The Journal of Geometric Analysis (2022) 32:22
https://doi.org/10.1007/s12220-021-00768-0

Regularity Theory for Mixed Local and Nonlocal Parabolic *p*-Laplace Equations

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Received: 2 August 2021 / Accepted: 11 October 2021 / Published online: 9 December 2021

Abstract

We investigate the mixed local and nonlocal parabolic p-Laplace equation

$$\partial_t u(x,t) - \Delta_p u(x,t) + \mathcal{L}u(x,t) = 0,$$

where Δ_p is the usual local p-Laplace operator and $\mathcal{L}$ is the nonlocal p-Laplace type operator. Based on the combination of suitable Caccioppoli-type inequality and Logarithmic Lemma with a De Giorgi–Nash–Moser iteration, we establish the local boundedness and Hölder continuity of weak solutions for such equations.

Keywords Local boundedness · Hölder continuity · Mixed local and nonlocal parabolic p-Laplace equation

Mathematics Subject Classification 35B45 · 35B65 · 35D30 · 35K55 · 35R11

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1 Introduction

In this paper, we are concerned with the local behaviour of weak solutions to the following mixed problem:

$$\partial_t u(x,t) - \Delta_p u(x,t) + \mathcal{L}u(x,t) = 0 \quad \text{in } Q_T, \quad 1 < p < \infty, \tag{1.1}$$

where $Q_T := \Omega \times (0, T)$ with $T > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. This kind of evolution equations arises from the Lévy process, image processing etc; see [16] and references therein. The local p-Laplace operator Δ_p is defined as follows:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

and $\mathcal{L}$ is a nonlocal p-Laplace operator given by

$$\mathcal{L}u(x,t) = \text{P.V.} \int_{\mathbb{R}^N} K(x,y,t)|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))\,\mathrm{d}y, \tag{1.2}$$

where the symbol P.V. stands for the Cauchy principal value. Here, K is a symmetric kernel fulfilling

$$K(x,y,t) = K(y,x,t)$$

and

$$\frac{\Lambda^{-1}}{|x-y|^{N+sp}} \le K(x,y,t) \le \frac{\Lambda}{|x-y|^{N+sp}} \tag{1.3}$$

with $\Lambda \ge 1$ and $0 < s < 1$ for all $x, y \in \mathbb{R}^N$ and $t \in (0, T)$.

Before stating our main results, let us mention some known results. For the nonlocal parabolic equations of p-Laplacian type,

$$\partial_t u(x,t) + \mathcal{L}u(x,t) = 0, \tag{1.4}$$

the existence and uniqueness of strong solutions were verified by Vázquez [27], where the author studied the long-time behaviours as well. Mazón–Rossi–Toledo [24] established the well-posedness of solutions to Eq. (1.4) together with the asymptotic property. When it comes to regularity theory of this equation, Strömqvist [26] obtained the existence and local boundedness of weak solutions provided $p \ge 2$. Hölder regularity with specific exponents in the case $p \ge 2$ was proved by Brasco–Lindgren–Strömqvist [5]. Furthermore, Ding–Zhang–Zhou [12] showed the local boundedness and Hölder continuity of weak solutions to the nonhomogeneous case under the conditions that $1 < p < \infty$ and $2 < p < \infty$, respectively. We refer the readers to [6,18,22,28,29] and references therein for more results.

In the mixed local and nonlocal setting, for the case $p = 2$,

$$-\Delta u + (-\Delta)^s u = 0, \tag{1.5}$$

Foondun [19] has derived Harnack inequality and interior Hölder estimates for nonnegative solutions, see also [8] for a diverse approach. In addition, the Harnack inequality regarding the parabolic version of (1.5) was established in [2,7], where, however, the authors only proved such inequality for globally nonnegative solutions. Very recently, Garain–Kinnunen [21] proved a weak Harnack inequality with a tail term for sign changing solutions to the parabolic problem of (1.5). For what concerns maximum principles, interior sobolev regularity along with symmetry results among many other quantitative and qualitative properties for solutions to (1.5), one can see for instance [3,4,13–15]. In the nonlinear framework (i.e. $p \neq 2$), Garain–Kinnunen [20] developed the local regularity theory for

$$-\Delta_p u + \text{P.V.} \int_{\mathbb{R}^N} K(x, y)|u(x) - u(y)|^{p-2}(u(x) - u(y))\, \mathrm{d}y = 0$$

with $K(x, y) \simeq |x - y|^{-(N+sp)}$, involving boundedness, Hölder continuity, Harnack inequality, as well as lower/upper semicontinuity of weak supersolutions/subsolutions. Nonetheless, to the best of our knowledge, there are few results concerning on the mixed local and nonlocal nonlinear parabolic problems. To this end, influenced by the ideas developed in [10,12,20], we aim to establish the local boundedness and interior Hölder regularity of weak solutions to Eq. (1.1). It is noteworthy that our results are new even for the case $p = 2$.

Before giving the notion of weak solutions to (1.1), let us recall the tail space

$$L^q_\alpha(\mathbb{R}^N) := \left\{ v \in L^q_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|v(x)|^q}{1 + |x|^{N+\alpha}}\, \mathrm{d}x < +\infty \right\}, \quad q > 0 \text{ and } \alpha > 0.$$

Then, we define the tail appearing in estimates throughout this article,

$$\begin{aligned} \mathrm{Tail}_\infty(v; x_0, r, I) &= \mathrm{Tail}_\infty(v; x_0, r, t_0 - T_1, t_0 + T_2) \\ &:= \operatorname*{ess\,sup}_{t \in I} \left(r^p \int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{|v(x, t)|^{p-1}}{|x - x_0|^{N+sp}}\, \mathrm{d}x \right)^{\frac{1}{p-1}}, \end{aligned} \tag{1.6}$$

where $(x_0, t_0) \in \mathbb{R}^N \times (0, T)$ and the interval $I = [t_0 - T_1, t_0 + T_2] \subseteq (0, T)$. This is a parabolic counterpart to the tail introduced in [10]. It is easy to check that $\mathrm{Tail}_\infty(v; x_0, r, I)$ is well-defined for any $v \in L^\infty(I; L^{p-1}_{sp}(\mathbb{R}^N))$.

For any $1 < p < \infty$ and $0 < s < 1$, the fractional Sobolev space is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y < \infty \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

which is a reflexive Banach space, see [11,25]. From [11, Proposition 2.2], we know that the classical Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the fractional Sobolev space $W^{s,p}(\Omega)$.

The notion of weak solutions to (1.1) is stated as follows.

Definition 1.1 A function $u \in L^p(I; W^{1,p}_{\mathrm{loc}}(\Omega)) \cap C(I; L^2_{\mathrm{loc}}(\Omega)) \cap L^\infty(I; L^{p-1}_{sp}(\mathbb{R}^N))$ is a local weak subsolution (super-) to (1.1) if for any closed interval $I := [t_1, t_2] \subseteq (0, T)$, there holds that

$$\begin{aligned} &\int_\Omega u(x,t_2)\varphi(x,t_2)\,\mathrm{d}x - \int_\Omega u(x,t_1)\varphi(x,t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega u(x,t)\partial_t\varphi(x,t)\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} \mathcal{E}(u,\varphi,t)\,\mathrm{d}t \le (\ge) 0, \end{aligned} \tag{1.7}$$

for every nonnegative test function $\varphi \in L^p(I; W^{1,p}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$ with the property that φ has spatial support compactly contained in Ω, where

$$\begin{aligned} \mathcal{E}(u,\varphi,t) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big[&|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t)) \\ &\times (\varphi(x,t)-\varphi(y,t))K(x,y,t) \Big] \mathrm{d}x\mathrm{d}y. \end{aligned}$$

A function u is a local weak solution to (1.1) if and only if u is a local weak subsolution and supersolution.

We now are in a position to state the main contribution of this work. First, we provide the local boundedness of weak solutions in the cases that $p > \frac{2N}{N+2}$ and $1 < p \le \frac{2N}{N+2}$. For two real numbers, set

$$a \vee b := \max\{a,b\}, \quad a_+ := \max\{a,0\}, \quad a_- := -\min\{a,0\}.$$

Theorem 1.2 (Local boundedness) *Let $p > 2N/(N+2)$ and $q := \max\{p,2\}$. Assume that u is a local weak subsolution to (1.1). Let $(x_0,t_0) \in Q_T$, $R \in (0,1)$ and $Q_R^- \equiv B_R(x_0) \times (t_0 - R^p, t_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - R^p, t_0] \subseteq (0,T)$. Then it holds that*

$$\operatorname*{ess\,sup}_{Q^-_{R/2}} u \le \mathrm{Tail}_\infty\big(u_+; x_0, R/2, t_0 - R^p, t_0\big) + C\left(\fint_{Q_R^-} u_+^q\,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{N(p\kappa - q)}} \vee 1,$$

where $\kappa := 1 + 2/N$ and $C > 0$ only depends on N, p, s and Λ.

In the scenario that $1 < p \leq 2N/(N+2)$, assuming that the weak subsolution has the following constructions: for $m > \max\left\{2, \frac{N(2-p)}{p}\right\}$, there exists a sequence of $\{u_k\}_{k\in\mathbb{N}}$ whose components are bounded subsolutions of (1.1) fulfilling

$$\|u_k\|_{L^\infty_{\mathrm{loc}}(0,T;L^{p-1}_{sp}(\mathbb{R}^N))} \leq C \tag{1.8}$$

and

$$u_k \to u \quad \text{in } L^m_{\mathrm{loc}}(Q_T) \text{ as } k \to \infty. \tag{1.9}$$

Theorem 1.3 (Local boundedness) *Let $1 < p \leq 2N/(N+2)$, $\kappa = 1 + 2/N$ and $m > \max\left\{2, \frac{N(2-p)}{p}\right\}$. Suppose that $u \in L^m_{\mathrm{loc}}(Q_T)$ with the properties (1.8) and (1.9) is a local weak subsolution to (1.1). Let $(x_0, t_0) \in Q_T$, $R \in (0,1)$, and $Q_R^- \equiv B_R(x_0) \times (t_0 - R^p, t_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - R^p, t_0] \subseteq (0, T)$. Then it holds that*

$$\operatorname*{ess\,sup}_{Q^-_{R/2}} u \leq \mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^p, t_0\right) + C\left(⨍_{Q_R^-} u_+^m \,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{(N+p)(m-2-\beta)}} \vee \left(⨍_{Q_R^-} u_+^m \,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{(N+p)(m-p-\beta)}},$$

where $\beta = N(m - p\kappa)/(N+p)$ and $C > 0$ only depends on N, p, s, m and Λ.

Based on the boundedness result (Theorem 1.2), we are able to deduce that the weak solutions are locally Hölder continuous for $p > 2$.

Theorem 1.4 (Hölder continuity) *Let $p > 2$. Assume that u is a local weak solution to (1.1). Let $(x_0, t_0) \in Q_T$, $R \in (0,1)$ and $Q_R \equiv B_R(x_0) \times (t_0 - R^p, t_0 + R^p)$ such that $\overline{Q}_R \subseteq Q_T$. Then there is a constant $\alpha \in (0, p/(p-1))$ such that for every $\rho \in (0, R/2)$,*

$$\operatorname*{ess\,osc}_{Q_{\rho, d\rho^p}} u < C\left(\frac{\rho}{R}\right)^\alpha \left[\mathrm{Tail}_\infty\left(u; x_0, R/2, t_0 - R^p, t_0 + R^p\right) + \left(⨍_{Q_R} |u|^p \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \vee 1\right]$$

with some $d \in (0,1)$ and $C \geq 1$ depending on N, p, s and Λ.

