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Polyharmonic Almost Complex Structures

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Abstract

We consider the existence and regularity of weakly polyharmonic almost complex structures on a compact almost Hermitian manifold M^{2m}. Such objects satisfy the elliptic system $[\Delta^m J, J] = 0$ weakly. We prove a general regularity theorem for semilinear systems in critical dimensions (with *critical growth nonlinearities*), which includes the system of polyharmonic almost complex structures in dimension four and six.

Keywords Polyharmonic almost complex structures · Regularity of Semilinear systems · Critical growth of nonlinearities

Mathematics Subject Classification 53C15 · 58E20 · 35J48

1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n with a compatible almost complex structure. Denote by $\mathcal{J}_g$ the space of smooth almost complex structures compatible with g, i.e., $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. Consider the following functional, for all $m \in \mathbb{N}^+$, $J \in \mathcal{J}_g$,

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$$\mathcal{E}_m(J) = \int_M \left|\Delta^{\frac{m}{2}} J\right|^2 dV := \begin{cases} \int_M |\nabla \Delta^{k-1} J|^2 \, dV, & m = 2k-1, \\ \int_M |\Delta^k J|^2 \, dV, & m = 2k, \end{cases} \tag{1}$$

where ∇ and Δ are Levi-Civita connection and Laplace-Beltrami operator on (M, g), respectively, and dV denotes the volume element of (M, g). We call the critical points of functional $\mathcal{E}_m(J)$ *m-harmonic almost complex structures*. These objects are tensor-valued version of polyharmonic maps which have attracted quite some attention in recent years. When $m = 1$, the critical points of functional $\mathcal{E}_m(J)$ are also called *harmonic almost complex structures* introduced by Wood [17] in 1990s. We refer the reader to the recent survey [3] for the background and results in this subject. The first author have studied the existence and regularity of harmonic almost complex structures [7] from the point of view of geometric analysis. In this paper, we focus on the case of polyharmonic almost complex structures with $m \geq 2$. Recall the definition of the Sobolev spaces of almost complex structures.

Definition 1 Suppose (M^n, g) be an almost Hermitian manifold with compatible almost complex structures in $\mathcal{J}_g$. We define $W^{k,p}(\mathcal{J}_g)$ to be the closed subspace of $W^{k,p}(T^*M \otimes TM)$ consisting of those sections $J \in W^{k,p}(T^*M \otimes TM)$, which satisfy $J^2 = -id, \quad g(J\cdot, J\cdot) = g(\cdot, \cdot)$ almost everywhere.

Now, we state our main results.

- **Theorem** 1 There always exists an energy-minimizer of $\mathcal{E}_m(J)$ in $W^{m,2}(\mathcal{J}_g)$.
- **Theorem** 2 Suppose $J \in W^{m,2}(\mathcal{J}_g)$ is a weakly m-harmonic almost complex structure on (M^{2m}, g) with $m \in \{2, 3\}$. Then J is Hölder-continuous.

For semilinear elliptic systems with *critical growth nonlinearities*, the most essential step towards the smoothness is to prove the Hölder continuity, such as the systems for (poly)harmonic maps, see for example [2,5,10,15,16] and references therein. It is well-known that a semilinear elliptic system with *critical growth nonlinearities* and at *critical dimension* might be singular [4,9]. For weakly harmonic map, it can be even singular everywhere [13] when the dimension is three and above. The smooth regularity starts with Helein's seminal result [10] for harmonic maps in dimension two where the special (algebraic) structure of the system plays a substantial role. New proofs and understanding of Helein's seminal results can be found [1,14]. The methods can be generalized to fourth order elliptic system in dimension four [2,11]. General smooth regularity for biharmonic maps and polyharmonic maps have been obtained by [16] and [5], respectively.

We shall briefly compare our results with the results in the theory of (poly)harmonic maps. Theorem 1 is a standard practice in calculus of variations, while the main point is that the absolute energy-minimizer is not trivial due to its tensor-valued nature. The main result is to prove the Hölder regularity in Theorem 2 and our method is motivated by the work in [2] and [5]. In [2], the authors explore a special divergence structure of the biharmonic system into the spheres and our elliptic system shares some similarities. On the other hand, the tensor-valued nature makes our arguments much more complicated, mainly due to the fact that matrix multiplication is not commutative. We certainly believe that this divergence structure should hold for all weakly polyharmonic almost complex structures but we do not find a systematic way to argue that. Instead we only show that the elliptic system for polyharmonic almost complex structures has a desired divergence structure when $m = 2, 3$ by brutal computations. Given this divergence structure, our argument for Hölder regularity is quite different from the method used in [2], but more like a generalization of [5]. We use extension of maps (almost complex structures) instead of solving boundary value problem. Our methods are very general and work for all dimensions. A main difficulty is that the background metric is not necessarily Euclidean, while most results in the setting of polyharmonic maps (see [2,5,16] etc) only consider the Euclidean case. Even though the methods for semilinear system are expected to work similarly, the non-Euclidean background metric really leads to complicated computations and presentations. Once the Hölder regularity is assured, the proof of smoothness follows the strategy in [5].

The paper is organized as follows. In Sect. 2, we collect some facts for Lorentz spaces and Green's functions. In Sect. 3, we establish the existence of the energy-minimizers and derive the Euler-Lagrange equations. Moreover we show that a weak limit of a sequence of weakly m-harmonic almost complex structures in $W^{m,2}$ is still m-harmonic. In Sect. 4, we prove decay estimates for a class of semilinear elliptic equations in critical dimension and obtain the Hölder regularity of weakly m-harmonic almost complex structures on (M^{2m}, g) for $m = 2, 3$. In Sect. 5, we generalize the higher regularity results of Gastel and Scheven [5] to prove smoothness of weakly m-harmonic almost complex structures. Appendix derive the divergence structures in detail for m-harmonic almost complex structures when $m = 2, 3$.

2 Preliminaries

In this section, we gather some facts that will be used later. First of all, let us denote by $G(x) = c_m \ln |x|$ the fundamental solution for Δ^m on $\mathbb{R}^{2m}$, where c_m is a suitable constant only dependent of m. We have the following lemma,

Lemma 1 *Suppose $k \in [1, 2m]$ is a positive integer and $p, q \in (1, \infty)$ satisfy*

$$1 + \frac{1}{p} = \frac{k}{2m} + \frac{1}{q}.$$

If $f \in L^q(\mathbb{R}^{2m})$, then we have

$$\left\| \int_{\mathbb{R}^{2m}} \nabla^k G(x-y) f(y) dy \right\|_{L^p(\mathbb{R}^{2m})} \leq C \|f\|_{L^q(\mathbb{R}^{2m})} \tag{2}$$

where C is a positive constant only dependent of m, k, q.

Proof Since $\nabla^{2m} G$ is a Calderón-Zygmund kernel, (2) holds for $k = 2m$ and all $p = q \in (1, \infty)$. For $k = 1, \cdots, 2m-1$, we have

$$\nabla^k G \in L^{\frac{2m}{k}, \infty}\left(\mathbb{R}^{2m}\right)$$

where $L^{\frac{2m}{k},\infty}(\mathbb{R}^{2m})$ is a Lorentz space. By the convolution inequality for Lorentz spaces (cf. [12] Theorem 2.6), we deduce that, for $s \le p$

$$\left\| \int_{\mathbb{R}^{2m}} \nabla^k G(x-y) f(y) dy \right\|_{L^{p,p}(\mathbb{R}^{2m})} \le C \|\nabla^k G\|_{L^{\frac{2m}{k},\infty}(\mathbb{R}^{2m})} \|f\|_{L^{q,s}(\mathbb{R}^{2m})}$$

The fact that $k \in [1, 2m-1]$ implies $\frac{1}{p} < \frac{1}{q}$. Thus, we can choose $s = q$. Moreover, there holds that $L^p(\mathbb{R}^{2m}) = L^{p,p}(\mathbb{R}^{2m})$ for all $p \in (1, \infty)$ (cf. [18] Lemma 1.8.10), which implies (2). □

For more details about Lorentz spaces, we refer the readers to [12,18]. We also need the following standard fact about the elliptic operator Δ^m.

Lemma 2 *Let B_1 be the unit ball of $\mathbb{R}^n$. Suppose $v(x) \in W^{m,2}(B_1) \cap L^\infty$ and $f \in L^\infty(B_1)$. If $v(x)$ satisfies $\Delta^m v(x) = f(x)$ in distributional sense, then*

$$\|v(x)\|_{L^\infty\left(B_{\frac{1}{2}}\right)} \le C \left(\|v(x)\|_{L^1(B_1)} + \|f(x)\|_{L^\infty(B_1)} \right), \tag{3}$$

where C is a positive constant only dependent of n

3 Existence of Energy-Minimizer and the Euler-Lagrange Equation

In this section, we establish the existence of the energy-minimizers of $\mathcal{E}_m(J)$, derive its Euler-Lagrange equation and define the weak solutions. Moreover we prove that a weak limit of a sequence of weakly m-harmonic almost complex structures with bounded $W^{m,2}$ norm is still m-harmonic.

Theorem 1 *There always exists an energy-minimizer of $\mathcal{E}_m(J)$ in $W^{m,2}(\mathcal{J}_g)$.*

Proof The proof is standard in calculus of variations. We include the details for completeness. Take a minimizing sequence $J_k \in W^{m,2}(\mathcal{J}_g)$ such that

$$\inf_{J \in W^{m,2}} \mathcal{E}_m(J) = \lim_{k \to \infty} \mathcal{E}_m(J_k).$$

Note that by interpolation inequality and integration by parts,

$$\|J\|^2_{W^{m,2}} \le C \left(\sum_{|\alpha|=m} \|\nabla^\alpha J\|^2_{L^2} + \|J\|^2_{L^\infty} \right) \le C\big(\mathcal{E}_m(J) + 1\big), \ \ \forall J \in W^{m,2}(\mathcal{J}_g).$$

Hence, the sequence $\{J_k\}$ is bounded in $W^{m,2}$. This implies that there exists a subsequence, still denoted by J_k, and $J_0 \in W^{m,2}$, such that J_k converges weakly to J_0 in $W^{m,2}$ and $\mathcal{E}_m(J_0) \le \underline{\lim}_{k\to\infty} \mathcal{E}_m(J_k)$. Moreover, J_k converges strongly to J_0 in $W^{m-1,2}$ and hence $J_0 \in \mathcal{J}_g$. It follows that J_0 is an energy-minimizer of the functional $\mathcal{E}_m(J)$. □

Denote by $T_q^p(M)$ the set of all (p, q) tensor fields on (M, g). There is a natural inner product on $T_q^p(M)$ induced by g, denoted by $\langle , \rangle$. In local coordinate $\{x^i\}_{i=1}^n$, $A \in T_q^p(M)$ can be expressed by

$$A = A_{i_1 \cdots i_q}^{j_1 \cdots j_p} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q}.$$

The inner product of $A, B \in T_q^p(M)$ is given by

$$\langle A, B \rangle = A_{i_1 \cdots i_q}^{k_1 \cdots k_p} B_{j_1 \cdots j_q}^{l_1 \cdots l_p} g^{i_1 j_1} \cdots g^{i_q j_q} g_{k_1 l_1} \cdots g_{k_p l_p},$$

where $g = g_{ij} dx^i \otimes dx^j$ and (g^{ij}) is the inverse of (g_{ij}). For $A \in T_1^1(M)$, define the adjoint operator A^* of A by

$$g(X, A^* Y) := g(AX, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

where $\mathfrak{X}(M)$ is the set of all smooth vector fields on (M, g). In local coordinates, if $A = A_i^j \partial_{x^j} \otimes dx^i$ we have $(A^*)_i^j = A_k^l g^{kj} g_{li}$.

Proposition 1 *We have the following standard facts,*

1. *For all* $A, B \in T_q^p(M)$*, there holds*

$$\int_M \langle \nabla A, \nabla B \rangle = -\int_M \langle A, \Delta B \rangle .$$

2. *For all* $A \in T_1^1(M)$ *and* $X \in \mathfrak{X}(M)$*, there holds* $(\nabla_X A)^* = \nabla_X (A^*)$.
3. *For all* $A, B \in T_1^1(M)$*, there holds* $\langle A, B \rangle = \langle A^*, B^* \rangle$.
4. *For all* $A, B, C \in T_1^1(M)$*, there holds* $\langle A, BC \rangle = \langle B^* A, C \rangle = \langle AC^*, B \rangle$.

For $A, B \in T_1^1(M)$, AB is regarded as the composition of linear maps, i.e., $AB \in T_1^1(M)$. In local coordinate we have $(AB)_i^j = A_s^j B_i^s$. With these notations, we have

$$\mathcal{J}_g = \left\{ J \in T_1^1(M) : \ J^2 = -id, \ J^* + J = 0 \right\}. \tag{4}$$

Let $\{J(t)\}_{t \in (-\delta, \delta)}$ be a C^1 curve in $\mathcal{J}_g$ with $J(0) = J$. Let $S = \frac{dJ}{dt}|_{t=0}$. Such S is called *an admissible variational direction* of J in $\mathcal{J}_g$. Denote $\mathcal{S}_J$ to be the set of all admissible variational directions of J.

Proposition 2 *We have*

$$\mathcal{S}_J = \left\{ S \in T_1^1(M) : SJ + JS = 0, S + S^* = 0 \right\}.$$

For any $J \in \mathcal{J}_g$, define the operator $\Phi_J : T_1^1(M) \to S_J$ by

$$\Phi_J(T) = \frac{1}{4}\Big((T + JTJ) - (T + JTJ)^* \Big).$$

On each fiber of $T_1^1(M)$, Φ_J is precisely the orthogonal projection onto $(S_J)_x$, satisfying that for all $T \in T_1^1(M)$ and $S \in \mathcal{S}_J$,

$$\langle T, S\rangle = \langle \Phi_J(T), S\rangle \,.$$

Proposition 3 *The Euler-Lagrange equation of functional $\mathcal{E}_m(J)$ is*

$$\left[\Delta^m J, J\right] = 0. \tag{5}$$

Proof Suppose $J \in \mathcal{J}_g$ is a critical point of $\mathcal{E}_m(J)$. For any $S \in \mathcal{S}_J$, we have

$$0 = \delta\mathcal{E}_m(J) = (-1)^m 2\int_M \left\langle \Delta^m J, S\right\rangle = (-1)^m 2\int_M \left\langle \Phi_J\left(\Delta^m J\right), S\right\rangle$$

which implies $\Phi_J(\Delta^m J) = 0$. Equivalently we have

$$\left[\Delta^m J, J\right] := \Delta^m J J - J\Delta^m J = 0.$$

□

An almost complex structure $J \in W^{m,2}(\mathcal{J}_g)$ satisfying (5) in distributional sense is called *weakly m-harmonic*.

