan) Let μ and ν be two Borel probability measures on $\mathbb{R}$. We say that a Borel measure on $\mathbb{R}^2$ is a *subtransport plan between* μ *and* ν if we have $(\mathrm{pr}_1)_*\pi \leq \mu$ and $(\mathrm{pr}_2)_*\pi \leq \nu$.

Proposition 3.30 *Let* μ *and* ν *be two Borel probability measures on* $\mathbb{R}$. *Then we have* $\mu \succ'_{(s,t)} \nu$ *if and only if there exists a subtransport plan* π *between* μ *and* ν *such that* $\operatorname{dev}_{\succ} \operatorname{supp} \pi \leq s$ *and* $1 - \pi(\mathbb{R}^2) \leq t$.

The proof of the above proposition is easy and omitted.

Theorem 3.31 *Let* μ_1, μ_2, *and* μ_3 *be three Borel probability measures on* $\mathbb{R}$ *and let* $s_i, t_i \geq 0$ *for* $i = 1, 2$. *If* $\mu_1 \succ'_{(s_1,t_1)} \mu_2$ *and if* $\mu_2 \succ'_{(s_2,t_2)} \mu_3$, *then we have* $\mu_1 \succ'_{(s_1+s_2,t_1+t_2)} \mu_3$.

Proof Suppose that $\mu_1 \succ'_{(s_1,t_1)} \mu_2$ and $\mu_2 \succ'_{(s_2,t_2)} \mu_3$. There exists a subtransport plan π_i between μ_i and μ_{i+1} such that $\mathrm{dev}_\succ \operatorname{supp}\pi_i \le s_i$ and $1 - \pi_i(\operatorname{supp}\pi_i) \le t_i$ for $i = 1, 2$. Put $\mu' := (\mathrm{pr}_2)_*\pi_1$ and $\mu'' := (\mathrm{pr}_1)_*\pi_2$. By the disintegration theorem (see III-70 in [18] or Theorem 5.3.1 in [1]), there exist two families $\{(\pi_1)_x\}_{x\in\mathbb{R}}$ and $\{(\pi_2)_x\}_{x\in\mathbb{R}}$ of Borel probability measures on $\mathbb{R}$ such that

$$\pi_1(A \times B) = \int_B (\pi_1)_x(A)d\mu'(x),$$
$$\pi_2(A \times B) = \int_A (\pi_2)_x(B)d\mu''(x)$$

for any Borel subsets A and B of $\mathbb{R}$. Now, we put

$$\pi_{123}(A \times B \times C) := \int_B (\pi_1)_x(A) \cdot (\pi_2)_x(C)d(\mu' \wedge \mu'')(x),$$
$$\pi_{13} := (\mathrm{pr}_{13})_*\pi_{123}$$

for any three Borel subsets A, B, and C of $\mathbb{R}$, where $\mu' \wedge \mu'' := \mu' - (\mu' - \mu'')_+$ and a measure $(\mu' - \mu'')_+$ is defined by

$$(\mu' - \mu'')_+(B) := \sup\{\, \mu'(B') - \mu''(B') \mid B' \subset B \text{ is a Borel set}\,\}$$

for any Borel set $B \subset \mathbb{R}$. Then we have

$$(\mathrm{pr}_{12})_*\pi_{123} \le \pi_1, \quad (\mathrm{pr}_{23})_*\pi_{123} \le \pi_2. \tag{3.5}$$

In particular, π_{13} is a subtransport plan between μ_1 and μ_3. Moreover, we obtain $1 - \pi_{13}(\operatorname{supp}\pi_{13}) \le t_1 + t_2$ because we have

$$\begin{aligned}
\pi_{13}(\mathbb{R}^2) &= \int_{\mathbb{R}} ((\pi_1)_x(\mathbb{R}) \cdot (\pi_2)_x(\mathbb{R}))\, d(\mu' \wedge \mu'')(x) \\
&= (\mu' \wedge \mu'')(\mathbb{R}) \\
&= \mu'(\mathbb{R}) - (\mu' - \mu'')_+(\mathbb{R}) \\
&\ge \mu'(\mathbb{R}) - (\mu_2 - \mu'')_+(\mathbb{R}) \\
&= \mu'(\mathbb{R}) - (\mu_2(\mathbb{R}) - \mu''(\mathbb{R})) \\
&= \mu'(\mathbb{R}) + \mu''(\mathbb{R}) - 1 \\
&\ge (1 - t_1) + (1 - t_2) - 1 = 1 - t_1 - t_2.
\end{aligned}$$

It remains to show $\mathrm{dev}_\succ \operatorname{supp}\pi_{13} \le s_1 + s_2$. By Proposition 3.24 and since

$$\operatorname{supp}\pi_{13} = \overline{\mathrm{pr}_{13}(\operatorname{supp}\pi_{123})},$$

it suffices to prove

$$\mathrm{dev}_\succ(\mathrm{pr}_{13}(\operatorname{supp}\pi_{123})) \le s_1 + s_2. \tag{3.6}$$

Take any $(x_i, z_i) \in \mathrm{pr}_{13}(\operatorname{supp}\pi_{123})$ for $i = 1, 2$. There exists a point $y_i \in \mathbb{R}$ such that $(x_i, y_i, z_i) \in \operatorname{supp}\pi_{123}$. By (3.5), we have

$$\operatorname{supp}\pi_{123} \subset \mathrm{pr}_{12}^{-1}(\overline{\mathrm{pr}_{12}(\operatorname{supp}\pi_{123})}) \subset \mathrm{pr}_{12}^{-1}(\operatorname{supp}\pi_1)$$

and

$$\operatorname{supp}\pi_{123} \subset \mathrm{pr}_{23}^{-1}(\overline{\mathrm{pr}_{23}(\operatorname{supp}\pi_{123})}) \subset \mathrm{pr}_{23}^{-1}(\operatorname{supp}\pi_2).$$

This implies that $(x_i, y_i) \in \operatorname{supp}\pi_1$ and $(y_i, z_i) \in \operatorname{supp}\pi_2$. Now, let us prove

$$\max\{\, y_1 - y_2, 0\,\} - \max\{\, x_1 - x_2, 0\,\} \le s_1. \tag{3.7}$$

In the case that $y_1 < y_2$, we have

$$\max\{\, y_1 - y_2, 0\,\} - \max\{\, x_1 - x_2, 0\,\} = -\max\{\, x_1 - x_2, 0\,\} \le 0.$$

In the case that $y_1 \ge y_2$, we have

$$\begin{aligned}\max\{\, y_1 - y_2, 0\,\} - \max\{\, x_1 - x_2, 0\,\} &= y_1 - y_2 - \max\{\, x_1 - x_2\,\} \\ &\le \operatorname{dev}_{\succ} \operatorname{supp}\pi_1 \le s_1.\end{aligned}$$

Combining (3.7) with $\operatorname{dev}_{\succ} \operatorname{supp}\pi_2 \le s_2$, we obtain

$$\begin{aligned}z_1 - z_2 - \max\{\, x_1 - x_2, 0\,\} &\le z_1 - z_2 - \max\{\, y_1 - y_2, 0\,\} + s_1 \\ &\le s_1 + s_2,\end{aligned}$$

which implies (3.6). This completes the proof. □

3.3 Isoperimetric Comparison Condition with an Error

In this section, we prove Theorem 1.7 to explain the relation between ε-iso-dominant and ICL_ε. We also explain the relation between $\mathrm{IC}_\varepsilon^+$ (Definition 3.35) and ICL_ε. The condition $\mathrm{IC}_\varepsilon^+$ is a discretization of IC (Definition 3.1). At the end of this section, we give some examples of these conditions.

Proposition 3.32 *Let ε be a non-negative real number. If a Borel probability measure ν on $\mathbb{R}$ is an ε-iso-dominant of an mm-space X, then $(t \cdot \mathrm{id}_{\mathbb{R}})_*\nu$ is a $t\varepsilon$-iso-dominant of tX.*

Remark 3.33 By Theorem 3.26, a Borel measure on $\mathbb{R}$ is a 0-iso-dominant if and only if it is an iso-dominant.