The paper is organized as follows. In Sect. 2, we collect some notations and auxiliary inequalities. Necessary energy estimates are showed in Sect. 3. Sections 4 and 5 are devoted to proving the local boundedness and Hölder regularity of weak solutions, respectively.

2 Preliminaries

In this section, we first give some notations for clarity and then provide some important inequalities to be used later.

2.1 Notation

Let $B_\rho(x)$ be the open ball with radius ρ and centred at $x \in \mathbb{R}^N$. We denote the parabolic cylinders by $Q_{\rho,r}(x,t) := B_\rho(x)\times(t-r,t+r)$, $Q_\rho(x,t) := Q_{\rho,\rho^p}(x,t) = B_\rho(x) \times (t-\rho^p, t+\rho^p)$ and $Q_\rho^-(x,t) := Q^-_{\rho,\rho^p}(x,t) = B_\rho(x) \times (t-\rho^p, t)$ with $r, \rho > 0$ and $(x,t) \in \mathbb{R}^N \times (0,T)$. If not important, or clear from the context, we simply write these symbols by $B_\rho = B_\rho(x)$, $Q_{\rho,r} = Q_{\rho,r}(x,t)$, $Q_\rho = Q_\rho(x,t)$ and $Q_\rho^- = Q_\rho^-(x,t)$. Moreover, for $g \in L^1(V)$, we denote the integral average of g by

$$(g)_V := \fint_V g(x)\,\mathrm{d}x := \frac{1}{|V|}\int_V g(x)\,\mathrm{d}x.$$

Define

$$J_p(a,b) = |a-b|^{p-2}(a-b)$$

for any $a, b \in \mathbb{R}$. We also use the notation

$$\mathrm{d}\mu = \mathrm{d}\mu(x,y,t) = K(x,y,t)\,\mathrm{d}x\mathrm{d}y.$$

It is worth mentioning that the constant C represents a general positive constant which may differ from each other.

Next, we will show several fundamental but very useful Sobolev inequalities. The similar results can be found in [1,9]. For the sake of readability and completeness, we give the proof of the last two lemmas.

2.2 Sobolev Inequalities

Lemma 2.1 *Let* $1 \le p, \ell \le q < \infty$ *satisfy* $\frac{N}{p} - \frac{N}{q} \le 1$ *and*

$$\theta\left(1 - \frac{N}{p} + \frac{N}{q}\right) + (1-\theta)\left(\frac{N}{q} - \frac{N}{\ell}\right) = 0$$

with $\theta \in (0,1)$. *Then there exists a constant* $C > 0$ *only depending on* N, p, q, ℓ *such that*

$$\|u\|_{L^q(B_1)} \le C\|Du\|^{\theta}_{L^p(B_1)}\|u\|^{1-\theta}_{L^\ell(B_1)} \tag{2.1}$$

for all $u \in W^{1,p}(B_1) \cap L^\ell(B_1)$.

Lemma 2.2 *Let $1 < p < N$. Then for every $u \in W^{1,p}(B_1)$, there holds that*

$$\|u\|_{L^{\frac{Np}{N-p}}(B_1)} \le C\|u\|_{W^{1,p}(B_1)},$$

where $C > 0$ only depends on N and p.

Lemma 2.3 *Let $0 < t_1 < t_2$ and $p \in (1, \infty)$. Then for every*

$$u \in L^p\left(t_1, t_2; W^{1,p}(B_r)\right) \cap L^\infty\left(t_1, t_2; L^2(B_r)\right),$$

it holds that

$$\begin{aligned}
&\int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^{p(1+\frac{2}{N})}\,\mathrm{d}x\mathrm{d}t \\
&\le C\left(r^p \int_{t_1}^{t_2} ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right) \\
&\quad \times \left(\operatorname*{ess\,sup}_{t_1<t<t_2} ⨍_{B_r} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}},
\end{aligned} \tag{2.2}$$

where $C > 0$ only depends on p and N.

Proof We divide the proof into the following two cases.

Case 1 $1 < p < N$. Applying Lemma 2.2 and Hölder inequality, we infer that

$$\begin{aligned}
&\int_{t_1}^{t_2} ⨍_{B_1} |u(x,t)|^{p(1+\frac{2}{N})}\,\mathrm{d}x\mathrm{d}t \\
&= \int_{t_1}^{t_2} ⨍_{B_1} |u(x,t)|^{\frac{2p}{N}} |u(x,t)|^p\,\mathrm{d}x\mathrm{d}t \\
&\le \int_{t_1}^{t_2} \left(⨍_{B_1} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}} \left(⨍_{B_1} |u(x,t)|^{\frac{pN}{N-p}}\,\mathrm{d}x\right)^{\frac{N-p}{N}} \mathrm{d}t \\
&\le C\left(\operatorname*{ess\,sup}_{t_1<t<t_2} ⨍_{B_1} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}} \int_{t_1}^{t_2} ⨍_{B_1} |u(x,t)|^p + |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By the scaling argument, we get

$$\begin{aligned}
&\int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^{p(1+\frac{2}{N})}\,\mathrm{d}x\mathrm{d}t \\
&\le C\left(r^p \int_{t_1}^{t_2} ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right) \\
&\quad \times \left(\operatorname*{ess\,sup}_{t_1<t<t_2} ⨍_{B_r} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}}.
\end{aligned}$$

Case 2: $p \geq N$. We can easily find that

$$1 \geq \frac{N}{p} - \frac{N}{p(1+\frac{2}{N})}, \quad p\left(1+\frac{2}{N}\right) > 2$$

and

$$\theta\left(1 - \frac{N}{p} + \frac{N}{p\left(1+\frac{2}{N}\right)}\right) + (1-\theta)\left(\frac{N}{p\left(1+\frac{2}{N}\right)} - \frac{N}{2}\right) = 0$$

with $\theta = \frac{N}{N+2}$. Thus by Lemma 2.1, it follows that

$$\|u\|_{L^{p\left(1+\frac{2}{N}\right)}(B_1)} \leq C\|Du\|_{L^p(B_1)}^{\frac{N}{N+2}}\|u\|_{L^2(B_1)}^{\frac{2}{N+2}}$$

for any $t \in (t_1, t_2)$. Using the rescaling argument, we have

$$⨍_{B_r} |u(x,t)|^{p(1+\frac{2}{N})}\,\mathrm{d}x \leq Cr^p ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x \left(⨍_{B_r} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}}$$

for any $t \in (t_1, t_2)$. Integrating the above inequality over (t_1, t_2), we get

$$\begin{aligned}
&\int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^{p(1+\frac{2}{N})}\,\mathrm{d}x\mathrm{d}t \\
&\leq Cr^p \int_{t_1}^{t_2} ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x \left(⨍_{B_r} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}}\,\mathrm{d}t \\
&\leq Cr^p \int_{t_1}^{t_2} ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t \left(\operatorname*{ess\,sup}_{t_1<t<t_2} ⨍_{B_r} |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p}{N}}.
\end{aligned}$$

□

Lemma 2.4 *Let* $0 < t_1 < t_2$ *and* $p \in (1,\infty)$. *Then for every*

$$u \in L^p\left(t_1, t_2; W^{1,p}(B_r)\right) \cap L^\infty\left(t_1, t_2; L^p(B_r)\right),$$

it holds that

$$\begin{aligned}
&\int_{t_1}^{t_2} ⨍_{B_r} |u(x,t)|^{p(1+\frac{p}{N})}\,\mathrm{d}x\mathrm{d}t \\
&\leq Cr^p \int_{t_1}^{t_2} ⨍_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t \left(\operatorname*{ess\,sup}_{t_1<t<t_2} ⨍_{B_r} |u(x,t)|^p\,\mathrm{d}x\right)^{\frac{p}{N}}, \qquad (2.3)
\end{aligned}$$

where $C > 0$ *only depends on* p *and* N.

Proof It follows from Lemma 2.1 that

$$\|u\|_{L^{p(1+\frac{p}{N})}(B_1)} \le C\|Du\|_{L^p(B_1)}^{\frac{N}{N+p}}\|u\|_{L^p(B_1)}^{\frac{p}{N+p}}$$

for all $t \in (t_1, t_2)$, where $C > 0$ only depends on p and N. Using the rescaling argument, we have

$$\fint_{B_r} |u(x,t)|^{p(1+\frac{p}{N})}\,\mathrm{d}x \le Cr^p \fint_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x \left(\fint_{B_r} |u(x,t)|^p\,\mathrm{d}x\right)^{\frac{p}{N}}.$$

Integrating the above inequality over (t_1, t_2), we get

$$\begin{aligned}
&\int_{t_1}^{t_2}\fint_{B_r} |u(x,t)|^{p(1+\frac{p}{N})}\,\mathrm{d}x\mathrm{d}t\\
&\le Cr^p \int_{t_1}^{t_2}\fint_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x \left(\fint_{B_r} |u(x,t)|^p\,\mathrm{d}x\right)^{\frac{p}{N}}\,\mathrm{d}t\\
&\le Cr^p \int_{t_1}^{t_2}\fint_{B_r} |\nabla u(x,t)|^p\,\mathrm{d}x\mathrm{d}t \left(\operatorname*{ess\,sup}_{t_1<t<t_2}\fint_{B_r} |u(x,t)|^p\,\mathrm{d}x\right)^{\frac{p}{N}}.
\end{aligned}$$

□

3 Energy Estimates

In this section, we will establish the Caccioppoli inequality and Logarithmic form inequality for Eq. (1.1). The first step of the proof should be the regularization procedure with respect to the time variable, which can be performed by straightforward adaptation of standard reasonings as used in [5,12,23]. We omit this step here.

Lemma 3.1 (Caccioppoli-type inequality) *Let $p > 1$ and u be a local subsolution to (1.1). Let $B_r \equiv B_r(x_0)$ satisfy $\overline{B}_r \subseteq \Omega$ and $0 < \tau_1 < \tau_2$, $\ell > 0$ satisfy $[\tau_1 - \ell, \tau_2] \subseteq (0, T)$. For any nonnegative functions $\psi \in C_0^\infty(B_r)$ and $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) \equiv 0$ if $t \le \tau_1 - \ell$ and $\eta(t) \equiv 1$ if $t \ge \tau_1$, it holds that*

$$\begin{aligned}
&\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} |\nabla w_+(x,t)|^p\psi^p(x)\eta^2(t)\,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_{B_r} w_+^2(x,t)\psi^p(x)\,\mathrm{d}x\\
&\quad+\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\int_{B_r} |w_+(x,t)\psi(x) - w_+(y,t)\psi(y)|^p\eta^2(t)\,\mathrm{d}\mu\mathrm{d}t\\
&\le C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} |\nabla\psi(x)|^p w_+^p(x,t)\eta^2(t)\,\mathrm{d}x\mathrm{d}t\\
&\quad+ C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\int_{B_r} (\max\{w_+(x,t), w_+(y,t)\})^p|\psi(x) - \psi(y)|^p\eta^2(t)\,\mathrm{d}\mu\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ C \operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<\tau_2 \\ x\in \operatorname{supp}\psi}} \int_{\mathbb{R}^N\setminus B_r} \frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}}\,\mathrm{d}y \int_{\tau_1-\ell}^{\tau_2}\int_{B_r} w_+(x,t)\psi^p(x)\eta^2(t)\,\mathrm{d}x\mathrm{d}t \\
&+ C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} w_+^2(x,t)\psi^p(x)\eta(t)|\partial_t\eta(t)|\,\mathrm{d}x\mathrm{d}t, \qquad (3.1)
\end{aligned}
$$

where $w_+(x,t) := (u(x,t)-k)_+$ and $C>0$ depends on N, p, s, Λ.