Proposition 4 *A weakly m-harmonic almost complex structure J satisfies the following in distributional sense,*

$$\Delta^m J = \sum_{s=0}^{m-1} (-1)^{m+1+s} \nabla^s \cdot g_s \tag{6}$$

where $g_s = \sum C_{k_1,k_2,k_3} \nabla^{k_1} J \nabla^{k_2} J \nabla^{k_3} J$ for nonnegative integers $k_1 + k_2 + k_3 = 2m - s$, $k_i \in [0, m]$ and $C_{k_1,k_2,k_3} \in \mathbb{Z}$. That is, for any $T \in T_1^1(M)$, there holds

1. when $m = 2k$, $k \in \mathbb{N}^+$,

$$\int_M \left\langle \Delta^k J, \Delta^k T\right\rangle + \sum_{s=0}^{m-1} \int_M \left\langle g_s, \nabla^s T\right\rangle = 0 \tag{7}$$

2. when $m = 2k - 1$, $k \in \mathbb{N}^+$,

$$\int_M \left\langle \nabla\Delta^{k-1} J, \nabla\Delta^{k-1} T\right\rangle + \sum_{s=0}^{m-1} \int_M \left\langle g_s, \nabla^s T\right\rangle = 0 \tag{8}$$

For simplicity, we will give the exact meaning of ∇^s in the proof.

Proof We focus on the case $m = 2k$ since the case $m = 2k - 1$ is similar. Suppose J is weakly m-harmonic. Then for any $S \in \mathcal{S}_J$, we have

$$0 = 2\int_M \left\langle \Delta^k J, \Delta^k S \right\rangle.$$

Taking $S = \Phi_J(T)$ for any $T \in T_1^1(M)$,

$$\begin{aligned}
0 &= 2\int_M \left\langle \Delta^k J, \Delta^k \Phi_J(T) \right\rangle \\
&= \frac{1}{2}\int_M \left\langle \Delta^k J\,,\ \Delta^k\big((T + JTJ) - (T + JTJ)^*\big) \right\rangle \\
&= \frac{1}{2}\int_M \left\langle \Delta^k J, \Delta^k\big(T + JTJ\big) \right\rangle - \left\langle \Delta^k J^*, \Delta^k\big(T + JTJ\big) \right\rangle \\
&= \int_M \left\langle \Delta^k J, \Delta^k\big(T + JTJ\big) \right\rangle \qquad (9) \\
&= \int_M \left\langle \Delta^k J, \Delta^k T \right\rangle + \left\langle \Delta^k J, J\Delta^k T J \right\rangle + \left\langle \Delta^k J, R_1 \right\rangle \\
&= \int_M \left\langle \Delta^k J, \Delta^k T \right\rangle + \left\langle J\Delta^k J J, \Delta^k T \right\rangle + \left\langle \Delta^k J, R_1 \right\rangle \\
&= \int_M \left\langle \Delta^k J, \Delta^k T \right\rangle - \left\langle \big(\Delta^k J J + R_2\big)J, \Delta^k T \right\rangle + \left\langle \Delta^k J, R_1 \right\rangle \\
&= \int_M 2\left\langle \Delta^k J, \Delta^k T \right\rangle - \left\langle R_2 J, \Delta^k T \right\rangle + \left\langle \Delta^k J, R_1 \right\rangle \\
&= \int_M 2\left\langle \Delta^k J, \Delta^k T \right\rangle + \left\langle \nabla(R_2 J), \nabla\Delta^{k-1} T \right\rangle + \left\langle \Delta^k J, R_1 \right\rangle \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \Delta^k(JTJ) - J\Delta^k T J, \\
R_2 &= \Delta^k(JJ) - \Delta^k J J - J\Delta^k J = -\Delta^k J J - J\Delta^k J
\end{aligned}$$

We describe the terms of R_1 and R_2 by taking a local orthonormal fields $\{e_i\}_{i=1}^n$ as follows,

$$\begin{aligned}
R_1 &= \Delta^k(JTJ) - J\Delta^k T J \\
&= \nabla_{i_1}^2 \cdots \nabla_{i_k}^2 (JTJ) - J\Delta^k T J \\
&= \sum_{\substack{\alpha<,\beta<,\gamma< \\ k_1+k_2+k_3=m \\ k_2\le m-1}} \nabla_{i_{\alpha_1}} \cdots \nabla_{i_{\alpha_{k_1}}} J\, \nabla_{i_{\beta_1}} \cdots \nabla_{i_{\beta_{k_2}}} T\, \nabla_{i_{\gamma_1}} \cdots \nabla_{i_{\gamma_{k_3}}} J \\
R_2 &= \Delta^k(JJ) - \Delta^k J J - J\Delta^k J
\end{aligned}$$

$$= \sum_{\substack{\alpha<,\beta<,\\ k_1+k_2=m\\ 1\le k_1,k_2\le m-1}} \nabla_{i_{\alpha_1}} \cdots \nabla_{i_{\alpha_{k_1}}} J \, \nabla_{i_{\beta_1}} \cdots \nabla_{i_{\beta_{k_2}}} J$$

where the symbol $\alpha <$ means $1 \le \alpha_1 \le \cdots \le \alpha_{k_1} \le k$ and we write

$$\nabla^{k_1} = \sum_{\alpha<} \nabla_{i_{\alpha_1}} \cdots \nabla_{i_{\alpha_{k_1}}}.$$

Then, we can rewrite R_1 and R_2 in the following,

$$R_1 = \sum_{\substack{k_1+k_2+k_3=m\\ k_2\le m-1}} \nabla^{k_1} J \, \nabla^{k_2} T \, \nabla^{k_3} J,$$
$$R_2 = \sum_{\substack{k_1+k_2=m\\ 1\le k_1,k_2\le m-1}} \nabla^{k_1} J \, \nabla^{k_2} J.$$

Substituting the above into (10), we get (7) as follows,

$$\begin{aligned} 0 &= \int_M 2\left\langle \Delta^k J, \Delta^k T\right\rangle + \int_M \sum_{\substack{k_1+k_2=m\\ 1\le k_1,k_2\le m-1}} \left\langle \nabla\left(\nabla^{k_1} J \, \nabla^{k_2} J J\right), \nabla \Delta^{k-1} T\right\rangle \\ &\quad + \int_M \sum_{\substack{k_1+k_2+k_3=m\\ k_2\le m-1}} \left\langle \Delta^k J, \nabla^{k_1} J \, \nabla^{k_2} T \, \nabla^{k_3} J\right\rangle \\ &= \int_M 2\left\langle \Delta^k J, \Delta^k T\right\rangle + \int_M \sum_{\substack{k_1+k_2=m\\ 1\le k_1,k_2\le m-1}} \left\langle \nabla\left(\nabla^{k_1} J \, \nabla^{k_2} J J\right), \nabla \Delta^{k-1} T\right\rangle \\ &\quad + \int_M \sum_{\substack{k_1+k_2+k_3=m\\ k_2\le m-1}} \left\langle \nabla^{k_1} J \, \Delta^k J \, \nabla^{k_3} J, \nabla^{k_2} T\right\rangle, \end{aligned}$$

□

Proposition 5 *A weak limit of a sequence of weakly m-harmonic almost complex structures with uniformly bounded $W^{m,2}$ norm is still m-harmonic.*

Proof Let J be weakly m-harmonic. Then for any $T \in T^1_1(M)$, there holds

1. when $m = 2k$, $k \in \mathbb{N}^+$

$$\int_M \left\langle \Delta^k J, \left[J, \Delta^k T\right]\right\rangle + \sum_{\substack{k_1+k_2=m\\ 1\le k_1,k_2\le m-1}} \left\langle \Delta^k J, \left[\nabla^{k_1} J, \nabla^{k_2} T\right]\right\rangle = 0, \qquad (11)$$

2. when $m = 2k-1, k \in \mathbb{N}^+$

$$\int_M \left\langle \nabla \Delta^{k-1} J, \left[J, \nabla \Delta^{k-1} T \right] \right\rangle + \sum_{\substack{k_1+k_2=m \\ 1\le k_1,k_2\le m-1}} \left\langle \nabla \Delta^{k-1} J, \left[\nabla^{k_1} J, \nabla^{k_2} T \right] \right\rangle = 0.$$

We only prove the case $m = 2k$. Recall (9) holds for all $T \in T_1^1(M)$,

$$\int_M \left\langle \Delta^k J, \Delta^k \big(T + JTJ \big) \right\rangle = 0$$

By replacing T by JT, we derive

$$\int_M \left\langle \Delta^k J, \Delta^k \big(JT - TJ \big) \right\rangle = \int_M \left\langle \Delta^k J, \Delta^k [J, T] \right\rangle = 0.$$

This implies (11) since

$$\begin{aligned} \Delta^k[J, T] &= \left[\Delta^k J, T \right] + \left[J, \Delta^k T \right] + \sum_{\substack{k_1+k_2=m \\ 1\le k_1,k_2\le m-1}} \left[\nabla^{k_1} J, \nabla^{k_2} T \right], \\ \left\langle \Delta^k J, \left[\Delta^k J, T \right] \right\rangle &= \left\langle \Delta^k J, \Delta^k J\, T \right\rangle - \left\langle \Delta^k J, T\, \Delta^k J \right\rangle \\ &= \left\langle \left(\Delta^k J \right)^* \Delta^k J, T \right\rangle - \left\langle \Delta^k J \left(\Delta^k J \right)^*, T \right\rangle \\ &= \left\langle \Delta^k J^* \Delta^k J, T \right\rangle - \left\langle \Delta^k J\, \Delta^k J^*, T \right\rangle = 0. \end{aligned}$$

Now, suppose $\{J_l\}$ is a sequence of weakly m-harmonic almost complex structures in $W^{m,2}$ such that $J_l \rightharpoonup J_0$ in $W^{m,2}$ and $\sup_l \|J_l\|_{W^{m,2}} < \infty$. By Rellich-Kondrachov theorem, we know that J_l converges to J_0 in $W^{m-1,2}$. Hence $J_0 \in W^{m,2}(\mathcal{J}_g)$. Since $J_l \rightharpoonup J_0$ in $W^{m,2}$, we have

$$\lim_{l\to\infty} \int_M \left\langle \Delta^k J_l, \left[J_0, \Delta^k T \right] \right\rangle = \int_M \left\langle \Delta^k J_0, \left[J_0, \Delta^k T \right] \right\rangle. \tag{12}$$

Since J_l converges to J_0 in $W^{m-1,2}$, $\sup_l \|J_l\|_{W^{m,2}} < \infty$ and

$$\left| \int_M \left\langle \Delta^k J_l, \left[J_l - J_0, \Delta^k T \right] \right\rangle \right| \le \|\Delta^k J_l\|_{L^2} \|J_l - J_0\|_{L^2} \|\Delta^k T\|_{L^\infty},$$

we have $\lim_{l\to\infty} \int_M \langle \Delta^k J_l, [J_l - J_0, \Delta^k T] \rangle = 0$. With (12) this implies

$$\lim_{l\to\infty} \int_M \left\langle \Delta^k J_l, \left[J_l, \Delta^k T \right] \right\rangle = \int_M \left\langle \Delta^k J_0, \left[J_0, \Delta^k T \right] \right\rangle.$$

Similarly we conclude that, for all $k_1 + k_2 = m$ and $1 \le k_1, k_2 \le m-1$,

$$\lim_{l\to\infty} \int_M \left\langle \Delta^k J_l, \left[\nabla^{k_1} J_l, \nabla^{k_2} T\right]\right\rangle = \int_M \left\langle \Delta^k J_0, \left[\nabla^{k_1} J_0, \nabla^{k_2} T\right]\right\rangle.$$

Hence J_0 is weakly m-harmonic and this completes the proof. □

4 Decay Estimates and Hölder Regularity

In this section, we establish decay estimates for a class of semilinear elliptic equations in critical dimension and deduce the Hölder regularity of $W^{m,2}$ m-harmonic almost complex structure on (M^{2m}, g) for $m = 2, 3$. For simplicity, we use C to denote a uniform positive constant.

4.1 Decay Estimates for $W^{2,2}$ Biharmonic Almost Complex Structure on $\mathbb{R}^4$

First, we consider decay estimates for biharmonic almost complex structure defined on B_1 in $\mathbb{R}^4$ as a special case. The presentation is much clearer and more streamlined for this case and the main ideas are essentially the same. Consider the biharmonic almost complex structure equation,

$$\Delta^2 J = J\left(\nabla \Delta J \nabla J + \nabla J \nabla \Delta J + \Delta J \Delta J + \Delta (\nabla J)^2\right)$$

where $J : B_1 \subset \mathbb{R}^4 \to M_4(\mathbb{R})$ (the set of all 4×4 real matrices) satisfies

$$J^2 = -id, \quad J + J^T = 0$$

Proposition 6 asserts that the biharmonic almost complex structure equation admit a good *divergence* form. That is, for any given constant matrix λ_0, biharmonic almost complex structure J satisfies

$$\Delta^2 J = T_{\lambda_0} \tag{13}$$

where T_{λ_0} is a linear combination of the following terms

$$\nabla^\alpha \Big((J - \lambda_0) * \nabla^\beta J * \nabla^\gamma J \Big), \qquad \lambda_0 * \nabla^\alpha \Big((J - \lambda_0) * \nabla^\delta J \Big),$$

where $\alpha, \beta, \gamma, \delta$ are multi-indices such that $1 \le |\alpha| \le 3$, $0 \le |\beta|, |\gamma|, |\delta| \le 2$, $|\alpha| + |\beta| + |\gamma| = 4$ and $|\alpha| + |\delta| = 4$. The notation $A * B$ means the composition of terms A and B, such as AB and BA. Then we have the following,

Lemma 3 *Suppose $J \in W^{2,2}(B_1, M_4(\mathbb{R}))$ is a weakly biharmonic almost complex structure on unit ball $B_1 \subset \mathbb{R}^4$. Then, given any $\tau \in (0, 1)$, there exists $\epsilon_0 > 0$ and*

$\theta_0 \in (0, \frac{1}{2})$ *such that if*

$$E(J,1) := \left(\int_{B_1} |\nabla J|^4\right)^{\frac{1}{4}} + \left(\int_{B_1} |\nabla^2 J|^2\right)^{\frac{1}{2}} \le \epsilon_0,$$

then we have

$$D_{p_0}(J,\theta_0) \le \theta_0^{\tau} D_{p_0}(J,1), \tag{14}$$

where $p_0 = \frac{8}{3}$ *and* $D_p(J,r) := \left(r^{p-4}\int_{B_r} |\nabla u|^p\right)^{\frac{1}{p}}$.