Let A be a subset of $\mathbb{R}$. We put

$$\delta^-(A; a) := \inf\{\, t > 0 \mid a - t \in A\,\}$$

for a point $a \in A$, where we define

$$\delta^-(A; a) := \infty$$

if $\{ t > 0 \mid a - t \in A \} = \emptyset$. We define $\Delta(A)$ by

$$\Delta(A) := \sup\{ \delta^-(A; a) \mid a \in A \setminus \{\inf A\} \}.$$

If A is a closed set, we have $a - \delta^-(A; a) \in A$.

Proof of Theorem 1.7 (1) Let V be the cumulative distribution function of ν. Take any 1-Lipschitz function $f : X \to \mathbb{R}$ and let $F : \mathbb{R} \to [0, 1]$ be the cumulative distribution function of $f_* m_X$. We put $\pi := (\tilde{V}, \tilde{F})_* \mathcal{L}^1|_{[0,1]}$ and see $\pi \in \Pi(\nu, f_* m_X)$. It suffices to prove $\mathrm{dev}_{\succ} \operatorname{supp} \pi \leq \varepsilon + \delta$, where $\delta := \Delta(\operatorname{supp} \nu)$. Take any points $(x_1, y_1), (x_2, y_2) \in \operatorname{supp} \pi$. Let us prove

$$y_2 - y_1 - \max\{ x_2 - x_1, 0 \} \leq \varepsilon + \delta. \tag{3.8}$$

Since $\{0, 1\}$ is a null set with respect to $\mathcal{L}^1$, we have

$$\begin{aligned} \operatorname{supp} \pi &= \operatorname{supp}(\tilde{V}, \tilde{F})_* \mathcal{L}^1|_{[0,1]} \\ &\subset \overline{(\tilde{V}, \tilde{F})(\operatorname{supp} \mathcal{L}^1|_{[0,1]} \setminus \{0, 1\})} \\ &= \overline{(\tilde{V}, \tilde{F})((0, 1))}. \end{aligned}$$

Then, there exists $\{t_i^n\}_{n=1}^\infty \subset (0, 1)$ such that $x_i = \lim_{n\to\infty} \tilde{V}(t_i^n)$ and $y_i = \lim_{n\to\infty} \tilde{F}(t_i^n)$ for $i = 1, 2$.

If $x_1 > x_2$, we have $y_1 \geq y_2$, which implies (3.8). In fact, we have

$$y_2 - y_1 - \max\{ x_2 - x_1, 0 \} = y_2 - y_1 \leq 0 \leq \varepsilon + \delta.$$

We assume $x_1 \leq x_2$. Let us prove

$$V \circ \tilde{V}(t_2^n) \leq F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \delta + \varepsilon) \tag{3.9}$$

for any positive integer n. In the case that $\tilde{V}(t_1^n) = \inf \operatorname{supp} \nu$, we have

$$0 < t_1^n \leq F \circ \tilde{F}(t_1^n) = m_X(f^{-1}((-\infty, \tilde{F}(t_1^n)])),$$

which implies

$$f^{-1}((-\infty, \tilde{F}(t_1^n)]) \neq \emptyset. \tag{3.10}$$

Since we have

$$\nu(\{\inf \operatorname{supp} \nu\}) \leq m_X(\{x\}) \text{ for any } x \in \operatorname{supp} m_X \tag{3.11}$$

and (3.10), we have

$$V \circ \tilde{V}(t_1^n) = \nu(\{\inf \operatorname{supp} \nu\}) \leq m_X(f^{-1}((-\infty, \tilde{F}(t_1^n)])),$$

where (3.11) is the assumption of this theorem. By using $\mathrm{ICL}_\varepsilon(\nu)$, we obtain

$$\begin{aligned} V \circ \tilde{V}(t_2^n) &\leq m_X(B_{\tilde{V}(t_2^n)-\tilde{V}(t_1^n)+\varepsilon}(f^{-1}((-\infty, \tilde{F}(t_1^n)]))) \\ &\leq m_X(f^{-1}(B_{\tilde{V}(t_2^n)-\tilde{V}(t_1^n)+\varepsilon}((-\infty, \tilde{F}(t_1^n)]))) \\ &= F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \varepsilon). \end{aligned}$$

In the case that $\tilde{V}(t_1^n) > \inf \operatorname{supp} \nu$, we have $\delta^-(\operatorname{supp} \nu; \tilde{V}(t_1^n)) < \infty$. By the definition of $\delta^-(\operatorname{supp} \nu; \tilde{V}(t_1^n))$, there exists a sequence $\{s_k^n\}_{k=1}^\infty$ of positive real numbers such that $\lim_{k\to\infty} s_k^n = \delta^-(\operatorname{supp} \nu; \tilde{V}(t_1^n))$ and $\tilde{V}(t_1^n) - s_k^n \in \operatorname{supp} \nu$ for any positive integer k. By the definition of $\tilde{V}$, we have $V(\tilde{V}(t_1^n) - s) < t_1^n$ for any real number $s > 0$, which implies

$$V(\tilde{V}(t_1^n) - s_k^n) < t_1^n \leq F \circ \tilde{F}(t_1^n) = m_X(f^{-1}((-\infty, \tilde{F}(t_1^n)])).$$

By $\mathrm{ICL}_\varepsilon(\nu)$, we have

$$\begin{aligned} V \circ \tilde{V}(t_2^n) &\leq m_X(B_{\tilde{V}(t_2^n)-\tilde{V}(t_1^n)+s_k^n+\varepsilon}(f^{-1}((-\infty, \tilde{F}(t_1^n)]))) \\ &\leq m_X(f^{-1}(B_{\tilde{V}(t_2^n)-\tilde{V}(t_1^n)+s_k^n+\varepsilon}((-\infty, \tilde{F}(t_1^n)]))) \\ &= F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + s_k^n + \varepsilon). \end{aligned}$$

By taking the limit as $k \to \infty$, we have

$$\begin{aligned} V \circ \tilde{V}(t_2^n) &\leq F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \delta^-(\operatorname{supp} \nu; \tilde{V}(t_1^n)) + \varepsilon) \\ &\leq F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \delta + \varepsilon). \end{aligned}$$

Hence we obtain (3.9).

By using (3.9), we have

$$t_2^n \leq V \circ \tilde{V}(t_2^n) = F(\tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \delta + \varepsilon).$$

By the definition of $\tilde{F}(t_2^n)$, we have

$$\tilde{F}(t_2^n) \leq \tilde{F}(t_1^n) + \tilde{V}(t_2^n) - \tilde{V}(t_1^n) + \delta + \varepsilon.$$

By taking the limit as $n \to \infty$, we obtain $y_2 - y_1 \leq x_2 - x_1 + \delta + \varepsilon$. This completes of proof. □

Proof of Theorem 1.7 (2) Take any two real numbers $a, b \in \operatorname{supp} \nu$ with $a \leq b$ and any Borel set $A \subset X$ with $m_X(A) > 0$ and $m_X(A) \geq V(a)$. We define a 1-Lipschitz function $f : X \to \mathbb{R}$ by $f(x) := d_X(x, A)$ for $x \in X$. Since ν is an ε-iso-dominant of X, there exists a transport plan π between ν and $f_* m_X$ such that $\operatorname{dev}_{\succ} \operatorname{supp} \pi \leq \varepsilon$. We put

$$a' := \sup\{\, x \mid (x, y) \in \operatorname{supp} \pi \cap (\mathbb{R} \times (-\infty, 0]) \text{ for some } y \,\},$$
$$b' := \sup\{\, x \mid (x, y) \in \operatorname{supp} \pi \cap (\mathbb{R} \times (-\infty, b - a + \varepsilon]) \text{ for some } y \,\}.$$

We remark that we have $a' \leq b'$ by the definition of a' and b'. We claim that we can assume that $b' < \infty$ because we have

$$m_X(B_{b-a+2\varepsilon}(A)) = 1 \text{ if } b' = \infty. \tag{3.12}$$

We now check (3.12). First, let us prove

$$\operatorname{supp} \pi \subset \mathbb{R} \times (-\infty, b - a + 2\varepsilon]. \tag{3.13}$$

We take any point $(x, y) \in \operatorname{supp} \pi$. By $b' = \infty$, there exists $(x', y') \in \operatorname{supp} \pi$ such that $x \leq x'$ and $y' \leq b - a + \varepsilon$. Then we have

$$y \leq y' + \operatorname{dev}_{\succ} \operatorname{supp} \pi \leq b - a + 2\varepsilon$$

because $\operatorname{dev}_{\succ} \operatorname{supp} \pi \leq \varepsilon$. This implies (3.13). Then (3.13) implies

$$\begin{aligned} m_X(B_{b-a+2\varepsilon}(A)) &= f_* m_X((-\infty, b - a + 2\varepsilon]) \\ &= \pi(\mathbb{R} \times (-\infty, b - a + 2\varepsilon]) \geq 1. \end{aligned}$$

This completes the proof of (3.12).