Proof Taking $\varphi(x,t) := w_+(x,t)\psi^p(x)\eta^2(t) = (u(x,t)-k)_+\,\psi^p(x)\eta^2(t)$ as a test function in (1.7), for $s\in[\tau_1,\tau_2]$, we can obtain

$$
\begin{aligned}
0 \geq& \int_{\tau_1-\ell}^{s}\int_{B_r} \partial_t u(x,t) w_+(x,t)\psi^p(x)\eta^2(t)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{\tau_1-\ell}^{s}\int_{B_r} |\nabla u(x,t)|^{p-2}\nabla u(x,t)\cdot\nabla(w_+(x,t)\psi^p(x)\eta^2(t))\,\mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{2}\int_{\tau_1-\ell}^{s}\int_{B_r}\int_{B_r} J_p(w(x,t),w(y,t))\Big(w_+(x,t)\psi^p(x)\eta^2(t) \\
&-w_+(y,t)\psi^p(y)\eta^2(t)\Big)\,\mathrm{d}\mu\mathrm{d}t \\
&+ \int_{\tau_1-\ell}^{s}\int_{\mathbb{R}^N\setminus B_r}\int_{B_r} J_p\left(w(x,t),w(y,t)\right) w_+(x,t)\psi^p(x)\eta^2(t)\,\mathrm{d}\mu\mathrm{d}t \\
=:& I_1 + I_2 + \frac{1}{2}I_3 + I_4.
\end{aligned}
$$

Then we are going to estimate I_1, I_2, I_3 and I_4. First, we evaluate

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_{\tau_1-\ell}^{s}\int_{B_r} \partial_t w_+^2(x,t)\psi^p(x)\eta^2(t)\,\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_{B_r} w_+^2(x,t)\psi^p(x)\eta^2(t)\mathrm{d}x\Big|_{\tau_1-\ell}^{s} \\
&\quad - \int_{\tau_1-\ell}^{s}\int_{B_r} w_+^2(x,t)\psi^p(x)\eta(t)\partial_t\eta(t)\,\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_{B_r} w_+^2(x,s)\psi^p(x)\,\mathrm{d}x - \int_{\tau_1-\ell}^{s}\int_{B_r} w_+^2(x,t)\psi^p(x)\eta(t)\partial_t\eta(t)\,\mathrm{d}x\mathrm{d}t, \qquad (3.2)
\end{aligned}
$$

where in the last line we note that $\eta(\tau_1-\ell)=0$ and $\eta(s)=1$ when $s\geq\tau_1$. We proceed with estimating I_2. By Young's inequality with ε, it yields that

$$
\begin{aligned}
I_2 &= \int_{\tau_1-\ell}^{s}\int_{B_r} \psi^p\eta^2|\nabla u|^{p-2}\nabla u\cdot\nabla w_+ + p\psi^{p-1}\eta^2 w_+|\nabla u|^{p-2}\nabla u\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{\tau_1-\ell}^{s}\int_{B_r} |\nabla w_+|^p\psi^p\eta^2\,\mathrm{d}x\mathrm{d}t - \int_{\tau_1-\ell}^{s}\int_{B_r} p\psi^{p-1}\eta^2 w_+|\nabla u|^{p-1}|\nabla\psi|\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{\tau_1-\ell}^{s}\int_{B_r} |\nabla w_+|^p\psi^p\eta^2\,\mathrm{d}x\mathrm{d}t - \varepsilon\int_{\tau_1-\ell}^{s}\int_{B_r} \psi^p\eta^2|\nabla w_+|^p\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
& - C(p,\varepsilon) \int_{\tau_1-\ell}^{s} \int_{B_r} |\nabla \psi|^p \eta^2 w_+^p \,\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2} \int_{\tau_1-\ell}^{s} \int_{B_r} |\nabla w_+|^p \psi^p \eta^2 \,\mathrm{d}x\mathrm{d}t - C(p) \int_{\tau_1-\ell}^{s} \int_{B_r} |\nabla \psi|^p \eta^2 w_+^p \,\mathrm{d}x\mathrm{d}t, \qquad (3.3)
\end{aligned}
$$

with $\varepsilon = \frac{1}{2}$. As the same proof of Lemma 3.3 in [12], we have

$$
\begin{aligned}
I_3 \geq & \frac{1}{2^{p+1}} \int_{\tau_1-\ell}^{s} \int_{B_r} \int_{B_r} |w_+(x,t)\psi(x) - w_+(y,t)\psi(y)|^p \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t \\
& - C \int_{\tau_1-\ell}^{s} \int_{B_r} \int_{B_r} (\max\{w_+(x,t), w_+(y,t)\})^p |\psi(x) - \psi(y)|^p \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t
\end{aligned} \tag{3.4}
$$

and

$$
I_4 \geq -C \operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<s \\ x\in \operatorname{supp}\psi}} \int_{\mathbb{R}^N\setminus B_r} \frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}} \,\mathrm{d}y \int_{\tau_1-\ell}^{s} \int_{B_r} w_+(x,t)\psi^p(x)\eta^2(t) \,\mathrm{d}x\mathrm{d}t. \tag{3.5}
$$

Merging the estimates on (3.2)–(3.5), we get

$$
\begin{aligned}
& \int_{\tau_1-\ell}^{s} \int_{B_r} |\nabla w_+(x,t)|^p \psi^p(x)\eta^2(t) \,\mathrm{d}x\mathrm{d}t + \int_{B_r} w_+^2(x,s)\psi^p(x) \,\mathrm{d}x \\
& \quad + \int_{\tau_1-\ell}^{s} \int_{B_r} \int_{B_r} |w_+(x,t)\psi(x) - w_+(y,t)\psi(y)|^p \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t \\
& \leq C \int_{\tau_1-\ell}^{\tau_2} \int_{B_r} |\nabla \psi(x)|^p w_+^p(x,t)\eta^2(t) \,\mathrm{d}x\mathrm{d}t \\
& \quad + C \int_{\tau_1-\ell}^{\tau_2} \int_{B_r} \int_{B_r} \max\{w_+(x,t), w_+(y,t)\}^p \, |\psi(x) - \psi(y)|^p \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t \\
& \quad + C \operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<\tau_2 \\ x\in \operatorname{supp}\psi}} \int_{\mathbb{R}^N\setminus B_r} \frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}} \,\mathrm{d}y \int_{\tau_1-\ell}^{\tau_2} \int_{B_r} w_+(x,t)\psi^p(x)\eta^2(t) \,\mathrm{d}x\mathrm{d}t \\
& \quad + C \int_{\tau_1-\ell}^{\tau_2} \int_{B_r} w_+^2(x,t)\psi^p(x)\eta(t)|\partial_t \eta(t)| \,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

which leads to the desired result. □

From the forthcoming lemma, one can interpret the reason why we only derive the Hölder continuity of weak solutions in the case $p > 2$. For the subquadratic case, we at present cannot show a similar logarithmic estimate.

Lemma 3.2 (Logarithmic estimates) *Let $p > 2$ and u be a local weak solution to (1.1). Let $B_r \equiv B_r(x_0)$ and $(x_0, t_0) \in Q_T$, $T_0 > 0$, $0 < r \leq R/2$. We also denote*

$\tilde{Q} \equiv B_R(x_0) \times (t_0 - 2T_0, t_0 + 2T_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - 2T_0, t_0 + 2T_0] \subseteq (0, T)$. If $u \in L^\infty(\tilde{Q})$ and $u \geq 0$ in $\tilde{Q}$, then for any $d > 0$, it holds that

$$\begin{aligned}
&\int_{t_0-T_0}^{t_0+T_0} \int_{B_r} |\nabla \log (u(x,t) + d)|^p \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_0-T_0}^{t_0+T_0} \int_{B_r} \int_{B_r} \left|\log \left(\frac{u(x,t)+d}{u(y,t)+d}\right)\right|^p \,\mathrm{d}\mu\mathrm{d}t \\
&\leq C T_0 r^N d^{1-p} R^{-p} \left[\mathrm{Tail}_\infty(u; x_0, R, t_0 - 2T_0, t_0 + 2T_0)\right]^{p-1} \\
&\quad + C r^N d^{2-p} + C T_0 r^{N-sp} + C T_0 r^{N-p},
\end{aligned} \tag{3.6}$$

where $C > 0$ depends on N, p, s and Λ.

Proof Let $\psi \in C_0^\infty(B_{3r/2})$ and $\eta \in C_0^\infty(t_0 - 2T_0, t_0 + 2T_0)$ fulfil

$$0 \leq \psi \leq 1, \quad |\nabla \psi| < C r^{-1} \text{ in } B_{2r}, \quad \psi \equiv 1 \text{ in } B_r,$$

and

$$0 \leq \eta \leq 1, \quad |\partial_t \eta| < C T_0^{-1} \text{ in } (t_0 - 2T_0, t_0 + 2T_0), \quad \eta \equiv 1 \text{ in } (t_0 - T_0, t_0 + T_0).$$

Choosing $\varphi(x,t) := (u(x,t) + d)^{1-p} \psi^p(x) \eta^2(t)$ to test the weak formulation of (1.1), we get

$$\begin{aligned}
0 &= -\int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} \partial_t \left((u(x,t)+d)^{1-p} \psi^p(x) \eta^2(t)\right) u(x,t) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} |\nabla u|^{p-2} \nabla u \cdot \nabla \left((u(x,t)+d)^{1-p} \psi^p(x) \eta^2(t)\right) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{2} \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} \int_{B_{2r}} J_p(u(x,t), u(y,t)) \left[\frac{\psi^p(x)}{(u(x,t)+d)^{p-1}} \right. \\
&\quad \left. - \frac{\psi^p(y)}{(u(y,t)+d)^{p-1}}\right] \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t \\
&\quad + \int_{t_0-2T_0}^{t_0+2T_0} \int_{\mathbb{R}^N \setminus B_{2r}} \int_{B_{2r}} J_p(u(x,t), u(y,t)) \frac{\psi^p(x)}{(u(x,t)+d)^{p-1}} \eta^2(t) \,\mathrm{d}\mu\mathrm{d}t \\
&=: I_1 + I_2 + \frac{1}{2} I_3 + I_4.
\end{aligned}$$

It follows from the proof of [12, Lemma 3.5] that

$$I_1 = \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} \left((u(x,t)+d)^{1-p} \psi^p(x) \eta^2(t)\right) \partial_t u(x,t) \,\mathrm{d}x\mathrm{d}t \leq C r^N d^{2-p}, \tag{3.7}$$

$$I_3 \le -C \int_{t_0-T_0}^{t_0+T_0} \int_{B_r} \int_{B_r} \left| \log \left(\frac{u(x,t)+d}{u(y,t)+d} \right) \right|^p \mathrm{d}x\mathrm{d}y\mathrm{d}t + C T_0 r^{N-sp}, \tag{3.8}$$

and

$$I_4 \le C T_0 r^{N-sp} + C T_0 r^N R^{-p} d^{1-p} \left[\mathrm{Tail}_\infty (u; x_0, R, t_0 - 2T_0, t_0 + 2T_0) \right]^{p-1}. \tag{3.9}$$

By Young's inequality with ε, we estimate the integral I_2 as

$$\begin{aligned} I_2 &= -(p-1) \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} (u(x,t)+d)^{-p} \psi^p(x) |\nabla u(x,t)|^p \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\quad + p \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} (u(x,t)+d)^{1-p} \psi^{p-1}(x) |\nabla u(x,t)|^{p-2} \nabla u(x,t) \\ &\quad \cdot \nabla \psi(x) \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\le -(p-1) \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} (u(x,t)+d)^{-p} \psi^p(x) |\nabla (u(x,t)+d)|^p \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\quad + \varepsilon \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} (u(x,t)+d)^{-p} \psi^p(x) |\nabla (u(x,t)+d)|^p \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\quad + C(\varepsilon) \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_{2r}} |\nabla \psi(x)|^p \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\le -C(p) \int_{t_0-2T_0}^{t_0+2T_0} \int_{B_r} (u(x,t)+d)^{-p} |\nabla (u(x,t)+d)|^p \eta^2(t)\, \mathrm{d}x\mathrm{d}t \\ &\quad + C(N,p) T_0 r^{N-p} \\ &\le -C(p) \int_{t_0-T_0}^{t_0+T_0} \int_{B_r} |\nabla \log(u(x,t)+d)|^p \, \mathrm{d}x\mathrm{d}t + C(N,p) T_0 r^{N-p} \end{aligned} \tag{3.10}$$

with ε satisfying that $\varepsilon < p-1$. Combining with (3.7)–(3.10), we can obtain the Logarithmic estimates. □

Next, we will give a corollary of Lemma 3.2, which plays a crucial role in obtaining the Hölder continuity.