Proof We extend J to $\widetilde{J} \in W^{2,2}(\mathbb{R}^4, M_4(\mathbb{R})) \cap L^\infty$ such that

$$\widetilde{J}|_{B_1} = J, \quad \widetilde{J}|_{\mathbb{R}^4\setminus B_2} = \lambda_0 := \frac{1}{|B_1|}\int_{B_1} J$$

$$\|\nabla \widetilde{J}\|_{L^{p_0}(\mathbb{R}^4)} \le C\,\|\nabla J\|_{L^{p_0}(B_1)} \tag{15}$$

$$E(\widetilde{J},\infty) \le C\,E(J,1), \tag{16}$$

By the standard extension to $J-\lambda_0$ in B_1, there exists a function $\widetilde{J}-\lambda_0$ on $\mathbb{R}^4$ with compact support contained in B_2 and satisfying

$$\|\widetilde{J}-\lambda_0\|_{L^\infty(\mathbb{R}^4)} \le C\|J-\lambda_0\|_{L^\infty(B_1)}, \tag{17}$$

$$\|\widetilde{J}-\lambda_0\|_{W^{1,p_0}(\mathbb{R}^4)} \le C\|J-\lambda_0\|_{W^{1,p_0}(B_1)} \tag{18}$$

$$\|\widetilde{J}-\lambda_0\|_{W^{2,2}(\mathbb{R}^4)} \le C\|J-\lambda_0\|_{W^{2,2}(B_1)}. \tag{19}$$

Since $\widetilde{J}-\lambda_0$ has a compact support, (17) implies $\widetilde{J}-\lambda_0 \in L^q(\mathbb{R}^4)$ for all $q \in [1,\infty]$. By Poincáre inequality (15) follows from (18). We obtain (16) by Poincáre inequality, Sobolev inequality and (19),

$$\begin{aligned} E(\widetilde{J},\infty) &= E(\widetilde{J}-\lambda_0,\infty) \\ &\le C\|\nabla^2\widetilde{J}\|_{L^2(\mathbb{R}^4)} \le C\|J-\lambda_0\|_{W^{2,2}(B_1)} \le C\|\nabla J\|_{W^{1,2}(B_1)} \le C E(J,1) \end{aligned}$$

Note that $\widetilde{J}$ may not be almost complex structure outside B_1. Now let $G(x) = c\ln|x|$ be the fundamental solution for Δ^2 on $\mathbb{R}^4$, where c is a constant. Then $\nabla^4 G$ is a Calderón-Zygmund kernel. Define

$$\omega(x) = \int_{\mathbb{R}^4} G(x-y)\,\widetilde{T}_{\lambda_0}(y)dy = \sum_{\alpha,\beta,\gamma} \omega_{\alpha,\beta,\gamma} + \sum_{\alpha,\delta} \omega_{\alpha,\delta},$$

where

$$\omega_{\alpha,\beta,\gamma} = \int_{\mathbb{R}^4} \nabla^\alpha G(x-y)\Big(\big(\widetilde{J}(y)-\lambda_0\big) * \nabla^\beta \widetilde{J}(y) * \nabla^\gamma \widetilde{J}(y)\Big)dy,$$
$$\omega_{\alpha,\delta} = \int_{\mathbb{R}^4} \nabla^\alpha G(x-y)\Big(\lambda_0 * \big(\widetilde{J}(y)-\lambda_0\big) * \nabla^\delta \widetilde{J}(y)\Big)dy,$$

and $\widetilde{T}_{\lambda_0}$ is defined by replacing J by $\widetilde{J}$ in T_{λ_0} (see (13)) , and $\alpha, \beta, \gamma, \delta$ are multi-indices such that $1 \le |\alpha| \le 3$, $0 \le |\beta|, |\gamma|, |\delta| \le 2$, $|\alpha| + |\beta| + |\gamma| = 4$ and $|\alpha| + |\delta| = 4$. We claim that for $E(J, 1) \le 1$, there holds

$$\|\nabla \omega\|_{L^{p_0}(B_1)} \le CE(J,1)\|\nabla J\|_{L^{p_0}(B_1)}. \tag{20}$$

We will prove the above inequality term by term. By Lemma 1, we have

$$\begin{aligned}
\|\nabla \omega_{\alpha,\beta,\gamma}\|_{L^{p_0}(\mathbb{R}^4)} &\le C\Big\| |\widetilde{J}-\lambda_0|\,|\nabla^\beta \widetilde{J}|\,|\nabla^\gamma \widetilde{J}|\Big\|_{L^{q_0}(\mathbb{R}^4)}\\
&\le C\|\widetilde{J}-\lambda_0\|_{L^{q_1}(\mathbb{R}^4)}\|\nabla^\beta \widetilde{J}\|_{L^{\frac{4}{|\beta|}}(\mathbb{R}^4)}\|\nabla^\gamma \widetilde{J}\|_{L^{\frac{4}{|\gamma|}}(\mathbb{R}^4)}\\
&\le C\|\nabla \widetilde{J}\|_{L^{p_0}(\mathbb{R}^4)}E(\widetilde{J},\infty)^{N_{\beta,\gamma}}\\
&\le C\|\nabla J\|_{L^{p_0}(B_1)}E(J,1)^{N_{\beta,\gamma}}
\end{aligned}$$

where we let $\frac{4}{s} := \infty$ for $s = 0$, $N_{\beta,\gamma}$ stands for the number of non-zero elements in $\{\beta, \gamma\}$ and $q_0, q_1 \in (1, \infty)$ satisfy

$$\frac{1}{p_0} + 1 = \frac{|\alpha|+1}{4} + \frac{1}{q_0}, \qquad \frac{1}{q_0} = \frac{1}{q_1} + \frac{|\beta|}{4} + \frac{|\gamma|}{4}.$$

Since $|\alpha| + |\beta| + |\gamma| = 4$ and $1 \le |\alpha| \le 3$, we know that $1 \le N_{\beta,\gamma} \le 2$ and hence such q_0 and q_1 exist. If $E(J, 1) \le 1$, there holds

$$\|\nabla \omega_{\alpha,\beta,\gamma}\|_{L^{p_0}(B_1)} \le \|\nabla \omega_{\alpha,\beta,\gamma}\|_{L^{p_0}(\mathbb{R}^4)} \le C\|\nabla J\|_{L^{p_0}(B_1)}E(J,1). \tag{21}$$

By a similar argument, we also have

$$\|\nabla \omega_{\alpha,\delta}\|_{L^{p_0}(B_1)} \le \|\nabla \omega_{\alpha,\delta}\|_{L^{p_0}(\mathbb{R}^4)} \le C\|\nabla J\|_{L^{p_0}(B_1)}E(J,1). \tag{22}$$

Combining (21) and (22), we deduce (20).

Finally, we turn to proving (14). Let $v(x) := J(x) - \omega(x)$, then we know $v(x)$ is biharmonic on unit ball B_1, i.e., $\Delta^2 v(x) = 0$. Since ∇v is also biharmonic, it follows from Lemma 2 (or see Lemma 6.2 in [5]) that there holds

$$\|\nabla v\|_{L^\infty\left(B_{\frac{1}{2}}\right)} \le C\|\nabla v\|_{L^1(B_1)}.$$

Hence, for any $\theta \in (0, \frac{1}{2})$ and $E(u,1) \le 1$, there holds

$$
\begin{aligned}
D_{p_0}(J,\theta) &= \theta^{1-\frac{4}{p_0}} \|\nabla J\|_{L^{p_0}(B_\theta)} \\
&\le \theta^{1-\frac{4}{p_0}} \left(\|\nabla \omega(x)\|_{L^{p_0}(B_\theta)} + \|\nabla v(x)\|_{L^{p_0}(B_\theta)}\right) \\
&\le C\theta^{1-\frac{4}{p_0}} \left(\|\nabla \omega(x)\|_{L^{p_0}(B_\theta)} + \theta^{\frac{4}{p_0}} \|\nabla v(x)\|_{L^{\infty}(B_\theta)}\right) \\
&\le C\theta^{1-\frac{4}{p_0}} \left(\|\nabla \omega(x)\|_{L^{p_0}(B_\theta)} + \theta^{\frac{4}{p_0}} \|\nabla v(x)\|_{L^{p_0}(B_1)}\right) \\
&\le C\theta^{1-\frac{4}{p_0}} \left(\|\nabla \omega(x)\|_{L^{p_0}(B_\theta)} + \theta^{\frac{4}{p_0}} \|\nabla \omega(x)\|_{L^{p_0}(B_1)}\right. \\
&\quad \left. +\theta^{\frac{4}{p_0}} \|\nabla J(x)\|_{L^{p_0}(B_1)}\right) \\
&\le C\theta^{1-\frac{4}{p_0}} \left(\|\nabla \omega(x)\|_{L^{p_0}(B_\theta)} + \theta^{\frac{4}{p_0}} \|\nabla J(x)\|_{L^{p_0}(B_1)}\right) \\
&\le C \left(\theta^{1-\frac{4}{p_0}} E(J,1) \|\nabla J(x)\|_{L^{p_0}(B_\theta)} + \theta \|\nabla J(x)\|_{L^{p_0}(B_1)}\right) \\
&\le C \left(\theta^{1-\frac{4}{p_0}} E(J,1) + \theta\right) D_{p_0}(J,1).
\end{aligned}
$$

Thus, for any given $\tau \in (0,1)$, by choosing $\theta = \theta_0$ and ϵ_0 sufficiently small, we obtain (14) for $E(J,1) \le \epsilon_0$. □

4.2 Decay Estimates for a Class of Semilinear Elliptic Equations

Consider the following semilinear elliptic equation for $u : B_1 \subset \mathbb{R}^n \to \mathbb{R}^K$,

$$
\Delta^m u = \Psi \left(x, \nabla u, \cdots, \nabla^{2m-1} u\right) \tag{23}
$$

where $\Psi : \mathbb{R}^n \times \mathbb{R}^{nK} \times \cdots \times \mathbb{R}^{n^{2m-1}K} \to \mathbb{R}^K$ is smooth and B_1 is the unit ball in $\mathbb{R}^n$ centered at origin. We can generalize the results in Sect,. 4.1 to (23) which admit a good *divergence* structure specified in the following,

Definition 2 We say that the equation (23) admits a good divergence form if for any fixed constant vector $\lambda_0 \in \mathbb{R}^K$, Ψ can be decomposed into $\Psi_H + \Psi_L$, the highest order term Ψ_H and the lower order term Ψ_L, which satisfy the following properties:

1. Ψ_H is a linear combination of the following terms

$$
\nabla^{\alpha}((u-\lambda_0) * h_{\alpha,\beta}), \quad \text{with } |h_{\alpha,\beta}| \le C \prod_{i=1}^{s} \left|\nabla^{\beta_i} u\right|, \tag{24}
$$

where α, β_i are multi-indices and $\beta = (\beta_1, \cdots, \beta_s)$ such that

$$|\alpha| + \sum_{i=1}^{s} |\beta_i| = 2m, \tag{25}$$

$$|\beta_i| \leq m, \quad i = 1, \cdots, s, \;\; s \in \mathbb{N}^+, \tag{26}$$

$$1 \leq \sum_{i=1}^{s} |\beta_i| \leq 2m - 1, \tag{27}$$

2. Ψ_L is a linear combination of the following three types of terms

$$\begin{aligned} &\nabla^\alpha (a_{\alpha,\gamma}(x) * \ell_{\alpha,\gamma}), \quad \text{with } |\ell_{\alpha,\gamma}| \leq C \prod_{i=1}^{s} \left|\nabla^{\gamma_i} u\right|, \\ &b_t(x) * \big(u(x) - \lambda_0\big) * \ell_{0,t}, \quad \text{with } |\ell_{0,t}| \leq C|u|^t, \quad t \in \mathbb{N}, \\ &c(x), \end{aligned} \tag{28}$$

where $\gamma = (\gamma_1, \cdots, \gamma_s)$, $a_{\alpha,\gamma}(x), b_t(x), c(x) \in C^{2m}(\overline{B_1}, \mathbb{R}^K)$ and

$$|\alpha| + \sum_{i=1}^{s} |\gamma_i| \leq 2m - 1, \tag{29}$$

$$|\gamma_i| \leq m, \quad i = 1, \cdots, s, \;\; s \in \mathbb{N}^+, \tag{30}$$

$$\sum_{i=1}^{s} |\gamma_i| \geq 1. \tag{31}$$

Remark 1 1. The condition (26) and (30) are natural for us to define the weak solution to (23) for $u \in W^{m,2}$.
2. The condition (27) plays an important role in proving the Hölder continuity of u in critical dimension $n = 2m$ under the structure (24) of Ψ_H.
3. A trivial verification shows that the terms in the form

$$g(x) * \nabla^{\alpha_1} u * \cdots * \nabla^{\alpha_t} u \quad \text{for } g(x) \in C^{4m}(\overline{B_1}, \mathbb{R}^K), \; \sum_i |\alpha_i| \leq 2m - 1$$

can always be rewritten as a linear combination of terms (28).

For any ball B_r of radius r centered at origin in $\mathbb{R}^n$, any $p > 1$, and $q_l \in (1, \infty)$ given by $\frac{1}{q_l} = \frac{1}{2} - \frac{m-l}{n}$ for $l = 1, \cdots, m$ and $n \geq 2m$, denote

$$E(u, r) = \sum_{l=1}^{m} \left(r^{lq_l - n} \int_{B_r} |\nabla^l u|^{q_l} \right)^{\frac{1}{q_l}}, \tag{32}$$

$$D_p(u, r) = \left(r^{p-n} \int_{B_r} |\nabla u|^p \right)^{\frac{1}{p}}. \tag{33}$$

Lemma 4 *Suppose $n = 2m$ and $u \in W^{m,2}(B_1, \mathbb{R}^K) \cap L^\infty$ satisfies (23) in distributional sense. If (23) admits a good divergence form and $\|u\|_{L^\infty(B_1)} \le \mathcal{B} < \infty$, then, given any $\tau \in (0, 1)$, there exists $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$, which are only dependent of $\tau, \mathcal{B}, m$, such that if $E(u, 1) \le \epsilon_0$, then we have*

$$D_{p_0}(u, \theta_0) \le \theta_0^\tau \big(D_{p_0}(u, 1) + \Lambda\big), \tag{34}$$

where $p_0 = \frac{4m}{3} \in (1, 2m)$ and

$$\Lambda := \sum_{\alpha,\gamma} \|a_{\alpha,\gamma}(x)\|_{L^\infty(B_1)} + \sum_t \|b_t(x)\|_{L^\infty(B_1)} + \|\nabla c(x)\|_{L^\infty(B_1)}$$

where $a_{\alpha,\gamma}(x), b_t(x), c(x)$ are from (28) in lower order terms Ψ_L of (23).