Now, we have

$$\begin{aligned} V(a) \leq m_X(A) &\leq f_* m_X((-\infty, 0]) = \pi(\mathbb{R} \times (-\infty, 0]) \\ &= \pi((-\infty, a'] \times (-\infty, 0]) \leq V(a'). \end{aligned}$$

In particular, we have

$$a' \geq \inf \operatorname{supp} \nu \tag{3.14}$$

because $V(a') \geq m_X(A) > 0$. Let us prove $a \leq a'$. By (3.14), we may assume $a > \inf \operatorname{supp} \nu$. If $a > a'$, then we have $V(a) > V(a')$ because we have $\nu(\{a\}) > 0$ or $\operatorname{supp} \nu$ is connected, which implies a contradiction.

Next, let us prove $b \leq b'$. We may assume $b \geq a'$ because $b \leq a' \leq b'$ if $b \leq a'$. Let us prove that

$$\text{there exists } y_0' \leq 0 \text{ such that } (a', y_0') \in \operatorname{supp} \pi. \tag{3.15}$$

Take any positive integer n. There exists $(s_n, t_n) \in \operatorname{supp} \pi$ such that $a' - 1/n < s_n \le a'$ and $t_n \le 0$. By Proposition 3.25, we have

$$|t_n - t_1| \le |s_n - s_1| + \operatorname{dev}_{\succ} \operatorname{supp} \pi \le 1 + \varepsilon.$$

Since the sequence $\{t_n\}$ is bounded, there exists a subsequence $\{n(i)\}$ and $y_0' \in \mathbb{R}$ such that $t_{n(i)} \to y_0'$ as $i \to \infty$. We have $y_0' \le 0$ because $t_n \le 0$ for any n. Since $(s_{n(i)}, t_{n(i)}) \to (a', y_0')$ as $i \to \infty$ and $\operatorname{supp} \pi$ is closed, we obtain $(a', y_0') \in \operatorname{supp} \pi$. Hence (3.15) is proved.

Similarly, let us prove that

$$\text{there exists } y_0' \le 0 \text{ such that } (a', y_0') \in \operatorname{supp} \pi. \tag{3.16}$$

By Proposition 2.3,

$$b \in \operatorname{supp} \nu = \operatorname{supp}(\mathrm{pr}_1)_* \pi = \overline{\mathrm{pr}_1(\operatorname{supp} \pi)}. \tag{3.17}$$

By (3.17), there exists a sequence $\{(s_n, t_n)\} \subset \operatorname{supp} \pi$ such that $s_n \to b$ as $n \to \infty$. The sequence $\{t_n\}$ is bounded because $\operatorname{dev}_{\succ} \operatorname{supp} \pi \le \varepsilon$. Hence there exists a subsequence $\{n(i)\}$ and $y_0 \in \mathbb{R}$ such that $t_{n(i)} \to y_0$ as $i \to \infty$. Since $(s_{n(i)}, t_{n(i)}) \to (b, y_0)$ as $i \to \infty$ and $\operatorname{supp} \pi$ is closed, we obtain $(b, y_0) \in \operatorname{supp} \pi$. Hence (3.16) is proved.

Now, we have

$$y_0 \le y_0 - y_0' \le b - a' + \varepsilon \le b - a + \varepsilon$$

since $\operatorname{dev}_{\succ} \operatorname{supp} \pi \le \varepsilon$. Therefore, we have $(b, y_0) \in \operatorname{supp} \pi \cap (\mathbb{R} \times (-\infty, b - a + \varepsilon])$, which implies $b \le b'$ by the definition of b'.

If we have

$$\operatorname{supp} \pi \cap ((-\infty, b'] \times \mathbb{R}) \subset (-\infty, b'] \times (-\infty, b - a + 2\varepsilon], \tag{3.18}$$

then we obtain

$$\begin{aligned} V(b) \le V(b') &= \pi((-\infty, b'] \times \mathbb{R}) \\ &\le \pi((-\infty, b'] \times (-\infty, b - a + 2\varepsilon]) \\ &\le \pi(\mathbb{R} \times (-\infty, b - a + 2\varepsilon]) \\ &= f_* m_X((-\infty, b - a + 2\varepsilon]) \\ &= m_X(B_{b-a+2\varepsilon}(A)). \end{aligned}$$

It remains to prove (3.18). Take any point $(x, y) \in \operatorname{supp} \pi \cap ((-\infty, b'] \times \mathbb{R})$. In the case that $x < b'$, there exists $(x', y') \in \operatorname{supp} \pi \cap (\mathbb{R} \times (-\infty, b - a + \varepsilon])$ such that $x' > x$ by the definition of b'. Now, we have $y - y' = y - y' - \max\{x - x', 0\} \le \operatorname{dev}_{\succ} \operatorname{supp} \pi \le \varepsilon$. Hence, we obtain $y \le y' + \varepsilon \le b - a + 2\varepsilon$.

In the case that $x = b'$, then for any positive integer n, there exists a point $(x_n, y_n) \in \operatorname{supp} \pi \cap (\mathbb{R} \times (-\infty, b - a + \varepsilon])$ such that $x - 1/n < x_n \leq x$. By $\operatorname{dev}_{\succ} \operatorname{supp} \pi \leq \varepsilon$, we obtain

$$\begin{aligned} y &\leq y_n + x - x_n + \varepsilon \\ &\leq x - x_n + b - a + 2\varepsilon \\ &\leq \frac{1}{n} + b - a + 2\varepsilon \to b - a + 2\varepsilon \text{ as } n \to \infty. \end{aligned}$$

Hence we have $(x, y) \in (-\infty, b'] \times (-\infty, b - a + 2\varepsilon]$. This completes the proof. □

Isoperimetric profiles are for non-discrete spaces. The following Definition 3.34 define isoperimetric profiles for discrete spaces.

Definition 3.34 (ε-discrete isoperimetric profile) Let X be an mm-space, and $\varepsilon \geq 0$ a real number. We define the *ε-discrete isoperimetric profile I_X^ε of X* by

$$I_X^\varepsilon(v) := \inf\{\, m_X(B_\varepsilon(A)) \mid m_X(A) = v \,\} \text{ for } v \in \operatorname{Im} m_X,$$

where $\operatorname{Im} m_X := \{\, m_X(A) \mid A \subset X \text{ is a Borel set.} \,\}$.

The following Definition 3.35 is a discrete version of $\mathrm{IC}(\nu)$ condition.

Definition 3.35 (Isoperimetric comparison condition with an error) We say that an mm-space X satisfies the condition $\mathrm{IC}_\varepsilon^+(\nu)$ for a Borel probability measure ν on $\mathbb{R}$ and a real number $\varepsilon \geq 0$ if we have

$$I_X^{\delta^+(t)+\varepsilon} \circ V(t) \geq V(t + \delta^+(t))$$

for any $t \in (\operatorname{supp} \nu \setminus \{\sup \operatorname{supp} \nu\}) \cap V^{-1}(\operatorname{Im} m_X \setminus \{0\})$, where $V(t) := \nu((-\infty, t])$ is the cumulative distribution function of ν, and where

$$\delta^+(t) := \inf\{\, s > 0 \mid t + s \in \operatorname{supp} \nu \,\}.$$

The following Propositions 3.36 and 3.37 explain the relation between $\mathrm{IC}_\varepsilon^+$ condition and ICL condition.