Corollary 3.3 *Let $p > 2$ and u be a local weak solution to (1.1). Let $B_r \equiv B_r(x_0)$ and $(x_0, t_0) \in Q_T$, $T_0 > 0$, $0 < r \le R/2$. We also denote $\tilde{Q} \equiv B_R(x_0) \times (t_0 - 2T_0, t_0 + 2T_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - 2T_0, t_0 + 2T_0] \subseteq (0, T)$. Suppose that $u \in L^\infty(\tilde{Q})$ and $u \ge 0$ in $\tilde{Q}$. Let $a, d > 0$, $b > 1$ and define*

$$v := \min \left\{ (\log(a+d) - \log(u+d))_+, \log b \right\}.$$

Then it holds that

$$\int_{t_0-T_0}^{t_0+T_0} \fint_{B_r} \left| v(x,t) - (v)_{B_r}(t) \right|^p \, \mathrm{d}x\mathrm{d}t$$

$$\leq CT_0 d^{1-p}\left(\frac{r}{R}\right)^p [\mathrm{Tail}_\infty(u; x_0, R, t_0 - 2T_0, t_0 + 2T_0)]^{p-1} + CT_0 + Cd^{2-p}r^p + CT_0 r^{p-sp}, \tag{3.11}$$

where $C > 0$ *depends on* N, p, s *and* Λ.

Proof By the Poincaré inequality (see for example [17, p. 276]), it yields that

$$⨍_{B_r} \left|v(x,t) - (v)_{B_r}(t)\right|^p \,\mathrm{d}x \leq Cr^{p-N}\int_{B_r} |\nabla v(x,t)|^p \,\mathrm{d}x$$

for any $t \in (t_0 - T_0, t_0 + T_0)$. Integrating the above inequality over $(t_0 - T_0, t_0 + T_0)$ leads to

$$\int_{t_0-T_0}^{t_0+T_0} ⨍_{B_r} \left|v(x,t) - (v)_{B_r}(t)\right|^p \,\mathrm{d}x\mathrm{d}t \leq Cr^{p-N}\int_{t_0-T_0}^{t_0+T_0}\int_{B_r} |\nabla v(x,t)|^p \,\mathrm{d}x\mathrm{d}t, \tag{3.12}$$

where $C > 0$ depends only on N, p. Observing that v is a truncation function of the sum of a constant and $\log(u + d)$, which gives that

$$\int_{t_0-T_0}^{t_0+T_0}\int_{B_r} |\nabla v(x,t)|^p \,\mathrm{d}x\mathrm{d}t \leq \int_{t_0-T_0}^{t_0+T_0}\int_{B_r} |\nabla \log(u(x,t) + d)|^p \,\mathrm{d}x\mathrm{d}t. \tag{3.13}$$

We can get the results from Lemma 3.2 along with (3.12) and (3.13). □

4 Local Boundedness

In this part, we are ready to study the local boundedness of weak solutions. To this end, we first introduce some notations. For $\sigma \in [1/2, 1)$, set

$$r_0 := r, \quad r_j := \sigma r + 2^{-j}(1-\sigma)r, \quad \tilde{r}_j := \frac{r_j + r_{j+1}}{2}, \quad j = 0, 1, 2, \ldots$$

and

$$Q_j^- := B_j \times \Gamma_j := B_{r_j}(x_0) \times \left(t_0 - r_j^p, t_0\right), \quad j = 0, 1, 2, \ldots,$$
$$\tilde{Q}_j^- := \tilde{B}_j \times \tilde{\Gamma}_j := B_{\tilde{r}_j}(x_0) \times \left(t_0 - \tilde{r}_j^p, t_0\right), \quad j = 0, 1, 2, \ldots.$$

Denote

$$k_j := \left(1 - 2^{-j}\right)\tilde{k}, \quad \tilde{k}_j := \frac{k_{j+1} + k_j}{2}, \quad j = 0, 1, 2, \ldots$$

with

$$\tilde{k} \geq \frac{\mathrm{Tail}_\infty\left(u_+; x_0, \sigma r, t_0 - r^p, t_0\right)}{2}.$$

Let

$$w_j := (u - k_j)_+, \quad \tilde{w}_j := \left(u - \tilde{k}_j\right)_+, \quad j = 0, 1, 2, \ldots.$$

We now provide a Caccioppoli-type inequality in a special cylinder, which leads to the recursive inequalities (see Lemmas 4.2 and 4.3).

Lemma 4.1 *Let $p > 1$ and u be a local weak subsolution to (1.1). Let $(x_0, t_0) \in Q_T$, $0 < r < 1$ and $Q_r^- = B_r(x_0) \times (t_0 - r^p, t_0)$ such that $\overline{B}_r(x_0) \subseteq \Omega$ and $\left[t_0 - r^p, t_0\right] \subseteq (0, T)$. Suppose q is a parameter satisfying $q \geq \max\{p, 2\}$, it holds that*

$$\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} |\nabla \tilde{w}_j(x,t)|^p \,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{t \in \Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t)\,\mathrm{d}x \\
&\quad + \int_{\Gamma_{j+1}} \int_{B_{j+1}} \fint_{B_{j+1}} \frac{|\tilde{w}_j(x,t) - \tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\leq \frac{C}{r^p} \left(\frac{1}{\sigma^p (1-\sigma)^{N+sp}} + \frac{1}{(1-\sigma)^p} \right) \left(\frac{2^{(p+q-2)j}}{\tilde{k}^{q-2}} + \frac{2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}} \right) \\
&\quad \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t, \qquad (4.1)
\end{aligned}$$

where $C > 0$ only depends on N, p, s, Λ and q.

Proof First we give a trivial but very useful inequality

$$\tilde{w}_j^\tau(x,t) \leq \frac{C 2^{(q-\tau)j}}{\tilde{k}^{q-\tau}} w_j^q(x,t) \quad \text{in } Q_T, \qquad (4.2)$$

where $0 \leq \tau < q$. We take the cut-off functions $\psi_j \in C_0^\infty(\tilde{B}_j)$ and $\eta_j \in C_0^\infty(\tilde{\Gamma}_j)$ such that

$$0 \leq \psi_j \leq 1, \quad \left|\nabla \psi_j\right| \leq \frac{C 2^j}{(1-\sigma) r} \text{ in } \tilde{B}_j, \quad \psi_j \equiv 1 \text{ in } B_{j+1}$$

and

$$0 \leq \eta_j \leq 1, \quad \left|\partial_t \eta_j\right| \leq \frac{C 2^{pj}}{(1-\sigma)^p r^p} \text{ in } \tilde{\Gamma}_j, \quad \eta_j \equiv 1 \text{ in } \Gamma_{j+1}.$$

Let $r = r_j$, $\tau_2 = t_0$, $\tau_1 = t_0 - r_{j+1}^p$ and $\ell = \tilde{r}_j^p - r_{j+1}^p$ in Lemma 3.1. Then we arrive at

$$
\begin{aligned}
&\int_{\tilde{\Gamma}_j}\int_{B_j} |\nabla \tilde{w}_j(x,t)|^p \psi_j^p(x)\eta_j^2(t)\,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} \int_{B_j} \tilde{w}_j^2(x,t)\psi_j^p(x)\,\mathrm{d}x \\
&\quad + \int_{\tilde{\Gamma}_j}\int_{B_j}\int_{B_j} \left|\tilde{w}_j(x,t)\psi_j(x) - \tilde{w}_j(y,t)\psi_j(y)\right|^p \eta_j^2(t)\,\mathrm{d}\mu\mathrm{d}t \\
&\le C\int_{\tilde{\Gamma}_j}\int_{B_j} |\nabla\psi_j(x)|^p \tilde{w}_j^p(x,t)\eta_j^2(t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_{\tilde{\Gamma}_j}\int_{B_j}\int_{B_j} (\max\{\tilde{w}_j(x,t),\tilde{w}_j(y,t)\})^p \left|\psi_j(x)-\psi_j(y)\right|^p \eta_j^2(t)\,\mathrm{d}\mu\mathrm{d}t \\
&\quad + C \operatorname*{ess\,sup}_{\substack{t\in\tilde{\Gamma}_j \\ x\in \operatorname{supp}\psi_j}} \int_{\mathbb{R}^N\setminus B_j} \frac{\tilde{w}_j^{p-1}(y,t)}{|x-y|^{N+sp}}\,\mathrm{d}y \int_{\tilde{\Gamma}_j}\int_{B_j} \tilde{w}_j(x,t)\psi_j^p(x)\eta_j^2(t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_{\tilde{\Gamma}_j}\int_{B_j} \tilde{w}_j^2(x,t)\psi_j^p(x)\eta_j(t)\left|\partial_t\eta_j(t)\right|\,\mathrm{d}x\mathrm{d}t \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.3}
$$

Using (4.2) and the definition of ψ_j, we estimate I_1 as

$$
\begin{aligned}
I_1 &\le \frac{C2^{pj}}{(1-\sigma)^p r^p}\int_{\Gamma_j}\int_{B_j} \tilde{w}_j^p(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\le \frac{C2^{qj}}{\tilde{k}^{q-p}(1-\sigma)^p r^p}\int_{\Gamma_j}\int_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.4}
$$

Analogous to the proof of Lemma 4.1 in [12], we have

$$
I_2 \le \frac{C2^{qj}}{\tilde{k}^{q-p}(1-\sigma)^p r^{sp}}\int_{\Gamma_j}\int_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t, \tag{4.5}
$$

$$
I_3 \le \frac{C2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}\sigma^p(1-\sigma)^{N+sp} r^p}\int_{\Gamma_j}\int_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t \tag{4.6}
$$

and

$$
I_4 \le \frac{C2^{(p+q-2)j}}{\tilde{k}^{q-2}(1-\sigma)^p r^p}\int_{\Gamma_j}\int_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{4.7}
$$

By virtue of the fact that $\psi_j \equiv 1$ in B_{j+1}, $\eta_j \equiv 1$ in Γ_{j+1} and (1.3), merging inequalities (4.3)–(4.7), we get

$$
\int_{\Gamma_{j+1}} \fint_{B_{j+1}} |\nabla\tilde{w}_j(x,t)|^p\,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t)\,\mathrm{d}x
$$

$$
\begin{aligned}
&+ \int_{\Gamma_{j+1}} \int_{B_{j+1}} \fint_{B_{j+1}} \frac{|\tilde{w}_j(x,t) - \tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\leq \frac{C}{r^p(1-\sigma)^p} \left(\frac{2^{qj}}{r^{(s-1)p}\tilde{k}^{q-p}} + \frac{2^{(N+sp+q-1)j}}{\sigma^p(1-\sigma)^{N+p(s-1)}\tilde{k}^{q-p}} + \frac{2^{qj}}{\tilde{k}^{q-p}} + \frac{2^{(p+q-2)j}}{\tilde{k}^{q-2}} \right) \\
&\times \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

After rearrangement, we get the desired result. □

The following two lemmas are the consequences of Lemmas 2.3 and 4.1 .