Proof For simplicity, we denote by C a positive constant only dependent of $\tau, \mathcal{B}, m$. Following the similar argument in the proof of Lemma 3, we can extend u to $\widetilde{u} \in W^{m,2}(\mathbb{R}^{2m}, \mathbb{R}^K) \cap L^\infty$ such that

$$\begin{aligned}
&\widetilde{u}|_{B_1} = u, \quad \widetilde{u}|_{\mathbb{R}^{2m}\setminus B_2} = \lambda_0 := \frac{1}{|B_1|}\int_{B_1} u \\
&\|\widetilde{u}\|_{L^\infty(\mathbb{R}^{2m})} \le C\,\|u\|_{L^\infty(B_1)} &(35)\\
&\|\nabla\widetilde{u}\|_{L^{p_0}(\mathbb{R}^{2m})} \le C\,\|\nabla u\|_{L^{p_0}(B_1)} &(36)\\
&E(\widetilde{u}, \infty) \le C\,E(u, 1), &(37)
\end{aligned}$$

Of course, by a standard extension theorem to the functions $a_{\alpha,\gamma}(x), b_t(x) \in C^{2m}(\overline{B_1}, \mathbb{R}^K)$ from the lower order term Ψ_L, there exist the corresponding functions $\widetilde{a}_{\alpha,\gamma}(x), \widetilde{b}_t(x) \in C_0^{2m}(\mathbb{R}^{2m}, \mathbb{R}^K)$ such that

$$\begin{aligned}
&\widetilde{a}_{\alpha,\gamma}(x)|_{B_1} = a_{\alpha,\gamma}(x), \quad \widetilde{b}_t(x)|_{B_1} = b_t(x),\\
&\widetilde{a}_{\alpha,\gamma}(x)|_{\mathbb{R}^{2m}\setminus B_2} = 0, \quad \widetilde{b}_t(x)|_{\mathbb{R}^{2m}\setminus B_2} = 0,\\
&\|\widetilde{a}_{\alpha,\gamma}(x)\|_{L^\infty(\mathbb{R}^{2m})} \le C\|a_{\alpha,\gamma}(x)\|_{L^\infty(B_1)},\\
&\|\widetilde{b}_t(x)\|_{L^\infty(\mathbb{R}^{2m})} \le C\|b_t(x)\|_{L^\infty(B_1)}.
\end{aligned}$$

Let $G(x) = c_m \ln|x|$ be the fundamental solution for Δ^m on $\mathbb{R}^{2m}$. Then $\nabla^{2m}G$ is a Calderón-Zygmund kernel. Let us define

$$\omega(x) = \sum_{\alpha,\beta} \omega_{\alpha,\beta}(x) + \sum_{\alpha,\gamma} \omega_{\alpha,\gamma}(x) + \sum_t \omega_{0,t}(x),$$

where

$$\omega_{\alpha,\beta}(x) = \int_{\mathbb{R}^{2m}} \nabla^\alpha G(x-y)\Big(\big(\widetilde{u}(y)-\lambda_0\big) * \widetilde{h}_{\alpha,\beta}(y)\Big)dy$$
$$\omega_{\alpha,\gamma}(x) = \int_{\mathbb{R}^{2m}} \nabla^\alpha G(x-y)\Big(\widetilde{a}_{\alpha,\gamma}(y) * \widetilde{\ell}_{\alpha,\gamma}(y)\Big)dy$$
$$\omega_{0,t}(x) = \int_{\mathbb{R}^{2m}} G(x-y)\Big(\widetilde{b}_t(y) * \big(\widetilde{u}(y)-\lambda_0\big) * \widetilde{\ell}_{0,t}(y)\Big)dy.$$

We claim that, for $p_0 = \frac{4m}{3}$ and $E(u,1) \leq 1$, there holds

$$\|\nabla\omega\|_{L^{p_0}(B_1)} \leq C\Big(E(u,1)\|\nabla u\|_{L^{p_0}(B_1)} + E(u,1)\cdot \Lambda\Big). \tag{38}$$

We will prove above inequality term by term. By Lemma 1, we have

$$\begin{aligned}
\|\nabla\omega_{\alpha,\beta}\|_{L^{p_0}(B_1)} \leq \|\nabla\omega_{\alpha,\beta}\|_{L^{p_0}(\mathbb{R}^{2m})} &\leq C\Big\|\big|\widetilde{u}-\lambda_0\big|\cdot\big|\widetilde{h}_{\alpha,\beta}\big|\Big\|_{L^{q_{\alpha,\beta}}(R^{2m})}\\
&\leq C\|\widetilde{u}-\lambda_0\|_{L^{4m}(\mathbb{R}^{2m})}\prod_{i=1}^{s}\|\nabla^{\beta_i}\widetilde{u}\|_{L^{\frac{2m}{|\beta_i|}}(\mathbb{R}^{2m})}\\
&\leq C\|\nabla\widetilde{u}\|_{L^{p_0}(\mathbb{R}^{2m})}\prod_{i=1}^{s}\|\nabla^{\beta_i}\widetilde{u}\|_{L^{\frac{2m}{|\beta_i|}}(\mathbb{R}^{2m})}\\
&\leq C\|\nabla u\|_{L^{p_0}(B_1)}E(u,1)^{n_\beta}\|u\|_{L^\infty(B_1)}^{s-n_\beta}\\
&\leq C\mathcal{B}^{s-n_\beta}\|\nabla u\|_{L^{p_0}(B_1)}E(u,1)^{n_\beta}
\end{aligned}$$

where $\frac{2m}{|\beta_i|} := \infty$ for $|\beta_i| = 0$, $n_\beta = \big|\{\beta_i : \beta_i \neq 0\}\big| \geq 1$, and

$$\frac{1}{q_{\alpha,\beta}} = \frac{1}{2m}\left(\sum_{i=1}^{s}|\beta_i| + \frac{1}{2}\right) = 1 + \frac{3}{4m} - \frac{|\alpha|+1}{2m}.$$

Note that (27) implies $q_{\alpha,\beta} \in (1,\infty)$. Hence, if $E(u,1) \leq 1$, there holds

$$\|\nabla\omega_{\alpha,\beta}\|_{L^{p_0}(B_1)} \leq CE(u,1)\|\nabla u\|_{L^{p_0}(B_1)}. \tag{39}$$

Similarly, by Lemma 1, we obtain that

$$\begin{aligned}\|\nabla\omega_{\alpha,\gamma}\|_{L^{p_0}(B_1)} \leq C\|\nabla\omega_{\alpha,\gamma}\|_{L^{4m}(\mathbb{R}^{2m})} &\leq C\Big\| |\widetilde{a}_{\alpha,\gamma}| \cdot |\widetilde{\ell}_{\alpha,\gamma}| \Big\|_{L^{q_{\alpha,\gamma}}(\mathbb{R}^{2m})} \\ &\leq C\|\widetilde{a}_{\alpha,\gamma}\|_{L^{q_1}(\mathbb{R}^{2m})} \prod_{i=1}^{s} \|\nabla^{\gamma_i}\widetilde{u}\|_{L^{\frac{2m}{|\gamma_i|}}(\mathbb{R}^{2m})} \\ &\leq C \cdot \Lambda \cdot E(u,1)^{n_\gamma} \|u\|^{s-n_\gamma}_{L^\infty(B_1)}\end{aligned}$$

where $\frac{2m}{|\gamma_i|} := \infty$ for $|\gamma_i| = 0$, $n_\gamma = \big|\{\gamma_i : \gamma_i \neq 0\}\big| \geq 1$ due to (31), and

$$\frac{1}{q_{\alpha,\gamma}} = \frac{1}{4m} + 1 - \frac{|\alpha|+1}{2m}, \qquad \frac{1}{q_1} = \frac{1}{4m} + 1 - \frac{1}{2m}\left(|\alpha| + \sum_{i=1}^{s}|\gamma_i| + 1\right).$$

Note that (29) implies $q_{\alpha,\gamma}, q_1 \in (1,\infty)$. Hence, if $E(u,1) \leq 1$, there holds

$$\|\nabla\omega_{\alpha,\gamma}\|_{L^{p_0}(B_1)} \leq C \cdot \Lambda \cdot E(u,1) \tag{40}$$

Similar argument applies to terms $\omega_{0,t}$ and yields

$$\|\nabla\omega_{0,t}\|_{L^{p_0}(B_1)} \leq C \cdot \Lambda \cdot E(u,1). \tag{41}$$

Combining (39), (40) and (41) gives (38). In the last we prove (34). Denote $v(x) := u(x) - \omega(x)$, then $v(x)$ satisfies $\Delta^m v(x) = c(x)$ on B_1 in distributional sense. By Lemma 2, we have

$$\|\nabla v(x)\|_{L^\infty\left(B_{\frac{1}{2}}\right)} \leq C\big(\|\nabla v(x)\|_{L^1(B_1)} + \|\nabla c(x)\|_{L^\infty(B_1)}\big).$$

Hence, for any $\theta \in (0, \frac{1}{2})$ and $E(u,1) \leq 1$, there holds

$$\begin{aligned}D_{p_0}(u,\theta) &= \theta^{1-\frac{2m}{p_0}}\|\nabla u\|_{L^{p_0}(B_\theta)} \\ &\leq \theta^{1-\frac{2m}{p_0}}\|\nabla v\|_{L^{p_0}(B_\theta)} + \theta^{1-\frac{2m}{p_0}}\|\nabla\omega\|_{L^{p_0}(B_\theta)} \\ &\leq C\theta\|\nabla v\|_{L^\infty(B_\theta)} + \theta^{1-\frac{2m}{p_0}}\|\nabla\omega\|_{L^{p_0}(B_1)} \\ &\leq C\theta\big(\|\nabla v\|_{L^{p_0}(B_1)} + \|\nabla c(x)\|_{L^\infty(B_1)}\big) + \theta^{1-\frac{2m}{p_0}}\|\nabla\omega\|_{L^{p_0}(B_1)} \\ &\leq C\theta\big(\|\nabla u\|_{L^{p_0}(B_1)} + \|\nabla\omega\|_{L^{p_0}(B_1)} + \Lambda\big) + \theta^{1-\frac{2m}{p_0}}\|\nabla\omega\|_{L^{p_0}(B_1)} \\ &\leq C\left(\theta\big(\|\nabla u\|_{L^{p_0}(B_1)} + \Lambda\big) + \theta^{1-\frac{2m}{p_0}}\|\nabla\omega\|_{L^{p_0}(B_1)}\right) \\ &\leq C\left(\theta\big(\|\nabla u\|_{L^{p_0}(B_1)} + \Lambda\big) + \theta^{1-\frac{2m}{p_0}}E(u,1)\big(\|\nabla u\|_{L^{p_0}(B_1)} + \Lambda\big)\right)\end{aligned}$$

$$\leq C\left(\theta + \theta^{1-\frac{2m}{p_0}} E(u,1)\right)\left(D_{p_0}(u,1) + \Lambda\right).$$

Thus, for any given $\tau \in (0,1)$, by choosing $\theta = \theta_0$ and ϵ_0 sufficiently small, we obtain (34) for $E(u,1) \leq \epsilon_0$. □

4.3 Hölder Regularity for *m*-Harmonic Almost Complex Structure

In this subsection, we prove the Hölder regularity using the decay estimates above,

Theorem 2 *Suppose* $J \in W^{m,2}(\mathcal{J}_g)$ *is a weakly* m*-harmonic almost complex structure on* (M^{2m}, g) *with* $m \in \{2,3\}$. *Then* J *is Hölder-continuous.*

Since the Hölder regularity is a local property, we work on local coordinates on (M^n, g). Let B_1 be the unit ball of $\mathbb{R}^n$ and write g as a smooth metric on B_1. First we consider the Euclidean case with $g = g_0 = \sum_i dx^i \otimes dx^i$ on B_1. The general case is a small perturbation of the Euclidean case.

4.3.1 The Euclidean Case (B_1, g_0)

In this case, an almost complex structure J on B_1 can be regarded as a function in $W^{m,2}(B_1, M_n(\mathbb{R}))$ such that $J^2 = -id$ and $J^T + J = 0$, where $M_n(\mathbb{R})$ is the set of all real $n \times n$ matrices and J^T is the transpose of matrix J. The inner product of $A, B \in T_1^1(B_1)$ reads $\langle A, B\rangle = \sum_{i,j=1}^n A_i^j B_i^j$ for $A = A_i^j dx^i \otimes \frac{\partial}{\partial x^j}$ and $B = B_i^j dx^i \otimes \frac{\partial}{\partial x^j}$. Thus, the inner product of $(1,1)$ tensor fields on B_1 can be viewed as the inner product of two vectors in Euclidean space $\mathbb{R}^{n^2}$.

First we need to write the Euler-Lagrange equation in a good divergence form in the sense of Definition 2.

Lemma 5 *Suppose* J *is a* $W^{m,2}$ *weakly* m*-harmonic almost structure on* (B_1, g_0), $m = 2, 3$. *Then* J *satisfies the following in distributional sense,*

$$\Delta^m J = \Psi\left(J, \nabla J, \cdots, \nabla^{2m-1} J\right) \tag{42}$$

where Ψ *can be rewritten as a linear combination of the following terms, for any fixed constant matrix* $\lambda_0 \in M_n(\mathbb{R})$,

$$\nabla^{\alpha} * \left((J-\lambda_0) * \nabla^{\beta} J * \nabla^{\gamma} J\right) \quad or \quad \lambda_0 * \nabla^{\alpha} * \left((J-\lambda_0) * \nabla^{\delta} J\right),$$

where $\alpha, \beta, \gamma, \delta$ *are multi-indices such that* $1 \leq |\alpha| \leq 2m-1$, $0 \leq |\beta|, |\gamma|, |\delta| \leq m$, $|\alpha| + |\beta| + |\gamma| = 2m$ *and* $|\alpha| + |\delta| = 2m$.