Proposition 3.36 *Let X be a finite mm-space equipped with the uniform measure, and ν a Borel probability measure on $\mathbb{R}$ with $N := \# \operatorname{supp} \nu < \infty$. Let ε be a non-negative real number. We assume that*

$$\operatorname{Im} \nu \subset (1/\#X)\mathbb{Z} := \left\{ \frac{1}{\#X} \cdot n \mid n \in \mathbb{Z} \right\}.$$

If X satisfies $\mathrm{IC}_\varepsilon^+(\nu)$, then it satisfies $\mathrm{ICL}_{(N-1)\varepsilon}(\nu)$.

Proof Suppose that X satisfies $\mathrm{IC}_\varepsilon^+(\nu)$. Take any two real numbers $a, b \in \operatorname{supp}\nu$ with $a \leq b$ and a Borel subset $A \subset X$ with $m_X(A) \geq V(a)$. We remark that $V(a) > 0$ because $\#\operatorname{supp}\nu < \infty$. We may assume $a < \sup\operatorname{supp}\nu$. We inductively define $\delta_n^+ : \mathbb{R} \to [0, \infty]$ by

$$\delta_1^+(t) := \delta^+(t) + t, \quad \delta_{n+1}^+(t) := \delta^+ \circ \delta_n^+(t) + \delta_n^+(t)$$

for any positive integer n. Now, there exists a positive integer n_0 such that $\delta_{n_0}^+(a) = b$ and $n_0 \leq N - 1$. Let us prove by induction

$$m_X(B_{\delta_n^+(a)-a+n\varepsilon}(A)) \geq V \circ \delta_n^+(a) \tag{3.19}$$

for any positive integer $n \leq n_0$.

First, we consider the case $n = 1$. Since m_X is the uniform measure and $\mathrm{Im}\nu \subset (1/\#X)\mathbb{Z}$, there exists a Borel set $\tilde{A}_1 \subset A$ such that $m_X(\tilde{A}_1) = V(a)$ because we have $m_X(A) \geq V(a)$. By the definition of $I_X^{\delta^+(a)+\varepsilon}$, we have

$$\begin{aligned} m_X(B_{\delta_1^+(a)-a+\varepsilon}(A)) &= m_X(B_{\delta^+(a)+\varepsilon}(A)) \\ &\geq m_X(B_{\delta^+(a)+\varepsilon}(\tilde{A}_1)) \\ &\geq I_X^{\delta^+(a)+\varepsilon} \circ V(a) \\ &\geq V \circ \delta_1^+(a), \end{aligned}$$

where we remark that X satisfies $\mathrm{IC}_\varepsilon^+(\nu)$.

Next, we assume (3.19) for $n = k$. Hence, we have

$$m_X(B_{\delta_k^+(a)-a+k\varepsilon}(A)) \geq V \circ \delta_k^+(a),$$

which implies that there exists a Borel subset

$$\tilde{A}_k \subset B_{\delta_k^+(a)-a+k\varepsilon}(A)$$

such that $m_X(\tilde{A}_k) = V \circ \delta_k^+(a)$. Therefore we have

$$\begin{aligned} m_X(B_{\delta_{k+1}^+(a)-a+(k+1)\varepsilon}(A)) &\geq m_X(B_{\delta_{k+1}^+(a)-\delta_k^+(a)+\varepsilon}(B_{\delta_k^+(a)-a+k\varepsilon}(A))) \\ &\geq m_X(B_{\delta^+\circ\delta_k^+(a)+\varepsilon}(\tilde{A}_k)) \\ &\geq I_X^{\delta^+\circ\delta_k^+(a)+\varepsilon} \circ V \circ \delta_k^+(a) \\ &\geq V \circ \delta_k^+(a) \end{aligned}$$

if $k + 1 \leq n_0$. Hence we obtain (3.19). In particular, we have

$$m_X(B_{\delta_{n_0}^+(a)-a+n_0\varepsilon}(A)) \geq V \circ \delta_{n_0}^+(a).$$

Therefore we obtain

$$\begin{aligned} m_X(B_{b-a+(N-1)\varepsilon}(A)) &\geq m_X(B_{\delta^+_{n_0}(a)-a+n_0\varepsilon}(A)) \\ &\geq V \circ \delta^+_{n_0}(a) = V(b). \end{aligned}$$

This completes the proof. □

Proposition 3.37 *Let X be an mm-space and ν a Borel probability measure on $\mathbb{R}$, and $\varepsilon \geq 0$ a real number. If X satisfies $\mathrm{ICL}_\varepsilon(\nu)$, then it satisfies $\mathrm{IC}^+_\varepsilon(\nu)$.*

Proof Take any $t \in (\operatorname{supp}\nu \setminus \{\sup\operatorname{supp}\nu\}) \cap V^{-1}(\operatorname{Im} m_X \setminus \{0\})$. Since $t \in \operatorname{supp}\nu \setminus \{\sup\operatorname{supp}\nu\}$, we have $\delta^+(t) < \infty$. Since $t \in V^{-1}(\operatorname{Im} m_X \setminus \{0\})$, we have $V(t) \in \operatorname{Im} m_X$ and $V(t) > 0$. Take any Borel set $A \subset X$ with $m_X(A) = V(t)$. By $\mathrm{ICL}_\varepsilon(\nu)$, we have

$$m_X(B_{\delta^+(t)+\varepsilon}(A)) \geq V(t + \delta^+(t))$$

because we have $t, t + \delta^+(t) \in \operatorname{supp}\nu$. This implies that

$$I_X^{\delta^+(t)+\varepsilon} \circ V(t) \geq V(t + \delta^+(t))$$

by Definition 3.34. This completes the proof. □

Example 3.38 Let $G_1, G_2, \ldots, G_n$ be connected graphs with same order $k \geq 2$. Let $\Pi_{i=1}^n G_i$ be the Cartesian product graph equipped with the path metric and the uniform measure. Let $d_0 : [k]^n \to \mathbb{R}$ be the l^1-distance function from the origin. Then $\Pi_{i=1}^n G_i$ satisfies $\mathrm{ICL}((d_0)_* m_{[k]^n})$ by Corollary 14 in [5] because

$$m_{[k]^n}(B_a(0)) = (d_0)_* m_{[k]^n}((-\infty, a]) =: V(a)$$

for any $a \in \operatorname{supp}(d_0)_* m_{[k]^n}$, where

$$B_a(0) := \left\{ x \in [k]^n \mid \sum_{i=1}^n x_i \leq a \right\}.$$

Hence the measure $(d_0)_* m_{[k]^n}$ is a 1-iso-dominant of $\Pi_{i=1}^n G_i$ by Theorem 1.7 (1). In particular, the measure $(d_0)_* m_{[k]^n}$ is a 1-iso-dominant of the discrete l^1-cube $[k]^n$.

Example 3.39 We assume that k is a positive even integer. Let $X := (\mathbb{Z}/(k\mathbb{Z}))^n$ be the discrete torus equipped with the l^1-metric and the uniform measure m_X, and $d_0 : X \to \mathbb{R}$ the l^1-distance function from the origin. Then it satisfies $\mathrm{ICL}((d_0)_* m_X)$ by Corollary 6 in [4] because

$$m_X(B_a(0)) = (d_0)_* m_X((-\infty, a]) =: V(a)$$

for any $a \in \operatorname{supp}(d_0)_* m_X$, where

$$B_a(0) := \{x \in X \mid d_0(x, 0) \leq a\}.$$

Hence the measure $(d_0)_* m_X$ is a 1-iso-dominant of X.

3.4 Stability of ε-Iso-Dominant

The aim of this subsection is to prove Theorem 1.8. We prepare some definitions and lemmas to prove it. The following Definition 3.40 is a generalization of ε-iso-dominant (Definition 1.4).

Definition 3.40 ((s, t)-iso-dominant) Let s and t be two non-negative real numbers. We call a Borel probability measure ν on $\mathbb{R}$ an (s, t)-*iso-dominant* of an mm-space X if we have $\nu \succ'_{(s,t)} \mu$ for all $\mu \in \mathcal{M}(X; 1)$.