Lemma 4.2 *Let $p > 2N/(N+2)$ and $\max\{p,2\} \leq q < p(N+2)/N$. Let $(x_0,t_0) \in Q_T$, $0 < r < 1$ and $Q_r^- = B_r(x_0) \times (t_0 - r^p, t_0)$ such that $\overline{B}_r(x_0) \subseteq \Omega$ and $[t_0 - r^p, t_0] \subseteq (0,T)$. Then for a local subsolution u to (1.1), we infer that*

$$
\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^q(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad \leq \frac{C2^{bj}}{r^{\frac{pq}{\kappa N}}} \left(\frac{1}{\sigma^{\frac{q(N+p)}{\kappa N}}(1-\sigma)^{\frac{q(N+p)(N+sp)}{p\kappa N}}} + \frac{1}{(1-\sigma)^{\frac{q(N+p)}{\kappa N}}} \right) \\
&\quad\quad \times \left(\frac{1}{\tilde{k}^{\frac{q}{\kappa}(\frac{q}{N}+1-\frac{2}{p})}} + \frac{1}{\tilde{k}^{\frac{q}{\kappa}(\frac{q}{N}+\frac{2}{N}-\frac{p}{N})}} \right) \left(\int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t \right)^{1+\frac{q}{\kappa N}}
\end{aligned}
\tag{4.8}
$$

for $j \in \mathbb{N}$, where $b := (1+p/N)(N+p+q)$, $\kappa := 1+2/N$ and $C > 0$ only depends on N, p, s, Λ and q.

Proof Since $q < p\kappa$, it follows from Hölder inequality that

$$
\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^q(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad \leq \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^q(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad \leq \left(\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)\,\mathrm{d}x\mathrm{d}t \right)^{\frac{q}{p\kappa}} \left(\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \chi_{\{u \geq \tilde{k}_j\}}(x,t)\,\mathrm{d}x\mathrm{d}t \right)^{1-\frac{q}{p\kappa}}.
\end{aligned}
\tag{4.9}
$$

From (4.2), we can get

$$
\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \chi_{\{u \geq \tilde{k}_j\}}(x,t)\,\mathrm{d}x\mathrm{d}t \leq \frac{C2^{qj}}{\tilde{k}^q} \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t \tag{4.10}
$$

and

$$\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^p(x,t)\,\mathrm{d}x\mathrm{d}t \le \frac{C2^{(q-p)j}}{\tilde{k}^{q-p}} \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{4.11}$$

Now by (4.11), Lemmas 2.3 and 4.1 we estimate

$$\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\le C\left(r^p \int_{\Gamma_{j+1}} \fint_{B_{j+1}} |\nabla \tilde{w}_j(x,t)|^p\,\mathrm{d}x\mathrm{d}t + \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^p(x,t)\,\mathrm{d}x\mathrm{d}t\right) \\
&\quad \times \left(\operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t)\,\mathrm{d}x\right)^{\frac{p}{N}} \\
&\le C r^{-\frac{p^2}{N}} \left(\frac{1}{\sigma^p(1-\sigma)^{N+sp}} + \frac{1}{(1-\sigma)^p}\right)^{\frac{p}{N}} \left(\frac{2^{(p+q-2)j}}{\tilde{k}^{q-2}} + \frac{2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}}\right)^{\frac{p}{N}} \\
&\quad \times \left[\left(\frac{1}{\sigma^p(1-\sigma)^{N+sp}} + \frac{1}{(1-\sigma)^p}\right)\left(\frac{2^{(p+q-2)j}}{\tilde{k}^{q-2}} + \frac{2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}}\right) + \frac{2^{(q-p)j}}{\tilde{k}^{q-p}}\right] \\
&\quad \times \left(\int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t\right)^{1+\frac{p}{N}} \\
&\le \frac{C2^{bj}}{r^{\frac{p^2}{N}}} \left(\frac{1}{\sigma^{\frac{p(N+p)}{N}}(1-\sigma)^{\frac{(N+p)(N+sp)}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+p)}{N}}}\right) \\
&\quad \times \left(\frac{1}{\tilde{k}^{q-2}} + \frac{1}{\tilde{k}^{q-p}}\right)^{1+\frac{p}{N}} \left(\int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)\,\mathrm{d}x\mathrm{d}t\right)^{1+\frac{p}{N}}
\end{aligned} \tag{4.12}$$

with $b = (1 + p/N)(N + p + q)$. Combining (4.9), (4.10) and (4.12), we get the desired result. □

Lemma 4.3 *Let $1 < p \le 2N/(N+2)$ and $m > \max\left\{2, \frac{N(2-p)}{p}\right\}$. Let $(x_0, t_0) \in Q_T$, $0 < r < 1$ and $Q_r^- = B_r(x_0) \times (t_0 - r^p, t_0)$ such that $\overline{B}_r(x_0) \subseteq \Omega$ and $\left[t_0 - r^p, t_0\right] \subseteq (0, T)$. Suppose that $u \in L^\infty_{\mathrm{loc}}(Q_T)$ is a local weak subsolution to (1.1). Then for any $j \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^m(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\le \frac{C2^{bj}}{r^{\frac{p^2}{N}}} \left(\frac{1}{\sigma^{\frac{p(N+p)}{N}}(1-\sigma)^{\frac{(N+p)(N+sp)}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+p)}{N}}}\right) \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}}\right)^{1+\frac{p}{N}}
\end{aligned}$$

$$\times \left\|\tilde{w}_j\right\|_{L^\infty\left(Q_{j+1}^-\right)}^{m-p\kappa}\left(\int_{\Gamma_j}\fint_{B_j} w_j^m(x,t)\,\mathrm{d}x\mathrm{d}t\right)^{1+\frac{p}{N}}, \tag{4.13}$$

where $b:=(1+p/N)(N+p+m)$, $\kappa:=1+2/N$ *and* $C>0$ *only depends on* N,p,s,m *and* Λ.

Proof Based on the assumptions, we have

$$\begin{aligned}\int_{\Gamma_{j+1}}\fint_{B_{j+1}} w_{j+1}^m(x,t)\,\mathrm{d}x\mathrm{d}t &\le \int_{\Gamma_{j+1}}\fint_{B_{j+1}} \tilde{w}_j^m(x,t)\,\mathrm{d}x\mathrm{d}t\\ &\le \left\|\tilde{w}_j\right\|_{L^\infty\left(Q_{j+1}^-\right)}^{m-p\kappa}\int_{\Gamma_{j+1}}\fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)\,\mathrm{d}x\mathrm{d}t. \quad (4.14)\end{aligned}$$

By utilizing Lemma 2.3, Lemma 4.1 with $q=m$ and the inequality (4.11) with $q=m$, it yields that

$$\begin{aligned}&\int_{\Gamma_{j+1}}\fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)\,\mathrm{d}x\mathrm{d}t\\ &\le C\left(r^p\int_{\Gamma_{j+1}}\fint_{B_{j+1}}|\nabla\tilde{w}_j(x,t)|^p\,\mathrm{d}x\mathrm{d}t+\int_{\Gamma_{j+1}}\fint_{B_{j+1}}\tilde{w}_j^p(x,t)\,\mathrm{d}x\mathrm{d}t\right)\\ &\quad\times\left(\operatorname*{ess\,sup}_{t\in\Gamma_{j+1}}\fint_{B_{j+1}}\tilde{w}_j^2(x,t)\,\mathrm{d}x\right)^{\frac{p}{N}}\\ &\le \frac{C2^{bj}}{r^{\frac{p^2}{N}}}\left[\frac{1}{\sigma^{\frac{p(N+p)}{N}}(1-\sigma)^{\frac{(N+p)(N+sp)}{N}}}+\frac{1}{(1-\sigma)^{\frac{p(N+p)}{N}}}\right]\left(\frac{1}{\tilde{k}^{m-p}}+\frac{1}{\tilde{k}^{m-2}}\right)^{1+\frac{p}{N}}\\ &\quad\times\left(\int_{\Gamma_j}\fint_{B_j} w_j^m(x,t)\,\mathrm{d}x\mathrm{d}t\right)^{1+\frac{p}{N}} \quad (4.15)\end{aligned}$$

with $b:=(1+p/N)(N+p+m)$. Thus combining (4.14) and (4.15), we get the desired inequality (4.13). □

Remark 4.4 In Lemma 4.3, the quantity $\sigma^{\frac{p(N+p)}{N}}$ can be removed, since $\sigma\in[1/2,1)$. In addition,

$$\max\left\{(1-\sigma)^{\frac{(N+p)(N+sp)}{N}},(1-\sigma)^{\frac{p(N+p)}{N}}\right\}\ge(1-\sigma)^{\frac{(N+p)^2}{N}}.$$

Thus, we can get

$$\int_{\Gamma_{j+1}}\fint_{B_{j+1}} w_{j+1}^m(x,t)\,\mathrm{d}x\mathrm{d}t$$

$$\leq \frac{C2^{bj}}{r^{\frac{p^2}{N}}(1-\sigma)^{\frac{(N+p)^2}{N}}}\left(\frac{1}{\tilde{k}^{m-p}}+\frac{1}{\tilde{k}^{m-2}}\right)^{1+\frac{p}{N}}$$
$$\times \left\|\tilde{w}_j\right\|^{m-p\kappa}_{L^\infty\left(Q^-_{j+1}\right)}\left(\int_{\Gamma_j}⨍_{B_j} w_j^m(x,t)\,\mathrm{d}x\mathrm{d}t\right)^{1+\frac{p}{N}}.$$

Next, we introduce an analysis lemma which will be used later.

Lemma 4.5 [9, Lemma 4.1] *Let $\{Y_j\}_{j=0}^\infty$ be a sequence of positive numbers such that*

$$Y_0 \leq K^{-\frac{1}{\delta}}b^{-\frac{1}{\delta^2}} \text{ and } Y_{j+1}\leq Kb^jY_j^{1+\delta},\quad j=0,1,2,\ldots$$

for some constants K, $b>1$ and $\delta>0$. Then we have $\lim_{j\to\infty}Y_j=0$.

Finally, we end this section by proving the results of local boundedness.