Lemma 5 will be proved in Appendix. Now we prove Theorem 2 for the Euclidean case (B_1, g_0). First, we use the normalized energy $E(J; x, r)$ defined by replacing

u,B_r by J, $B_r(x)$ respectively in (32). That is, due to $n = 2m$,

$$E(J; x, r) = \sum_{l=1}^{m} \left(\int_{B_r(x)} |\nabla^l J|^{\frac{2m}{l}} \right)^{\frac{l}{2m}}.$$

For any fixed $R_0 \in (0, 1)$, we have that for every $\epsilon_0 > 0$, there exists $r_0 \in (0, 1 - R_0)$ such that

$$\sup_{x \in \overline{B}_{R_0}} E(J; x, r_0) < \epsilon_0. \tag{43}$$

For $x_0 \in \overline{B}_{R_0}$, $J_{x_0,r_0}(x) := J(x_0 + r_0 x)$ is also a $W^{m,2}$ m-harmonic almost structure on (B_1, g_0) with

$$E(J_{x_0,r_0}; 0, 1) = E(J; x_0, r_0) < \epsilon_0.$$

By Lemma 5, J_{x_0,r_0} admits a good divergence form (see Definition 2) with $\Psi_L = 0$. Then it follows from Lemma 4 that by choosing suitable $\epsilon_0 > 0$ in (43), there exists $\theta_0 \in (0, \frac{1}{2})$ and $p_0 = \frac{4m}{3}$ such that

$$\begin{aligned} D_{p_0}(J; x_0, \theta_0 r_0) &= D_{p_0}(J_{x_0,r_0}; 0, \theta_0) \\ &\le \sqrt{\theta_0} D_{p_0}(J_{x_0,r_0}; 0, 1) = \sqrt{\theta_0} D_{p_0}(J; x_0, r_0). \end{aligned}$$

A standard iteration argument shows that there exists $\alpha \in (0, 1)$ such that

$$D_{p_0}(J; x_0, r) \le C r^{\alpha}, \quad \forall r \in (0, r_0).$$

This, combined with the Morrey's lemma, yields that $J \in C^{0,\alpha}(\overline{B}_{R_0})$, hence that $J \in C^{0,\alpha}(B_1)$.

4.3.2 The General Case (B_1, g)

In this subsection, we prove the Hölder regularity of the general case on (B_1, g) by a perturbation method. We start by recalling the scaling invariance of the functional $\mathcal{E}_m(J)$ in critical dimension $n = 2m$. If $g_\lambda := \lambda^2 g$ for some positive real number λ, then $\mathcal{E}_m(J, g) = \mathcal{E}_m(J, g_\lambda)$, where $\mathcal{E}_m(J, g) = \int_M \left|\Delta_g^{\frac{m}{2}} J\right|^2 dV_g$. It follows that if J is a weakly m-harmonic on (M, g), then J is also m-harmonic on (M, g_λ). If we take the geodesic normal coordinates on the unit geodesic ball centered at fixed point in (M, g_λ), then the metric g_λ in such local coordinates converges to the Euclidean metric in $C^\infty(B_1)$ as λ goes to infinity. Hence, we can assume that, by a scaling if necessary, the metric g on B_1 is sufficiently close to the Euclidean metric in the sense

$$|g_{ij}(x) - \delta_{ij}| + \sum_{k=1}^{2m} |D^k g_{ij}(x)| \le \delta_0, \quad \forall x \in B_1 \tag{44}$$

where δ_0 is sufficiently small and will be determined later. Now we prove Theorem 2 in the general case (B_1, g).

Firstly, we introduce an operator $\mathfrak{m}$ which maps a (1, 1) tensor field A on (B_1, g) to a $n \times n$ real matrix valued function,

$$A_{\mathfrak{m}} := \mathfrak{m}(A) = (A_i^j)$$

where $A = A_i^j dx^i \otimes \frac{\partial}{\partial x^j}$. In other words, A denotes tensor field and $A_{\mathfrak{m}}$ denotes its coefficient matrix. Let us denote by ∇ the covariant derivative on (B_1, g) and D the ordinary derivatives (i.e., $D_k = \partial_k$). Here it is necessary to emphasize the difference between the derivatives on tensor fields and matrix valued functions. For example, for $A = A_i^j dx^i \otimes \partial_j$, we have

$$(\nabla_{\partial_k} A)_i^j = D_k A_i^j + A_i^s \Gamma_{ks}^j - A_s^j \Gamma_{ki}^s$$

where Γ_{ij}^k denote the Christoffel symbols with respect to metric g. To simplify notation, we rewrite above equation as

$$\left(\nabla_{\partial_k} A\right)_{\mathfrak{m}} = D_k A_{\mathfrak{m}} + Dg * A_{\mathfrak{m}}$$

where $D_k A_{\mathfrak{m}} = D_k(A_i^j) = (D_k A_i^j)$. Similarly, there holds

$$\left(\Delta A\right)_{\mathfrak{m}} = \Delta A_{\mathfrak{m}} + Dg * DA_{\mathfrak{m}} + \left(D^2 g + Dg * Dg\right) * A_{\mathfrak{m}}. \tag{45}$$

Recall the m-harmonic almost complex structure equation (61), i.e.,

$$\Delta^m J = T(J, \nabla J, \cdots, \nabla^{2m-1} J).$$

We will reduce above equation to a perturbation form of the Euclidean case step by step. As a example, we show how to handle the term $\Delta^m J$. Repeated application of (45) yields

$$(\Delta^m J)_{\mathfrak{m}} = \Delta^m J_{\mathfrak{m}} + L_1(D^i g, D^j J_{\mathfrak{m}})$$

where L_1 stands for the lower order terms in the following form

$$L_1 = \sum D^{i_1} g * \cdots * D^{i_s} g * D^j J_{\mathfrak{m}}$$

with $i_\mu \geq 1, \mu = 1, \cdots, s, j \geq 0$ and $j + \sum_{\mu=1}^{s} i_\mu = 2m$. Let $\Delta_0 = \sum_{i=1}^{2m} \partial_i^2$ be the standard Laplace operator on Euclidean space $\mathbb{R}^{2m}$. Recall that for any smooth function f,

$$\Delta f = g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ij} \Gamma_{ij}^s \frac{\partial f}{\partial x^s} =: g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + Dg * Df,$$

where we omit the terms g^{ij} in the expression $Dg * Df$ due to boundedness of g. Then we have

$$\left(\Delta^m - \Delta_0^m\right) J_{\mathfrak{m}} = P_1 + L_2(D^i g, D^j J_{\mathfrak{m}}),$$

where P_1 stands for the perturbation term in the following form

$$\begin{aligned} P_1 &= \left(g^{i_1 j_1} \cdots g^{i_m j_m} - \delta^{i_1 j_1} \cdots \delta^{i_m j_m}\right) D^{2m}_{i_1 j_1 \cdots i_m j_m} J_{\mathfrak{m}} \\ &=: \sum_{|\beta|=2m} a_\beta(x) * D^\beta J_{\mathfrak{m}}, \end{aligned} \tag{46}$$

and L_2 also stands for the lower order terms and has the similar expression as L_1. Hence, we obtain

$$\left(\Delta^m J\right)_{\mathfrak{m}} = \Delta_0^m J_{\mathfrak{m}} + P_1 + \widetilde{L}_1 \left(D^i g, D^j J_{\mathfrak{m}}\right),$$

where $\widetilde{L}_1 = L_1 + L_2$ has the following form

$$\widetilde{L}_1 = \sum D^{i_1} g * \cdots * D^{i_s} g * D^j J_{\mathfrak{m}}$$

with $i_\mu \geq 1, \mu = 1, \cdots, s,\ j \geq 0$ and $(\sum_{\mu=1}^s i_\mu) + j = 2m$. Similar arguments apply to the nonlinear terms and yield

$$T\left(J, \nabla J, \cdots, \nabla^{2m-1} J\right) = T_s\left(J_{\mathfrak{m}}, DJ_{\mathfrak{m}}, \cdots, D^{2m-1} J_{\mathfrak{m}}\right) + P_2 + \widetilde{L}_2$$

where T_s admits a good divergence form as Ψ in Lemma 5, P_2 stands for the perturbation terms

$$P_2 = \sum b_{ijk} * D^i\left((J_{\mathfrak{m}} - \lambda_0) * D^j J_{\mathfrak{m}} * D^k J_{\mathfrak{m}}\right) \tag{47}$$

where b_{ijk} consists of $|g^{st} - \delta^{st}|$, $0 \leq j, k \leq m$ and $i + j + k = 2m$, and $\widetilde{L}_2$ stands for the lower order terms in the following form

$$\widetilde{L}_2 = \sum D^{i_1} g * \cdots * D^{i_s} g * D^j J_{\mathfrak{m}} * D^k J_{\mathfrak{m}} * D^l J_{\mathfrak{m}}$$

with $i_\mu \geq 1$ $\mu = 1, \cdots, s, 0 \leq j+k+l \leq 2m-1$ and $\left(\sum_{\mu=1}^s i_\mu\right) + j+k+l = 2m$. By the arguments above, we get the final reduced equation about $J_{\mathfrak{m}}$

$$\Delta_0^m J_{\mathfrak{m}} = T_s + \mathcal{P} + \mathcal{L} \tag{48}$$

where $\mathcal{P} = P_2 - P_1$ and $\mathcal{L} = \widetilde{L}_2 - \widetilde{L}_1$. In other words, the nonlinear part of (48) consists of three types of terms: terms that admit a good divergence form, the perturbation terms and the lower order terms. Now recall the definition of $E(u, r)$ and $D_p(u, r)$ in (32) and (33) respectively. Then, we claim that for any given $\tau \in (0,1)$, there exists $\delta_0 > 0$, $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if the metric g satisfies (44) and $E(J_{\mathfrak{m}}, 1) < \epsilon_0$, then we have

$$D_{p_0}(J_{\mathfrak{m}}, \theta_0) \leq \theta_0^{\tau}\big(D_{p_0}(J_{\mathfrak{m}}, 1) + \|Dg\|_{C^{2m-1}(B_1)}\big). \tag{49}$$

where $p_0 = \frac{4m}{3}$. The above claim is a direct consequence of Lemma 4 provided $\mathcal{P} \equiv 0$. Hence the key point is to prove that the inequality (38) in Lemma 4 still holds with additional nonlinear terms $\mathcal{P}$. We claim that, there holds

$$\|D\omega_{\mathcal{P}}\|_{L^{p_0}(B_1)} \leq C\big(\delta_0\|DJ_{\mathfrak{m}}\|_{L^{p_0}(B_1)} + E(J_{\mathfrak{m}}, 1)\|Dg\|_{C^{2m-1}(B_1)}\big), \tag{50}$$

where $\omega_{\mathcal{P}}(x) := \int_{\mathbb{R}^{2m}} G(x-y)\mathcal{P}(\widetilde{J}_{\mathfrak{m}})(y)dy$. We now turn to proving (50). Since

$$a(x)D^{2m}J_{\mathfrak{m}} = D^{2m-1}\big(a(x)DJ_{\mathfrak{m}}\big) + \text{Lower order terms},$$

and

$$\begin{aligned}
&\left\|D\int_{\mathbb{R}^{2m}} D^{2m-1}G(x-y)\widetilde{a}(y)D\widetilde{J}_{\mathfrak{m}}(y)dy\right\|_{L^{p_0}(\mathbb{R}^{2m})} \\
&\quad = \left\|\int_{\mathbb{R}^{2m}} D^{2m}G(x-y)\widetilde{a}(y)D\widetilde{J}_{\mathfrak{m}}(y)dy\right\|_{L^{p_0}(\mathbb{R}^{2m})} \\
&\quad \leq C\|\widetilde{a}(x)D\widetilde{J}_{\mathfrak{m}}(x)\|_{L^{p_0}(\mathbb{R}^{2m})} \\
&\quad \leq C\|a(x)\|_{L^{\infty}(B_1)}\|DJ_{\mathfrak{m}}(x)\|_{L^{p_0}(B_1)} \\
&\quad \leq C\,\delta_0\,\|DJ_{\mathfrak{m}}(x)\|_{L^{p_0}(B_1)},
\end{aligned}$$

it follows from estimates for lower order terms in Lemma 4 that (50) holds for the terms $a(x)D^{2m}J_{\mathfrak{m}}$ in (46). In the same manner, (50) also holds for the terms in (47). Hence the decay estimate (49) holds and it implies $J_{\mathfrak{m}} \in C^{0,\alpha}(B_1)$ for some $\alpha \in (0,1)$.

5 Higher Regularity for *m*-Harmonic Almost Complex Structures

We state the higher regularity results for a class of semilinear elliptic equations as a generalization in [5]. The proof follows essentially Proposition 7.1 in [5].

Theorem 3 *Suppose $n \geq 2m$ and $u \in C^{0,\mu} \cap W^{m,2}(B_1, \mathbb{R}^K)$ satisfies*

$$\Delta^m u = \Psi\left(x, \nabla u, \cdots, \nabla^{2m-1}u\right) \tag{51}$$

in distributional sense, where Ψ can be divided into two parts: the highest order terms H and lower order terms L, i.e., $\Psi = H + L$, which admit the following structures:

$$H = \sum_{k=0}^{m-1} \nabla^k \cdot g_k, \ \textit{where}\ |g_k| \le C \sum_{l=1}^{m} |\nabla^l u|^{\frac{2m-k}{l}},$$
$$L = \sum_{k=0}^{m-1} \nabla^k \cdot \widetilde{g}_k, \ \textit{where}\ |\widetilde{g}_k| \le C \sum_{\gamma} \left(\prod_i |\nabla^{\gamma_i} u| \right) \textit{for} \sum_i |\gamma_i| \le 2m - 1 - k.$$

Then, $u \in C^\infty(B_1, \mathbb{R}^K)$.

Proof Gastel and Scheven in [5] proved the theorem in the case $\Psi = H$. According to the proof of Proposition 7.1 in [5], it suffices to prove the following two claims in the case $\Psi = L$:

(1)
$$\sup_{B_\rho(x) \subset B_R} \rho^{2m-n-2\mu} \int_{B_\rho(x)} |\nabla^m u|^2 < \infty, \quad \forall\, 0 < R < 1, \tag{52}$$

(2) For every non-integer $\nu := [\nu] + \sigma \in (0, m)$, if $u \in C^{[\nu],\sigma}(B_1, \mathbb{R}^K)$ and

$$\sup_{B_\rho(x) \subset B_R} \rho^{2m-n-2\nu} \int_{B_\rho(x)} |\nabla^m u|^2 < \infty, \quad \forall\, 0 < R < 1, \tag{53}$$

then we have that, for $0 \le k \le m-1$ and $B_\rho(x) \subset B_R$, there holds

$$\left(\rho^{2m-n} \int_{B_\rho(x)} |\widetilde{g}_k|^{\frac{2m}{2m-k}} \right)^{\frac{2m-k}{2m}} \le C \rho^{\frac{m+1}{m} \nu} \tag{54}$$

Before proceeding to prove claims, we make some conventions: fix $R \in (0, 1)$, always assume $B_\rho(x) \subset B_R$, and C stand for the positive constants only dependent of $m, n, \|u\|_{C^{0,\mu}(B_R)}$.