Definition 3.41 (Distortion from the diagonal) Let (X, d_X) be a metric space. We define *the distortion from the diagonal* of a subset $S \subset X \times X$ by

$$\operatorname{dis}_\Delta S := \sup\{ d_X(x, y) \mid (x, y) \in S \}$$

if S is nonempty. We define $\operatorname{dis}_\Delta \emptyset := 0$. Let μ and ν be two Borel probability measures on X. We define *the distortion from the diagonal* of a transport plan $\pi \in \Pi(\mu, \nu)$ between μ and ν by

$$\operatorname{dis}_\Delta \pi := \inf_S \max\{ \operatorname{dis}_\Delta S, 1 - \pi(S) \}$$

where $S \subset X \times X$ is a closed subset.

Theorem 3.42 *(Strassen's theorem; cf. [26, Corollary 1.28]) Let μ and ν be two Borel probability measures on a metric space X. Then we have*

$$d_P(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \operatorname{dis}_\Delta \pi.$$

Lemma 3.43 *For a subset $S \subset \mathbb{R}^2$, we have*

$$\operatorname{dev}_\succ S \leq 2 \operatorname{dis}_\Delta S.$$

Proof Take any two points $(x, y), (x', y') \in S$. If $x - x' \geq 0$, then we have

$$\begin{aligned} y - y' - \max\{ x - x', 0 \} &= y - y' - |x - x'| \\ &\leq |y - y'| - |x - x'| \\ &\leq |x - y| + |x' - y'| \leq 2 \operatorname{dis}_\Delta S. \end{aligned}$$

If $x - x' < 0$, then we have

$$\begin{aligned} y - y' - \max\{ x - x', 0 \} &= y - y' \\ &< y - y' + x' - x \\ &\leq |y - x| + |x' - y'| \leq 2 \operatorname{dis}_\Delta S. \end{aligned}$$

Hence we obtain $\operatorname{dev}_\succ S \leq 2 \operatorname{dis}_\Delta S$. This completes the proof. □

Lemma 3.44 *Let μ and ν be two Borel probability measures on $\mathbb{R}$. If $d_P(\mu,\nu)<\varepsilon$, then we have $\mu \succ'_{(2\varepsilon,\varepsilon)} \nu$.*

Proof This follows from Theorem 3.42 and Lemma 3.43. □

Lemma 3.45 *Let μ and ν be two Borel probability measures on $\mathbb{R}$, and X an mm-space. If μ is an (s,t)-iso-dominant of X and we have $d_P(\mu,\nu)<\varepsilon$, then ν is an $(s+2\varepsilon, t+\varepsilon)$-iso-dominant of X.*

Proof This follows from Lemma 3.44 and Theorem 3.31. □

Lemma 3.46 *Let X and Y be two mm-spaces, and ν a Borel probability measure on $\mathbb{R}$. If ν is an (s,t)-iso-dominant of X and we have $d_{\mathrm{conc}}(X,Y)<\varepsilon$, then ν is an $(s+2\varepsilon, t+\varepsilon)$-iso-dominant of Y.*

Proof Take any $g \in \mathcal{L}ip_1(Y)$. By $d_{\mathrm{conc}}(X,Y)<\varepsilon$, there exists two parameters $\varphi : I \to X$ and $\psi : I \to Y$ such that

$$d_{\mathrm{H}}^{\mathrm{KF}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(Y)) < \varepsilon.$$

Hence there exists $f \in \mathcal{L}ip_1(X)$ such that $d_{\mathrm{KF}}(\varphi^* f, \psi^* g) < \varepsilon$. By Lemma 2.20, we have

$$\begin{aligned} d_{\mathrm{P}}(f_* m_X, g_* m_Y) &= d_{\mathrm{P}}(f_*(\varphi_*\mathcal{L}^1), g_*(\psi_*\mathcal{L}^1)) \\ &\le d_{\mathrm{KF}}(\varphi^* f, \psi^* g) < \varepsilon. \end{aligned}$$

Therefore we have $f_* m_X \succ'_{(2\varepsilon,\varepsilon)} g_* m_Y$ by Lemma 3.44. Since ν is an (s,t)-iso-dominant of X, we have $\nu \succ'_{(s,t)} f_* m_X$, which implies $\nu \succ'_{(s+2\varepsilon,t+\varepsilon)} g_* m_Y$ by Theorem 3.31. □

Proof of Theorem 1.8 Without loss of generality, we assume

$$d_{\mathrm{conc}}(X_n, X) < \varepsilon_n \text{ and } d_{\mathrm{P}}(\nu_n, \nu) < \varepsilon_n \quad \text{for any positive integer } n.$$

Take any positive integer n. Since the measure ν_n is an $(s+\varepsilon_n, t+\varepsilon_n)$-iso-dominant of X_n, the measure ν is an $(s+3\varepsilon_n, t+2\varepsilon_n)$-iso-dominant of X_n by Lemma 3.45. By Lemma 3.46, the measure ν is an $(s+5\varepsilon_n, t+3\varepsilon_n)$-iso-dominant of X. Hence, we have $\nu \succ'_{(s+5\varepsilon_n,t+3\varepsilon_n)} f_* m_X$ for any $f \in \mathcal{L}ip_1(X)$. By Theorem 3.28, we obtain $\nu \succ'_{(s,t)} f_* m_X$. This completes the proof. □

We apply Theorem 1.8 for the space of pyramids Π. The space Π is a natural compactificaion of the set $\mathcal{X}$ of mm-spaces. We refer to [12,21] for the theory of pyramids.

Definition 3.47 (Pyramid, cf. Definition 6.3 in [21]) A subset $\mathcal{P} \subset \mathcal{X}$ is called a *pyramid* if it satisfies the following (1), (2), and (3).

(1) If $X \in \mathcal{P}$ and if $Y \prec X$, then $Y \in \mathcal{P}$.
(2) For any two mm-spaces $X, X' \in \mathcal{P}$, there exists an mm-space $Y \in \mathcal{P}$ such that $X \prec Y$ and $X' \prec Y$.
(3) $\mathcal{P}$ is nonempty and $\square$-closed.

We denote the set of pyramids by Π. The set Π is equipped with the weak Hausdorff convergence (Definition 6.4 in [21]). About the weak Hausdorff convergence, we introduce the following useful proposition (cf. Proposition 6.9 in [21]).

Proposition 3.48 *(Down-to-earth criterion for weak convergence) For given $\square$-closed subset $\mathcal{Y}_n, \mathcal{Y} \subset \mathcal{X}$, $n = 1, 2, \ldots$, the following (1) and (2) are equivalent to each other.*

(1) $\mathcal{Y}_n$ converges weakly to $\mathcal{Y}$.
(2) Let $\underline{\mathcal{Y}}_\infty$ be the set of the limits of convergent subsequences $Y_n \in \mathcal{Y}_n$, and $\overline{\mathcal{Y}}_\infty$ the set of the limits of convergent subsequences of $Y_n \in \mathcal{Y}_n$. Then we have

$$\mathcal{Y} = \underline{\mathcal{Y}}_\infty = \overline{\mathcal{Y}}_\infty$$

To apply Theorem 1.8 for pyramids, we consider the following Propositions 3.49 and 3.51, and Definition 3.50.

Proposition 3.49 *Let X and Y be two mm-spaces. If a Borel probability measure ν on $\mathbb{R}$ is an (s, t)-iso-dominant of X for $s, t \geq 0$ and $X \succ Y$, then ν is an (s, t)-iso-dominant of Y.*

Definition 3.50 Let $\mathcal{Y} \subset \mathcal{X}$. We say that a Borel probability measure ν on $\mathbb{R}$ is *an (s, t)-iso-dominant of $\mathcal{Y}$* if ν is an (s, t)-iso-dominant of X for any mm-space $X \in \mathcal{Y}$.

Proposition 3.51 *Let X be an mm-space, and ν a Borel probability measure on $\mathbb{R}$. Then, ν is an (s, t)-iso-dominant of X if and only if ν is an (s, t)-iso-dominant of $\mathcal{P}_X := \{ Y \in \mathcal{X} \mid Y \prec X \}$.*

The following Theorem 3.52 is the stability of isoperimetric inequalities for weak Hausdorff convergence. This is generalization of Theorem 1.8.