Proof of Theorem 1.2 Let $r=R,\sigma=\frac{1}{2}$, then $r_j=\frac{R}{2}+2^{-j-1}R$. Fix $q=\max\{2,p\}$. We denote

$$Y_j=\int_{\Gamma_j}⨍_{B_j}\left(u-k_j\right)_+^q\,\mathrm{d}x\mathrm{d}t,\quad j=0,1,2,\ldots.$$

Supposing $\tilde{k}\geq 1$ and recalling $r<1$, we derive from Lemma 4.2 that

$$\begin{aligned}\frac{Y_{j+1}}{r^p}&\leq \frac{C2^{bj}Y_j^{1+\frac{q}{N\kappa}}}{r^{p(1+\frac{q}{N\kappa})}\tilde{k}^{\frac{q}{\kappa}\left(\frac{q}{N}+\frac{2}{N}-\frac{p}{N}\right)}}+\frac{C2^{bj}Y_j^{1+\frac{q}{N\kappa}}}{r^{p(1+\frac{q}{N\kappa})}\tilde{k}^{\frac{q}{\kappa}\left(\frac{q}{N}+1-\frac{2}{p}\right)}}\\&\leq\frac{C2^{bj}}{\tilde{k}^{q\left(1-\frac{q}{p\kappa}\right)}}\left(\frac{Y_j}{r^p}\right)^{1+\frac{q}{N\kappa}},\end{aligned}\tag{4.16}$$

where $b:=(1+p/N)(N+p+q)$, $\kappa:=1+2/N$ and $C>0$ only depends on N,p,s,Λ. For any $j\in\mathbb{N}$, define $W_j=Y_j/r^p$. Thus we get

$$W_{j+1}\leq\frac{C2^{bj}}{\tilde{k}^{q\left(1-\frac{q}{p\kappa}\right)}}W_j^{1+\frac{q}{N\kappa}},$$

where $\tilde{k}$ is such that

$$\tilde{k}\geq\max\left\{\mathrm{Tail}_\infty\left(u_+;x_0,R/2,t_0-R^p,t_0\right),C\left(⨍_{t_0-R^p}^{t_0}⨍_{B_R}u_+^q\,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{N(p\kappa-q)}}\vee 1\right\}$$

with C only depending on N, p, s and Λ. Thus along with Lemma 4.5 we can deduce that $\lim_{j\to\infty} W_j = 0$. Consequently, we get the desired result

$$\operatorname*{ess\,sup}_{Q^-_{R/2}} u \le \mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^p, t_0\right) + C\left(⨍_{Q^-_R} u_+^q \,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{N(p\kappa-q)}} \vee 1.$$

We now finish the proof. □

Proof of Theorem 1.3 Under the hypotheses (1.8) and (1.9), we may suppose, by a proper approximation procedure that u is locally bounded in advance. Indeed, observe that the approximation subsolutions u_k are bounded (i.e. $\operatorname{ess\,sup}_{Q^-_n}(u_k)_+ < \infty$). Therefore, we could substitute u by u_k to perform the proof below. In other words, the estimate (4.20) below still holds true for u replaced by u_k, which along with (1.8) and (1.9) leads to a k-independent bound on u_k in L^∞. We eventually find that u is qualitatively locally bounded via the a.e. convergence of u_k. That is, $\operatorname{ess\,sup}_{Q^-_n} u_+$ is finite for $n = 0, 1, 2, 3, \ldots$.

Let $R_0 = R/2$ and $R_n = R/2 + \sum_{i=1}^n 2^{-i-1}R$ for $n \in \mathbb{N}^+$, and $Q^-_n = B_{R_n}(x_0) \times \left(t_0 - R_n^p, t_0\right)$. Set

$$M_n = \operatorname*{ess\,sup}_{Q^-_n} u_+, \quad n = 0, 1, 2, 3, \ldots.$$

Choosing $r = R_{n+1}$ and $\sigma r = R_n$, then

$$\sigma = \frac{1/2 + \sum_{i=1}^n 2^{-i-1}}{1/2 + \sum_{i=1}^{n+1} 2^{-i-1}} \ge \frac{1}{2}.$$

We denote

$$Y_j = \int_{\Gamma_j} ⨍_{B_j} \left(u - k_j\right)_+^m \,\mathrm{d}x\mathrm{d}t, \quad j = 0, 1, 2, \ldots.$$

Due to Lemma 4.3, we obtain

$$\begin{aligned} Y_{j+1} &\le \frac{C2^{bj}}{R_{n+1}^{\frac{p^2}{N}}} \|u_+\|^{m-p\kappa}_{L^\infty(Q^-_{n+1})} \left(\frac{1}{(1-\sigma)^{\frac{(N+p)^2}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+p)}{N}}}\right) \\ &\quad \times \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}}\right)^{1+\frac{p}{N}} Y_j^{1+\frac{p}{N}} \\ &\le \frac{C2^{bj+dn}}{R_{n+1}^{\frac{p^2}{N}}} M_{n+1}^{m-p\kappa} \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}}\right)^{1+\frac{p}{N}} Y_j^{1+\frac{p}{N}} \end{aligned} \tag{4.17}$$

with $b := (1 + p/N)(N + p + m)$ and $d = (N + p)^2/N$. For any $j \in \mathbb{N}$, define $W_j = Y_j/R_n^p$. Then we get

$$W_{j+1} \le C2^{bj+dn} M_{n+1}^{m-p\kappa} \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}} \right)^{1+\frac{p}{N}} W_j^{1+\frac{p}{N}}.$$

According to Lemma 4.5, we can see that

$$\lim_{j\to\infty} Y_j = 0$$

if

$$W_0 \le C2^{-\frac{dnN}{p}-\frac{bN^2}{p^2}} M_{n+1}^{-\frac{N(m-p\kappa)}{p}} \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}} \right)^{-\frac{p+N}{p}}. \tag{4.18}$$

To ensure the above inequality, we need choose $\tilde{k}$ properly large. Indeed,

$$\begin{aligned} W_0 = \frac{Y_0}{R_n^p} &= R_n^{-p} \int_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \\ &\le 2^p ⨍_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Namely,

$$2^p ⨍_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \le C2^{-\frac{dnN}{p}-\frac{bN^2}{p^2}} M_{n+1}^{-\frac{N(m-p\kappa)}{p}} \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}} \right)^{-\frac{p+N}{p}},$$

which implies that

$$C2^{\frac{dnN}{N+p}} M_{n+1}^{\frac{N(m-p\kappa)}{N+p}} \left(⨍_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{N+p}} \le \left(\frac{1}{\tilde{k}^{m-p}} + \frac{1}{\tilde{k}^{m-2}} \right)^{-1}.$$

Therefore, we take

$$\begin{aligned} \tilde{k} &= C2^{\frac{dnN}{(N+p)(m-p)}} M_{n+1}^{\frac{N(m-p\kappa)}{(N+p)(m-p)}} \left(⨍_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-p)}} \\ &\quad + C2^{\frac{dnN}{(N+p)(m-2)}} M_{n+1}^{\frac{N(m-p\kappa)}{(N+p)(m-2)}} \left(⨍_{t_0-R_{n+1}^p}^{t_0} ⨍_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-2)}} \\ &\quad + \frac{\mathrm{Tail}_\infty \left(u_+; x_0, R_n, t_0 - R_{n+1}^p, t_0 \right)}{2}, \end{aligned}$$

which makes (4.18) hold true. Here the constant C only depends on N, p, s, m and Λ. Under this choice, it follows from Lemma 4.5 that

$$\begin{aligned} M_n &= \operatorname*{ess\,sup}_{Q_{R_n}^-} u_+ \\ &\leq C2^{\frac{dnN}{(N+p)(m-p)}} M_{n+1}^{\frac{N(m-p\kappa)}{(N+p)(m-p)}} \left(\fint_{t_0-R_{n+1}^p}^{t_0} \fint_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-p)}} \\ &\quad + C2^{\frac{dnN}{(N+p)(m-2)}} M_{n+1}^{\frac{N(m-p\kappa)}{(N+p)(m-2)}} \left(\fint_{t_0-R_{n+1}^p}^{t_0} \fint_{B_{R_{n+1}}} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-2)}} \\ &\quad + \frac{\mathrm{Tail}_\infty\left(u_+; x_0, R_n, t_0 - R_{n+1}^p, t_0\right)}{2}. \end{aligned} \tag{4.19}$$

Obverse $m > \max\left\{2, \frac{N(2-p)}{p}\right\}$ and $\kappa = 1 + 2/N$, which indicates that

$$0 < \frac{N(m-p\kappa)}{(N+p)(m-p)}, \frac{N(m-p\kappa)}{(N+p)(m-2)} < 1.$$

Now we apply the Young's inequality with ε to (4.19), arriving at

$$\begin{aligned} M_n &\leq \varepsilon M_{n+1} + C2^{\frac{dnN}{(N+p)(m-p-\beta)}} \varepsilon^{-\frac{\beta}{m-p-\beta}} \left(\fint_{t_0-R^p}^{t_0} \fint_{B_R} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-p-\beta)}} \\ &\quad + C2^{\frac{dnN}{(N+p)(m-2-\beta)}} \varepsilon^{-\frac{\beta}{m-2-\beta}} \left(\fint_{t_0-R^p}^{t_0} \fint_{B_R} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-2-\beta)}} \\ &\quad + \frac{\mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^p, t_0\right)}{2} \end{aligned}$$

with $\beta = (m - p\kappa)N/(p + N)$, where we used the fact $R/2 \leq R_n < R$. Via the induction argument, we can derive

$$\begin{aligned} M_0 &\leq \varepsilon^{n+1} M_{n+1} \\ &\quad + C\varepsilon^{-\frac{\beta}{m-p-\beta}} \left(\fint_{t_0-R^p}^{t_0} \fint_{B_R} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-p-\beta)}} \sum_{i=0}^{n} \left(2^{\frac{dN}{(N+p)(m-p-\beta)}} \varepsilon \right)^i \\ &\quad + C\varepsilon^{-\frac{\beta}{m-2-\beta}} \left(\fint_{t_0-R^p}^{t_0} \fint_{B_R} u_+^m \,\mathrm{d}x\mathrm{d}t \right)^{\frac{p}{(N+p)(m-2-\beta)}} \sum_{i=0}^{n} \left(2^{\frac{dN}{(N+p)(m-2-\beta)}} \varepsilon \right)^i \\ &\quad + \frac{\mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^p, t_0\right)}{2} \sum_{i=0}^{n} \varepsilon^i, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

It is easy to see that the sum on the right-hand side could be revised by a convergent series, provided that we take

$$\varepsilon = 2^{-\left[\frac{dN}{(N+p)(m-2-\beta)}+1\right]}.$$

Finally, letting $n \to \infty$, we deduce that

$$\begin{aligned} \operatorname*{ess\,sup}_{Q^-_{R/2}} u &\leq \mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^p, t_0\right) \\ &\quad + C\left(\fint_{Q^-_R} u_+^m \,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{(N+p)(m-2-\beta)}} \vee \left(\fint_{Q^-_R} u_+^m \,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{(N+p)(m-p-\beta)}}, \end{aligned} \tag{4.20}$$

where $C > 0$ depends only on N, p, s, m and Λ. □

5 Local Hölder Continuity

In this section, we aim at establishing the Hölder continuity of weak solutions to (1.1) in the case that $p > 2$, based on the local boundedness results. Before verifying this conclusion, we introduce some notations.