We first prove the Claim (1) by standard integral estimates. Since $u \in C^{0,\mu}(B_1)$, we have

$$\|u - \overline{u}\|_{L^\infty(B_\rho(x))} \le C[u]_{\mu; B_R} \rho^\mu \le C\rho^\mu,$$

where $\overline{u} = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} u(y)$. To simplify the proof in the following estimate, we assume $\|u - \overline{u}\|_{L^\infty(B_\rho(x))} \le 1$.

By Gagliardo-Nirenberg interpolation inequality, we have that, for $1 \le l \le m-1$, there holds

$$\left(\rho^{2m-n}\int_{B_\rho(x)}|\nabla^l u|^{\frac{2m}{l}}\right)^{\frac{l}{2m}} \leq C\|u-\overline{u}\|_{L^\infty}^{1-\frac{l}{m}}\left(\rho^{2m-n}\int_{B_\rho(x)}|\nabla^m u|^2\right)^{\frac{l}{2m}} + C\|u-\overline{u}\|_{L^\infty}. \tag{55}$$

It follows that

$$\begin{aligned}\int_{B_\rho(x)}|\nabla^l u|^{\frac{2m}{l}} &\leq C\|u-\overline{u}\|_{L^\infty}^{\left(1-\frac{l}{m}\right)\frac{2m}{l}}\int_{B_\rho(x)}|\nabla^m u|^2 + C\rho^{n-2m}\|u-\overline{u}\|_{L^\infty}^{\frac{2m}{l}}\\ &\leq C\|u-\overline{u}\|_{L^\infty}^{\frac{2}{m}}\int_{B_\rho(x)}|\nabla^m u|^2 + C\rho^{n-2m}\|u-\overline{u}\|_{L^\infty}^2.\end{aligned} \tag{56}$$

On the other hand, by Hölder's inequality, it follows from (55) that, for $1 \leq l \leq m-1$ and $q \in [1, \frac{2m}{l}]$

$$\left(\rho^{lq-n}\int_{B_\rho(x)}|\nabla^l u|^q\right)^{\frac{1}{q}} \leq C\|u-\overline{u}\|_{L^\infty}^{1-\frac{l}{m}}\left(\rho^{2m-n}\int_{B_\rho(x)}|\nabla^m u|^2\right)^{\frac{l}{2m}} + C\|u-\overline{u}\|_{L^\infty}. \tag{57}$$

We choose a cut-off function $\eta \in C_0^\infty(B_\rho(x), [0, 1])$ such that

$$\eta|_{B_{\frac{\rho}{2}}(x)} \equiv 1 \quad \text{and} \quad \|\nabla^l \eta\|_{L^\infty} \leq C\rho^{-l}, \quad \forall l \in \mathbb{N}.$$

Testing (51) with $\eta^{2m}(u-\overline{u})$, we compute

$$\begin{aligned}\int \eta^{2m}|\nabla^m u|^2 dy &\leq C\sum_{k=0}^{m-1}\int|\nabla^m u|\cdot|\nabla^k(u-\overline{u})|\cdot|\nabla^{m-k}\eta^{2m}|\\ &\quad + C\sum_{k=0}^{m-1}\sum_{j=0}^{k}\int|\nabla^j(u-\overline{u})|\cdot|\nabla^{k-j}\eta^{2m}|\cdot|\widetilde{g}_k|\\ &=: \sum_{k=0}^{m-1} I_k + \sum_{k=0}^{m-1}\sum_{j=0}^{k} II_{kj}.\end{aligned}$$

Let $\epsilon_1, \epsilon_2 \in (0, 1)$ be constants to be chosen later. For I_0, we obtain

$$
\begin{aligned}
I_0 &\leq C \int |\nabla^m u| \cdot |u - \overline{u}| \cdot |\nabla^m \eta^{2m}| \\
&\leq C \rho^{\frac{n}{2} - m} \|u - \overline{u}\|_{L^\infty} \left(\int \eta^{2m} |\nabla^m u|^2 \right)^{\frac{1}{2}} \\
&\leq \epsilon_1 \int \eta^{2m} |\nabla^m u|^2 + C_{\epsilon_1} \rho^{n-2m} \|u - \overline{u}\|^2_{L^\infty} \\
&\leq \epsilon_1 \int \eta^{2m} |\nabla^m u|^2 + C_{\epsilon_1} \rho^{n-2m+2\mu}.
\end{aligned}
$$

For $1 \leq k \leq m - 1$, we obtain

$$
\begin{aligned}
I_k &\leq C \rho^{k-m} \int |\nabla^m u| \cdot |\nabla^k u| \cdot \eta^{m+k} \\
&\leq C \rho^{k-m} \left(\int \eta^{2m} |\nabla^m u|^2 \right)^{\frac{1}{2}} \left(\int \eta^{2k} |\nabla^k u|^2 \right)^{\frac{1}{2}} \\
&\leq \epsilon_1 \int \eta^{2m} |\nabla^m u|^2 + C_{\epsilon_1} \rho^{2k-2m} \int \eta^{2k} |\nabla^k u|^2 \\
&\leq \epsilon_1 \int \eta^{2m} |\nabla^m u|^2 \\
&\quad + C_{\epsilon_1} \rho^{2k-2m} \left(\epsilon_2 \rho^{2m-2k} \int_{B_\rho(x)} |\nabla^m u|^2 + C_{\epsilon_2} \rho^{n-2k} \|u - \overline{u}\|^2_{L^\infty} \right) \\
&\leq \epsilon_1 \int \eta^{2m} |\nabla^m u|^2 + \epsilon_2 C_{\epsilon_1} \int_{B_\rho(x)} |\nabla^m u|^2 + C_{\epsilon_1} C_{\epsilon_2} \rho^{n-2m} \|u - \overline{u}\|^2_{L^\infty} \\
&\leq C\epsilon_1 \int \eta^{2m} |\nabla^m u|^2 + \epsilon_2 C_{\epsilon_1} \int_{B_\rho(x)} |\nabla^m u|^2 + C_{\epsilon_1} C_{\epsilon_2} \rho^{n-2m+2\mu}
\end{aligned}
$$

where we use (57) with $q = 2$ and Young's inequality in the fourth inequality. Next, we estimate II_{00} as follows

$$
\begin{aligned}
II_{00} &\leq C \sum_\gamma \int |u - u| \cdot \eta^{2m} \cdot \prod_i |\nabla^{\gamma_i} u| \\
&\leq C \sum_\gamma \rho^{\frac{n}{p_0}} \cdot \|u - \overline{u}\|_{L^\infty} \prod_i \|\eta^{|\gamma_i|} \nabla^{\gamma_i} u\|_{L^{\frac{2m}{|\gamma_i|}}} \\
&\leq C \|u - \overline{u}\|_{L^\infty} \left(\rho^n + \sum_{\gamma_i \neq 0} \int \eta^{2m} |\nabla^{|\gamma_i|} u|^{\frac{2m}{|\gamma_i|}} \right) \\
&\leq C \|u - \overline{u}\|_{L^\infty} \left(\rho^n + \sum_{l=1}^m \int_{B_\rho(x)} |\nabla^l u|^{\frac{2m}{l}} \right)
\end{aligned}
$$

$$\begin{aligned}&\leq C\|u-\overline{u}\|_{L^\infty}\int_{B_\rho(x)}|\nabla^m u|^2+C\rho^n\|u-\overline{u}\|_{L^\infty}\\&\quad+C\left(\|u-\overline{u}\|_{L^\infty}^{1+\frac{2}{m}}\int_{B_\rho(x)}|\nabla^m u|^2+\rho^{n-2m}\|u-\overline{u}\|_{L^\infty}^3\right)\\&\leq C\rho^\mu\int_{B_\rho(x)}|\nabla^m u|^2+C\rho^{n-2m+2\mu}\end{aligned}$$

where we use (56) in the fifth inequality, and $1=\frac{1}{p_0}+\frac{1}{2m}\sum_i|\gamma_i|$. Note that, due to $\sum_i|\gamma_i|\leq 2m-1$, it follows that $p_0\in[1,2m]$. Similar arguments apply to $II_{k,0}$ and we obtain, for $1\leq k\leq m-1$,

$$II_{k0}\leq C\rho^\mu\int_{B_\rho(x)}|\nabla^m u|^2+C\rho^{n-2m+2\mu}.$$

By Hölder's inequality and Young's inequality, we have that, for $1\leq k\leq m-1$,

$$\begin{aligned}\int\eta^{2m}|\widetilde{g}_k|^{\frac{2m}{2m-k}}&\leq\epsilon_2\int_{B_\rho(x)}|\nabla^m u|^2+C_{\epsilon_2}\left(\rho^n+\sum_{l=1}^{m-1}\int_{B_\rho(x)}|\nabla^l u|^{\frac{2m}{l}}\right)\\&\leq\left(\epsilon_2+C_{\epsilon_2}\rho^{\frac{2\mu}{m}}\right)\int_{B_\rho(x)}|\nabla^m u|^2+C_{\epsilon_2}\rho^{n-2m+2\mu},\end{aligned}\tag{58}$$

where we apply (56) in second inequality. Now for $1\leq j<k\leq m-1$, we have

$$\begin{aligned}II_{kj}&\leq C\rho^{j-k}\int|\widetilde{g}_k|\cdot|\nabla^j u|\cdot\eta^{2m+j-k}\\&\leq C\rho^{j-k}\|\eta^j\nabla^j u\|_{L^{\frac{2m}{k}}}\|\eta^{2m-k}\widetilde{g}_k\|_{L^{\frac{2m}{2m-k}}}\\&\leq C\rho^{(j-k)\frac{2m}{k}}\int_{B_\rho(x)}|\nabla^j u|^{\frac{2m}{k}}+C\int\eta^{2m}|\widetilde{g}_k|^{\frac{2m}{2m-k}}\\&\leq C\left(\epsilon_2+C_{\epsilon_2}\rho^{\frac{2\mu}{m}}\right)\int_{B_\rho(x)}|\nabla^m u|^2+C_{\epsilon_2}\rho^{n-2m+2\mu}\end{aligned}$$

where we use (57) with $q=\frac{2m}{k}$ and (58) in the last inequality. Similarly, we obtain, for $1\leq k\leq m-1$

$$II_{kk}\leq(\epsilon_2+C_{\epsilon_2}\rho^{\frac{2\mu}{m}})\int_{B_\rho(x)}|\nabla^m u|^2+C_{\epsilon_2}\rho^{n-2m+2\mu}.$$

Combining above all estimates, we deduce that

$$\begin{aligned}\int\eta^{2m}|\nabla^m u|^2dy\leq C\epsilon_1\int\eta^{2m}|\nabla^m u|^2+C\left(\epsilon_2+C_{\epsilon_2}\rho^{\frac{2\mu}{m}}\right)\int_{B_\rho(x)}|\nabla^m u|^2\\+C_{\epsilon_1,\epsilon_2}\rho^{n-2m+2\mu}.\end{aligned}$$

Thus, by choosing $\epsilon_1, \epsilon_2, \rho_0$ small enough, we have that, for all $\rho \leq \rho_0$, there holds

$$\int_{B_{\frac{\rho}{2}}} |\nabla^m u|^2 \leq \varepsilon \int_{B_\rho} |\nabla^m u|^2 + C\rho^{n-2m+2\mu}$$

where $\varepsilon < 2^{2m-n-2\mu}$ is a fixed positive number. A standard iteration argument implies (52).

The task is now to prove Claim (2). Since $u \in C^{[\nu],\sigma}(B_1)$ with $\nu = [\nu] + \sigma$, we know that, there exists a Taylor polynomials P_x at the points x such that

$$\|u\|_{C^{[\nu],\sigma}(B_R)} \leq C < \infty, \quad \|u - P_x\|_{L^\infty(B_\rho(x))} \leq C\rho^\nu.$$

By Gagliardo-Nirenberg interpolation inequality and (53), we have that, for $\nu < l \leq m$, there holds

$$\begin{aligned}
&\left(\rho^{2m-n}\int_{B_\rho(x)} |\nabla^l u|^{\frac{2m}{l}}\right)^{\frac{l}{2m}} \\
&\leq C\|u - P_x\|_{L^\infty(B_\rho(x))}^{1-\frac{l}{m}}\left(\rho^{2m-n}\int_{B_\rho(x)} |\nabla^m u|^2\right)^{\frac{l}{2m}} + C\|u - P_x\|_{L^\infty(B_\rho(x))} \\
&\leq C\rho^\nu.
\end{aligned}$$

Let us compute

$$\begin{aligned}
\left(\rho^{2m-n}\int_{B_\rho(x)} |\widetilde{g}_k|^{\frac{2m}{2m-k}}\right)^{\frac{2m-k}{2m}} &\leq C\rho^{(2m-n)\frac{2m-k}{2m}} \cdot \rho^{\frac{n}{q_0}} \prod_{|\gamma_i|>\nu} \left\|\nabla^{|\gamma_i|}u\right\|_{L^{\frac{2m}{|\gamma_i|}}(B_\rho(x))} \\
&\leq C\rho^\tau
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{q_0} + \frac{1}{2m}\sum_{|\gamma_i|>\nu} |\gamma_i| &= \frac{2m-k}{2m}, \\
\tau &= (2m-n)\cdot\frac{2m-k}{2m} + \frac{n}{q_0} + \sum_{|\gamma_i|>\nu}\left(\nu + \frac{n-2m}{2m}|\gamma_i|\right). \qquad (59)
\end{aligned}$$

Combining above two identities yields

$$\begin{aligned}
\tau &= (2m-n)\cdot\frac{2m-k}{2m} + \frac{n}{q_0} + n_\nu\nu + (n-2m)\left(\frac{2m-k}{2m} - \frac{1}{q_0}\right) \\
&= \frac{2m}{q_0} + n_\nu\nu.
\end{aligned}$$

where $n_\nu = \left|\{\gamma_i : |\gamma_i| > \nu\}\right|$.