Theorem 3.52 *Let $\mathcal{Y}_n \subset \mathcal{X}$ be a $\square$-closed subset, and $\overline{\mathcal{Y}}_\infty$ the set of the limits of convergent subsequences of $Y_n \in \mathcal{Y}_n$. We assume that a sequence $\{\nu_n\}_{n=1}^\infty$ of Borel probability measures on $\mathbb{R}$ converges weakly to a Borel probability measure ν, and a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of non-negative real numbers converges to 0. If ν_n is an $(s + \varepsilon_n, t + \varepsilon_n)$-iso-dominant of $\mathcal{Y}_n$ for any positive integer n, then ν is an (s, t)-iso-dominant of $\overline{\mathcal{Y}}_\infty$.*

Proof This theorem follows from Theorem 1.8 and Proposition 2.23. □

We obtain the following corollary by Proposition 3.48.

Corollary 3.53 *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of pyramids, and $\{\nu_n\}_{n=1}^{\infty}$ a sequence of Borel probability measures on $\mathbb{R}$. We assume that $\{\mathcal{P}_n\}_{n=1}^{\infty}$ converges weakly to a pyramid $\mathcal{P}$ and $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure ν on $\mathbb{R}$, and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of non-negative real numbers converges to 0. If ν_n is an $(s+\varepsilon_n, t+\varepsilon_n)$-iso-dominant of $\mathcal{P}_n$, then ν is an (s,t)-iso-dominant of $\mathcal{P}$.*

4 Applications of Iso-Lipschitz Order

4.1 Isoperimetric Inequality of Non-discrete l^1-Cubes

In this subsection, we assume that $[0,1]^n$ is equipped with the l^1-metric d_{l^1} and the uniform measure $m_{[0,1]^n} := \mathcal{L}^n|_{[0,1]^n}$, where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure. Put $[k] := \{0, 1, 2, \dots, k-1\}$. We have

$$\frac{1}{k}[k] = \left\{0, \frac{1}{k}, \frac{2}{k}, \dots, 1 - \frac{1}{k}\right\} \subset [0,1].$$

We assume that $\frac{1}{k}[k]^n$ is equipped with the l^1-metric d_{l^1} and the uniform measure $m_{\frac{1}{k}[k]^n} := \frac{1}{k^n}\sum_{x\in\frac{1}{k}[k]^n}\delta_x$.

Lemma 4.1 *The sequence $\{m_{\frac{1}{k}[k]^n}\}_{k=1}^{\infty}$ converges weakly to $m_{[0,1]^n}$ as $k \to \infty$.*

Proof Define a function $f : [0,1]^n \to \frac{1}{k}[k]^n$ by $f((x_i)_{i=1}^n) := (\frac{1}{k}\lfloor kx_i \rfloor)_{i=1}^n$, where $\lfloor\cdot\rfloor$ is the floor function. We set $\pi := (\mathrm{id}_{[0,1]^n}, f)_* m_{[0,1]^n}$. Then we have $\pi \in \Pi(m_{[0,1]^n}, m_{\frac{1}{k}[k]^n})$. Take any point $(x, f(x)) \in \operatorname{supp}\pi = (\mathrm{id}_{[0,1]^n}, f)([0,1]^n)$ and put $x := (x_i)_{i=1}^n$. Since

$$d_{l^1}(x, f(x)) = \sum_{i=1}^{n}\left|x_i - \frac{1}{k}\lfloor kx_i \rfloor\right| \le \frac{n}{k},$$

we have $\mathrm{dis}_\Delta \operatorname{supp}\pi \le \frac{n}{k}$. By Theorem 3.42, we obtain

$$d_{\mathrm{P}}(m_{[0,1]^n}, m_{\frac{1}{k}[k]^n}) \le \frac{n}{k} \to 0$$

as $k \to \infty$. This completes the proof. □

Proof of Theorem 1.9 We define a function $d_0 : \mathbb{R}^n \to \mathbb{R}$ by $d_0((x_i)_{i=1}^n) := \sum_{i=1}^n |x_i|$. By Example 3.38, the measure $(d_0)_* m_{[k]^n}$ is a 1-iso-dominant of $[k]^n$. Hence, the measure $(\frac{1}{k}d_0)_* m_{[k]^n}$ is a $\frac{1}{k}$-iso-dominant of $\frac{1}{k}[k]^n$ by Proposition 3.32. Since d_0 is 1-Lipschitz, we have

$$\begin{aligned} d_{\mathrm{P}}((\frac{1}{k}d_0)_* m_{[k]^n}, (d_0)_* m_{[0,1]^n}) &= d_{\mathrm{P}}((d_0)_* m_{\frac{1}{k}[k]^n}, (d_0)_* m_{[0,1]^n}) \\ &\le d_{\mathrm{P}}(m_{\frac{1}{k}[k]^n}, m_{[0,1]^n}) \to 0 \end{aligned}$$

as $n \to \infty$ by Lemma 4.1. By Theorem 1.8, the measure $(d_0)_* m_{[0,1]^n}$ is an iso-dominant of $[0,1]^n$. This completes the proof. □

We obtain Theorem 1.10 in the same way as in the proof of Theorem 1.9 by using Example 3.39.

As another application of Theorem 1.8, we obtain the following, which is a variant of normal law à la Lévy (see Theorem 2.2 in [21]) by using Theorem 13 in [5].

Theorem 4.2 *(Normal law à la Lévy on product graphs) Let $G_1, G_2, \ldots, G_n, \ldots$ be connected graphs with same order $k \geq 2$. Put*

$$\varepsilon_n := \sqrt{\frac{12}{(k^2 - 1)n}}.$$

Let $X_n := (\prod_{i=1}^n G_i, d_{X_n}, m_{X_n})$ be the Cartesian product graph equipped with the path metric d_{X_n} and the uniform measure m_{X_n}. Put $Y_n := (\prod_{i=1}^n G_i, \varepsilon_n \cdot d_{X_n}, m_{X_n})$. Let $\{f_{n_i}\}$ be a subsequence of a sequence of 1-Lipschitz functions $f_n : Y_n \to \mathbb{R}, n = 1, 2, \ldots$. If $(f_{n_i})_* m_{Y_{n_i}}$ converges weakly to a Borel probability measure σ, then we have $\gamma^1 \succ' \sigma$, where γ^1 is the 1-dimensional standard Gaussian measure.

In the case that $k = 2$, we see that X_n is the n-dimensional Hamming cube. If we replace X_n by n-dimensional (non-discrete) l^1-cube or n-dimensional (non-discrete) l^1-torus, we obtain the normal law à la Lévy respectively.

Proof of Theorem 4.2 We define a function $d_0 : \mathbb{R}^n \to \mathbb{R}$ by

$$d_0((x_i)_{i=1}^n) := \sum_{i=1}^n |x_i|.$$

By Example 3.38, the measure $(d_0)_* m_{[k]^n}$ is a 1-iso-dominant of X_n, which implies that $(\varepsilon_n \cdot d_0)_* m_{[k]^n}$ is an ε_n-iso-dominant of Y_n by Proposition 3.32. By Proposition 3.51, the measure $(\varepsilon_n \cdot d_0)_* m_{[k]^n}$ is an ε_n-iso-dominant of $\mathcal{P}_{Y_n} := \{ Y \in \mathcal{X} \mid Y \prec Y_n \}$. By the central limit theorem, $(\varepsilon_n \cdot d_0)_* m_{[k]^n}$ converges weakly to γ^1 as $n \to \infty$. Putting $\mathcal{Y}_n := \mathcal{P}_{Y_n}$, γ^1 is an iso-dominant of $\overline{\mathcal{Y}}_\infty$ by Theorem 3.52.

Take any sequence of 1-Lipschitz functions $f_n : Y_n \to \mathbb{R}, n = 1, 2, \ldots$. We assume that a subsequence $\{n(i)\}_{i=1,2,\ldots}$ satisfies that $(f_{n(i)})_* m_{Y_{n(i)}}$ converges weakly to a measure σ as $i \to \infty$. Since $(f_n)_* m_{Y_n} \in \mathcal{P}_{Y_n}$, we have $\sigma \in \overline{\mathcal{Y}}_\infty$. Then we obtain $\gamma^1 \succ' \sigma$ because γ^1 is an iso-dominant of $\overline{\mathcal{Y}}_\infty$. This completes the proof. □

4.2 Comparison Theorem for Observable Diameter

In this subsection, we evaluate the observable diameter by using the iso-dominant.