Let $(\overline{x}_0, \overline{t}_0) \in Q_T$ and $r \in (0, R]$ for some $R \in (0, 1)$. Let also $\alpha \in (0, \frac{p}{p-1})$ and $\sigma \in (0, \sigma_0)$ with $\sigma_0^{\frac{p}{p-1}} \leq \frac{1}{4}$ be two constants to be determined later. Set

$$r_j := \frac{\sigma^j r}{2}, \quad \omega(r_0) = \omega(r/2) := M, \quad \omega(r_j) := \left(\frac{r_j}{r_0}\right)^\alpha \omega(r_0), \quad j = 0, 1, 2, 3\ldots \tag{5.1}$$

and

$$M := C\left[\mathrm{Tail}_\infty\left(u; \overline{x}_0, r/2, \overline{t}_0 - r^p, \overline{t}_0 + r^p\right) + \left(\fint_{Q_r} |u|^p \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \vee 1\right] \tag{5.2}$$

with C depending on N, p, s, Λ. Define

$$d_j := \begin{cases} \left[\varepsilon\sigma^{(j-1)\alpha} M\right]^{2-p} & \text{if } j \geq 1, \\ 1 & \text{if } j = 0, \end{cases}$$

where

$$\varepsilon = \sigma^{\frac{p}{p-1}-\alpha}.$$

Thus, it is easy to obtain

$$\frac{1}{d_{j+1}} = \left[\varepsilon\omega(r_j)\right]^{p-2} \text{ for all } j \geq 0. \tag{5.3}$$

Denote

$$B_j := B_{r_j}(\overline{x}_0) \text{ and } t_j := d_j r_j^p$$

and

$$Q_j := Q_{r_j,t_j}\left(\overline{x}_0, \overline{t}_0\right) = B_j \times \left(\overline{t}_0 - t_j, \overline{t}_0 + t_j\right).$$

Hence, for $j \geq 1$, we have

$$4\left(\sigma^{\frac{p}{p-1}-\alpha}\right)^{2-p} r_1^p \leq r_0^p \quad \text{and} \quad 4\sigma^{\alpha(2-p)} r_{j+1}^p \leq r_j^p.$$

The above inequalities combine with the definitions of d_j and t_j gives that

$$4t_{j+1} \leq t_j \text{ for all } j \geq 0. \tag{5.4}$$

Now we are going to deduce an oscillation reduction on weak solutions.

Lemma 5.1 *Let $p > 2$ and u be a local weak solution to (1.1). Set $\left(\overline{x}_0, \overline{t}_0\right) \in Q_T$, $r \in (0, R]$ for some $R \in (0, 1)$ and $Q_R \equiv B(\overline{x}_0) \times \left(\overline{t}_0 - R^p, \overline{t}_0 + R^p\right)$ such that $\overline{Q}_R \subseteq Q_T$. Then*

$$\operatorname*{essosc}_{Q_j} u \leq \omega(r_j) \textit{ for all } j = 0, 1, 2, \ldots. \tag{5.5}$$

Proof We prove this Lemma by the induction argument. It follows from Theorem 1.2 and the definition of $\omega(r_0)$ that (5.5) holds true for $j = 0$. We may assume that (5.5) is valid for each $i \in \{0, \ldots, j\}$ with some $j \geq 0$. Then we devote to proving it holds for $i = j + 1$.

Denote

$$2B_{j+1} := B_{2r_{j+1}}(\overline{x}_0), \quad 2Q_{j+1} := B_{2r_{j+1}}(\overline{x}_0) \times (\overline{t}_0 - 2t_{j+1}, \overline{t}_0 + 2t_{j+1}).$$

It is obvious that either

$$\frac{\left|2Q_{j+1} \cap \left\{u \geq \operatorname*{ess\,inf}_{Q_j} u + \omega(r_j)/2\right\}\right|}{\left|2Q_{j+1}\right|} \geq \frac{1}{2} \tag{5.6}$$

or

$$\frac{\left|2Q_{j+1}\cap\left\{u\le \operatorname*{ess\,inf}_{Q_j} u+\omega(r_j)/2\right\}\right|}{\left|2Q_{j+1}\right|}\ge\frac{1}{2} \tag{5.7}$$

must hold. In the case of (5.6), we set $u_j := u - \operatorname*{ess\,inf}_{Q_j} u$. In the case of (5.7), we set $u_j := \omega(r_j) - \left(u - \operatorname*{ess\,inf}_{Q_j} u\right)$. In all cases, we have

$$\frac{\left|2Q_{j+1}\cap\left\{u_j\ge\omega(r_j)/2\right\}\right|}{\left|2Q_{j+1}\right|}\ge\frac{1}{2} \tag{5.8}$$

and

$$0\le \operatorname*{ess\,sup}_{Q_i} u_j\le 2\omega(r_i)\quad \text{for } i=0,\dots,j. \tag{5.9}$$

Now we provide an important estimate to be used later,

$$\begin{aligned}&\left[\mathrm{Tail}_\infty\left(u_j;\overline{x}_0,r_j,\overline{t}_0-t_j,\overline{t}_0+t_j\right)\right]^{p-1}\\&\quad\le C\sigma^{-\alpha(p-1)}\left[\omega(r_j)\right]^{p-1}\ \text{ for } j=0,1,2\dots,\end{aligned} \tag{5.10}$$

where C only depends on N, p, s, the difference of $p/(p-1)$ and α. Indeed, it is easy to see the claim is true when $j=0$. For $j\ge 1$, we have

$$\begin{aligned}&\left[\mathrm{Tail}_\infty\left(u_j;\overline{x}_0,r_j,\overline{t}_0-t_j,\overline{t}_0+t_j\right)\right]^{p-1}\\&=r_j^p\operatorname*{ess\,sup}_{t\in(\bar{t}_0-t_j,\bar{t}_0+t_j)}\sum_{i=1}^{j}\int_{B_{i-1}\setminus B_i}\frac{\left|u_j(x,t)\right|^{p-1}}{|x-\overline{x}_0|^{N+sp}}\,\mathrm{d}x\\&\quad+r_j^p\operatorname*{ess\,sup}_{t\in(\bar{t}_0-t_j,\bar{t}_0+t_j)}\int_{\mathbb{R}^N\setminus B_0}\frac{\left|u_j(x,t)\right|^{p-1}}{|x-\overline{x}_0|^{N+sp}}\,\mathrm{d}x\\&\le r_j^p\sum_{i=1}^{j}\left(\operatorname*{ess\,sup}_{Q_{i-1}} u_j\right)^{p-1}\int_{\mathbb{R}^N\setminus B_i}\frac{1}{|x-\overline{x}_0|^{N+sp}}\,\mathrm{d}x\\&\quad+r_j^p\operatorname*{ess\,sup}_{t\in(\bar{t}_0-t_j,\bar{t}_0+t_j)}\int_{\mathbb{R}^N\setminus B_0}\frac{\left|u_j(x,t)\right|^{p-1}}{|x-\overline{x}_0|^{N+sp}}\,\mathrm{d}x\\&\le C\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^p[\omega(r_{i-1})]^{p-1},\end{aligned}$$

where in the last line we used (5.9) and the definition of u_j. Since $\sigma \le 1/4$ and $\alpha < p/(p-1)$, we estimate the right-hand side as follows:

$$
\begin{aligned}
&\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^p [\omega(r_{i-1})]^{p-1} \\
&= [\omega(r_0)]^{p-1}\left(\frac{r_j}{r_0}\right)^{\alpha(p-1)} \sum_{i=1}^{j}\left(\frac{r_{i-1}}{r_i}\right)^{\alpha(p-1)}\left(\frac{r_j}{r_i}\right)^{p-\alpha(p-1)} \\
&= \left[\omega(r_j)\right]^{p-1}\sigma^{-\alpha(p-1)}\sum_{i=0}^{j-1}\sigma^{i(p-\alpha(p-1))} \\
&\le \left[\omega(r_j)\right]^{p-1}\frac{\sigma^{-\alpha(p-1)}}{1-\sigma^{p-\alpha(p-1)}} \\
&\le \frac{4^{p-\alpha(p-1)}}{(p-\alpha(p-1))\log 4}\sigma^{-\alpha(p-1)}\left[\omega(r_j)\right]^{p-1}.
\end{aligned}
$$

Thus, we have proved (5.10) with the constant C depending on N, p, s, the difference of $p/(p-1)$ and α. Next, we define

$$
v := \min\left\{\left[\log\left(\frac{\omega(r_j)/2+d}{u_j+d}\right)\right]_+, k\right\} \quad \text{for } k > 0. \tag{5.11}
$$

It follows from Corollary 3.3 with $a \equiv \omega(r_j)/2$ and $b \equiv \exp(k)$ that

$$
\begin{aligned}
&\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}} \fint_{2B_{j+1}} \left|v(x,t)-(v)_{2B_{j+1}}(t)\right|^p \,\mathrm{d}x\mathrm{d}t \\
&\le Ct_{j+1}d^{1-p}\left(\frac{r_{j+1}}{r_j}\right)^p \left[\mathrm{Tail}_\infty\left(u_j;\bar{x}_0,r_j,\bar{t}_0-4t_{j+1},\bar{t}_0+4t_{j+1}\right)\right]^{p-1} \\
&\quad + Ct_{j+1} + Cd^{2-p}r_{j+1}^p + Ct_{j+1}r_{j+1}^{p-sp} \\
&\le Ct_{j+1}d^{1-p}\left[\varepsilon\omega(r_j)\right]^{p-1} + Ct_{j+1} + Cd^{2-p}r_{j+1}^p,
\end{aligned}
$$

where in the last line we used the fact $4t_{j+1} \le t_j$ and (5.10). Choosing $d = \varepsilon\omega(r_j)$ in (5.11), we get

$$
d^{2-p} = d_{j+1}.
$$

Since $\alpha < p/(p-1)$, we can verify

$$
d^{1-p} \le r_{j+1}^{-p}.
$$

Thus, we arrive at

$$\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}} \fint_{2B_{j+1}} \left|v(x,t)-(v)_{2B_{j+1}}(t)\right|^p \,\mathrm{d}x\mathrm{d}t \le Ct_{j+1},$$

where C depends on N, p, s, Λ, the difference of $p/(p-1)$ and α. Following the calculation in pp. 37–38 in [12], there holds that

$$\frac{\left|2Q_{j+1}\cap\left\{u_j\le 2\varepsilon\omega(r_j)\right\}\right|}{\left|2Q_{j+1}\right|} \le \frac{\overline{C}}{\log\left(\frac{1}{\sigma}\right)}, \tag{5.12}$$

where $\overline{C}>0$ depends on N, p, s, Λ, the difference of $p/(p-1)$ and α.