We claim that

$$\tau \geq \frac{m+1}{m}\nu. \tag{60}$$

which implies (54). Obviously, (60) holds for $n_\nu \geq 2$. For $n_\nu = 0$, (59) implies $\frac{1}{q_0} = \frac{2m-k}{2m}$. Hence, for $\nu \in (0, m)$

$$\tau = \frac{2m}{q_0} = 2m - k \geq m + 1 \geq \frac{m+1}{m}\nu.$$

For $n_\nu = 1$, (59) and the fact $k + \sum_i |\gamma_i| \leq 2m - 1$ imply $\frac{1}{q_0} \geq \frac{1}{2m}$. Hence, for $\nu \in (0, m)$,

$$\tau = \frac{2m}{q_0} + \nu \geq 1 + \nu \geq \frac{m+1}{m}\nu.$$

Thus, the claim (60) is proved. □

As a direct consequence, Theorem 3 implies the smoothness of weakly m-harmonic almost complex structures.

Corollary 1 *Suppose $n \geq 2m$ and $J \in C^{0,\alpha} \cap W^{m,2}$ is a weakly m-harmonic almost complex structure on (M^n, g). Then J is smooth.*

6 Appendix

In this section, we will rewrite m-harmonic almost complex structure equation in a good divergence form in the spirit of [2] to prove Lemma 5. First we have the following,

Lemma 6 *The Euler-Lagrange equation $[\Delta^m J, J] = 0$ is equivalent to*

$$\Delta^m J = \frac{1}{4} T_m \left(J, \nabla J, \cdots, \nabla^{2m-1} J\right) \tag{61}$$

where $T_m = J Q_m + Q_m J$ and

$$Q_m = \Delta^m \left(J^2\right) - \Delta^m J\, J - J\, \Delta^m J.$$

Proof This is a direct computation using the fact $\Delta(J^2) = 0$. □

Lemma 5 can be stated as follows,

Proposition 6 *For $m = 2, 3$, T_m in Lemma 6 can be rewritten as*

$$T_m = T_{\lambda_0} - \left[J - \lambda_0, \left[\Delta^m J, J\right]\right] \tag{62}$$

where T_{λ_0} is a linear combination of the following terms

$$\nabla^{\alpha}\Big((J-\lambda_0)*\nabla^{\beta}J*\nabla^{\gamma}J\Big) \quad or \quad \lambda_0*\nabla^{\alpha}\Big((J-\lambda_0)*\nabla^{\delta}J\Big),$$

where $\alpha, \beta, \gamma, \delta$ are multi-indices such that $1 \le |\alpha| \le 2m-1$, $0 \le |\beta|, |\gamma|, |\delta| \le m$, $|\alpha|+|\beta|+|\gamma| = 2m$ and $|\alpha|+|\delta| = 2m$.

In what follows, we always assume J is a square matrix valued function and satisfies $J^2 = -id$. In this situation, we know $\nabla\lambda_0 = 0$ for every constant matrix λ_0. The reason for emphasizing this point is that if we consider the constant matrix λ_0 as a (1,1) tensor field on (B_1, g), then $(1, 2)$ tensor field $\nabla\lambda_0$ might not be zero.

6.1 The case m=2: biharmonic almost complex structure

By the definition of T_m in Theorem 6, we have

$$T_2 = JQ_2 + Q_2J,$$

where $Q_2 = 2\nabla\Delta J\nabla J + 2\nabla J\nabla\Delta J + 2\Delta J\Delta J + 2\Delta(\nabla J)^2$. Set

$$\begin{aligned}
\mathbf{I} &= J\big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\big) + \big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\big)J \\
\mathbf{II} &= J(\Delta J)^2 + (\Delta J)^2 J \\
\mathbf{III} &= J\Delta(\nabla J)^2 + \Delta(\nabla J)^2 J.
\end{aligned}$$

Thus, we obtain $T_2 = 2\mathbf{I} + 2\mathbf{II} + 2\mathbf{III}$. Firstly, we compute the term $\mathbf{I}$:

$$\begin{aligned}
\mathbf{I} &= J\nabla\Delta J\nabla J + J\nabla J\nabla\Delta J + \nabla\Delta J\nabla JJ + \nabla J\nabla\Delta JJ \\
&= J\nabla\Delta J\nabla J - \nabla JJ\nabla\Delta J - \nabla\Delta JJ\nabla J + \nabla J\nabla\Delta JJ \\
&= [J, \nabla\Delta J]\nabla J + \nabla J[\nabla\Delta J, J] \\
&= [\nabla J, [\nabla\Delta J, J]].
\end{aligned}$$

Since $\nabla\Big([\nabla\Delta J, J] - [\Delta J, \nabla J]\Big) = [\Delta^2 J, J]$, we have

$$\begin{aligned}
&\nabla[J-\lambda_0, [\nabla\Delta J, J] - [\Delta J, \nabla J]] \\
&\quad = [\nabla J, [\nabla\Delta J, J] - [\Delta J, \nabla J]] + [J-\lambda_0, [\Delta^2 J, J]]. \qquad (63)
\end{aligned}$$

Now we compute the left-hand side of above equality:

$$\begin{aligned}
&\nabla[J-\lambda_0,[\nabla\Delta J,J]-[\Delta J,\nabla J]]\\
&\quad=\nabla[J-\lambda_0,\nabla[\Delta J,J]]-2\nabla[J-\lambda_0,[\Delta J,\nabla J]]\\
&\quad=\nabla[J-\lambda_0,\nabla[\Delta J,J]]+T_{\lambda_0}\\
&\quad=\Delta[J-\lambda_0,[\Delta J,J]]-\nabla[\nabla J,[\Delta J,J]]+T_{\lambda_0}\\
&\quad=-\nabla[\nabla J,[\Delta J,J]]+T_{\lambda_0}\\
&\quad=-[\Delta J,[\Delta J,J]]-[\nabla J,[\nabla\Delta J,J]]-[\nabla J,[\Delta J,\nabla J]]+T_{\lambda_0}
\end{aligned}$$

Substituting above equality into (63) yields

$$2\mathbf{I}=2[\nabla J,[\nabla\Delta J,J]]=-[\Delta J,[\Delta J,J]]-[J-\lambda_0,[\Delta^2 J,J]]+T_{\lambda_0} \tag{64}$$

We now turn to compute the term **III**. Since

$$\begin{aligned}
J\Delta(\nabla J)^2&=(J-\lambda_0)\Delta(\nabla J)^2+\lambda_0\Delta(\nabla J)^2\\
&=(J-\lambda_0)\Delta(\nabla J)^2+\lambda_0\Delta\nabla\big((J-\lambda_0)\nabla J\big)-\lambda_0\Delta\big((J-\lambda_0)\Delta J\big)\\
&=(J-\lambda_0)\Delta(\nabla J)^2+T_{\lambda_0}\\
&=\nabla_p\big((J-\lambda_0)\nabla_p(\nabla J)^2\big)-\nabla_p J\nabla_p(\nabla J)^2+T_{\lambda_0}\\
&=-\nabla_p J\nabla_p(\nabla J)^2+T_{\lambda_0}\\
&=-\nabla_p\big(\nabla_p J(\nabla J)^2\big)+\Delta J(\nabla J)^2+T_{\lambda_0}\\
&=\Delta J(\nabla J)^2+T_{\lambda_0}
\end{aligned}$$

and similarly $\Delta(\nabla J)^2 J=(\nabla J)^2\Delta J+T_{\lambda_0}$, we have

$$\mathbf{III}=\Delta J(\nabla J)^2+(\nabla J)^2\Delta J+T_{\lambda_0} \tag{65}$$

Now let us proceed to compute **II**:

$$\begin{aligned}
\mathbf{II}&=J(\Delta J)^2+(\Delta J)^2 J=-\big(\Delta J J+2(\nabla J)^2\big)\Delta J+\Delta J\Delta J J\\
&=\Delta J[\Delta J,J]-2(\nabla J)^2\Delta J
\end{aligned}$$

where we used the fact $\Delta(J^2)=0$ which implies

$$\Delta J J=-J\Delta J-2\nabla J\nabla J. \tag{66}$$

On the other hand, we also have

$$\begin{aligned}
\mathbf{II}&=J(\Delta J)^2+(\Delta J)^2 J=J\Delta J\Delta J-\Delta J\big(J\Delta J+2(\nabla J)^2\big)\\
&=[J,\Delta J]\Delta J-2\Delta J(\nabla J)^2
\end{aligned}$$

Hence, we obtain

$$\begin{aligned} 2\mathbf{II} &= \Delta J[\Delta J, J] + [J, \Delta J]\Delta J - 2\big((\nabla J)^2 \Delta J + \Delta J(\nabla J)^2\big) \\ &= [\Delta J, [\Delta J, J]] - 2(\nabla J)^2 \Delta J - 2\Delta J(\nabla J)^2 \\ &= [\Delta J, [\Delta J, J]] - 2\mathbf{III} \end{aligned} \tag{67}$$

where in the last equality we used (65). Substituting (67) into (64), we get

$$T_2 = 2\mathbf{I} + 2\mathbf{II} + 2\mathbf{III} = T_{\lambda_0} - \left[J - \lambda_0, \left[\Delta^2 J, J\right]\right]$$

which is the desired conclusion.

6.2 The Case m=3: 3-Harmonic Almost Complex Structure

By the definition of T_m in Theorem 6, we have

$$T_3 = JQ_3 + Q_3 J,$$

where

$$\begin{aligned} Q_3 = {} & 2\nabla\Delta^2 J\nabla J + 2\nabla J\nabla\Delta^2 J + \Delta^2 J\Delta J + \Delta J\Delta^2 J \\ & + 2\Delta\big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\big) + 2\Delta\left(\Delta J\right)^2 + 2\Delta^2\left(\nabla J\right)^2. \end{aligned}$$

For simplicity, we collect some terms which are T_{λ_0} type and appear frequently in the following proof.

Lemma 7 *The following terms are T_{λ_0} type terms for any given constant matrix λ_0:*

$$\nabla\left(\nabla J * \nabla^2 J * \nabla^2 J\right), \nabla^2\left(\nabla J * \nabla J * \nabla^2 J\right), \nabla\left(\nabla J * \nabla J * \nabla^3 J\right), \nabla^4\left(\nabla J\right)^2.$$

Proof For simplicity, we only show how to rewrite the first term and the third term. Other terms can be handled in much the same way. The first term:

$$\begin{aligned} \nabla\big(\nabla J * \nabla^2 J * \nabla^2 J\big) &= \nabla\left(\nabla\left(J - \lambda_0\right) * \nabla^2 J * \nabla^2 J\right) \\ &= \nabla^2\left(\left(J - \lambda_0\right) * \nabla^2 J * \nabla^2 J\right) - \nabla\left(\left(J - \lambda_0\right) * \nabla^3 J * \nabla^2 J\right) \\ &\quad - \nabla\left(\left(J - \lambda_0\right) * \nabla^2 J * \nabla^3 J\right) \\ &= T_{\lambda_0}. \end{aligned}$$

The third term:

$$
\begin{aligned}
&\nabla\big(\nabla J * \nabla J * \nabla^3 J\big) \\
&\quad = \nabla^2\big(\nabla J * \nabla J * \nabla^2 J\big) - \nabla\big(\nabla^2 J * \nabla J * \nabla^2 J\big) - \nabla\big(\nabla J * \nabla^2 J * \nabla^2 J\big) \\
&\quad = T_{\lambda_0}.
\end{aligned}
$$

□

Note that we will emphasize the terms of T_{λ_0} type by underlining it in the following proof. Set

$$
\begin{aligned}
\mathbf{I} &= J\nabla\Delta^2 J\nabla J + J\nabla J\nabla\Delta^2 J + \nabla\Delta^2 J\nabla J\, J + \nabla J\nabla\Delta^2 J\, J, \\
\mathbf{II} &= J\big(\Delta^2 J\Delta J + \Delta J\Delta^2 J\big) + \big(\Delta^2 J\Delta J + \Delta J\Delta^2 J\big)J, \\
\mathbf{III} &= J\Delta\big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\big) + \Delta\big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\big)J, \\
\mathbf{IV} &= J\Delta\,(\Delta J)^2 + \Delta\,(\Delta J)^2\, J, \\
\mathbf{V} &= J\Delta^2\,(\nabla J)^2 + \Delta^2\,(\nabla J)^2\, J.
\end{aligned}
$$

Then, we obtain $T_3 = 2\mathbf{I} + \mathbf{II} + 2\mathbf{III} + 2\mathbf{IV} + 2\mathbf{V}$.
Step One: dealing with I. Now Let us compute the first term **I**:

$$
\begin{aligned}
\mathbf{I} &= J\nabla\Delta^2 J\nabla J + J\nabla J\nabla\Delta^2 J + \nabla\Delta^2 J\nabla J J + \nabla J\nabla\Delta^2 J J \\
&= J\nabla\Delta^2 J\nabla J - \nabla J J\nabla\Delta^2 J - \nabla\Delta^2 J J\nabla J + \nabla J\nabla\Delta^2 J J \\
&= \left[J, \nabla\Delta^2 J\right]\nabla J + \nabla J\left[\nabla\Delta^2 J, J\right] \\
&= \left[\nabla J, \left[\nabla\Delta^2 J, J\right]\right],
\end{aligned}
$$

Since $\nabla\Big([\nabla\Delta^2 J, J] - [\Delta^2 J, \nabla J] + [\nabla\Delta J, \Delta J]\Big) = [\Delta^3 J, J]$, we have

$$
\begin{aligned}
&\nabla\left[J - \lambda_0, \left[\nabla\Delta^2 J, J\right] - \left[\Delta^2 J, \nabla J\right] + [\nabla\Delta J, \Delta J]\right] \\
&\quad = \left[\nabla J, \left[\nabla\Delta^2 J, J\right] - \left[\Delta^2 J, \nabla J\right] + [\nabla\Delta J, \Delta J]\right] + \left[J - \lambda_0, \left[\Delta^3 J, J\right]\right].
\end{aligned}
\tag{68}
$$

Now we compute the left-hand side of above equality.