Proposition 4.3 *Let μ and ν be two Borel probability measures on $\mathbb{R}$. If $\mu \succ'_{(s,t)} \nu$, then we have*

$$\operatorname{diam}(\mu; 1 - \kappa) + s \geq \operatorname{diam}(\nu; 1 - \kappa - t) \quad \text{for any } \kappa > 0.$$

Proof By $\mu \succ'_{(s,t)} \nu$, there exist $\pi \in \Pi(\mu,\nu)$ and a Borel set $S \subset \mathbb{R}^2$ such that $\mathrm{dev}_{\succ} S \le s$ and $1-\pi(S) \le t$. Take any Borel set $A \subset \mathbb{R}$ with $\mu(A) \ge 1-\kappa$. Put $B := \mathrm{pr}_2(S \cap (\mathrm{pr}_1)^{-1}(A))$. Since

$$\begin{aligned}\nu(\overline{B}) &\ge \pi(S \cap (\mathrm{pr}_1)^{-1}(A)) \\ &= \pi((\mathrm{pr}_1)^{-1}(A)) - \pi(S^c \cap (\mathrm{pr}_1)^{-1}(A)) \\ &\ge \mu(A) - \pi(S^c) \ge 1-\kappa-t,\end{aligned}$$

we have $\mathrm{diam}(\nu; 1-\kappa-t) \le \mathrm{diam} B$. By Proposition 3.25, we have $\mathrm{diam} B \le \mathrm{diam} A + \mathrm{dev}_{\succ} S$, which implies $\mathrm{diam}(\nu; 1-\kappa-t) \le \mathrm{diam} A + s$. Then we obtain $\mathrm{diam}(\nu; 1-\kappa-t) \le \mathrm{diam}(\mu; 1-\kappa) + s$. This completes the proof. □

Proposition 4.4 *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel probability measures on $\mathbb{R}$ and κ a positive real number. We assume that $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure μ on $\mathbb{R}$ and that the function $t \mapsto \mathrm{diam}(\mu; 1-t)$ is continuous at κ. Then we have*

$$\lim_{n\to\infty} \mathrm{diam}(\mu_n; 1-\kappa) = \mathrm{diam}(\mu; 1-\kappa).$$

Proof Put $\varepsilon_n := d_{\mathrm{P}}(\mu_n, \mu) + \frac{1}{n}$. By Lemma 3.44, we have $\mu \succ'_{(2\varepsilon_n,\varepsilon_n)} \mu_n$ and $\mu_n \succ'_{(2\varepsilon_n,\varepsilon_n)} \mu$. Since $\kappa - \varepsilon_n > 0$ for sufficiently large n, we have

$$\begin{aligned}\mathrm{diam}(\mu; 1-(\kappa-\varepsilon_n)) + 2\varepsilon_n &\ge \mathrm{diam}(\mu_n; 1-(\kappa-\varepsilon_n)-\varepsilon_n) \\ &\ge \mathrm{diam}(\mu; 1-\kappa-\varepsilon_n) - 2\varepsilon_n\end{aligned}$$

by Proposition 4.3. Since $t \mapsto \mathrm{diam}(\mu; 1-t)$ is continuous, we obtain $\lim_{n\to\infty} \mathrm{diam}(\mu_n; 1-\kappa) = \mathrm{diam}(\mu; 1-\kappa)$. This completes the proof. □

Lemma 4.5 *Let μ be a probability measure on $\mathbb{R}$ with $d\mu = \varphi\, d\mathcal{L}^1$, where $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure and $\varphi : \mathbb{R} \to (0,\infty)$ is a continuous function. We assume that φ is even on $\mathbb{R}$ and strictly decreasing on $[0,\infty)$. Then we have*

$$\mathrm{diam}(\mu; 1-\kappa) = F^{-1}\left(1-\frac{\kappa}{2}\right) - F^{-1}\left(\frac{\kappa}{2}\right)$$

for any $\kappa \in (0,1]$, where $F(t) := \mu((-\infty, t])$ is the cumulative distribution function of μ. In particular, the function $\kappa \mapsto \mathrm{diam}(\mu; 1-\kappa)$ is continuous on $(0,\infty)$.

Proof We may assume that $\kappa \in (0,1)$ because the case $\kappa = 1$ is clear. Since μ is a measure on $\mathbb{R}$ and has no atom, $\mathrm{diam}(\mu; 1-\kappa)$ is equals to the infimum of $\mathrm{diam} I$ where $I \subset \mathbb{R}$ is a closed interval with $\mu(I) = 1-\kappa$. We have $-\infty < \inf I < F^{-1}(\kappa)$ because we may assume that $\mathrm{diam} I < \infty$. If $\inf I = t$, we have

$$\sup I = F^{-1}(1 - \kappa + F(t))$$

because $\mu(I) = 1 - \kappa$. This implies

$$\operatorname{diam}(\mu; 1 - \kappa) = \inf\{D(t) \mid t \leq F^{-1}(\kappa)\},$$

where we put $D(t) := F^{-1}(1 - \kappa + F(t)) - t$.

Since $F'(t) = \varphi(t)$, we have

$$\frac{dD}{dt}(t) = \frac{\varphi(t)}{\varphi \circ F^{-1}(1 - \kappa + F(t))} - 1. \tag{4.1}$$

We have $D'(t) = 0$ if and only if

$$\varphi(t) = \varphi \circ F^{-1}(1 - \kappa + F(t)). \tag{4.2}$$

Since φ is even on $\mathbb{R}$ and strictly decreasing on $[0, \infty)$, the Eq. (4.2) implies that

$$t = -F^{-1}(1 - \kappa + F(t)) \tag{4.3}$$

because $t < F^{-1}(1 - \kappa + F(t))$. We have

$$F(-t) = 1 - F(t) \tag{4.4}$$

for any real number t because φ is even. By (4.3) and (4.4), we have $F(t) = \kappa/2$. We put $t_0 := F^{-1}(\kappa/2)$. By (4.1), we have $D'(t) < 0$ if $t < t_0$, and $D'(t) > 0$ if $t > t_0$. This implies that $D(t_0)$ is the minimum of D. Then we have

$$\operatorname{diam}(\mu; 1 - \kappa) = D(t_0) = F^{-1}(1 - \frac{\kappa}{2}) - F^{-1}(\frac{\kappa}{2}).$$

Since F^{-1} is continuous on $(0, 1)$, the function $\kappa \mapsto \operatorname{diam}(\mu; 1-\kappa)$ is also continuous on $(0, 1)$. Since $\operatorname{diam}(\mu; 1 - \kappa) = 0$ for any $\kappa \geq 1$, the function $\kappa \mapsto \operatorname{diam}(\mu; 1 - \kappa)$ is continuous on $[1, \infty)$. This completes the proof. □

Theorem 4.6 *Let s and t be two non-negative real numbers. If a Borel probability measure ν on $\mathbb{R}$ is an (s, t)-iso-dominant of an mm-space X, then we have* $\operatorname{ObsDiam}(X; -\kappa - t) \leq \operatorname{diam}(\nu; 1 - \kappa) + s$ *for any $\kappa \geq 0$.*

Proof Take any 1-Lipschitz function $f : X \to \mathbb{R}$. Since ν is an (s, t)-iso-dominant of X, we have $\nu \succ'_{(s,t)} f_* m_X$. By Proposition 4.3, we have $\operatorname{diam}(f_* m_X; 1 - \kappa - t) \leq \operatorname{diam}(\nu; 1 - \kappa) + s$. Hence, we obtain $\operatorname{ObsDiam}(X; 1 - \kappa - t) \leq \operatorname{diam}(\nu; 1 - \kappa) + s$. This completes the proof. □

Let $G_1, G_2, \ldots, G_n, \ldots$ be connected graphs with same order $k \geq 2$. Put $\varepsilon_{k,n} := \sqrt{\frac{12}{(k^2-1)n}}$. We define a function $d_{0,n} : \mathbb{R}^n \to \mathbb{R}$ by $d_{0,n}((x_i)_{i=1}^n) := \sum_{i=1}^n |x_i|$. Put $\nu_{k,n} := (\varepsilon_{k,n} \cdot d_{0,n})_* m_{[k]^n}$.