In what follows, we will proceed by a suitable iteration to infer the desired oscillation decay over the domain Q_{j+1}. For any $i=0,1,2,\ldots$, we define

$$\begin{aligned}
&\varrho_i = r_{j+1}+2^{-i}r_{j+1}, \quad \tilde{\varrho}_i := \frac{\varrho_i+\varrho_{i+1}}{2},\\
&\theta_i := t_{j+1}+2^{-i}t_{j+1}, \quad \tilde{\theta}_i := \frac{\theta_i+\theta_{i+1}}{2},\\
&Q^i := B^i\times\Gamma_i := B_{\varrho_i}(\overline{x}_0)\times(\bar{t}_0-\theta_i,\bar{t}_0+\theta_i),\\
&\tilde{Q}^i := \tilde{B}^i\times\tilde{\Gamma}_i := B_{\tilde{\Gamma}_i}(\overline{x}_0)\times(\bar{t}_0-\tilde{\theta}_i,\bar{t}_0+\tilde{\theta}_i).
\end{aligned}$$

Then we take the cut-off function $\psi_i\in C_0^\infty(\tilde{B}^i)$ and $\eta_i\in C_0^\infty(\tilde{\Gamma}_i)$ such that

$$0\le\psi_i\le 1, \quad |\nabla\psi_i|\le C2^i r_{j+1}^{-1} \text{ in } \tilde{B}^i, \quad \psi_i\equiv 1 \text{ in } B^{i+1}$$

and

$$0\le\eta_i\le 1, \quad |\partial_t\eta_i|\le C2^i t_{j+1}^{-1} \text{ in } \tilde{\Gamma}_i, \quad \eta_i\equiv 1 \text{ in } \Gamma_{i+1}.$$

Define

$$k_i := (1+2^{-i})\varepsilon\omega(r_j), \quad v_i := (k_i-u_j)_+.$$

Taking $\ell=\theta_i-\theta_{i+1}$, $\tau_1=\bar{t}_0-\theta_{i+1}$ and $\tau_2=\bar{t}_0+\theta_{i+1}$ in Lemma 3.1, we get

$$\begin{aligned}
&\int_{\Gamma_{i+1}}\fint_{B^i}|\nabla v_i(x,t)|^p\psi_i^p(x)\,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}\fint_{B^i} v_i^2(x,t)\psi_i^p(x)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_{i+1}}\int_{B^i}\fint_{B^i}\frac{|v_i(x,t)\psi_i(x)-v_i(y,t)\psi_i(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\le C\int_{\Gamma_i}\fint_{B^i}|\nabla\psi_i(x)|^p v_i^p(x,t)\eta_i^2(t)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ C \int_{\Gamma_i} \int_{B^i} \fint_{B^i} \max\{v_i(x,t), v_i(y,t)\}^p \, |\psi_i(x) - \psi_i(y)|^p \, \eta_i^2(t) \, \mathrm{d}\mu \mathrm{d}t \\
&+ C \operatorname*{ess\,sup}_{\substack{t \in \Gamma_i \\ x \in \operatorname{supp}\psi_i}} \int_{\mathbb{R}^N \setminus B^i} \frac{v_i^{p-1}(y,t)}{|x-y|^{N+sp}} \, \mathrm{d}y \int_{\Gamma_i} \fint_{B^i} v_i(x,t) \psi_i^p(x) \eta_i^2(t) \, \mathrm{d}x \mathrm{d}t \\
&+ C \int_{\Gamma_i} \fint_{B^i} v_i^2(x,t) \psi_i^p(x) \eta_i(t) \, |\partial_t \eta_i(t)| \, \mathrm{d}x \mathrm{d}t \\
=&: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We estimate I_1 as

$$
\begin{aligned}
I_1 &\le C k_i^p 2^{pi} r_{j+1}^{-p} \int_{\Gamma_i} \fint_{B_i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t \\
&\le C 2^{pi} r_{j+1}^{-p} [\varepsilon \omega(r_j)]^p \int_{\Gamma_i} \fint_{B_i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t,
\end{aligned}
$$

where we used the properties of ψ_i. As the computations in pp. 39–40 in [12], we derive

$$
\begin{aligned}
I_2 &\le C 2^{pi} r_{j+1}^{-p} [\varepsilon \omega\left(r_j\right)]^p \int_{\Gamma_i} \fint_{B_i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t, \\
I_3 &\le C 2^{(N+sp)i} r_{j+1}^{-p} [\varepsilon \omega\left(r_j\right)]^p \int_{\Gamma_i} \fint_{B_i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t
\end{aligned}
$$

and

$$
I_4 \le C 2^{spi} r_{j+1}^{-p} [\varepsilon \omega\left(r_j\right)]^p \int_{\Gamma_i} \fint_{B_i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t.
$$

Since $\psi_i \equiv 1$ in B^{i+1}, we can deduce that

$$
\begin{aligned}
&\int_{\Gamma_{i+1}} \fint_{B^{i+1}} |\nabla v_i(x,t)|^p \, \mathrm{d}x \mathrm{d}t + \int_{\Gamma_{i+1}} \int_{B^{i+1}} \fint_{B^{i+1}} \frac{|v_i(x,t) - v_i(y,t)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \mathrm{d}y \mathrm{d}t \\
&\quad + \operatorname*{ess\,sup}_{t \in \Gamma_{i+1}} \fint_{B^{i+1}} v_i^2(x,t) \, \mathrm{d}x \\
&\le C 2^{(N+p)i} r_{j+1}^{-p} [\varepsilon \omega(r_j)]^p \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) \, \mathrm{d}x \mathrm{d}t. \qquad (5.13)
\end{aligned}
$$

From (5.3), we get

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t \in \Gamma_{i+1}} \fint_{B^{i+1}} v_i^p(x,t) \, \mathrm{d}x &\le k_i^{p-2} \operatorname*{ess\,sup}_{t \in \Gamma_{i+1}} \fint_{B^{i+1}} v_i^2(x,t) \, \mathrm{d}x \\
&\le C d_{j+1}^{-1} \operatorname*{ess\,sup}_{t \in \Gamma_{i+1}} \fint_{B^{i+1}} v_i^2(x,t) \, \mathrm{d}x. \qquad (5.14)
\end{aligned}
$$

Combining (5.13) with (5.14) gives that

$$
\begin{aligned}
& r_{j+1}^{p} \fint_{\Gamma_{i+1}} \fint_{B^{i+1}} |\nabla v_i(x,t)|^p \,\mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{t\in\Gamma_{i+1}} \fint_{B^{i+1}} v_i^p(x,t)\,\mathrm{d}x \\
&\quad \le d_{j+1}^{-1} \int_{\Gamma_{i+1}} \fint_{B^{i+1}} |\nabla v_i(x,t)|^p \,\mathrm{d}x\mathrm{d}t + d_{j+1}^{-1} \operatorname*{ess\,sup}_{t\in\Gamma_{i+1}} \fint_{B^{i+1}} v_i^2(x,t)\,\mathrm{d}x \\
&\quad \le C 2^{(N+p)i} r_{j+1}^{-p} d_{j+1}^{-1} [\varepsilon\omega(r_j)]^p \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad \le C 2^{(N+p)i} [\varepsilon\omega(r_j)]^p A_i, \qquad (5.15)
\end{aligned}
$$

where A_i is denoted by

$$
A_i := \frac{\left|Q_i \cap \{u_j \le k_i\}\right|}{|Q_i|}.
$$

In view of Lemma 2.4, we can see

$$
\begin{aligned}
\int_{\Gamma_{i+1}} \fint_{B^{i+1}} v_i^{p(1+\frac{p}{N})}(x,t)\,\mathrm{d}x\mathrm{d}t &\le C r_{j+1}^{p} \int_{\Gamma_{i+1}} \fint_{B^{i+1}} |\nabla v_i(x,t)|^p \,\mathrm{d}x\mathrm{d}t \\
&\quad \times \left(\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}} \fint_{B^{i+1}} v_i^p(x,t)\,\mathrm{d}x \right)^{\frac{p}{N}}. \qquad (5.16)
\end{aligned}
$$

Merging (5.15) and (5.16) leads to

$$
\begin{aligned}
& A_{i+1}\,(k_i - k_{i+1})^{p(1+\frac{p}{N})} \\
&\quad \le \fint_{\Gamma_{i+1}} \fint_{B^{i+1}\cap\{u_j \le k_{i+1}\}} v_i^{p(1+\frac{p}{N})}(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad \le C r_{j+1}^{p} \fint_{\Gamma_{i+1}} \fint_{B^{i+1}} |\nabla v_i(x,t)|^p \,\mathrm{d}x\mathrm{d}t \left(\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}} \fint_{B^{i+1}} v_i^p(x,t)\,\mathrm{d}x \right)^{\frac{p}{N}} \\
&\quad \le C \left[2^{(N+p)i} (\varepsilon\omega(r_j))^p A_i \right]^{1+\frac{p}{N}}.
\end{aligned}
$$

We can readily get the recursive inequality

$$
A_{i+1} \le \tilde{C} 2^{(N+p)(1+\frac{p}{N})i} A_i^{1+\frac{p}{N}},
$$

where $\tilde{C}$ depends on N, p, s, Λ, the difference of $p/(p-1)$ and α. Set

$$
\nu^* := \tilde{C}^{-\frac{N}{p}} 2^{\frac{-N(N+p)^2}{p^2}}.
$$

Then we take

$$\sigma = \min\left\{\frac{1}{4}, \sigma_0, \exp\left(-\frac{\overline{C}}{\nu^*}\right)\right\},$$

where we need to note that σ_0 only depends on p. Utilizing the definition of A_i, we obtain

$$A_0 = \frac{\left|2Q_{j+1} \cap \left\{u_j \le 2\varepsilon\omega(r_j)\right\}\right|}{\left|2Q_{j+1}\right|} \le \nu^*.$$

It follows from Lemma 4.5 that

$$\lim_{i\to\infty} A_i = 0,$$

which implies that

$$u_j(x,t) \ge \varepsilon\omega(r_j) \text{ in } Q_{j+1}.$$

Thereby, recalling (5.1) and the definition of u_j, we have

$$\begin{aligned} \operatorname*{ess\,\mathit{osc}}_{Q_{j+1}} u &= \operatorname*{ess\,sup}_{Q_{j+1}} u_j - \operatorname*{ess\,inf}_{Q_{j+1}} u_j \le (1-\varepsilon)\omega(r_j) \\ &= (1-\varepsilon)\sigma^{-\alpha}\omega(r_{j+1}). \end{aligned} \tag{5.17}$$

Now, we pick $\alpha \in (0, p/(p-1))$ such that

$$\sigma^{\alpha} \ge 1-\varepsilon = 1-\sigma^{\frac{p}{p-1}-\alpha},$$

which together with (5.17) ensures that

$$\operatorname*{essosc}_{Q_{j+1}} u \le \omega(r_{j+1}).$$

Now we finish the proof. □

Proof of Theorem 1.4 Let $p > 2$. Suppose that u is a local weak solution of (1.1). Let $(x_0, t_0) \in Q_T$, $R \in (0, 1)$ and $Q_R \equiv B_R(x_0) \times (t_0 - R^p, t_0 + R^p)$ such that $\overline{Q}_R \subseteq Q_T$. Taking $r = R$ in Lemma 5.1 we can get

$$\operatorname*{essosc}_{Q_j} u \le C\left(\frac{r_j}{R}\right)^{\alpha} \omega\left(\frac{R}{2}\right) \text{ for all } j \in \mathbb{N} \tag{5.18}$$

with $\alpha < p/(p-1)$, $\sigma < 1/4$, where $C \ge 1$ depends on N, p, s, Λ, and

$$\omega\left(\frac{R}{2}\right) = \mathrm{Tail}_{\infty}\left(u; x_0, R/2, t_0 - R^p, t_0 + R^p\right) + \left(⨍_{Q_R} |u|^p \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \vee 1. \tag{5.19}$$

For every $\rho \in (0, R/2]$, we have $\rho \in \left(r_{j_0+1}, r_{j_0}\right]$ for $j_0 \in \mathbb{N}$. Choosing $d = [C\omega(R/2)]^{2-p}$, it follows that $Q_{\rho,d\rho^p} \subseteq Q_{j_0}$. By applying (5.18), we get

$$\operatorname*{ess\,osc}_{Q_{\rho,d\rho^p}} u \leq \operatorname*{ess\,osc}_{Q_{j_0}} u \leq C\sigma^{-\alpha}\left(\frac{r_{j_0+1}}{R}\right)^{\alpha}\omega\left(\frac{R}{2}\right) \leq C\sigma^{-\alpha}\left(\frac{\rho}{R}\right)^{\alpha}\omega\left(\frac{R}{2}\right).$$

We can obtain the Hölder continuity from the above inequality along with (5.19). □

Acknowledgements The authors wish to thank the anonymous reviewers for many valuable comments and suggestions to improve the manuscript. This work was supported by the National Natural Science Foundation of China (No. 12071098).

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