$$
\begin{aligned}
&\nabla\left[J - \lambda_0, \left[\nabla\Delta^2 J, J\right] - \left[\Delta^2 J, \nabla J\right] + \underline{[\nabla\Delta J, \Delta J]}\right] \\
&\quad = \nabla\left[J - \lambda_0, \left[\nabla\Delta^2 J, J\right] - \left[\Delta^2 J, \nabla J\right]\right] + T_{\lambda_0} \\
&\quad = \nabla[J - \lambda_0, \nabla\left[\Delta^2 J, J\right] - 2\left[\Delta^2 J, \nabla J\right]] + T_{\lambda_0} \\
&\quad = \Delta\left[J - \lambda_0, [\Delta^2 J, J]\right] - \nabla\big[\nabla J, [\Delta^2 J, J]\big] - 2\nabla[J - \lambda_0, \left[\Delta^2 J, \nabla J\right]] + T_{\lambda_0}
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta\big[J-\lambda_0, \nabla[\nabla\Delta J, J] - \underline{[\nabla\Delta J, \nabla J]}\big]\\
&\quad - 2\nabla_p\big[J-\lambda_0, \nabla_q[\nabla_q\Delta J, \nabla_p J] - \underline{\left[\nabla_q\Delta J, \nabla^2_{qp} J\right]}\big]\\
&\quad -\nabla\left[\nabla J, \left[\Delta^2 J, J\right]\right] + T_{\lambda_0}\\
&= \underline{\Delta\nabla\left[J-\lambda_0, [\nabla\Delta J, J]\right]} - \Delta[\nabla J, [\nabla\Delta J, J]]\\
&\quad - \underline{2\nabla^2_{pq}\big[J-\lambda_0, [\nabla_q\Delta J, \nabla J]\big]} + 2\underline{\nabla_p[\nabla_q J, [\nabla_q\Delta J, \nabla_p J]]}\\
&\quad -\nabla[\nabla J, [\Delta^2 J, J]] + T_{\lambda_0}\\
&= -\Delta[\nabla J, [\nabla\Delta J, J]] - \nabla[\nabla J, [\Delta^2 J, J]] + T_{\lambda_0},
\end{aligned}
$$

where in the second equality from bottom we employ lemma 7. By substituting above equality into (68), we obtain

$$
\begin{aligned}
\mathbf{I} &= [\nabla J, [\nabla\Delta^2 J, J]]\\
&= [\nabla J, [\Delta^2 J, \nabla J]] - [\nabla J, [\nabla\Delta J, \Delta J]] - \Delta[\nabla J, [\nabla\Delta J, J]] - \nabla\left[\nabla J, \left[\Delta^2 J, J\right]\right]\\
&\quad + T_{\lambda_0} - \left[J-\lambda_0, \left[\Delta^3 J, J\right]\right].
\end{aligned}
$$

Since $\nabla[\nabla J, [\Delta^2 J, J]] = [\Delta J, [\Delta^2 J, J]] + [\nabla J, [\nabla\Delta^2 J, J]] + [\nabla J, [\Delta^2 J, \nabla J]]$, we deduce

$$
\begin{aligned}
2\mathbf{I} = &-[\nabla J, [\nabla\Delta J, \Delta J]] - \Delta[\nabla J, [\nabla\Delta J, J]] - \left[\Delta J, \left[\Delta^2 J, J\right]\right]\\
&+ T_{\lambda_0} - \left[J-\lambda_0, \left[\Delta^3 J, J\right]\right].
\end{aligned} \tag{69}
$$

By lemma 7, we can derive

$$
\begin{aligned}
&[\nabla J, [\nabla\Delta J, \Delta J]]\\
&= \nabla J\left(\nabla\Delta J\Delta J - \Delta J\nabla\Delta J\right) - \left(\nabla\Delta J\Delta J - \Delta J\nabla\Delta J\right)\nabla J\\
&= \nabla J\nabla\Delta J\Delta J + \Delta J\nabla\Delta J\nabla J - \nabla J\Delta J\nabla\Delta J - \nabla\Delta J\Delta J\nabla J\\
&= \underline{\nabla\left(\nabla J\Delta J\Delta J\right)} + \underline{\nabla\left(\Delta J\Delta J\nabla J\right)} - 2\left(\Delta J\right)^3 - 2\nabla J\Delta J\nabla\Delta J - 2\nabla\Delta J\Delta J\nabla J\\
&= -2\left(\Delta J\right)^3 - 2\nabla J\Delta J\nabla\Delta J - 2\nabla\Delta J\Delta J\nabla J + T_{\lambda_0}
\end{aligned} \tag{70}
$$

and

$$
\begin{aligned}
&\Delta[\nabla J, [\nabla\Delta J, J]]\\
&\quad = \Delta\big[\nabla J, \nabla[\Delta J, J] - [\Delta J, \nabla J]\big]\\
&\quad = \Delta[\nabla J, \nabla[\Delta J, J]] - \underline{\Delta[\nabla J, [\Delta J, \nabla J]]}\\
&\quad = \underline{\Delta\nabla[\nabla J, [\Delta J, J]]} - \Delta[\Delta J, [\Delta J, J]] + T_{\lambda_0}\\
&\quad = -\Delta\left(\left(\Delta J\right)^2 J + J\left(\Delta J\right)^2 - 2\Delta J J\Delta J\right) + T_{\lambda_0}
\end{aligned}
$$

$$= -\Delta\left(2(\Delta J)^2 J + 2J(\Delta J)^2 + 2\underline{(\nabla J)^2 \Delta J} + 2\underline{\Delta J (\nabla J)^2}\right) + T_{\lambda_0}$$
$$= -2\Delta\left((\Delta J)^2 J + J(\Delta J)^2\right) + T_{\lambda_0}. \tag{71}$$

where in the second equality from bottom we used (66). Substituting equalities (70) and (71) into equality (69) yields

$$2\mathbf{I} = 2(\Delta J)^3 + 2\nabla J \Delta J \nabla \Delta J + 2\nabla \Delta J \Delta J \nabla J + 2\Delta\left((\Delta J)^2 J + J(\Delta J)^2\right)$$
$$-\left[\Delta J, \left[\Delta^2 J, J\right]\right] + T_{\lambda_0} - \left[J - \lambda_0, \left[\Delta^3 J, J\right]\right]. \tag{72}$$

Step Two: dealing with V and II. Firstly, we deal with fifth term **V**. It follows from Lemma 7 that

$$\begin{aligned}
\mathbf{V} &= J\Delta^2(\nabla J)^2 + \Delta^2(\nabla J)^2 J \\
&= (J-\lambda_0)\Delta^2(\nabla J)^2 + \Delta^2(\nabla J)^2(J-\lambda_0) + \underline{\lambda_0\Delta^2(\nabla J)^2} + \underline{\Delta^2(\nabla J)^2\lambda_0} \\
&= \nabla\left((J-\lambda_0)\nabla\Delta(\nabla J)^2\right) - \nabla J\nabla\Delta(\nabla J)^2 \\
&\quad + \nabla\left(\nabla\Delta(\nabla J)^2(J-\lambda_0)\right) - \nabla\Delta(\nabla J)^2\nabla J + T_{\lambda_0} \\
&= \underline{\Delta\left((J-\lambda_0)\Delta(\nabla J)^2\right)} - \nabla\left(\nabla J\Delta(\nabla J)^2\right) - \nabla J\nabla\Delta(\nabla J)^2 \\
&\quad + \underline{\Delta\left(\Delta(\nabla J)^2(J-\lambda_0)\right)} - \nabla\left(\Delta(\nabla J)^2\nabla J\right) - \nabla\Delta(\nabla J)^2\nabla J + T_{\lambda_0} \\
&= -\nabla\left(\nabla J\Delta(\nabla J)^2\right) - \nabla J\nabla\Delta(\nabla J)^2 - \nabla\left(\Delta(\nabla J)^2\nabla J\right) \\
&\quad - \nabla\Delta(\nabla J)^2\nabla J + T_{\lambda_0}.
\end{aligned}$$

Since

$$\nabla_p\left(\nabla_p J\Delta(\nabla J)^2\right) = \nabla^2_{pq}\left(\nabla_p J\nabla_q(\nabla J)^2\right) - \nabla_p\left(\nabla^2_{qp} J\nabla_q(\nabla J)^2\right) = T_{\lambda_0}$$

and

$$\begin{aligned}
\nabla_p J\nabla_p\Delta(\nabla J)^2 &= \underline{\nabla_p\left(\nabla_p J\Delta(\nabla J)^2\right)} - \Delta J\Delta(\nabla J)^2 \\
&= -\underline{\nabla_p\left(\Delta J\nabla_p(\nabla J)^2\right)} + \nabla_p\Delta J\nabla_p(\nabla J)^2 + T_{\lambda_0} \\
&= \underline{\nabla_p\left(\nabla_p\Delta J(\nabla J)^2\right)} - \Delta^2 J(\nabla J)^2 + T_{\lambda_0} \\
&= -\Delta^2 J(\nabla J)^2 + T_{\lambda_0},
\end{aligned}$$

we have

$$\mathbf{V} = \Delta^2 J\,(\nabla J)^2 + (\nabla J)^2\,\Delta^2 J + T_{\lambda_0}. \tag{73}$$

Next, we deal with the second term

$$\begin{aligned}\mathbf{II} &= J\left(\Delta^2 J \Delta J + \Delta J \Delta^2 J\right) + \left(\Delta^2 J \Delta J + \Delta J \Delta^2 J\right) J \\ &= \left[\Delta J, \left[\Delta^2 J, J\right]\right] - 2\,(\nabla J)^2\,\Delta^2 J - 2\Delta^2 J\,(\nabla J)^2 \\ &= \left[\Delta J, \left[\Delta^2 J, J\right]\right] - 2\mathbf{V} + T_{\lambda_0},\end{aligned} \tag{74}$$

where we have used (66) and (73).

Step Three: dealing with III Here we begin to deal with the third term:

$$\begin{aligned}\mathbf{III} &= J\Delta\Big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\Big) + \Delta\Big(\nabla\Delta J\nabla J + \nabla J\nabla\Delta J\Big)J \\ &= J\Delta\left(\nabla\big(\Delta J\nabla J + \nabla J\Delta J\big) - 2\,(\Delta J)^2\right) \\ &\quad + \Delta\left(\nabla\big(\Delta J\nabla J + \nabla J\Delta J\big) - 2\,(\Delta J)^2\right) J \\ &= J\Delta\nabla\Big(\Delta J\nabla J + \nabla J\Delta J\Big) + \Delta\nabla\Big(\Delta J\nabla J + \nabla J\Delta J\Big)J \\ &\quad - 2\left(J\Delta\,(\Delta J)^2 + \Delta\,(\Delta J)^2\,J\right) \\ &= J\Delta\nabla\Big(\Delta J\nabla J + \nabla J\Delta J\Big) + \Delta\nabla\Big(\Delta J\nabla J + \nabla J\Delta J\Big)J - 2\mathbf{IV}.\end{aligned}$$

Since

$$\begin{aligned}&J\Delta\nabla\big(\Delta J\nabla J\big) \\ &= \big(J-\lambda_0\big)\Delta\nabla\,(\Delta J\nabla J) + \lambda_0\Delta\nabla\,(\Delta J\nabla J) \\ &= (J-\lambda_0)\,\Delta\nabla\,(\Delta J\nabla J) + \underline{\lambda_0\Delta\nabla\Big(\nabla\left(\Delta J\big(J-\lambda_0\big)\right) - \nabla\Delta J\big(J-\lambda_0\big)\Big)} \\ &= (J-\lambda_0)\,\Delta\nabla\big(\Delta J\nabla J\big) + T_{\lambda_0} \\ &= \nabla_p\Big((J-\lambda_0)\nabla^2_{pq}\big(\Delta J\nabla_q J\big)\Big) - \nabla_p J\nabla^2_{pq}\big(\Delta J\nabla_q J\big) + T_{\lambda_0} \\ &= \underline{\Delta\Big((J-\lambda_0)\nabla_q\big(\Delta J\nabla_q J\big)\Big)} - \underline{\nabla_p\Big(\nabla_p J\nabla_q\big(\Delta J\nabla_q J\big)\Big)} \\ &\quad - \nabla_p J\nabla^2_{pq}\big(\Delta J\nabla_q J\big) + T_{\lambda_0} \\ &= -\underline{\nabla_p\Big(\nabla_p J\nabla_q\big(\Delta J\nabla_q J\big)\Big)} + \Delta J\nabla_q\big(\Delta J\nabla_q J\big) + T_{\lambda_0}\end{aligned}$$

$$
\begin{aligned}
&= \underline{\nabla_q\Big(\Delta J \Delta J \nabla_q J\Big)} - \nabla \Delta J \Delta J \nabla J + T_{\lambda_0} \\
&= -\nabla \Delta J \Delta J \nabla J + T_{\lambda_0},
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathbf{III} &= -\nabla \Delta J\Big(\Delta J \nabla J + \nabla J \Delta J\Big) - \Big(\Delta J \nabla J + \nabla J \Delta J\Big)\nabla \Delta J - 2\mathbf{IV} + T_{\lambda_0} \\
&= -\nabla \Delta J \Delta J \nabla J - \nabla J \Delta J \nabla \Delta J - \Big(\nabla \Delta J \nabla J \Delta J + \Delta J \nabla J \nabla \Delta J\Big) \\
&\quad - 2\mathbf{IV} + T_{\lambda_0} \\
&= -\nabla \Delta J \Delta J \nabla J - \nabla J \Delta J \nabla \Delta J - \underline{\nabla\big(\Delta J \nabla J \Delta J\big)} + (\Delta J)^3 - 2\mathbf{IV} + T_{\lambda_0} \\
&= -\nabla \Delta J \Delta J \nabla J - \nabla J \Delta J \nabla \Delta J + (\Delta J)^3 - 2\mathbf{IV} + T_{\lambda_0}. \qquad (75)
\end{aligned}
$$

Step Four: dealing with IV Since

$$
\begin{aligned}
J \Delta (\Delta J)^2 &= \nabla_p \left(J \nabla_p (\Delta J)^2\right) - \nabla_p J \nabla_p (\Delta J)^2 \\
&= \Delta \left(J (\Delta J)^2\right) - \underline{\nabla_p \left(\nabla_p J (\Delta J)^2\right)} - \nabla_p J \nabla_p (\Delta J)^2 \\
&= \Delta \left(J (\Delta J)^2\right) - \underline{\nabla_p \left(\nabla_p J (\Delta J)^2\right)} + (\Delta J)^3 + T_{\lambda_0} \\
&= \Delta \left(J (\Delta J)^2\right) + (\Delta J)^3 + T_{\lambda_0},
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathbf{IV} &= J \Delta (\Delta J)^2 + \Delta (\Delta J)^2 J \\
&= \Delta\Big(J (\Delta J)^2 + (\Delta J)^2 J\Big) + 2 (\Delta J)^3 + T_{\lambda_0}. \qquad (76)
\end{aligned}
$$

Step Five: divergence forms of nonlinearity Combining the equalities (72), (74), (75) and (76), we derive that

$$
2\mathbf{I} + \mathbf{II} + 2\mathbf{III} + 2\mathbf{IV} + 2\mathbf{V} = T_{\lambda_0},
$$

which completes the proof.

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