Theorem 4.7 *We have*

$$\operatorname{ObsDiam}(\varepsilon_{k,n}\prod_{i=1}^{n}G_i;-\kappa) \le \operatorname{diam}(\nu_{k,n};1-\kappa)+\varepsilon_{k,n} \tag{4.5}$$

$$\le \operatorname{ObsDiam}(\varepsilon_{k,n}[k]^n;-\kappa)+\varepsilon_{k,n} \tag{4.6}$$

$$\le \operatorname{diam}(\nu_{k,n};1-\kappa)+2\varepsilon_{k,n}. \tag{4.7}$$

Proof By Theorem 4.6 and Example 3.38, and Proposition 3.32, we have (4.5) and (4.7). Since $\nu_{k,n}\in\mathcal{M}(\varepsilon_{k,n}[k]^n;1)$, we have (4.6). This completes the proof. □

Lemma 4.8 *We have*

$$\lim_{n\to\infty}\operatorname{diam}(\nu_{k,n};1-\kappa)=\operatorname{diam}(\gamma^1;1-\kappa) \tag{4.8}$$

for any $\kappa>0$.

Proof By Lemma 4.5, the function $\kappa\mapsto\operatorname{diam}(\gamma^1;1-\kappa)$ is continuous on $(0,\infty)$. Applying Proposition 4.4, we obtain (4.8) since the sequence $\{\nu_{k,n}\}$ converges to γ^1 weakly as $n\to\infty$. This completes the proof. □

Corollary 4.9 *We have*

$$\limsup_{n\to\infty}\operatorname{ObsDiam}\left(\varepsilon_{k,n}\prod_{i=1}^{n}G_i;-\kappa\right)\le\operatorname{diam}(\gamma^1;1-\kappa)\text{ for }\kappa>0.$$

Proof Since $\varepsilon_{k,n}\to 0$ as $n\to\infty$, we have

$$\limsup_{n\to\infty}\operatorname{ObsDiam}\left(\varepsilon_{k,n}\prod_{i=1}^{n}G_i;-\kappa\right)\le\lim_{n\to\infty}\operatorname{diam}(\nu_{k,n};1-\kappa)$$
$$=\operatorname{diam}(\gamma^1;1-\kappa)$$

by (4.5) in Theorem 4.7 and Lemma 4.8. □

Corollary 4.10 *We have*

$$\lim_{n\to\infty}\operatorname{ObsDiam}(\varepsilon_{k,n}[k]^n;-\kappa)=\operatorname{diam}(\gamma^1;1-\kappa)\text{ for }\kappa>0.$$

In particular, we obtain

$$\lim_{n\to\infty}\operatorname{ObsDiam}(\frac{2}{\sqrt{n}}Q^n;-\kappa)=\operatorname{diam}(\gamma^1;1-\kappa)\text{ for }\kappa>0$$

in the case $k=2$.

Proof Take any $\kappa > 0$. By (4.6) in Theorem 4.7 and Lemma 4.8, we have

$$\liminf_{n\to\infty} \mathrm{ObsDiam}(\varepsilon_{k,n}[k]^n; -\kappa) \geq \mathrm{diam}(\gamma^1; 1-\kappa).$$

By (4.7) in Theorem 4.7 and Lemma 4.8, we also have

$$\limsup_{n\to\infty} \mathrm{ObsDiam}(\varepsilon_{k,n}[k]^n; -\kappa) \leq \mathrm{diam}(\gamma^1; 1-\kappa).$$

These imply

$$\lim_{n\to\infty} \mathrm{ObsDiam}(\varepsilon_{k,n}[k]^n; -\kappa) = \mathrm{diam}(\gamma^1; 1-\kappa).$$

This completes the proof. □

4.3 Concentration on the l^1-Hyper Cube

The concentration function is an important function in geometry and probability. In this section, we evaluate the concentration function of the n-dimensional l^1-hyper cube $[0, D]^n$ as an application of Theorem 1.1. Evaluations of concentration functions has been researched in [12,17]. We obtain a better upper bound of the concentration function of l^1-hyper cube than the former result.

Definition 4.11 (Concentration function [2]) Let X be an mm-space. We define the *concentration function of* X

$$\alpha_X(r) := \sup\left\{1 - m_X(U_r(A)) \mid A \subset X \text{ is Borel with } m_X(A) \geq \frac{1}{2}\right\}$$

for any real number $r > 0$.

We set $x_+ := \max\{x, 0\}$ for the positive part of the real number x.

Theorem 4.12 *(Theorem 1 (9.5) in Chapter I in [9]) Let S_n be the sum of n independent random variables distributed uniformly over* $[0, D]$ *for any real number $D > 0$. Then we have*

$$\mathrm{P}(S_n \leq x) = \frac{1}{D^n n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-kD)_+^n$$

for any positive integer n and any real number $x \geq 0$.

Theorem 4.13 *(Hoeffding's inequality [14]) Let $X_1, \ldots, X_n$ be independent random variables with $a_i \leq X_i \leq b_i$, where a_i and b_i are two real numbers with $a_i \leq b_i$ for all $i = 1, \ldots, n$. We set $S_n := \sum_{i=1}^n X_i$. Then we have*

$$\mathrm{P}(S_n - \mathrm{E}[S_n] \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

We have the following Theorem 4.14 as an application of Theorem 1.1.

Theorem 4.14 *Let D be positive real number, and let $([0, D]^n, d_{l^1})$ be the n-dimensional hyper cube equipped with the l^1-metric and the normalized uniform measure. Then we have*

$$\alpha_{([0,D]^n,d_{l^1})}(r) = 1 - \frac{1}{D^n n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\left(\frac{n}{2} - k\right)D + r\right)_+^n \tag{4.9}$$

and

$$\alpha_{([0,D]^n,d_{l^1})}(r) \le \exp\left(-\frac{2r^2}{nD^2}\right) \tag{4.10}$$

for any real number $r > 0$.

Proof Let $d_0 : [0, D]^n \to \mathbb{R}$ be the distance function from the origin. By Theorem 1.1, we have

$$\alpha_{([0,D]^n,d_{l^1})}(r) = (d_0)_* m\left(\left[\frac{nD}{2} + r, \infty\right)\right) \tag{4.11}$$

for $r > 0$. By (4.11) and Theorem 4.12, we obtain (4.9). By (4.11) and Theorem 4.13, we obtain (4.10). □

We compare Theorem 4.14 with the following Theorems 4.15 and 4.16.

Theorem 4.15 *(cf. Corollary 1.17 in [17]) Let $m_X = \otimes_{i=1}^n m_{X_i}$ be the product probability measure on the Cartesian product $X = \Pi_{i=1}^n X_i$ of mm-spaces (X_i, d_{X_i}, m_{X_i}) with finite diameters D_i, $i = 1, \ldots, n$, equipped with the l^1-metric $d_X = \sum_{i=1}^n d_{X_i}$. Then we have*

$$\alpha_X(r) \le \exp\left(-\frac{r^2}{8\sum_{i=1}^n D_i^2}\right)$$

for any real number $r > 0$.

In the case where X_i is the interval $[0, D]$ equipped with the Lebesgue measure, Theorem 4.15 evaluates the concentration function of l^1-hyper cube $[0, D]^n$. Theorem 4.14 is a better evaluation than it.

The following Theorem 4.16 is an evaluation of the concentration function of the n-dimensional Hamming cube. In the Hamming cube case, Theorem 4.16 is a better evaluation than Theorem 4.15.

Theorem 4.16 *(cf. Theorem 2.11 in [17]) Let $m_{\{0,D\}^n}$ be the uniform measure on $\{0, D\}^n$ equipped with the l^1-metric d_{l^1}. Then we have*

$$\alpha_{(\{0,D\}^n,d_{l^1})}(r) \le \exp\left(-\frac{2r^2}{nD^2}\right)$$

for any real number $r > 0$.

The proof of Theorem 4.16 uses the isoperimetric inequality of the n-dimensional Hamming cube. Theorem 4.14 also uses the isoperimetric inequality.

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